REAL REPRESENTATIONS

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The goal of these notes is to explain the classification of real representations of a finite group. Throughout, G is a finite group, W is a $\mathbb{R}$-vector space or $\mathbb{R}G$-module, and V is a $\mathbb{C}$-vector space or $\mathbb{C}G$-module (except in Section 2, where V is over any field). Vector spaces and representations are assumed to be finite-dimensional.

1. Vector spaces over $\mathbb{R}$ and $\mathbb{C}$

1.1. **Constructions.** To get from $\mathbb{R}^n$ to $\mathbb{C}^n$, we can tensor with $\mathbb{C}$. In a more coordinate-free manner, if W is an $\mathbb{R}$-vector space, then its complexification $W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-vector space. We can view W as an $\mathbb{R}$-subspace of $W_{\mathbb{C}}$ by identifying each $w \in W$ with $w \otimes 1 \in W_{\mathbb{C}}$. Then an $\mathbb{R}$-basis of W is also a $\mathbb{C}$-basis of $W_{\mathbb{C}}$. In particular, $W_{\mathbb{C}}$ has the same dimension as W (but is a vector space over a different field).

Conversely, we can view $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ if we forget how to multiply by complex scalars that are not real. In a more coordinate-free manner, if V is a $\mathbb{C}$-vector space, then its restriction of scalars is the $\mathbb{R}$-vector space ${}_{\mathbb{R}}V$ with the same underlying abelian group but with only scalar multiplication by real numbers. If $v_1, \ldots, v_n$ is a $\mathbb{C}$-basis of V, then $v_1, iv_1, \ldots, v_n, iv_n$ is an $\mathbb{R}$-basis of ${}_{\mathbb{R}}V$. In particular, $\dim({}_{\mathbb{R}}V) = 2 \dim V$.

Also, if V is a $\mathbb{C}$-vector space, then the complex conjugate vector space $\overline{V}$ has the same underlying group but a new scalar multiplication $\cdot$ defined by $\lambda \cdot v := \bar{\lambda} v$, where $\bar{\lambda} v$ is defined using the original scalar multiplication.

Complexification and restriction of scalars are not inverse constructions. Instead:

Proposition 1.1 (Complexification and restriction of scalars)**.**

(a) *If V is a $\mathbb{C}$-vector space, then the map*

$$\begin{aligned} ({}_{\mathbb{R}}V)_{\mathbb{C}} &\longrightarrow V \oplus \overline{V} \\ v \otimes c &\longmapsto (cv, \bar{c}v) \end{aligned}$$

is an isomorphism of $\mathbb{C}$-vector spaces.

(b) *If W is an $\mathbb{R}$-vector space, then*

$$_{\mathbb{R}}(W_{\mathbb{C}}) \simeq W \oplus W.$$

Proof.

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(a) The map is $\mathbb{C}$-linear, by definition of the scalar multiplication on $\overline{V}$. It sends $x\otimes 1+y\otimes i$ to $(x+iy, x-iy)$, and one can recover $x, y \in V$ uniquely from $(x+iy, x-iy)$, so the map is an isomorphism.

(b) We have ${}_{\mathbb{R}}(W \otimes_{\mathbb{R}} \mathbb{C}) = W \otimes_{\mathbb{R}} (\mathbb{R} \oplus i\mathbb{R}) = W \oplus iW \simeq W \oplus W$. $\square$

1.2. **Linear maps between complexifications.** Tensoring $\mathrm{M}_{m,n}(\mathbb{R})$ with $\mathbb{C}$ yields $\mathrm{M}_{m,n}(\mathbb{C})$. The coordinate-free version of this is:

Proposition 1.2. *If W and X are $\mathbb{R}$-vector spaces, then*

$$\mathrm{Hom}_{\mathbb{R}}(W, X) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{C}}(W_{\mathbb{C}}, X_{\mathbb{C}}).$$

Corollary 1.3. *If W is an $\mathbb{R}$-vector space, then*

$$\mathrm{End}_{\mathbb{R}}(W) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{End}_{\mathbb{C}}(W_{\mathbb{C}}).$$

1.3. **Descent theory.** Let V and X be $\mathbb{C}$-vector spaces. A homomorphism $J\colon V \to X$ of abelian groups is called $\mathbb{C}$-antilinear if $J(\lambda v) = \bar{\lambda}\, J(v)$ for all $\lambda \in \mathbb{C}$ and $v \in V$; to give such a J is equivalent to giving a $\mathbb{C}$-linear map $V \to \overline{X}$.

