

GONALITY OF DYNATOMIC CURVES AND STRONG UNIFORM BOUNDEDNESS OF PREPERIODIC POINTS

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ABSTRACT. Fix $d \geq 2$ and a field k such that $\text{char } k \nmid d$. Assume that k contains the d th roots of 1. Then the irreducible components of the curves over k parameterizing preperiodic points of polynomials of the form $z^d + c$ are geometrically irreducible and have gonality tending to ∞ . This implies the function field analogue of the strong uniform boundedness conjecture for preperiodic points of $z^d + c$. It also has consequences over number fields: it implies strong uniform boundedness for preperiodic points of bounded eventual period, which in turn reduces the full conjecture for preperiodic points to the conjecture for periodic points.

1. INTRODUCTION

1.1. **Dynatomic curves.** Fix an integer $d \geq 2$. Let k be a field such that $\text{char } k \nmid d$. View $f = f_c := z^d + c$ as a polynomial in z with coefficients in $k[c]$. Let $f^n(z)$ be the n th iterate of f ; in particular, $f^0(z) := z$. If n and m are nonnegative integers with $n > m$, then any irreducible factor of $f^n(z) - f^m(z) \in k[z, c]$ defines an affine curve over k . By a **dynatomic curve** over k , we mean any such curve, or its smooth projective model. Any k -point on such a curve yields $c_0 \in k$ equipped with a **preperiodic point** in k , that is, an element $z_0 \in k$ that under iteration of $x^d + c_0$ eventually enters a cycle; the length of the cycle is called the **eventual period**. We consider two dynatomic curves to be different if the corresponding closed subschemes of \mathbb{A}_k^2 are distinct. Section 2 describes all dynatomic curves in characteristic 0 explicitly.

1.2. **Gonality.** Let \bar{k} be an algebraic closure of k . Let $\mu_d = \{x \in \bar{k} : x^d = 1\}$.

For a curve X over k , let $X_{\bar{k}} = X \times_k \bar{k}$. If X is irreducible, define the **gonality** $\gamma(X)$ of X as the least possible degree of a dominant rational map $X \dashrightarrow \mathbb{P}_k^1$. If X is geometrically irreducible, define its **\bar{k} -gonality** as $\gamma(X_{\bar{k}})$.

The following theorem, and its consequence, Theorem 1.7, are our main results.

Theorem 1.1. *Fix $d \geq 2$ and k such that $\text{char } k \nmid d$. Suppose that $\mu_d \subset k$.*

- (a) *Every dynatomic curve over k is geometrically irreducible.*
- (b) *If the dynatomic curves over k are listed in any order, their gonalitys tend to ∞ .*

Remark 1.2. Part (a) of Theorem 1.1 can fail if $\mu_d \not\subset k$. See Remark 2.2.

Remark 1.3. In proving Theorem 1.1 in positive characteristic, we face the challenge that we do not know explicitly what the dynatomic curves are, since we are not sure whether

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the known factors of the polynomials $f^n(z) - f^m(z)$ are irreducible. Some partial results regarding the irreducibility of dynatomic curves in positive characteristic may be found in [DKO⁺17], but we will not use them. Instead, to overcome the difficulties, we use a novel argument that is specific to finite fields to prove that the degrees of dynatomic curves over the c -line must tend to infinity even though we do not know precisely what these curves are; see Section 4.

Let us now introduce notation for our next result. Let $\mu(n)$ denote the Möbius μ -function. Then $f^n(z) - z = \prod_{e|n} \Phi_e(z, c)$, where

$$(1) \quad \Phi_n(z, c) := \prod_{e|n} (f^e(z) - z)^{\mu(n/e)} \in k[z, c].$$

Let $Y_1^{\text{dyn}}(n)$ be the curve defined by $\Phi_n(z, c) = 0$ in \mathbb{A}_k^2 , and let $X_1^{\text{dyn}}(n)$ be the normalization of its projective closure. To simplify notation, we omit the superscript dyn from now on. General points of $X_1(n)$ parametrize polynomials of the form $z^d + c$ equipped with a point of exact order n .

The morphism $(z, c) \mapsto (f(z), c)$ restricts to an order n automorphism of $Y_1(n)$, so it induces an order n automorphism σ of $X_1(n)$. The quotient of $X_1(n)$ by the cyclic group generated by σ is called $X_0(n)$. If $\text{char } k = 0$, it is known that $X_1(n)$ is geometrically irreducible, so $X_0(n)$ is too.

