THE ANALYTIC CLASS NUMBER FORMULA FOR 1-DIMENSIONAL AFFINE SCHEMES

BRUCE W. JORDAN AND BJORN POONEN

Abstract. We derive an analytic class number formula valid for an order in a product of S-integers in global fields, or equivalently for reduced finite-type affine schemes of pure dimension 1 over $\mathbb{Z}$.

1. Introduction

Let K be a finite extension of $\mathbb{Q}$. Let $\mathcal{O}_K$ be its ring of integers. Let $\zeta_K(s)$ be the Dedekind zeta function of K, which is the zeta function of $\operatorname{Spec} \mathcal{O}_K$. Dedekind [Dir94, Supplement XI, §184, IV], generalizing work of Dirichlet, proved the analytic class number formula, which expresses the residue of $\zeta_K(s)$ at $s = 1$ in terms of arithmetic invariants (see also Hilbert's *Zahlbericht* [Hil97, Theorem 56]). More precisely, he proved that

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}\,(2\pi)^{r_2}\,hR}{w\,|\operatorname{Disc} K|^{1/2}}, \tag{1}$$

where r_1 is the number of real places, r_2 is the number of complex places, h is the class number, R is the unit regulator, w is the number of roots of unity, and $\operatorname{Disc} K$ is the discriminant. F. K. Schmidt [Sch31, Satz 21] proved an analogue for a global function field.

This could be generalized in several ways:

- Replace $\mathcal{O}_K$ by a non-maximal order.
- Replace $\mathcal{O}_K$ by a ring of S-integers for some finite set S of places of K.
- Allow $\mathbb{Z}$-algebras of Krull dimension 1 that are not necessarily integral domains.

We will generalize simultaneously in all of these directions, by proving a version of (1) for an order $\mathcal{O}$ in a product of S-integers in global fields, or equivalently for a reduced affine finite-type $\mathbb{Z}$-scheme of pure dimension 1 (for the equivalence, see Proposition 4.2). Our main result, expressing the leading term of the arithmetic zeta function of [Ser65, p. 83] in terms of quantities to be defined in Section 5, is this:

Theorem 1.1 (Generalized analytic class number formula)**.** *Let X be a reduced affine finite-type $\mathbb{Z}$-scheme of pure dimension 1, say $X = \operatorname{Spec} \mathcal{O}$. Let m be the number of irreducible components of X. Then*

$$\lim_{s\to 1}(s-1)^m\zeta_X(s) = \frac{2^{r_1}\,(2\pi)^{r_2}\,h(\mathcal{O})\,R(\mathcal{O})\prod_{v\in S_{\text{nonarch}}}\left(\left(1-q_v^{-1}\right)/\log q_v\right)}{w(\mathcal{O})\,|\operatorname{Disc}\mathcal{O}|^{1/2}}. \tag{2}$$

Date: April 2, 2020.

2010 *Mathematics Subject Classification.* Primary 11R54; Secondary 11R29.

The second author was supported in part by National Science Foundation grant DMS-1069236 and DMS-1601946 and grants from the Simons Foundation (#340694, #402472, and #550033).

Remark 1.2. Theorem 1.1 seems to be new even for orders in rings of integers of number fields and for rings of S-integers in number fields.

Remark 1.3. Replacing a finite-type $\mathbb{Z}$-scheme X by its associated reduced subscheme X_{red} does not change $\zeta_X(s)$. On the other hand, if the scheme $X = \operatorname{Spec} \mathcal{O}$ in Theorem 1.1 is not assumed to be reduced, then $\mathcal{O}^\times$ need not be finitely generated, so defining the regulator $R(\mathcal{O})$ is problematic: for example, if $\mathcal{O} := \mathbb{F}_2[t, \epsilon]/(\epsilon^2)$, then $\mathcal{O}^\times$ is isomorphic to the additive group of $\mathbb{F}_2[t]$ via the homomorphism $1 + f\epsilon \mapsto f$.

Remark 1.4. Various authors defined other zeta functions attached to a singular curve over a finite field and computed their leading terms at $s = 0$ or $s = 1$ [Gal73, Gre89, ZG97a, ZG97b, Stö98], but these zeta functions are different from the usual zeta function in [Ser65] in general.

1.1. **Outline of the article.** Section 2 proves Theorem 1.1 for a ring of S-integers. Because we want a formula involving the S-class group and S-regulator instead of the usual class group and regulator, it is not convenient to deduce this from the classical formula for the ring of integers. Instead we redo the calculations of Tate's thesis for S-integers.

Sections 3 and 4 characterize the rings $\mathcal{O}$ such that $\operatorname{Spec} \mathcal{O}$ is a reduced affine finite-type $\mathbb{Z}$-scheme of pure dimension 1. In particular, the normalization $\widetilde{\mathcal{O}}$ of such a ring $\mathcal{O}$ is a product of rings of S-integers.

Section 5 defines all the quantities that appear in (2).

Section 6 proves Theorem 1.1. The formula for $\widetilde{\mathcal{O}}$ follows from the case proved in Section 2, so our strategy is to determine how each term in (2) changes when $\mathcal{O}$ is replaced by $\widetilde{\mathcal{O}}$. In particular, we use the Leray spectral sequence to determine how the unit group and Picard group change.

Finally Sections 7 and 8 illustrate (2) in examples exhibiting the phenomena that can arise in our context: both number fields and function fields, S-integers instead of just integers, multiple irreducible components, and non-maximal orders (and hence singular points of the scheme).

2. Tate's thesis for S-integers

Let K be a global field. Let μ be the torsion subgroup of $K^\times$, and let $w = \#\mu$. For each place v of K, let K_v be the completion of K at v. If $K_v \simeq \mathbb{R}$, equip it with Lebesgue measure. If $K_v \simeq \mathbb{C}$, equip it with 2 times Lebesgue measure. For each nonarchimedean v, let $\mathcal{O}_v$ be the valuation ring in K_v, let q_v be the size of its residue field, and equip K_v with the Haar measure dx for which $\operatorname{vol}(\mathcal{O}_v) = 1$; here we follow [Wei67, p. 95] instead of taking the self-dual measure as in [Tat67, p. 310]. We write $\operatorname{vol}(T)$ for the measure of a set T with respect to a measure that is implied by context.

If $a \in K_v$ for some v, let $|a|_v \in \mathbb{R}_{\geq 0}$ be the factor by which multiplication-by-a scales the Haar measure on K_v. The measure we use on $K_v^\times$ is not the restriction of the measure dx on K_v. If v is archimedean, equip $K_v^\times$ with the Haar measure $dx/|x|_v$. If v is nonarchimedean, equip $K_v^\times$ with the Haar measure for which $\operatorname{vol}(\mathcal{O}_v^\times) = 1$.

Let $K_{v,1}^\times := \{x \in K_v^\times : |x|_v = 1\}$. If v is nonarchimedean, $K_{v,1}^\times = \mathcal{O}_v^\times$, which has volume 1 for the Haar measure on $K_v^\times$. If v is archimedean, then equip $K_{v,1}^\times$ with the Haar measure compatible with the Haar measure on $K_v^\times$ and Lebesgue measure on $\mathbb{R}$ in the exact sequence

$$1 \longrightarrow K_{v,1}^\times \longrightarrow K_v^\times \xrightarrow{\log|\,|_v} \mathbb{R} \longrightarrow 0$$

(for the notion of compatibility, see, e.g., [DE14, Theorem 1.53]; the notion can be extended to exact sequences of arbitrary finite length by breaking them into short exact sequences).

Define the adèle ring $\mathbb{A}$ as the restricted product $\prod'_v(K_v, \mathcal{O}_v)$ with the product measure. Equip the idèle group $\mathbb{A}^\times = \prod'_v(K_v^\times, \mathcal{O}_v^\times)$ with the product of the multiplicative Haar measures. The field K embeds diagonally in $\mathbb{A}$, and $K^\times$ embeds diagonally in $\mathbb{A}^\times$. Equip the discrete groups K and $K^\times$ and their subgroups with the counting measure, in order to equip $\mathbb{A}/K$ and $\mathbb{A}^\times/K^\times$ with measures.