To recover $\mathbb{R}^n$ from its complexifcation $\mathbb{C}^n$ one takes the vectors fixed by coordinate-wise complex conjugation. More generally, given a $\mathbb{C}$-vector space V, finding a $\mathbb{R}$-vector space W such that $W_{\mathbb{C}} \simeq V$ is equivalent to finding a "complex conjugation" on V; more precisely:

Proposition 1.4. *There is an equivalence of categories*

$$\begin{aligned} \{\mathbb{R}\text{-vector spaces}\} &\leftrightarrow \{\mathbb{C}\text{-vector spaces equipped with } \mathbb{C}\text{-antilinear } J\colon V \to V \text{ such that } J^2 = 1\} \\ W &\mapsto (W_{\mathbb{C}}, 1_W \otimes (\text{complex conjugation})) \\ V^J := \{v \in V : Jv = v\} &\leftarrow\!\shortmid (V, J). \end{aligned}$$

Sketch of proof. The only tricky part is to show that given (V, J), the map $V^J \otimes_{\mathbb{R}} \mathbb{C} \to V$ sending $v \otimes c$ to cv is an isomorphism. For this, one can write down the inverse: map $v \in V$ to $\frac{1}{2}(v + Jv) \otimes 1 + \frac{1}{2i}(v - Jv) \otimes i \in V^J \otimes_{\mathbb{R}} \mathbb{C}$. $\square$

Remark 1.5. More generally, given any Galois extension of fields L/k, an action of $\mathrm{Gal}(L/k)$ on an L-vector space V is called semilinear if scalar multiplication is compatible with the actions of $\mathrm{Gal}(L/k)$ on L and V, that is, if ${}^g(\ell v) = ({}^g\ell)({}^g v)$ for all $g \in \mathrm{Gal}(L/k)$, $\ell \in L$ and $v \in V$. Then the category of k-vector spaces is equivalent to the category of L-vector spaces equipped with a semilinear $\mathrm{Gal}(L/k)$-action. This is called descent, since it specifies what extra structure is needed on an L-vector space to make it "descend" to a k-vector space.

1.4. **Representations.** All the constructions and propositions above are natural. In particular, if G acts on W, then it acts on any of the spaces constructed from W, and likewise for V. In particular,

- If W is an $\mathbb{R}G$-module, then $W_{\mathbb{C}}$ is a $\mathbb{C}G$-module, and the matrix of $g \in G$ acting on W with respect to a basis is the same as the matrix of g acting on $W_{\mathbb{C}}$, so $\chi_{W_{\mathbb{C}}} = \chi_W$.
- If V is a $\mathbb{C}G$-module, then $\overline{V}$ is another $\mathbb{C}G$-module, and $\chi_{\overline{V}} = \overline{\chi}_V$.
- If V is a $\mathbb{C}G$-module, then ${}_{\mathbb{R}}V$ is an $\mathbb{R}G$-module. Taking the characters of both sides in Proposition 1.1 shows that $\chi_{{}_{\mathbb{R}}V} = \chi_V + \overline{\chi}_V$.

A $\mathbb{C}$-representation V of G is said to be realizable over $\mathbb{R}$ if $V \simeq W_{\mathbb{C}}$ for some $\mathbb{R}$-representation W of G. This implies that χ_V is real-valued, but we will see that the converse can fail.

2. Pairings

2.1. **Bilinear forms.** Let V be a (finite-dimensional) vector space over any field k. A function $B\colon V \times V \to k$ is bi-additive if it is an additive homomorphism in each argument when the other is fixed; that is, $B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$ for all $v_1, v_2, w \in V$, and $B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2)$ for all $v, w_1, w_2 \in V$. The left kernel of B is $\{v \in V : B(v, w) = 0 \text{ for all } w \in V\}$, and the right kernel is defined similarly.