Theorem 1.4. *Fix $d \geq 2$ and a field k of characteristic 0. Then*

$$\gamma(X_0(n)) > \left(\frac{1}{2} - \frac{1}{2d} - o(1) \right) n$$

as $n \rightarrow \infty$. In particular, $\gamma(X_0(n)) \rightarrow \infty$.

Remark 1.5. Our definition of dynatomic curve does not include quotient curves such as $X_0(n)$, so the conclusion $\gamma(X_0(n)) \rightarrow \infty$ of Theorem 1.4 does not follow from Theorem 1.1(b). In fact, our logic runs in the opposite direction: we use Theorem 1.4 in the proof of the characteristic 0 case of Theorem 1.1(b).

Remark 1.6. Although the lower bound in Theorem 1.4 is linear in n , the best upper bound we know, $\gamma(X_0(n)) \leq (1 + o(1))d^n/n$ (see Proposition 3.1(b)), is exponential in n .

To prove Theorem 1.4, we use that $X_0(n)$ already has a morphism to \mathbb{P}^1 of degree lower than expected for its genus, namely $X_0(n) \xrightarrow{c} \mathbb{P}^1$. If it also had a morphism to \mathbb{P}^1 of bounded degree, then the Castelnuovo–Severi inequality would make the genus of $X_0(n)$ smaller than it actually is, a contradiction. See Section 3 for details.

To prove Theorem 1.1(b), we use different arguments in characteristic 0 and characteristic > 0 .

In characteristic 0, we use that each dynatomic curve dominates $X_1(n)$ and hence also $X_0(n)$ for some n , so by Theorem 1.4 its gonality is large when n is large; this lets us reduce to proving a gonality lower bound for the dynatomic curves above $X_1(n)$ for each fixed n . The latter curves for fixed n come in towers and we use the Castelnuovo–Severi inequality to work our way up each tower. See Section 3.

In characteristic p , we prove that the irreducible components of $f^n(z) - f^m(z)$ have large degree over the c -line, and we use that to prove that over the finite field $\mathbb{F}_q := \mathbb{F}_p(\mu_d)$ their

smooth projective models have so many \mathbb{F}_q -points over $c = \infty$ that their \mathbb{F}_q -gonalities must be large. Finally, we use a result controlling how gonality of a curve changes when the base field is enlarged. See Section 4.

1.3. Uniform boundedness of preperiodic points. The growth of gonality of classical modular curves implies the strong uniform boundedness theorem for torsion points on elliptic curves over function fields (the function field analogue of Merel’s theorem [Mer96]); see [NS96, Theorem 0.3]. Similarly, from Theorem 1.1 we will deduce the following function field analogue of a case of the Morton–Silverman conjecture [MS94, p. 100]:

Theorem 1.7 (Strong uniform boundedness theorem for preperiodic points over function fields). *Fix $d \geq 2$ and a field k such that $\text{char } k \nmid d$. Let K be the function field of an integral curve over k . Fix a positive integer D . Then there exists $B = B(d, K, D) > 0$ such that for every field extension $L \supseteq K$ of degree $\leq D$ and every $c \in L$ not algebraic over k , the number of preperiodic points of $z^d + c$ in L is at most B . If k is finite, the same holds with the words “not algebraic over k ” deleted.*

As far as we know, Theorem 1.7 is the first theorem proving strong uniform boundedness of preperiodic points for all members of a nontrivial algebraic family of maps over a global field. See [DKO⁺17, Section 4] for some related results and arguments.

Another application of Theorem 1.1 is the following, which over *number fields* provides a uniform bound on preperiodic points having a bounded eventual period:

Theorem 1.8. *Fix integers $d \geq 2$, $D \geq 1$, and $N \geq 1$. Then there exists $B = B(d, D, N) > 0$ such that for every number field K satisfying $[K : \mathbb{Q}] \leq D$ and every $c \in K$, the number of preperiodic points of $z^d + c$ in K with eventual period at most N is at most B .*

Theorems 1.7 and 1.8 are proved in Section 5. Theorem 1.8 implies that the strong uniform boundedness conjecture for *periodic* points over number fields implies the strong uniform boundedness conjecture for *preperiodic* points over number fields, as we now explain:

Corollary 1.9. *Fix integers $d \geq 2$ and $D \geq 1$. Suppose that there exists a bound $N = N(d, D)$ such that for every number field K satisfying $[K : \mathbb{Q}] \leq D$ and every $c \in K$, every periodic point of $z^d + c$ in K has period at most N . Then there exists a bound $B' = B'(d, D)$ such that for every number field K satisfying $[K : \mathbb{Q}] \leq D$ and every $c \in K$, the number of preperiodic points of $z^d + c$ in K is at most B' .*

Proof. By assumption, if $[K : \mathbb{Q}] \leq D$ and $c \in K$, then the preperiodic points of $z^d + c$ in K having eventual period at most N are *all* the preperiodic points in K . Therefore the bound $B(d, D, N)$ of Theorem 1.8 is actually a bound on the *total* number of preperiodic points in K . Take $B' = B(d, D, N) = B(d, D, N(d, D))$. \square

2. CLASSIFICATION OF DYNATOMIC CURVES

For $m, n \geq 1$, let $Y_1(m, n)$ be the curve over k whose general points parametrize polynomials $z^d + c$ equipped with a preperiodic point that after exactly m steps enters an n -cycle. This curve is the zero locus in \mathbb{A}_k^2 of the polynomial

$$\Phi_{m,n}(z, c) := \frac{\Phi_n(f^m(z), c)}{\Phi_n(f^{m-1}(z), c)}.$$

For a general point $(z, c) \in Y_1(1, n)$, the elements z and $f^n(z)$ are distinct preimages of $f(z)$, so $z = \zeta f^n(z)$ for some $\zeta \in \mu_d - \{1\}$. Suppose that $\mu_d \subseteq k$. For each $\zeta \in \mu_d - \{1\}$, let $Y_1(1, n)^\zeta$ be the subscheme of $Y_1(1, n)$ defined by the condition $z = \zeta f^n(z)$, so

$$Y_1(1, n) = \bigcup_{\zeta \in \mu_d - \{1\}} Y_1(1, n)^\zeta.$$

Both $(z, c) \mapsto (f(z), c)$ and $(z, c) \mapsto (\zeta^{-1}z, c)$ define isomorphisms $Y_1(1, n)^\zeta \rightarrow Y_1(n)$. In particular, $Y_1(1, n)^\zeta$ equals the curve $\Phi_n(\zeta^{-1}z, c) = 0$ in \mathbb{A}_k^2 . For $m \geq 2$, let $Y_1(m, n)^\zeta$ be the inverse image of $Y_1(1, n)^\zeta$ under

$$\begin{aligned} Y_1(m, n) &\longrightarrow Y_1(1, n) \\ (z, c) &\longmapsto (f^{m-1}(z), c). \end{aligned}$$

Then for any $m, n \geq 1$,

$$(2) \quad Y_1(m, n) = \bigcup_{\zeta \in \mu_d - \{1\}} Y_1(m, n)^\zeta,$$

and $Y_1(m, n)^\zeta$ equals the curve $\Phi_n(\zeta^{-1}f^{m-1}(z), c) = 0$ in \mathbb{A}_k^2 . The decomposition (2) corresponds to a factorization

$$(3) \quad \Phi_{m,n}(z, c) = \prod_{\zeta \in \mu_d - \{1\}} \Phi_n(\zeta^{-1}f^{m-1}(z), c).$$

The following is a collection of results from [Bou92, LS94, Mor96, Gao16].

Theorem 2.1. *Let k be a field of characteristic 0 such that $\mu_d \subset k$. Then the curves $Y_1(n)$ for $n \geq 1$ and the curves $Y_1(m, n)^\zeta$ for $m, n \geq 1$ and $\zeta \in \mu_d - \{1\}$ are irreducible, so they are all the dynatomic curves over k .*

Computer experiments (at least for $d = 2$) suggest that Theorem 2.1 holds for any field k such that $\text{char } k \nmid d$ and $\mu_d \subset k$, but in positive characteristic this remains unproved. See [DKO⁺17], especially Theorems B and D, for some progress in this direction.

Remark 2.2. If in Theorem 2.1 we drop the hypothesis that $\mu_d \subset k$, then the irreducible components of $Y_1(m, n)$ over k are in bijection with the $\text{Gal}(k(\mu_d)/k)$ -orbits in $\mu_d - \{1\}$. An irreducible component corresponding to an orbit of size greater than 1 is not geometrically irreducible.