Let S be a finite nonempty set of places of K containing all the archimedean places. Let S_{nonarch} be the set of nonarchimedean places in S. Define the ring of S-integers by

$$\mathcal{O} := \{x \in K : v(x) \geq 0 \text{ for all } v \notin S\}.$$

Lemma 2.1. *We have a measure-compatible isomorphism*

$$\frac{\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v}{\mathcal{O}} \xrightarrow{\sim} \frac{\mathbb{A}}{K}$$

and a measure-compatible short exact sequence

$$0 \longrightarrow \prod_{v\notin S} \mathcal{O}_v \longrightarrow \frac{\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v}{\mathcal{O}} \longrightarrow \frac{\prod_{v\in S} K_v}{\mathcal{O}} \longrightarrow 0.$$

Proof. Since S is nonempty, strong approximation [Cas67, §15] implies that any $x \in \mathbb{A}$ can be written as $y + \epsilon$ with $y \in K$ and $\epsilon = (\epsilon_v) \in \mathbb{A}$ such that $\epsilon_v \in \mathcal{O}_v$ for all $v \notin S$. In other words, the homomorphism

$$\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v \longrightarrow \frac{\mathbb{A}}{K}$$

is surjective. Its kernel is $K \cap \left(\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v\right) = \mathcal{O}$, so we obtain the claimed isomorphism. By definition of the measures, the upper three homomorphisms in the diagram

$$\begin{tikzcd}
\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v \arrow[r, hook] \arrow[d] & \mathbb{A} \arrow[d] \\
\frac{\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v}{\mathcal{O}} \arrow[r, "\sim"] & \frac{\mathbb{A}}{K}
\end{tikzcd}$$

respect the measures, so the induced isomorphism at the bottom does too.

The exact sequence is obtained from the measure-compatible split exact sequence

$$0 \longrightarrow \prod_{v\notin S} \mathcal{O}_v \longrightarrow \prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v \longrightarrow \prod_{v\in S} K_v \longrightarrow 0$$

by dividing each of the last two terms by the image of $\mathcal{O}$ with the counting measure. $\square$

Define the S-Arakelov divisor group (cf. the usual Arakelov divisor group in [Neu99, III.1.8]) by

$$\widehat{\operatorname{Div}}\,\mathcal{O} := \bigoplus_{v\in S} \mathbb{R} \times \bigoplus_{v\notin S} (\mathbb{Z}\log q_v),$$

where each $\mathbb{R}$ has Lebesgue measure and each $\mathbb{Z}\log q_v$ has the counting measure. We have a homomorphism $\deg\colon \widehat{\operatorname{Div}}\,\mathcal{O} \to \mathbb{R}$ that sums the components, and its kernel is denoted $\widehat{\operatorname{Div}}^0\,\mathcal{O}$, which acquires a measure compatible with the sequence $0 \to \widehat{\operatorname{Div}}^0\,\mathcal{O} \to \widehat{\operatorname{Div}}\,\mathcal{O} \to \mathbb{R} \to 0$.

The homomorphism $\mathbb{A}^\times \to \widehat{\operatorname{Div}}\,\mathcal{O}$ sending (x_v) to $(\log |x_v|_v)$ restricts to a homomorphism $K^\times \to \widehat{\operatorname{Div}}^0\,\mathcal{O}$ whose cokernel is called the S-Arakelov class group $\widehat{\operatorname{Pic}}^0\,\mathcal{O}$. Let $\mathbb{A}_1^\times$ be the kernel of the composition $\mathbb{A}^\times \to \widehat{\operatorname{Div}}\,\mathcal{O} \stackrel{\deg}{\to} \mathbb{R}$. Equip $\mathbb{R}^S := \bigoplus_{v\in S} \mathbb{R}$ and $\mathbb{R}$ with Lebesgue measure. Equip the sum-zero hyperplane $\mathbb{R}^S_0 := \ker\left(\mathbb{R}^S \xrightarrow{\text{sum}} \mathbb{R}\right)$ with the measure compatible with those (equivalently, identify $\mathbb{R}^S_0$ with its projection under the forget-one-coordinate map and use Lebesgue measure on the image).

Lemma 2.2. *We have a measure-preserving exact sequence*

$$0 \longrightarrow \frac{\mathbb{R}^S_0}{\operatorname{im}\mathcal{O}^\times} \longrightarrow \widehat{\operatorname{Pic}}^0\,\mathcal{O} \longrightarrow \operatorname{Pic}\mathcal{O} \longrightarrow 1.$$

Proof. Apply the snake lemma to

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}^\times & \longrightarrow & K^\times & \longrightarrow & K^\times/\mathcal{O}^\times & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{R}^S_0 & \longrightarrow & \widehat{\operatorname{Div}}^0\,\mathcal{O} & \longrightarrow & \operatorname{Div}\mathcal{O} & \longrightarrow & 0. \end{array}$$

$\square$

If $\operatorname{char} K > 0$, let q be the size of the constant field of K. In all the formulas below, the term in square brackets involving $\log q$ should be present only if $\operatorname{char} K > 0$. Each copy of $\mathbb{R}$ has Lebesgue measure, which induces a measure on quotients such as $\mathbb{R}/(\mathbb{Z}\log q)$.

Lemma 2.3. *We have a measure-compatible exact sequence*

$$1 \longrightarrow \frac{\prod_v K^\times_{v,1}}{\mu} \longrightarrow \frac{\mathbb{A}_1^\times}{K^\times} \longrightarrow \widehat{\operatorname{Pic}}^0\,\mathcal{O} \longrightarrow \bigoplus_{v\in S_{\text{nonarch}}} \frac{\mathbb{R}}{\mathbb{Z}\log q_v} \longrightarrow \left[\frac{\mathbb{R}}{\mathbb{Z}\log q}\right] \longrightarrow 0.$$

Proof. Since S contains all archimedean places, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus\limits_{v \text{ arch}} \mathbb{R} \times \bigoplus\limits_{v \text{ nonarch}} \mathbb{Z}\log q_v & \longrightarrow & \bigoplus\limits_{v\in S} \mathbb{R} \times \bigoplus\limits_{v\notin S} \mathbb{Z}\log q_v & \longrightarrow & \bigoplus\limits_{v\in S_{\text{nonarch}}} \dfrac{\mathbb{R}}{\mathbb{Z}\log q_v} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\text{sum}} & & \big\downarrow{\scriptstyle\text{sum}} & & \big\downarrow & & \\ 0 & \longrightarrow & \mathbb{R} & = & \mathbb{R} & \longrightarrow & 0 & & \end{array}$$

with exact measure-compatible rows. The second vertical homomorphism is $\deg\colon \widehat{\operatorname{Div}}\,\mathcal{O} \to \mathbb{R}$, so it is surjective with kernel $\widehat{\operatorname{Div}}^0\,\mathcal{O}$. The upper left group is $\operatorname{im}\left(\mathbb{A}^\times \to \widehat{\operatorname{Div}}\,\mathcal{O}\right)$, so the kernel of the first vertical homomorphism is $\operatorname{im}\left(\mathbb{A}_1^\times \to \widehat{\operatorname{Div}}^0\,\mathcal{O}\right)$. If K is a number field, then the first vertical homomorphism is surjective. If K is a function field, then its image is $\mathbb{Z}\log q$ since a smooth projective curve over $\mathbb{F}_q$ has closed points of every sufficiently large degree. Thus the snake lemma explains exactness at the last three nontrivial positions in the exact sequence

$$1 \longrightarrow \prod_v K^\times_{v,1} \longrightarrow \mathbb{A}_1^\times \longrightarrow \widehat{\operatorname{Div}}^0\,\mathcal{O} \longrightarrow \bigoplus_{v\in S_{\text{nonarch}}} \frac{\mathbb{R}}{\mathbb{Z}\log q_v} \xrightarrow{\text{snake}} \left[\frac{\mathbb{R}}{\mathbb{Z}\log q}\right] \longrightarrow 0,$$

and exactness at the first two positions follows since $K^\times_{v,1}$ is the kernel of $\log|\ |_v\colon K_v^\times \to \mathbb{R}$. The discrete subgroup $K^\times$ and compact subgroup $\prod_v K^\times_{v,1}$ of $\mathbb{A}_1^\times$ intersect in μ, so forming quotients yields the claimed exact sequence. By construction, each of the exact sequences above is measure-compatible. □