A function $B\colon V \times V \to k$ is a bilinear form (or bilinear pairing) if it is k-linear in each argument; that is, B is bi-additive and $B(\lambda v, w) = \lambda\, B(v, w)$ and $B(v, \lambda w) = \lambda\, B(v, w)$ for all $\lambda \in k$ and $v, w \in V$. We have

$$\{\text{bilinear forms on } V\} \simeq \operatorname{Hom}(V \otimes V, k) \simeq (V \otimes V)^* \simeq V^* \otimes V^* \simeq \operatorname{Hom}(V, V^*).$$

(here Hom is Hom_k, and $\otimes$ is $\otimes_k$).

Let B be a bilinear form.

- Call B symmetric if $B(v, w) = B(w, v)$ for all $v, w \in V$.
- Call B skew-symmetric if $B(v, w) = -B(w, v)$ for all $v, w \in V$.
- Call B alternating if $B(v, v) = 0$ for all $v \in V$.

If $\operatorname{char} k \neq 2$, then alternating and skew-symmetric are equivalent. (If $\operatorname{char} k = 2$, then alternating is the stronger and better-behaved condition.) The map sending B to the pairing $(x, y) \mapsto B(y, x)$ is a linear automorphism of order 2 of the space of bilinear forms, so if $\operatorname{char} k \neq 2$, it decomposes the space into $+1$ and -1 eigenspaces:

$$\{\text{bilinear forms}\} = \{\text{symmetric bilinear forms}\} \oplus \{\text{skew-symmetric bilinear forms}\},$$

which is the same as the decomposition

$$(V \otimes V)^* \simeq (\operatorname{Sym}^2 V)^* \oplus (\textstyle\bigwedge^2 V)^*.$$

2.2. **Sesquilinear and hermitian forms.** Now let V be a $\mathbb{C}$-vector space.

- A sesquilinear form (or sesquilinear pairing) is a bi-additive pairing $(\ ,\ )$ that is $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-antilinear in the second variable; that is $(\lambda v, w) = \lambda (v, w)$

and $(v, \lambda w) = \bar{\lambda}(v, w)$ for all $\lambda \in \mathbb{C}$ and $v, w \in V$. (The prefix "sesqui" means $1\frac{1}{2}$: the form is only $\mathbb{R}$-linear in the second argument.)

- A hermitian form (or hermitian pairing) is a bi-additive pairing $(\ ,\ )$ such that $(\lambda v, w) = \lambda(v, w)$ and $(w, v) = \overline{(v, w)}$ for all $\lambda \in \mathbb{C}$ and $v, w \in V$.

A hermitian pairing is sesquilinear. We have

$$\{\text{sesquilinear forms on } V\} \simeq \operatorname{Hom}(V \otimes \overline{V}, \mathbb{C}) \simeq (V \otimes \overline{V})^* \simeq V^* \otimes \overline{V}^* \simeq \operatorname{Hom}(\overline{V}, V^*).$$

2.3. **Nondegenerate and positive definite forms.** A bilinear form (or sesquilinear form) is called nondegenerate if its left kernel is 0, or equivalently its right kernel is 0, or equivalently the associated homomorphism $V \to V^*$ (respectively, $\overline{V} \to V^*$) is an isomorphism.

Suppose that $(\ ,\ )$ is either a bilinear form on an $\mathbb{R}$-vector space or a hermitian form on a $\mathbb{C}$-vector space. Then $(v, v) \in \mathbb{R}$ for all v. Call $(\ ,\ )$ positive definite if $(v, v) > 0$ for all nonzero $v \in V$. Positive definite forms are automatically nondegenerate.

3. Characters of symmetric and alternating squares

Let V be an n-dimensional $\mathbb{C}$-representation of G. If $g \in G$ acts on V with eigenvalues $\lambda_1, \ldots, \lambda_n$ (listed with multiplicity), then the eigenvalues of g acting on associated vector spaces are as follows:

Representation	Dimension	Eigenvalues
V	n	$\lambda_1, \ldots, \lambda_n$
$\overline{V}$	n	$\bar{\lambda}_1, \ldots, \bar{\lambda}_n$
V^*	n	$\bar{\lambda}_1, \ldots, \bar{\lambda}_n$
$V \otimes V$	n^2	$\lambda_i \lambda_j$ for all (i, j)
$\operatorname{Sym}^2 V$	$n(n+1)/2$	$\lambda_i \lambda_j$ for $i \le j$
$\bigwedge^2 V$	$n(n-1)/2$	$\lambda_i \lambda_j$ for $i < j$

These are obvious if V has a basis of eigenvectors (i.e., $\rho(g)$ is diagonalizable). In general, we have the Jordan decomposition $\rho(g) = d + n$, where d is diagonalizable and n is nilpotent, and $dn = nd$; then d and n induce commuting diagonalizable endomorphisms and nilpotent endomorphisms of each of the other representations, so the eigenvalues of g are the same as the eigenvalues of d on each of them.