3. GONALITY IN CHARACTERISTIC 0

Given a geometrically irreducible curve X over k , let $g(X)$ denote the genus of its smooth projective model. Let $D_0(n)$ be the degree of the morphism $X_0(n) \xrightarrow{c} \mathbb{P}^1$. Similarly, let $D_1(n) = \deg \left(X_1(n) \xrightarrow{c} \mathbb{P}^1 \right)$.

Proposition 3.1. *Fix $d \geq 2$, and fix a field k of characteristic 0.*

- (a) *We have $D_1(n) = (1 + o(1))d^n$ as $n \rightarrow \infty$.*
- (b) *We have $D_0(n) = (1 + o(1))d^n/n$ as $n \rightarrow \infty$.*
- (c) *We have $g(X_0(n)) > \left(\frac{1}{2} - \frac{1}{2d} - o(1)\right) d^n$ as $n \rightarrow \infty$.*
- (d) *The Galois group of (the Galois closure of) the covering $X_0(n) \xrightarrow{c} \mathbb{P}^1$ is the full symmetric group $S_{D_0(n)}$.*

Proof. We may assume $k = \mathbb{C}$.

- (a) See the proof of [Mor96, Theorem 13(d)]. The estimate is deduced from $D_1(n) = \deg_z \Phi_n = \sum_{e|n} \mu(n/e)d^e$, which follows from (1).
- (b) The first morphism in the tower $X_1(n) \rightarrow X_0(n) \xrightarrow{c} \mathbb{P}^1$ has degree n , so $D_0(n) = D_1(n)/n$. Substitute (a) into this.
- (c) More precisely, $g(X_0(n)) > \left(\frac{1}{2} - \frac{1}{2d} - \frac{1}{n}\right) d^n + O(nd^{n/2})$ as $n \rightarrow \infty$, by [Mor96, Theorem 13(d)].
- (d) This is a consequence of work of Bousch [Bou92] for $d = 2$, and Lau and Schleicher [LS94] for $d > 2$. See also [Mor98, Theorem B] and [Sch17]. \square

A well-known strategy for obtaining lower bounds on gonality (cf. [NS96] and [Poo07, §2]) involves the following, which will tell us roughly that if a high genus curve has a relatively low degree map to a low genus curve, it cannot have a second such map that is independent of the first.

Proposition 3.2 (Castelnuovo–Severi inequality). *Let F, F_1, F_2 be function fields of curves over k , of genera g, g_1, g_2 , respectively. Suppose that $F_i \subseteq F$ for $i = 1, 2$ and the compositum of F_1 and F_2 in F equals F . Let $d_i = [F : F_i]$ for $i = 1, 2$. Then*

$$g \leq d_1 g_1 + d_2 g_2 + (d_1 - 1)(d_2 - 1).$$

Proof. See [Sti93, III.10.3]. \square

Proof of Theorem 1.4. Let $X_0(n) \xrightarrow{h} \mathbb{P}^1$ be a dominant rational map of minimal degree.
Case I: h factors through $X_0(n) \xrightarrow{c} \mathbb{P}^1$. Then

$$\deg h \geq D_0(n) = (1 + o(1)) \frac{d^n}{n}$$

by Proposition 3.1(b), so $\deg h$ is much larger than n when n is large.

Case II: h does not factor through $X_0(n) \xrightarrow{c} \mathbb{P}^1$. Then the compositum of $k(c)$ and $k(h)$ in the function field $k(X_0(n))$ is strictly larger than $k(c)$. Because of the Galois group (Proposition 3.1(d)), the only nontrivial extension of $k(c)$ in $k(X_0(n))$ is the whole field $k(X_0(n))$. Thus $k(c)$ and $k(h)$ generate $k(X_0(n))$. By Proposition 3.2,

$$g(X_0(n)) \leq (D_0(n) - 1)(\deg h - 1).$$

Thus

$$\deg h \geq 1 + \frac{g(X_0(n))}{D_0(n) - 1} = \left(\frac{1}{2} - \frac{1}{2d} - o(1)\right) n$$

as $n \rightarrow \infty$, by Proposition 3.1(b,c). \square

The following lemma, which says in particular that $Y_1(m, n)^\zeta \xrightarrow{z} \mathbb{A}^1$ is étale above 0, will yield genus inequalities to combine with the Castelnuovo–Severi inequality in the proof of Theorem 1.1.