If K is a number field, let $(e_i)_{1\le i\le n}$ be a basis for the ring of integers of K, and define $\operatorname{Disc} K := \det(\operatorname{Tr}(e_i e_j))_{1\le i,j\le n} \in \mathbb{Z}$. If K is a global function field of genus g over $\mathbb{F}_q$, define $\operatorname{Disc} K := q^{2g-2}$; this is so that in both cases it is $|\operatorname{Disc} K|^{s/2}$ times the completed zeta function below that is symmetric with respect to $s \mapsto 1-s$. Define the S-class number $h(\mathcal{O}) := \#\operatorname{Pic}\mathcal{O}$. By the proof of the Dirichlet S-unit theorem, the image of $\mathcal{O}^\times \to \mathbb{R}^S_0$ is a full lattice; its covolume is called the S-regulator $R(\mathcal{O})$.

Lemma 2.4. *We have*

$$\begin{aligned}
\operatorname{vol}\left(\frac{\mathbb{A}}{K}\right) &= |\operatorname{Disc} K|^{1/2} \\
\operatorname{vol}\left(\frac{\prod_{v\in S} K_v}{\mathcal{O}}\right) &= |\operatorname{Disc} K|^{1/2} \\
\operatorname{vol}\left(K_{v,1}^\times\right) &= \begin{cases} 2, & \text{if } K_v \simeq \mathbb{R}; \\ 2\pi, & \text{if } K_v \simeq \mathbb{C}; \\ 1, & \text{if } K_v \text{ is nonarchimedean.} \end{cases} \\
\operatorname{vol}\left(\widehat{\operatorname{Pic}}^0\,\mathcal{O}\right) &= h(\mathcal{O})\,R(\mathcal{O}) \\
\operatorname{vol}\left(\frac{\mathbb{A}_1^\times}{K^\times}\right) &= \frac{2^{r_1}(2\pi)^{r_2}\,h(\mathcal{O})\,R(\mathcal{O})\,[\log q]}{w(\mathcal{O})\prod_{v\in S_{\text{nonarch}}}\log q_v}.
\end{aligned}$$

(*Again, the term in square brackets should be present only if* $\operatorname{char} K > 0$.)

Proof. The first formula can be found in [Wei82, §2.1.3]. It implies the second, by Lemma 2.1. For $\operatorname{vol}(K_{v,1}^\times)$ for archimedean v, see Tate's thesis [Tat67, p. 337]. For nonarchimedean v, the group $K_{v,1}^\times = \mathcal{O}_v^\times$ has volume 1 by definition of the measure. Lemma 2.2 computes $\operatorname{vol}\left(\widehat{\operatorname{Pic}}^0\,\mathcal{O}\right)$. Lemma 2.3 yields the last formula. □

Define gamma factors $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := (2\pi)^{1-s}\Gamma(s)$, and define the completed zeta function by

$$\widehat{\zeta}_K(s) := \Gamma_{\mathbb{R}}(s)^{r_1}\,\Gamma_{\mathbb{C}}(s)^{r_2}\,\zeta_K(s).$$

(Warning: We have used the definitions of [Wei67, VII.§6], but other authors use definitions of $\Gamma_{\mathbb{R}}(s)$ and $\Gamma_{\mathbb{C}}(s)$ that are nonzero constant multiples of these; cf. [Del73, §3.2] and [Den91, §2]. Some authors also include a factor $|\operatorname{Disc} K|^{s/2}$ in the definition of $\widehat{\zeta}_K(s)$ [Neu99, p. 467].)

Lemma 2.5. *We have*

$$\lim_{s\to 1}(s-1)\widehat{\zeta}_K(s) = \frac{\operatorname{vol}\left(\frac{\mathbb{A}_1^\times}{K^\times}\right)}{|\operatorname{Disc} K|^{1/2}\,[\log q]}.$$

Proof. See the proofs of [Wei67, Theorems 3 and 4]. □

Theorem 2.6 (Analytic class number formula for S-integers)**.** *We have*

$$\lim_{s\to 1}(s-1)\zeta_{\mathcal{O}}(s) = \frac{2^{r_1}\,(2\pi)^{r_2}\,h(\mathcal{O})\,R(\mathcal{O})\,\prod_{v\in S_{\text{nonarch}}}\left((1-q_v^{-1})\,/\log q_v\right)}{w(\mathcal{O})\,|\operatorname{Disc} K|^{1/2}}.$$

Proof. The functions $\widehat{\zeta}_K(s)$ and $\zeta_{\mathcal{O}}(s)$ differ only in that the former contains

- gamma factors $\Gamma_{\mathbb{R}}(s)$ and $\Gamma_{\mathbb{C}}(s)$, which take the value 1 at $s = 1$, and
- Euler factors $(1 - q_v^{-s})^{-1}$ for each $v \in S_{\text{nonarch}}$, which take the value $(1 - q_v^{-1})^{-1}$ at $s = 1$.

The formula follows from this, the last formula of Lemma 2.4, and Lemma 2.5. $\square$

3. Characterizing rings of S-integers

Lemma 3.1. *Let $\mathcal{O}$ be an integrally closed domain that is finitely generated as a $\mathbb{Z}$-algebra. Let $K = \operatorname{Frac} \mathcal{O}$. If the Krull dimension $\dim \mathcal{O}$ is 1, then K is a global field and $\mathcal{O}$ is a ring of S-integers in K in the sense of Section 2.*

Proof. Case 1: The image of $\operatorname{Spec} \mathcal{O} \to \operatorname{Spec} \mathbb{Z}$ *is a closed point.* Then $\mathcal{O}$ is a 1-dimensional algebra over $\mathbb{F}_p$, so $\operatorname{Spec} \mathcal{O}$ is a regular curve, equal to $C - S$, where C is the smooth projective curve with function field K, and S is a nonempty finite set of places.

Case 2: The morphism $\operatorname{Spec} \mathcal{O} \to \operatorname{Spec} \mathbb{Z}$ *is dominant.* Then it is of relative dimension 0 since $\dim \mathcal{O} = 1 = \dim \mathbb{Z}$. Thus K is a finite extension of $\mathbb{Q}$. Since $\mathcal{O}$ is integrally closed, $\mathcal{O}$ contains the integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in K. Thus $\mathcal{O} = \mathcal{O}_K[S^{-1}]$ for some set S of places of K. Since $\mathcal{O}$ is finitely generated, S is finite. $\square$

4. Describing 1-dimensional schemes

From now on, X is a reduced affine finite-type $\mathbb{Z}$-scheme of pure dimension 1, say $X = \operatorname{Spec} \mathcal{O}$. Let $(C_i)_{i \in I}$ be the 1-dimensional irreducible components of X. Let K_i be the function field of C_i. Let K be the total quotient ring of $\mathcal{O}$, obtained by inverting all non-zerodivisors, so $K = \prod K_i$.

Let $|X|$ be the set of closed points of X. These points correspond to maximal ideals of $\mathcal{O}$, which are exactly the prime ideals with finite residue field since $\mathcal{O}$ is a finitely generated $\mathbb{Z}$-algebra [EGA IV$_3$, 10.4.11.1(i)].

Let $\pi \colon \widetilde{X} \to X$ and $\pi_i \colon \widetilde{C}_i \to C_i$ be the normalization morphisms. Then $\widetilde{X}$ is the *disjoint* union $\coprod \widetilde{C}_i$. Correspondingly, the integral closure $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ in K is a finite product of rings $\widetilde{\mathcal{O}}_i$. By Lemma 3.1, K_i is a global field and there exists S_i such that $\widetilde{\mathcal{O}}_i$ is the ring of S_i-integers in K_i. Since π is an isomorphism above the generic point of each C_i, it is an isomorphism above $X - Z$ for some finite subset $Z \subset |X|$.