4. Classification of division algebras over $\mathbb{R}$

Lemma 4.1. *The only finite-dimensional field extensions of $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$.*

Proof. The fundamental theorem of algebra states that $\mathbb{C}$ is algebraically closed, so every finite extension of $\mathbb{R}$ embeds in $\mathbb{C}$. Since $[\mathbb{C} : \mathbb{R}] = 2$, there is no room for other fields in between. ☐

Theorem 4.2 (Frobenius 1877). *The only finite-dimensional (associative) division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.*

Proof. Let D be a finite-dimensional (associative) division algebras over $\mathbb{R}$ not equal to $\mathbb{R}$ or $\mathbb{C}$. For any $d \in D - \mathbb{R}$, the $\mathbb{R}$-subalgebra $\mathbb{R}[d] \subseteq D$ generated by d is a commutative domain of finite dimension over a field, so it is a field extension of finite degree over $\mathbb{R}$, hence a copy of $\mathbb{C}$. Fix one such copy, and let i be a $\sqrt{-1}$ in it. View D as a left $\mathbb{C}$-vector space. Conjugation by i on D (the map $x \mapsto ixi^{-1}$) is a $\mathbb{C}$-linear automorphism of D, and it is of order 2 since conjugation by $i^2 = -1$ is the identity, so it decomposes D into $+1$ and -1 eigenspaces D^+ and D^-. Explicitly,

$$D^+ = \{x : ixi^{-1} = x\} = \{x \text{ that commute with } i\} \supseteq \mathbb{C}$$
$$D^- = \{x : ixi^{-1} = -x\}.$$

If $x \in D^+$, then $\mathbb{C}[x]$ is commutative, hence a finite field extension of $\mathbb{C}$, but $\mathbb{C}$ is algebraically closed, so $\mathbb{C}[x] = \mathbb{C}$, so $x \in \mathbb{C}$. Thus $D^+ = \mathbb{C}$.

Since $D \neq \mathbb{C}$, we have $D^- \neq 0$. Choose $j \in D^-$ such that $j \neq 0$. Right multiplication by j defines a $\mathbb{C}$-linear map $D^+ \to D^-$ (if $d \in D^+$, then $i(dj)i^{-1} = (idi^{-1})(iji^{-1}) = d(-j) = -dj$, so $dj \in D^-$), and it is injective since D is a division algebra. Thus $\dim_{\mathbb{C}} D^- \leq \dim_{\mathbb{C}} D^+ = 1$. Hence $D^- = \mathbb{C}j$. Since $\mathbb{R}[j]$ is another copy of $\mathbb{C}$, we have $j^2 \in \mathbb{R} + \mathbb{R}j$. On the other hand $j^2 \in D^+ = \mathbb{C}$. Thus $j^2 \in (\mathbb{R} + \mathbb{R}j) \cap \mathbb{C}$, which is $\mathbb{R}$, since $\mathbb{R} + \mathbb{R}j$ and $\mathbb{C}$ are different 2-dimensional subspaces in D. Also, $j^2 \neq 0$.

If $j^2 > 0$, then $j^2 = r^2$ for some $r \in \mathbb{R}$, so $(j+r)(j-r) = 0$, so $j = \pm r \in \mathbb{R}$, a contradiction since $D^- \cap \mathbb{R} = 0$.

Thus $j^2 < 0$. Scale j to assume that $j^2 = -1$. Then $D = \mathbb{C} + \mathbb{C}j = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}ij$ with $i^2 = -1$, $j^2 = -1$, and $ij = -ji$, so $D \simeq \mathbb{H}$. $\square$

If D is an $\mathbb{R}$-algebra, then $D \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-algebra.