Lemma 3.3. *Let k be a field of characteristic 0 such that $\mu_d \subseteq k$. Let m and n be positive integers, and let $\zeta \in \mu_d - \{1\}$. Then the polynomial $\Phi_n(\zeta^{-1} f^{m-1}(0), c) \in k[c]$ has only simple roots, and their number is $d^{m-2} D_1(n)$ if $m \geq 2$.*

Proof. First suppose that n does not divide $m - 1$. In this case, the roots of $\Phi_{m,n}(0, c)$ are distinct by [HT15, Theorem 1.1], and therefore the roots of $\Phi_n(\zeta^{-1}f^{m-1}(0), c)$ are distinct by (3).

Now suppose that n divides $m - 1$. By [Eps12, Proposition A.1], the roots of $\Phi_n(0, c)$ are simple. If c is such a root, then the polynomial $f = f_c$ satisfies $f^n(0) = 0$, so $f^{m-1}(0) = 0$ and $\Phi_n(\zeta^{-1}f^{m-1}(0), c) = 0$. Thus $\Phi_n(0, c)$ divides $\Phi_n(\zeta^{-1}f^{m-1}(0), c)$. The factorization (3) yields

$$(4) \quad \frac{\Phi_{m,n}(0, c)}{\Phi_n(0, c)^{d-1}} = \prod_{\zeta \in \mu_{d-1}} \frac{\Phi_n(\zeta^{-1}f^{m-1}(0), c)}{\Phi_n(0, c)}.$$

By [HT15, Theorem 1.1], $\Phi_{m,n}(0, c)/\Phi_n(0, c)^{d-1}$ has only simple roots, none of which are also roots of $\Phi_n(0, c)$. Combining this with (4) shows that $\Phi_n(\zeta^{-1}f^{m-1}(0), c)$ has only simple roots.

It remains to prove $\deg \Phi_n(\zeta^{-1}f^{m-1}(0), c) = d^{m-2}D_1(n)$. In fact, this is [Gao16, Lemma 4.8]. In our notation, the argument is as follows. By induction on m , the degree of the polynomial $f^{m-1}(0) \in k[c]$ is d^{m-2} if $m \geq 2$. Hence, by induction on e , we have $\deg f^e(\zeta^{-1}f^{m-1}(0)) = d^e d^{m-2}$ for each $e \geq 0$. Thus the c -degree of $f^e(\zeta^{-1}f^{m-1}(0)) - \zeta^{-1}f^{m-1}(0)$ is d^{m-2} times the z -degree of $f^e(z) - z$ for each $e \geq 1$. Substituting $\zeta^{-1}f^{m-1}(0)$ for z in (1) shows that

$$\deg_c \Phi_n(\zeta^{-1}f^{m-1}(0), c) = d^{m-2} \deg_z \Phi_n(z, c) = d^{m-2}D_1(n). \quad \square$$

Proof of Theorem 1.1 in characteristic 0.

- (a) By Theorem 2.1, the dynatomic curves over k are the curves $Y_1(n)$ and $Y_1(m, n)^\zeta$, and they are geometrically irreducible.
- (b) The curves $Y_1(n)$ and $Y_1(m, n)^\zeta$ dominate $X_0(n)$, so their gonality is at least the gonality of $X_0(n)$, by [Poo07, Proposition A.1(vii)]. In light of Theorem 1.4, it remains to prove that in each tower

$$\cdots \longrightarrow Y_1(m, n)^\zeta \longrightarrow Y_1(m-1, n)^\zeta \longrightarrow \cdots \longrightarrow Y_1(1, n)^\zeta$$

for fixed n and ζ , the gonality tends to ∞ as $m \rightarrow \infty$.

Let $g_m = g(Y_1(m, n)^\zeta)$ and $\gamma_m = \gamma(Y_1(m, n)^\zeta)$. Let $Y_1(m, n)^\zeta \xrightarrow{h} \mathbb{P}^1$ be a dominant rational map of the minimal degree γ_m .

Case I: h factors through a curve Z with $k(Y_1(m-1, n)^\zeta) \subseteq k(Z) \subsetneq k(Y_1(m, n)^\zeta)$. These inclusions imply $\gamma_{m-1} \leq \gamma(Z)$ (by [Poo07, Proposition A.1(vii)]) and $2\gamma(Z) \leq \deg h = \gamma_m$, respectively. Combining these yields $\gamma_m \geq 2\gamma_{m-1}$. Thus γ_m grows, by induction on m .