Lemma 4.1. *The index $(\widetilde{\mathcal{O}} : \mathcal{O})$ is finite.*

Proof. Since $\mathcal{O}$ is a finitely generated $\mathbb{Z}$-algebra, $\widetilde{\mathcal{O}}$ is a finite $\mathcal{O}$-module. The finite $\mathcal{O}$-module $\widetilde{\mathcal{O}}/\mathcal{O}$ is supported on Z, and each $\mathfrak{p} \in Z$ has finite residue field. Thus $\widetilde{\mathcal{O}}/\mathcal{O}$ has a filtration with quotients that are finite as sets, so $\widetilde{\mathcal{O}}/\mathcal{O}$ is finite as a set. $\square$

Our work so far proves the following.

Proposition 4.2. *A scheme X is a reduced affine finite-type $\mathbb{Z}$-scheme of pure dimension 1 if and only if $X = \operatorname{Spec} \mathcal{O}$ for some finite-index subring $\mathcal{O}$ of a finite product of rings of S_i-integers in global fields K_i.*

5. Invariants

We retain the notation of Section 4.

5.1. **The invariants m, r_1, r_2 of K.** Let $m = m(K)$ be the number of irreducible components of X, so $m = \# I$.

Define $r_1 = r_1(K)$ and $r_2 = r_2(K)$ so that r_1 is the number of ring homomorphisms $K \to \mathbb{R}$, and $2r_2$ is the number of ring homomorphisms $K \to \mathbb{C}$ whose image is not contained in $\mathbb{R}$. If K is a number field, these are the usual r_1 and r_2. If K is a global function field, then $r_1 = r_2 = 0$. In the general case $K = \prod K_i$, we have $r_1(K) = \sum r_1(K_i)$ and $r_2(K) = \sum r_2(K_i)$.

5.2. **The unit group $\mathcal{O}^\times$ and roots of unity.** Let $\mathcal{O}^\times$ be the unit group of $\mathcal{O}$. Later we will prove that $\mathcal{O}^\times$ is a finitely generated abelian group. Let $\mu(\mathcal{O})$ be the torsion subgroup of $\mathcal{O}^\times$. Let $w(\mathcal{O}) := \#\mu(\mathcal{O})$.

5.3. **The Picard group $\operatorname{Pic}\mathcal{O}$ and the class number $h(\mathcal{O})$.** Let $\operatorname{Pic}\mathcal{O} := \operatorname{Pic} X = H^1(X, \mathscr{O}_X^*)$ [Har77, Exercise III.4.5]. Later we will prove that $\operatorname{Pic}\mathcal{O}$ is finite. Let $h(\mathcal{O}) := \# \operatorname{Pic}\mathcal{O}$.

5.4. **The discriminant $\operatorname{Disc}\mathcal{O}$.** For each global field K_i, we defined $\operatorname{Disc} K_i$ in Section 2. Define $\operatorname{Disc}\mathcal{O} := (\widetilde{\mathcal{O}} : \mathcal{O})^2 \prod \operatorname{Disc} K_i$; this is so that in the case where $\mathcal{O}$ is an order in the ring of integers of a number field, $\operatorname{Disc}\mathcal{O} = \det\left(\operatorname{Tr}_{K/\mathbb{Q}}(e_i e_j)\right)_{1 \le i,j \le n}$ for any $\mathbb{Z}$-basis (e_i) of $\mathcal{O}$.

5.5. **The logarithmic embedding and the regulator $R(\mathcal{O})$.** Let $S = \coprod S_i$ and $S_{\text{nonarch}} = \coprod (S_i)_{\text{nonarch}}$. For $v \in S_i \subseteq S$, let K_v be the completion of K_i at v. Taking the product of the homomorphisms $K_v^\times \xrightarrow{\log|\,|_v} \mathbb{R}$ yields a homomorphism $\prod_{v \in S} K_v^\times \xrightarrow{\lambda} \mathbb{R}^S$. Let $\widetilde{\phi}$ be the composition

$$\widetilde{\mathcal{O}}^\times \longrightarrow \prod_{v \in S} K_v^\times \xrightarrow{\lambda} \mathbb{R}^S.$$

Let $\phi = \widetilde{\phi}|_{\mathcal{O}^\times}$. Since $\ker\lambda$ is bounded in $\prod_{v\in S} K_v$ while $\mathcal{O}^\times$ is a discrete closed subset of $\prod_{v \in S} K_v$, $\ker\phi$ is finite; on the other hand, the codomain of ϕ is torsion-free; thus $\ker\phi = \mu(\mathcal{O})$.

The group $\widetilde{\phi}(\widetilde{\mathcal{O}}^\times)$ is a direct product of lattices in $\prod \mathbb{R}_0^{S_i}$. Later we will prove that $\mathcal{O}^\times$ is of finite index in $\widetilde{\mathcal{O}}^\times$, so $\phi(\mathcal{O}^\times)$ is again a full lattice $L(\mathcal{O})$ in $\prod \mathbb{R}_0^{S_i}$. The covolume of $L(\mathcal{O})$ is called the regulator, $R(\mathcal{O})$.

5.6. **The zeta function.** Since $\operatorname{Spec}\mathcal{O}$ is of finite type over $\mathbb{Z}$, it has a zeta function defined as an Euler product, as in [Ser65, p. 83]:

$$\zeta_X(s) := \prod_{\mathfrak{p} \in |X|} \left(1 - q_{\mathfrak{p}}^{-s}\right)^{-1}.$$

The product converges only for $s \in \mathbb{C}$ with sufficiently large real part, but as is well known and as we will explain, $\zeta_{\mathcal{O}}(s)$ admits a meromorphic continuation to the whole complex plane and has a pole at $s = 1$ of order m.

We have now defined all the quantities appearing in Theorem 1.1.

6. Relating the invariants of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$

Theorem 1.1 for a product of rings of S-integers follows from Theorem 2.6. In particular, it holds for $\widetilde{\mathcal{O}}$. To prove it for $\mathcal{O}$, we compare the formulas for $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ term by term.

For maximal ideals $\mathfrak{p} \subseteq \mathcal{O}$ and $\mathfrak{P} \subseteq \widetilde{\mathcal{O}}$, we write $\mathfrak{P}|\mathfrak{p}$ when π maps the closed point $\mathfrak{P} \in \widetilde{X}$ to $\mathfrak{p} \in X$.

6.1. **The zeta functions of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.** Hecke [Hec17], generalizing Riemann's work, proved that the Dedekind zeta function of a number field has a meromorphic continuation to the entire complex plane and has a simple pole at $s = 1$. The analogous result for global function fields was proved by F. K. Schmidt [Sch31]. These imply the analogue for a ring of S-integers in a global field. Taking a product yields the corresponding result of products of m rings of S-integers, except that now the pole has order m; this applies in particular to $\widetilde{\mathcal{O}}$. Next, by definition,

$$\frac{\zeta_{\widetilde{\mathcal{O}}}(s)}{\zeta_{\mathcal{O}}(s)} = \prod_{\mathfrak{p}} \frac{\prod_{\mathfrak{P}|\mathfrak{p}} (1 - q_{\mathfrak{P}}^{-s})^{-1}}{(1 - q_{\mathfrak{p}}^{-s})^{-1}},$$

where, for all but finitely many $\mathfrak{p}$, the fraction on the right is 1; cf. [Jen69, Theorem]. Thus $\zeta_{\mathcal{O}}(s)$ too is meromorphic with a pole of order m at 1, and we deduce the following.

Proposition 6.1. *We have*

$$\lim_{s \to 1} \frac{\zeta_{\widetilde{\mathcal{O}}}(s)}{\zeta_{\mathcal{O}}(s)} = \prod_{\mathfrak{p}} \frac{\prod_{\mathfrak{P}|\mathfrak{p}} (1 - q_{\mathfrak{P}}^{-1})^{-1}}{\left(1 - q_{\mathfrak{p}}^{-1}\right)^{-1}}.$$

6.2. **The discriminants of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.** Our definition of $\operatorname{Disc} \mathcal{O}$ immediately implies the following.