Proposition 4.3. *We have*

$$\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}$$
$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$$
$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{M}_2(\mathbb{C}).$$

Proof. The first isomorphism is a special case of the general isomorphism $A \otimes_A B \simeq B$.

The map $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ sending $a \otimes b$ to $(ab, a\bar{b})$ is an isomorphism by Proposition 1.1, and it respects multiplication.

There is a $\mathbb{C}$-algebra homomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{M}_2(\mathbb{C})$. sending $h \otimes 1$ for each $h \in \mathbb{H}$ to the linear endomorphism $x \mapsto hx$ of the 2-dimensional right $\mathbb{C}$-vector space $\mathbb{H}$ with basis $1, j$.

Explicitly, we have

$$\begin{aligned}
1 \otimes 1 &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
i \otimes 1 &\longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
j \otimes 1 &\longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
ij \otimes 1 &\longmapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.
\end{aligned}$$

For example, to get the image of $i \otimes 1$, observe that

$$\begin{aligned}
i1 &= 1 \cdot i + j \cdot 0 \\
ij &= 1 \cdot 0 + j \cdot (-i).
\end{aligned}$$

The four matrices on the right are linearly independent over $\mathbb{C}$, so $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{M}_2(\mathbb{C})$ is an isomorphism of 4-dimensional $\mathbb{C}$-algebras. □

5. Real and complex representations

Let G be a finite group. Let W be an irreducible $\mathbb{R}$-representation of G. Let V be one irreducible $\mathbb{C}$-subrepresentation of $W_{\mathbb{C}}$. The following table gives facts about this situation.

D	$\mathrm{End}_G(W_{\mathbb{C}})$	$W_{\mathbb{C}}$	${}_{\mathbb{R}}V$	$\dim_{\mathbb{R}} W$	$\dim_{\mathbb{C}} V$	V realiz. over $\mathbb{R}$?	$V \simeq \overline{V}$? χ_V real-valued?	$V \simeq V^*$? $\exists$ G-inv. B?	$\mathrm{FS}(V)$
$\mathbb{R}$	$\mathbb{C}$	V	$W \oplus W$	n	n	YES	YES	YES (symmetric)	1
$\mathbb{C}$	$\mathbb{C} \times \mathbb{C}$	$V \oplus \overline{V}$	W	$2n$	n	NO	NO	NO	0
$\mathbb{H}$	$\mathrm{M}_2(\mathbb{C})$	$V \oplus V$	W	$4n$	$2n$	NO	YES	YES (skew-sym.)	-1

The columns are as follows:

- First, $D := \mathrm{End}_G W$. By Schur's lemma, D is a division algebra over $\mathbb{R}$, so D is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Accordingly, V is said to be of real type, complex type, or quaternionic type. Let n be the dimension of W as a right D-vector space.
- We have $\mathrm{End}_G(W_{\mathbb{C}}) \simeq (\mathrm{End}_G W) \otimes_{\mathbb{R}} \mathbb{C} = D \otimes_{\mathbb{R}} \mathbb{C}$ by taking G-invariants in Corollary 1.3.
- The $W_{\mathbb{C}}$ column gives the decomposition of $W_{\mathbb{C}}$ into irreducible $\mathbb{C}$-representations.
- The ${}_{\mathbb{R}}V$ column gives the decomposition of ${}_{\mathbb{R}}V$ into irreducible $\mathbb{R}$-representations.
- The $\dim_{\mathbb{R}} W$ column gives $\dim_{\mathbb{R}} W = [D : \mathbb{R}] \dim_D W = [D : \mathbb{R}]n$.

- The $\dim_{\mathbb{C}} V$ column entries follow from the $W_{\mathbb{C}}$ column and the column giving $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}}(W_{\mathbb{C}})$.
- Is V realizable over $\mathbb{R}$? That is, is $V \simeq X_{\mathbb{C}}$ for some $\mathbb{R}$-representation X of G?
- Is $V \simeq \overline{V}$ as a $\mathbb{C}$-representation of G? Equivalently, is $\chi_V = \overline{\chi}_V$? That is, is it true that $\chi_V(g) \in \mathbb{R}$ for all $g \in G$?
- Is $V \simeq V^*$ as a $\mathbb{C}$-representation of G? Since