Case II: h does not factor through any such curve Z . Denote by π_m the degree d map

$$\begin{aligned} \pi_m: Y_1(m, n)^\zeta &\longrightarrow Y_1(m-1, n)^\zeta \\ (z, c) &\longmapsto (f(z), c), \end{aligned}$$

and let $\tilde{\pi}_m: X_1(m, n)^\zeta \rightarrow X_1(m-1, n)^\zeta$ be its extension to the smooth projective models. Let R_m be the ramification divisor of $\tilde{\pi}_m$.

Applying Proposition 3.2 to π_m and h yields

$$g_m \leq dg_{m-1} + (d-1)(\gamma_m - 1),$$

so it suffices to show that $g_m - dg_{m-1} \rightarrow \infty$ as $m \rightarrow \infty$. By Riemann–Hurwitz, this is equivalent to showing that $\deg R_m \rightarrow \infty$ as $m \rightarrow \infty$. In the fiber product diagram

$$\begin{array}{ccc} Y_1(m, n)^\zeta & \xrightarrow{\pi_m} & Y_1(m-1, n)^\zeta \\ z \downarrow & & \downarrow z \\ \mathbb{A}^1 & \xrightarrow{f} & \mathbb{A}^1, \end{array}$$

both vertical morphisms are étale above 0 by Lemma 3.3, while f has ramification index d at 0, so $\tilde{\pi}^m$ has ramification index d at each point of $Y_1(m, n)^\zeta$ where $z = 0$. For $m \geq 2$, the number of such points is $d^{m-2}D_1(n)$ by Lemma 3.3. Thus $\deg R_m \geq (d-1)d^{m-2}D_1(n)$, which tends to ∞ as $m \rightarrow \infty$. \square

4. GONALITY IN CHARACTERISTIC p

4.1. Reduction to the case of a finite field. Let $\mathbb{F}_q = \mathbb{F}_p(\mu_d)$.

Lemma 4.1. *Theorem 1.1 for \mathbb{F}_q implies Theorem 1.1 for any characteristic p field k containing μ_d .*

Proof. Theorem 1.1(a) for \mathbb{F}_q implies that the dynatomic curves over k are just the base extensions of the dynatomic curves X over \mathbb{F}_q , and that they are geometrically irreducible too. We will prove in Section 4.4 that the smooth projective model of each X has an \mathbb{F}_q -point. Then Theorem 2.5(iii) and Proposition A.1(ii) of [Poo07] imply $\gamma(X_k) \geq \sqrt{\gamma(X)}$, which implies Theorem 1.1(b) for k . \square

4.2. Symbolic dynamics. In order to prove the finiteness of the set of dynatomic curves of bounded degree over the c -line, and to prove that the dynatomic curves of higher degree have many \mathbb{F}_q -points above $c = \infty$, we need to analyze the splitting above $c = \infty$ in dynatomic curves. This splitting is given in Lemma 4.2 below. The symbolic dynamics approach yielding Lemma 4.2 is standard (cf. [Mor96, Lemma 1]), but the ways that Lemma 4.2 will be applied in subsequent sections are not standard.

View $f(z)$ as a polynomial in z over the local field $\mathbb{F}_q((c^{-1}))$. Normalize the valuation v on $\mathbb{F}_q((c^{-1}))$ so that $v(c^{-1}) = 1$, and extend v to an algebraic closure. Let $t = c^{-1/d}$, so $\mathbb{F}_q((t))$ is a degree d totally tamely ramified extension of $\mathbb{F}_q((c^{-1}))$.

Lemma 4.2.

- (a) *For any nonnegative integers $n > m$, the polynomial $f^n(z) - f^m(z)$ over $\mathbb{F}_q(c)$ is separable and splits completely over $\mathbb{F}_q((t))$.*
- (b) *Each zero of $f^n(z) - f^m(z)$ has valuation $-1/d$ and generates $\mathbb{F}_q((t))$ over $\mathbb{F}_q((c^{-1}))$.*

Proof. The ideas in the following argument are well known; cf. [Mor96, Lemma 1].