Proposition 6.2. *We have* $\dfrac{\operatorname{Disc} \widetilde{\mathcal{O}}}{\operatorname{Disc} \mathcal{O}} = (\widetilde{\mathcal{O}} : \mathcal{O})^{-2}$.

6.3. **Local units.** Let $\mathfrak{c} \subseteq \mathcal{O}$ be the annihilator of the $\mathcal{O}$-module $\widetilde{\mathcal{O}}/\mathcal{O}$. Then $\mathfrak{c}$ is also an $\widetilde{\mathcal{O}}$-ideal, called the conductor of $\mathcal{O}$. It is the largest $\widetilde{\mathcal{O}}$-ideal contained in $\mathcal{O}$.

Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}$. Let $\widetilde{\mathcal{O}}_{\mathfrak{p}}$ be the localization of the $\mathcal{O}$-algebra $\widetilde{\mathcal{O}}$ at $\mathfrak{p}$. Then $\widetilde{\mathcal{O}}_{\mathfrak{p}}$ is a semilocal ring whose maximal ideals correspond to the finitely many maximal ideals $\mathfrak{P} \subset \widetilde{\mathcal{O}}$ lying above $\mathfrak{p}$. Since $\widetilde{\mathcal{O}}/\mathcal{O}$ is finite, $\widetilde{\mathcal{O}}/\mathcal{O} \simeq \prod_{\mathfrak{p}} \widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}$, and $\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}$ is nontrivial for only finitely many $\mathfrak{p}$. Let $\mathfrak{c}_{\mathfrak{p}}$ be the localization of $\mathfrak{c}$ at $\mathfrak{p}$.

Lemma 6.3. *The natural map*

$$\frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}^{\times}}{\mathcal{O}_{\mathfrak{p}}^{\times}} \to \frac{\left(\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}}\right)^{\times}}{(\mathcal{O}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}})^{\times}}$$

is an isomorphism.

Proof. Case 1: $\mathfrak{c}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$. Then $1 \in \mathfrak{c}_{\mathfrak{p}}$, so $\mathfrak{c}_{\mathfrak{p}} = \widetilde{\mathcal{O}}_{\mathfrak{p}}$ too; thus both sides are trivial.

Case 2: $\mathfrak{c}_{\mathfrak{p}} \neq \mathcal{O}_{\mathfrak{p}}$. Then $\mathfrak{c}_{\mathfrak{p}} \subseteq \mathfrak{p}\mathcal{O}_{\mathfrak{p}} \subset \mathfrak{P}$ for every maximal ideal $\mathfrak{P}$ of $\widetilde{\mathcal{O}}_{\mathfrak{p}}$. If an element $\bar{a} \in (\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}})^{\times}$ is lifted to an element $a \in \widetilde{\mathcal{O}}_{\mathfrak{p}}$, then a lies outside each $\mathfrak{P}$, so $a \in \widetilde{\mathcal{O}}_{\mathfrak{p}}^{\times}$. Thus $\widetilde{\mathcal{O}}_{\mathfrak{p}}^{\times} \to (\widetilde{\mathcal{O}}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}})^{\times}$ is surjective. Similarly, $\mathcal{O}_{\mathfrak{p}}^{\times} \to (\mathcal{O}_{\mathfrak{p}}/\mathfrak{c}_{\mathfrak{p}})^{\times}$ is surjective. Both surjections have the same kernel $1 + \mathfrak{c}_{\mathfrak{p}}$, so the result follows. ∎

Lemma 6.4. *If $\mathfrak{c}_\mathfrak{p} \neq \mathcal{O}_\mathfrak{p}$, then*

$$\# \left(\widetilde{\mathcal{O}}_\mathfrak{p}/\mathfrak{c}_\mathfrak{p}\right)^\times = \# \left(\widetilde{\mathcal{O}}_\mathfrak{p}/\mathfrak{c}_\mathfrak{p}\right) \prod_{\mathfrak{P}|\mathfrak{p}} \left(1 - q_\mathfrak{P}^{-1}\right),$$
$$\# \left(\mathcal{O}_\mathfrak{p}/\mathfrak{c}_\mathfrak{p}\right)^\times = \# \left(\mathcal{O}_\mathfrak{p}/\mathfrak{c}_\mathfrak{p}\right) \left(1 - q_\mathfrak{p}^{-1}\right).$$

Proof. The maximal ideals of $\widetilde{\mathcal{O}}_\mathfrak{p}/\mathfrak{c}_\mathfrak{p}$ are the ideals $\mathfrak{P}\widetilde{\mathcal{O}}_\mathfrak{p}/\mathfrak{c}_\mathfrak{p}$ for $\mathfrak{P}|\mathfrak{p}$. An element of $\widetilde{\mathcal{O}}_\mathfrak{p}/\mathfrak{c}_\mathfrak{p}$ is a unit if and only if it lies outside each maximal ideal. The probability that a random element of the finite group $\widetilde{\mathcal{O}}_\mathfrak{p}/\mathfrak{c}_\mathfrak{p}$ lies outside $\mathfrak{P}\widetilde{\mathcal{O}}_\mathfrak{p}/\mathfrak{c}_\mathfrak{p}$ is $1 - q_\mathfrak{P}^{-1}$, and these events for different $\mathfrak{P}$ are independent by the Chinese remainder theorem, so the first equation follows. The second equation is similar (but easier). $\square$

Lemma 6.5. *We have*

$$\# \prod_\mathfrak{p} \frac{\widetilde{\mathcal{O}}_\mathfrak{p}^\times}{\mathcal{O}_\mathfrak{p}^\times} = \# \frac{\widetilde{\mathcal{O}}}{\mathcal{O}} \cdot \prod_\mathfrak{p} \frac{\prod_{\mathfrak{P}|\mathfrak{p}}(1 - q_\mathfrak{P}^{-1})}{1 - q_\mathfrak{p}^{-1}}.$$

Proof. By Lemmas 6.3 and 6.4,

$$\# \frac{\widetilde{\mathcal{O}}_\mathfrak{p}^\times}{\mathcal{O}_\mathfrak{p}^\times} = \# \frac{\widetilde{\mathcal{O}}_\mathfrak{p}}{\mathcal{O}_\mathfrak{p}} \cdot \frac{\prod_{\mathfrak{P}|\mathfrak{p}}(1 - q_\mathfrak{P}^{-1})}{1 - q_\mathfrak{p}^{-1}};$$

this holds even if $\mathfrak{c}_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$ since both sides are 1 in that case. Now take the product of both sides and use the isomorphism of finite groups

$$\frac{\widetilde{\mathcal{O}}}{\mathcal{O}} \simeq \prod_\mathfrak{p} \frac{\widetilde{\mathcal{O}}_\mathfrak{p}}{\mathcal{O}_\mathfrak{p}}.$$

$\square$

6.4. **Using the Leray spectral sequence to relate units and Picard groups for $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.** Let $\mathscr{O}_X$ be the structure sheaf of X, and let $\mathscr{O}_X^\times$ be the sheaf of units of $\mathscr{O}_X$. Define $\mathscr{O}_{\widetilde{X}}^\times$ similarly.