- (a) For each d th root of $-c$, interpreting $(-c + z)^{1/d}$ as $(-c)^{1/d}(1 - c^{-1}z)^{1/d}$ and expanding $(1 - c^{-1}z)^{1/d}$ in a binomial series defines a branch of the inverse of $z^d + c$ on the open disk $D := \{z \in \mathbb{F}_q((t)) : v(z) > v(c)\}$. Taking the derivative of $(-c + z)^{1/d}$ shows that each branch is a contracting map $D \rightarrow D$. These branches have disjoint images, each a smaller open disk around a different d th root of $-c$. Let S be the set of these d functions. Each finite sequence of elements of S defines a composition of functions, and for each m , the images of the different m -fold compositions are disjoint open disks. For each infinite

sequence s_1, s_2, \dots of elements of S , the images of $s_1 \cdots s_m$ for $m \geq 1$ are nested open disks whose radii tend to 0, so they have a unique point in their intersection; denote it $[s_1 s_2 \cdots]$. Any two distinct infinite sequences yield two points in disjoint disks, so these points are distinct. Since $f \circ s_1$ is the identity, f maps $[s_1 s_2 \cdots]$ to $[s_2 s_3 \cdots]$. For fixed nonnegative integers $n > m$, any n -long sequence s_1, \dots, s_n in S extends uniquely to an infinite sequence (s_i) satisfying $s_{i+n} = s_{i+m}$ for all $i \geq 1$, and then $[s_1 s_2 \cdots]$ is a zero of $f^n(z) - f^m(z)$. There are d^n of these, so they are *all* the zeros. In particular, these zeros are distinct elements of $\mathbb{F}_q((t))$. This implies that $f^n(z) - f^m(z)$ is separable.

- (b) The image of each $s \in S$ consists of elements of valuation exactly $-1/d$. Thus each element $[s_1 s_2 \cdots]$ has valuation $-1/d$. In particular, each zero of $f^n(z) - f^m(z)$ generates an extension field of $\mathbb{F}_q((c^{-1}))$ of ramification index divisible by d ; this extension field can only be the whole field $\mathbb{F}_q((t))$. \square

4.3. Dynatonic curves of low degree. It is here that we use the trick requiring the ground field to be finite.

Lemma 4.3. *For each $e \geq 1$, the set of dynatonic curves X over \mathbb{F}_q such that $\deg(X \xrightarrow{c} \mathbb{P}^1) = e$ is finite.*

Proof. Suppose that $Q(z) = \sum_{r=0}^e q_r z^{e-r}$ is a monic degree e factor of $f^n(z) - f^m(z)$ over $\mathbb{F}_q(c)$ for some n and m . For each r , the coefficient q_r is the r th elementary symmetric polynomial evaluated at the negatives of the zeros of Q ; those zeros have valuation $-1/d$ by Lemma 4.2(b), so $v(q_r) \geq -r/d$. On the other hand, by Gauss's lemma, $q_r \in \mathbb{F}_q[c]$, so $\deg q_r \leq r/d$. Thus there are only finitely many possibilities for each q_r , and hence finitely many possibilities for Q , each of which yields one dynatonic curve. \square

4.4. Gonality of dynatonic curves.

Proof of Theorem 1.1. By Lemma 4.1, we may assume that $k = \mathbb{F}_q$.

- (a) Let X be the smooth projective model of a dynatonic curve, corresponding to a factor of $f^n(z) - f^m(z)$ for some n and m . By Lemma 4.2(b), $f^n(z) - f^m(z)$ splits completely over $\mathbb{F}_q((t))$, and the preimage of ∞ under $X \xrightarrow{c} \mathbb{P}^1$ consists of \mathbb{F}_q -points, each of ramification index d . Every irreducible component Z of $X_{\overline{\mathbb{F}}_q}$ dominates \mathbb{P}^1 via c and hence must contain one of those \mathbb{F}_q -points, say x . Then each $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ -conjugate of Z contains x . On the other hand, since $X_{\overline{\mathbb{F}}_q}$ is smooth, its irreducible components are disjoint. Thus Z equals each of its conjugates, so Z descends to an irreducible component of X , which must be X itself. (This proves also that X has an \mathbb{F}_q -point, as promised in the proof of Lemma 4.1.)
- (b) To bound gonality from below, we use Ogg's method of counting points over a finite field; cf. [Ogg74], [Poo07, Section 3], and [DKO⁺17, Section 4]. Let X be the smooth projective model of a dynatonic curve. Let $e = \deg(X \xrightarrow{c} \mathbb{P}^1)$. Each preimage of ∞ in X is an \mathbb{F}_q -point of ramification index d , so there are e/d such points. On the other hand, \mathbb{P}^1 has only $q + 1$ points over \mathbb{F}_q , so any nonconstant morphism $X \rightarrow \mathbb{P}^1$ has degree at least $e/(d(q + 1))$. As X varies, $e \rightarrow \infty$ by Lemma 4.3. \square