Lemma 6.6. *The sheaf $R^1\pi_*\mathscr{O}_{\widetilde{X}}^\times$ on X is 0.*

Proof. By [Har77, Proposition III.8.1], its stalk $(R^1\pi_*\mathscr{O}_{\widetilde{X}}^\times)_\mathfrak{p}$ at a closed point $\mathfrak{p}$ of X is $\varinjlim_U \operatorname{Pic} \pi^{-1}U$, where U ranges over open neighborhoods of $\mathfrak{p}$ in X. Since $\pi^{-1}(\mathfrak{p})$ is finite, every line bundle on $\pi^{-1}U$ becomes trivial on $\pi^{-1}U'$ for some smaller neighborhood U' of $\mathfrak{p}$ in X. Thus $\varinjlim_U \operatorname{Pic} \pi^{-1}U = 0$. $\square$

Lemma 6.7. *We have $H^1(X, \pi_*\mathscr{O}_{\widetilde{X}}^\times) \simeq \operatorname{Pic} \widetilde{X}$.*

Proof. The Leray spectral sequence

$$H^p\left(X, R^q\pi_*\mathscr{F}\right) \implies H^{p+q}\left(\widetilde{X}, \mathscr{F}\right)$$

with $\mathscr{F} = \mathscr{O}_{\widetilde{X}}^\times$ yields an exact sequence

$$0 \longrightarrow H^1\left(X, \pi_*\mathscr{O}_{\widetilde{X}}^\times\right) \longrightarrow \operatorname{Pic} \widetilde{X} \longrightarrow H^0\left(X, R^1\pi_*\mathscr{O}_{\widetilde{X}}^\times\right).$$

Lemma 6.6 above completes the proof. $\square$

Proposition 6.8. *The following is an exact sequence of finite groups:*

$$0 \longrightarrow \frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} \longrightarrow \bigoplus_{\mathfrak{p}} \frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}^\times}{\mathcal{O}_{\mathfrak{p}}^\times} \longrightarrow \operatorname{Pic}\mathcal{O} \longrightarrow \operatorname{Pic}\widetilde{\mathcal{O}} \longrightarrow 0. \tag{3}$$

Proof. View $\widetilde{\mathcal{O}}_{\mathfrak{p}}^\times/\mathcal{O}_{\mathfrak{p}}^\times$ as a skyscraper sheaf on X supported at $\mathfrak{p}$. Then we have an exact sequence of sheaves on X

$$0 \longrightarrow \mathscr{O}_X^\times \longrightarrow \pi_*\mathscr{O}_{\widetilde{X}}^\times \longrightarrow \bigoplus_{\mathfrak{p}} \frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}^\times}{\mathcal{O}_{\mathfrak{p}}^\times} \longrightarrow 0.$$

The corresponding long exact sequence in cohomology is

$$0 \longrightarrow \mathcal{O}^\times \longrightarrow \widetilde{\mathcal{O}}^\times \longrightarrow \bigoplus_{\mathfrak{p}} \frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}^\times}{\mathcal{O}_{\mathfrak{p}}^\times} \longrightarrow \operatorname{Pic}X \longrightarrow H^1\left(X, \pi_*\mathscr{O}_{\widetilde{X}}^\times\right) \longrightarrow 0.$$

By definition, $\operatorname{Pic}X = \operatorname{Pic}\mathcal{O}$. By Lemma 6.7, the last term is $\operatorname{Pic}\widetilde{X} = \operatorname{Pic}\widetilde{\mathcal{O}}$.

The second term in (3) is finite by Lemma 6.5. Finally, $\widetilde{\mathcal{O}}$ is a finite product of rings of S-integers, each of which has finite Picard group, so $\operatorname{Pic}\widetilde{\mathcal{O}}$ is finite. Thus all four groups in (3) are finite. □

Remark 6.9. For a more elementary derivation of (3), at least in the case where $\mathcal{O}$ is an integral domain, see [Neu99, Proposition I.12.9].

Proposition 6.10. *We have*

$$\#\frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} = \frac{h(\widetilde{\mathcal{O}})}{h(\mathcal{O})} \cdot \#\frac{\widetilde{\mathcal{O}}}{\mathcal{O}} \cdot \prod_{\mathfrak{p}} \frac{\prod_{\mathfrak{P}|\mathfrak{p}}(1-q_{\mathfrak{P}}^{-1})}{1-q_{\mathfrak{p}}^{-1}}.$$

Proof. Take the alternating product of the orders of the groups in (3) and use Lemma 6.5. □

Remark 6.11. Finiteness of $(\widetilde{\mathcal{O}}^\times : \mathcal{O}^\times) < \infty$ can also be viewed as a consequence of the finiteness of $(\widetilde{\mathcal{O}} : \mathcal{O})$, by [BL17, Theorem 1.3].

6.5. **The regulators of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.** Let $L = L(\mathcal{O})$ be as in Section 5.5, and define $\widetilde{L} = L(\widetilde{\mathcal{O}})$ similarly. The group $\widetilde{L}$ is a full lattice in $\prod \mathbb{R}_0^{S_i}$. By Proposition 6.8, $(\widetilde{\mathcal{O}}^\times : \mathcal{O}^\times)$ is finite, so L is a full lattice in $\prod \mathbb{R}_0^{S_i}$ too.

Proposition 6.12. *We have*

$$\frac{R(\widetilde{\mathcal{O}})}{R(\mathcal{O})} \cdot \#\frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} = \frac{w(\widetilde{\mathcal{O}})}{w(\mathcal{O})}.$$

Proof. Applying the snake lemma to

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu(\mathcal{O}) & \longrightarrow & \mathcal{O}^\times & \longrightarrow & L & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \mu(\widetilde{\mathcal{O}}) & \longrightarrow & \widetilde{\mathcal{O}}^\times & \longrightarrow & \widetilde{L} & \longrightarrow & 0
\end{array}$$

yields an exact sequence

$$1 \longrightarrow \frac{\mu(\widetilde{\mathcal{O}})}{\mu(\mathcal{O})} \longrightarrow \frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} \longrightarrow \frac{\widetilde{L}}{L} \longrightarrow 0$$

of finite groups, the last of which has order $R(\mathcal{O})/R(\widetilde{\mathcal{O}})$. $\square$

6.6. **Conclusion of the proof.** To complete the proof of Theorem 1.1, we compare (2) for $\widetilde{\mathcal{O}}$ to (2) for $\mathcal{O}$. The ratio of the left side of (2) for $\widetilde{\mathcal{O}}$ to the left side of (2) for $\mathcal{O}$ is

$$\lim_{s\to 1} \frac{\zeta_{\widetilde{\mathcal{O}}}(s)}{\zeta_{\mathcal{O}}(s)}.$$

The ratio of the right sides is

$$\left|\frac{\operatorname{Disc}\widetilde{\mathcal{O}}}{\operatorname{Disc}\mathcal{O}}\right|^{-1/2} \left(\frac{w(\widetilde{\mathcal{O}})}{w(\mathcal{O})}\right)^{-1} \cdot \frac{h(\widetilde{\mathcal{O}})}{h(\mathcal{O})} \cdot \frac{R(\widetilde{\mathcal{O}})}{R(\mathcal{O})}.$$

By Propositions 6.1, 6.2, 6.10, and 6.12 and the definition of $\operatorname{Disc}\mathcal{O}$, both ratios equal

$$\prod_{\mathfrak{p}} \frac{\prod_{\mathfrak{P}|\mathfrak{p}} (1-q_{\mathfrak{P}}^{-1})^{-1}}{\left(1-q_{\mathfrak{p}}^{-1}\right)^{-1}}.$$

7. Example 1: a fiber product of rings

Let p be an odd prime. Consider the fiber product of *rings*[1]

$$\mathcal{O} := \mathbb{Z}[1/2] \times_{\mathbb{F}_p} \mathbb{F}_p[t] = \{\, (a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] : a \equiv f(0) \pmod{p} \,\}.$$

Then $\widetilde{\mathcal{O}} = \mathbb{Z}[1/2] \times \mathbb{F}_p[t]$ inside $K = \mathbb{Q} \times \mathbb{F}_p(t)$. Thus $\widetilde{X} := \operatorname{Spec}\widetilde{\mathcal{O}}$ is the disjoint union of two "curves" $\operatorname{Spec}\mathbb{Z}[1/2] \amalg \operatorname{Spec}\mathbb{F}_p[t]$, and $X := \operatorname{Spec}\mathcal{O}$ is the same except that the points $(p) \in \operatorname{Spec}\mathbb{Z}[1/2]$ and $(t) \in \operatorname{Spec}\mathbb{F}_p[t]$ are attached. Define

$$\begin{aligned} \mathfrak{p} &:= \{\, (a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] : a \equiv f(0) \equiv 0 \pmod{p} \,\} \\ \mathfrak{P} &:= \{\, (a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] : a \equiv 0 \pmod{p} \,\} \\ \mathfrak{P}' &:= \{\, (a,f) \in \mathbb{Z}[1/2] \times \mathbb{F}_p[t] : f(0) \equiv 0 \pmod{p} \,\}. \end{aligned}$$

Then $\mathfrak{p}$ is a prime of $\mathcal{O}$ (the point of attachment), and $\mathfrak{P}$ and $\mathfrak{P}'$ are the primes of $\widetilde{\mathcal{O}}$ lying above $\mathfrak{p}$. The conductor of $\mathcal{O}$ is $\mathfrak{p}$ viewed as an $\widetilde{\mathcal{O}}$-ideal.