5. STRONG UNIFORM BOUNDEDNESS OF PREPERIODIC POINTS

Proof of Theorem 1.7. Without loss of generality, $K = k(u)$ for some indeterminate u . Given L , let Y be the smooth projective integral curve over k with function field L . The condition $[L : K] \leq D$ implies that Y has gonality at most D , so each irreducible component of $Y_{\bar{k}}$ has gonality at most D . For $c \in L$ not algebraic over k , if $z \in L$ and $n > m$ satisfy $f^n(z) - f^m(z) = 0$, then (z, c) is a nonconstant and hence smooth L -point of the curve $f^n(z) - f^m(z) = 0$ in \mathbb{A}^2 , so it yields an L -point on a dynatomic curve X corresponding to a factor of $f^n(z) - f^m(z)$. This L -point defines a nonconstant k -morphism $Y \rightarrow X$, so $\gamma(X) \leq \gamma(Y) \leq D$. By Theorem 1.1(b), this places a uniform bound on n . For each n , the number of preperiodic points of $z^d + c$ corresponding to that value of n is uniformly bounded by $\sum_{m=0}^{n-1} \deg(f^n(z) - f^m(z)) = nd^n$, so bounding n bounds the number of preperiodic points too.

If k is finite and c lies in the maximal algebraic extension ℓ of k in L , then all the preperiodic points of $x^d + c$ in L are in ℓ , but $[\ell : k] \leq D$, so the number of preperiodic points is uniformly bounded by $(\#k)^D$. \square

To prove Theorem 1.8, we need the following result of Frey [Fre94, Proposition 2], which in turn relies on Faltings's theorems on rational points on subvarieties of abelian varieties.

Lemma 5.1. *Let C be a curve defined over a number field K . Let $D \geq 1$. If there are infinitely many points $P \in C(\bar{K})$ of degree $\leq D$ over K , then $\gamma(C) \leq 2D$.*

Proof of Theorem 1.8. For each (m, n) , let $S_{m,n}$ be the set of $(z_0, c) \in \bar{\mathbb{Q}} \times \bar{\mathbb{Q}}$ such that z_0 and c belong to some number field K of degree $\leq D$ and iteration of $z^d + c$ maps z_0 into a cycle of length exactly n after exactly m steps; thus $S_{m,n} \subseteq Y_1(m, n)(\bar{\mathbb{Q}})$. Suppose that the conclusion fails; then for some $n \leq N$, there exist infinitely many $m \geq 1$ for which $S_{m,n}$ is nonempty. Fix such an n .

Let $m_0 \geq 1$. By choice of n , the disjoint union $\coprod_{m \geq m_0} S_{m,n}$ is infinite. For each $m \geq m_0$, the c -coordinate map $Y_1(m, n) \rightarrow \mathbb{A}^1$ factors through $Y_1(m_0, n)$, so we obtain maps of sets

$$(5) \quad \coprod_{m \geq m_0} S_{m,n} \longrightarrow S_{m_0,n} \longrightarrow \bar{\mathbb{Q}}.$$

On the other hand, by Northcott's theorem [Nor50, Theorem 3], for any given $c \in \bar{\mathbb{Q}}$, the set of preperiodic points of $x^d + c$ of degree $\leq D$ over \mathbb{Q} is finite; thus each $c \in \bar{\mathbb{Q}}$ has finite preimage under the composition (5). Hence the image of (5) is infinite, so $S_{m_0,n}$ is infinite. Thus some $\mathbb{Q}(\mu_d)$ -irreducible component $Y_1(m_0, n)^\zeta$ of $Y_1(m_0, n)$ contains infinitely many points of degree $\leq D$ over $\mathbb{Q}(\mu_d)$. By Lemma 5.1, $\gamma(Y_1(m_0, n)^\zeta) \leq 2D$.

The previous paragraph applies for every integer $m_0 \geq 1$, contradicting Theorem 1.1(b). \square

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