Propositions 7.1 and 7.2 below verify Theorem 1.1 for $\mathcal{O}$ by computing the two sides of (2) independently.

Proposition 7.1. *We have*

$$\lim_{s\to 1} (s-1)^2 \zeta_X(s) = \frac{1-p^{-1}}{2\log p}.$$

[1]A fiber product of rings does not correspond to a fiber product of schemes.

Proof. We have

$$\lim_{s\to 1}(s-1)\zeta_{\mathbb{Z}}(s) = 1,$$
$$\lim_{s\to 1}(s-1)\zeta_{\mathbb{F}_p[t]}(s) = \lim_{s\to 1}\frac{s-1}{1-p^{1-s}} = \frac{1}{\log p}.$$

Taking zeta functions of

$$X - \{\mathfrak{p}\} = \widetilde{X} - \{\mathfrak{P}, \mathfrak{P}'\} = (\operatorname{Spec}\mathbb{Z} - \{(2),(p)\}) \amalg (\operatorname{Spec}\mathbb{F}_p[t] - \{(t)\})$$

yields

$$\begin{aligned}
(1-p^{-s})\,\zeta_X(s) &= (1-2^{-s})(1-p^{-s})\,\zeta_{\mathbb{Z}}(s)\cdot(1-p^{-s})\,\zeta_{\mathbb{F}_p[t]}(s)\\
(s-1)^2\,\zeta_X(s) &= (1-2^{-s})(1-p^{-s})\,\left((s-1)\,\zeta_{\mathbb{Z}}(s)\right)\big((s-1)\,\zeta_{\mathbb{F}_p[t]}(s)\big)\\
\lim_{s\to 1}(s-1)^2\zeta_X(s) &= (1-2^{-1})(1-p^{-1})\cdot 1\cdot\frac{1}{\log p} = \frac{1-p^{-1}}{2\log p}.
\end{aligned}$$

□

Proposition 7.2. *We have*

$$\frac{2^{r_1}\,(2\pi)^{r_2}\,h(\mathcal{O})\,R(\mathcal{O})\prod_{v\in S_{\text{nonarch}}}\left((1-q_v^{-1})\,/\log q_v\right)}{w(\mathcal{O})\,|\operatorname{Disc}\mathcal{O}|^{1/2}} = \frac{1-p^{-1}}{2\log p}.$$

Proof. First, $r_1 = 1$ and $r_2 = 0$. The set S_{nonarch} consists of the place 2 of $\mathbb{Q}$ and the place ∞ of $\mathbb{F}_p(t)$. By definition,

$$\operatorname{Disc}\mathcal{O} = (\widetilde{\mathcal{O}}:\mathcal{O})^2\,(\operatorname{Disc}\mathbb{Q})\,(\operatorname{Disc}\mathbb{F}_p(t)) = p^2\cdot 1\cdot p^{2\cdot 0-2} = 1.$$

Inside $\widetilde{\mathcal{O}}^\times = \mathbb{Z}[1/2]^\times \times \mathbb{F}_p^\times = \pm\{2^n\}_{n\in\mathbb{Z}} \times \mathbb{F}_p^\times$, we have

$$\mathcal{O}^\times = \{\pm(2^n,\, 2^n \bmod p) : n \in \mathbb{Z}\}.$$

In particular, $\mu(\mathcal{O}) = \{\pm 1\}$, so $w(\mathcal{O}) = 2$. Since $\mathcal{O}^\times$ and $\widetilde{\mathcal{O}}^\times$ agree modulo torsion,

$$R(\mathcal{O}) = R(\widetilde{\mathcal{O}}) = R(\mathbb{Z}[1/2])\,R(\mathbb{F}_p[t]) = (\log 2)\cdot 1 = \log 2.$$

By Lemma 6.3,

$$\frac{\widetilde{\mathcal{O}}_{\mathfrak{p}}^\times}{\mathcal{O}_{\mathfrak{p}}^\times} \simeq \frac{(\widetilde{\mathcal{O}}/\mathfrak{P})^\times \times (\widetilde{\mathcal{O}}/\mathfrak{P}')^\times}{(\mathcal{O}/\mathfrak{p})^\times} \simeq \frac{\mathbb{F}_p^\times \times \mathbb{F}_p^\times}{\mathbb{F}_p^\times},$$

in which the denominator $\mathbb{F}_p^\times$ is embedded diagonally in $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$. Substituting into (3) yields

$$1 \longrightarrow \frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} \longrightarrow \frac{\mathbb{F}_p^\times \times \mathbb{F}_p^\times}{\mathbb{F}_p^\times} \longrightarrow \operatorname{Pic}\mathcal{O} \longrightarrow \operatorname{Pic}\widetilde{\mathcal{O}} \longrightarrow 1.$$

The subgroup $1\times\mathbb{F}_p^\times$ of $\widetilde{\mathcal{O}}^\times$ surjects onto $\dfrac{\mathbb{F}_p^\times \times \mathbb{F}_p^\times}{\mathbb{F}_p^\times}$, and $\operatorname{Pic}\widetilde{\mathcal{O}} = \operatorname{Pic}\mathbb{Z}[1/2]\times\operatorname{Pic}\mathbb{F}_p[t] = \{1\}$, so $\operatorname{Pic}\mathcal{O} = \{1\}$. Thus $h(\mathcal{O}) = 1$.

Substituting all these values shows that the expression to be computed equals

$$\frac{2^1\,(2\pi)^0\cdot 1\cdot(\log 2)\cdot\left((1-2^{-1})/\log 2\right)\left((1-p^{-1})/\log p\right)}{2\cdot 1^{1/2}} = \frac{1-p^{-1}}{2\log p}.$$

□

8. Example 2: a non-maximal order in a real quadratic number field

Let $\mathcal{O} = \mathbb{Z}[\sqrt{d}]$, where d is an integer such that $d \geq 5$ and $d \equiv 5 \pmod 8$ (other cases could be handled similarly). Then $\widetilde{\mathcal{O}} = \mathbb{Z}[(1+\sqrt{d})/2]$. Above the prime ideal $\mathfrak{p} := (2, 1+\sqrt{d})$ of $\mathcal{O}$ with residue field $\mathbb{F}_2$ lies the prime ideal (2) of $\widetilde{\mathcal{O}}$ with residue field $\mathbb{F}_4$. The scheme $X := \operatorname{Spec} \mathcal{O}$ is analogous to a nodal curve for which the two slopes at the node $\mathfrak{p}$ are conjugate in the quadratic extension $\mathbb{F}_4$ of $\mathbb{F}_2$. Let $\tilde{\epsilon}$ be the fundamental unit of $\widetilde{\mathcal{O}}^\times$, so $\tilde{\epsilon} > 1$ for the standard real embedding $\widetilde{\mathcal{O}} \hookrightarrow \mathbb{R}$. Let n be the order of the image of $\tilde{\epsilon}$ in $(\widetilde{\mathcal{O}}/(2))^\times \simeq \mathbb{F}_4^\times$, so n is 1 or 3. Since $\mathcal{O}$ is the preimage of $\mathbb{F}_2$ under $\widetilde{\mathcal{O}} \twoheadrightarrow \mathbb{F}_4$, the element $\epsilon := \tilde{\epsilon}^n$ is the smallest power of $\tilde{\epsilon}$ lying in $\mathcal{O}^\times$.

Propositions 8.1 and 8.2 below verify Theorem 1.1 for $\mathcal{O}$ by computing the two sides of (2) independently.

Proposition 8.1. *We have*

$$\lim_{s \to 1} (s-1)\zeta_{\mathcal{O}}(s) = \frac{3\, h(\widetilde{\mathcal{O}})\, \log \tilde{\epsilon}}{2\sqrt{d}}.$$

Proof. The classical analytic class number formula for $\widetilde{\mathcal{O}}$, with $r_1 = 1$, $r_2 = 0$, $R(\widetilde{\mathcal{O}}) = \log \tilde{\epsilon}$, $w(\widetilde{\mathcal{O}}) = 2$, $\operatorname{Disc} \widetilde{\mathcal{O}} = d$, yields

$$\lim_{s \to 1} (s-1)\zeta_{\widetilde{\mathcal{O}}}(s) = \frac{h(\widetilde{\mathcal{O}})\, \log \tilde{\epsilon}}{\sqrt{d}}.$$

On the other hand, Proposition 6.1 with $q_{\mathfrak{p}} = 2$ and $q_{\mathfrak{P}} = 4$ yields

$$\lim_{s \to 1} \frac{\zeta_{\widetilde{\mathcal{O}}}(s)}{\zeta_{\mathcal{O}}(s)} = \frac{(1-1/4)^{-1}}{(1-1/2)^{-1}} = \frac{2}{3}.$$

Dividing the first equation by the second gives the result. □

Proposition 8.2. *We have*

$$\frac{2^{r_1}\, (2\pi)^{r_2}\, h(\mathcal{O})\, R(\mathcal{O}) \prod_{v \in S_{\text{nonarch}}} \left((1 - q_v^{-1}) \,/\, \log q_v \right)}{w(\mathcal{O})\, |\operatorname{Disc} \mathcal{O}|^{1/2}} = \frac{3\, h(\widetilde{\mathcal{O}})\, \log \tilde{\epsilon}}{2\sqrt{d}}.$$

Proof. First, $r_1 = 1$, $r_2 = 0$, and $w(\mathcal{O}) = 2$. By definition, $S_{\text{nonarch}} = \emptyset$. By Proposition 6.2, $\operatorname{Disc} \mathcal{O} = (\widetilde{\mathcal{O}} : \mathcal{O})^2 \operatorname{Disc} \widetilde{\mathcal{O}} = 4d$, and $R(\mathcal{O}) = \log \epsilon = n \log \tilde{\epsilon}$. The exact sequence (3) is

$$1 \longrightarrow \frac{\widetilde{\mathcal{O}}^\times}{\mathcal{O}^\times} \longrightarrow \frac{\mathbb{F}_4^\times}{\mathbb{F}_2^\times} \longrightarrow \operatorname{Pic} \mathcal{O} \longrightarrow \operatorname{Pic} \widetilde{\mathcal{O}} \longrightarrow 1,$$

so $h(\mathcal{O}) = (3/n)\, h(\widetilde{\mathcal{O}})$. Thus the expression to be computed equals

$$\frac{2^1\, (2\pi)^0 \cdot (3/n)\, h(\widetilde{\mathcal{O}}) \cdot n \log \tilde{\epsilon}}{2 \cdot (4d)^{1/2}} = \frac{3\, h(\widetilde{\mathcal{O}})\, \log \tilde{\epsilon}}{2\sqrt{d}}.$$ □

Acknowledgments

It is a pleasure to thank Tony Scholl for helpful discussions. We thank Carlos J. Moreno for bringing the article [Stö98] to our attention, and John Voight for suggesting the reference [Wei82]. Finally, we thank the referee for several excellent suggestions.

References

[BL17] Alex Bartel and Hendrik W. Lenstra Jr., *Commensurability of automorphism groups*, Compos. Math. **153** (2017), no. 2, 323–346, DOI 10.1112/S0010437X1600823X. MR3705226

[Cas67] J. W. S. Cassels, *Global fields*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, pp. 42–84. MR0222054

[DE14] Anton Deitmar and Siegfried Echterhoff, *Principles of harmonic analysis*, 2nd ed., Universitext, Springer, Cham, 2014. MR3289059

[Del73] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math., vol. 349, Springer, Berlin, 1973, pp. 501–597 (French). MR0349635 (50 #2128)

[Den91] Christopher Deninger, *On the Γ-factors attached to motives*, Invent. Math. **104** (1991), no. 2, 245–261, DOI 10.1007/BF01245075. MR1098609

[Dir94] P. G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie*, 4th ed., Braunschweig, 1894. Edited by and with supplements by R. Dedekind.

[Gal73] V. M. Galkin, *Zeta-functions of certain one-dimensional rings*, Izv. Akad. Nauk SSSR Ser. Mat. **37** (1973), 3–19 (Russian). MR0332729

[Gre89] Barry Green, *Functional equations for zeta functions of non-Gorenstein orders in global fields*, Manuscripta Math. **64** (1989), no. 4, 485–502, DOI 10.1007/BF01170941. MR1005249

[EGA IV_3] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966). Written in collaboration with J. Dieudonné. MR0217086 (36 #178)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[Hec17] E. Hecke, *Über die Zetafunktion beliebiger algebraischer Zahlkörper*, Nachr. d. K. Gesellschaft d. Wiss. zu Göttingen, Math.-Phys. Kl. **1** (1917), 77–89.

[Hil97] David Hilbert, *Die Theorie der algebraische Zahlkörper*, Jahresbericht der Deutschen Mathematiker-Vereinigung **4** (1897), 175–546; English transl., David Hilbert, *The theory of algebraic number fields* (1998), xxxvi+350. Translated from the German and with a preface by Iain T. Adamson; With an introduction by Franz Lemmermeyer and Norbert Schappacher. MR1646901 (99j:01027).

[Jen69] W. E. Jenner, *On zeta functions of number fields*, Duke Math. J. **36** (1969), 669–671. MR0249394

[Neu99] Jürgen Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher; With a foreword by G. Harder. MR1697859 (2000m:11104)

[Sch31] Friedrich Karl Schmidt, *Analytische Zahlentheorie in Körpern der Charakteristik p*, Math. Z. **33** (1931), no. 1, 1–32, DOI 10.1007/BF01174341 (German). MR1545199

[Ser65] Jean-Pierre Serre, *Zeta and L functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper & Row, New York, 1965, pp. 82–92. MR0194396 (33 #2606)

[Stö98] Karl-Otto Stöhr, *Local and global zeta-functions of singular algebraic curves*, J. Number Theory **71** (1998), no. 2, 172–202, DOI 10.1006/jnth.1998.2240. MR1633801

[Tat67] J. T. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, pp. 305–347. MR0217026

[Wei67] André Weil, *Basic number theory*, Die Grundlehren der mathematischen Wissenschaften, Band 144, Springer-Verlag New York, Inc., New York, 1967. MR0234930 (38 #3244)

[Wei82] André Weil, *Adeles and algebraic groups*, Progress in Mathematics, vol. 23, Birkhäuser, Boston, Mass., 1982. With appendices by M. Demazure and Takashi Ono. MR670072

[ZG97a] W. A. Zúñiga Galindo, *Zeta functions and Cartier divisors on singular curves over finite fields*, Manuscripta Math. **94** (1997), no. 1, 75–88, DOI 10.1007/BF02677839. MR1468935

[ZG97b] W. A. Zúñiga-Galindo, *Zeta functions of singular curves over finite fields*, Rev. Colombiana Mat. **31** (1997), no. 2, 115–124. MR1667594

Department of Mathematics, Baruch College, The City University of New York, One Bernard Baruch Way, New York, NY 10010-5526, USA
Email address: bruce.jordan@baruch.cuny.edu

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA
Email address: poonen@math.mit.edu
URL: http://math.mit.edu/~poonen/