$$\operatorname{Hom}(V, V^*) \simeq \{\text{bilinear forms on } V\}$$

as a $\mathbb{C}$-representation of G, and since isomorphisms correspond to nondegenerate bilinear forms, taking G-invariants shows that this question is the same as asking whether there exists a nondegenerate G-invariant bilinear form $B \colon V \times V \to \mathbb{C}$. We will show that in the cases where B exists, B is either symmetric or skew-symmetric.
- The Frobenius–Schur indicator of a $\mathbb{C}$-representation V of G is defined by

$$\operatorname{FS}(V) := \frac{1}{\#G} \sum_{g \in G} \chi_V(g^2).$$

Proof that the table is correct. Some columns have already been checked above. Let us now verify the rest.

$W_{\mathbb{C}}$ *column*: In general, if $V_1, \ldots, V_r$ are the irreducible $\mathbb{C}$-representations of G, and $X = \bigoplus_{i=1}^r n_i V_i$, then $\operatorname{End}_G X = \prod_{i=1}^r \operatorname{M}_{n_i}(\mathbb{C})$. Thus the $\operatorname{End}_G(W_{\mathbb{C}})$ column implies the $W_{\mathbb{C}}$ column, except that in the $\mathbb{C}$ case, we deduce only that $W_{\mathbb{C}} \simeq V \oplus V'$ for some distinct $\mathbb{C}$-representations V and V'. In that case, W has an action of $D = \mathbb{C}$, and hence $W = {}_{\mathbb{R}}\mathbb{W}$ for some $\mathbb{C}$-vector space $\mathbb{W}$; then $W_{\mathbb{C}} = ({}_{\mathbb{R}}\mathbb{W})_{\mathbb{C}} \simeq \mathbb{W} + \overline{\mathbb{W}}$, but then the Jordan–Hölder theorem implies that V, V' must be $\mathbb{W}, \overline{\mathbb{W}}$ in some order, so $V' \simeq \overline{V}$.

χ_V *real-valued column*: In the $\mathbb{R}$ case, $\chi_V = \chi_{W_{\mathbb{C}}} = \chi_W$, which is real-valued. In the $\mathbb{C}$ case, $V \not\simeq \overline{V}$, so χ_V is not real-valued. In the $\mathbb{H}$ case, $2\chi_V = \chi_{V \oplus V} = \chi_{W_{\mathbb{C}}} = \chi_W$, so χ_V is real-valued.

${}_{\mathbb{R}}V$ *column*: Since V is a subrepresentation of $W_{\mathbb{C}}$, the restriction of scalars ${}_{\mathbb{R}}V$ is a subrepresentation of ${}_{\mathbb{R}}(W_{\mathbb{C}})$, which is isomorphic to $W \oplus W$ by Proposition 1.1(b). Thus ${}_{\mathbb{R}}V$ is a direct sum of copies of W. If $D = \mathbb{R}$, then $V = W_{\mathbb{C}}$, so ${}_{\mathbb{R}}V \simeq W \oplus W$. If D is $\mathbb{C}$ or $\mathbb{H}$, then V is half the dimension of $W_{\mathbb{C}}$, so $V \simeq W$.

Realizability over $\mathbb{R}$: In the $\mathbb{R}$ case, $V \simeq W_{\mathbb{C}}$, so V is realizable by definition. In the $\mathbb{C}$ and $\mathbb{H}$ cases, if $V \simeq X_{\mathbb{C}}$ for some $\mathbb{R}$-representation X, then $W \simeq {}_{\mathbb{R}}V \simeq {}_{\mathbb{R}}(X_{\mathbb{C}}) \simeq X \oplus X$ by Proposition 1.1(b), contradicting the irreducibility of W.

Nondegenerate G-invariant bilinear form: The averaging argument shows that there exists a positive definite G-invariant hermitian form $(\ ,\ )$ on V. Fix one; it defines an isomorphism $\overline{V} \to V^*$. Thus $V \simeq \overline{V}$ if and only if $V \simeq V^*$, so these two columns have the same YES/NO answers. By Section 2.1, we have isomorphisms

$$\operatorname{Hom}(V, V^*) \simeq \{\text{symmetric bilinear forms}\} \oplus \{\text{skew-symmetric bilinear forms}\}.$$

Taking G-invariants yields

$$\operatorname{Hom}_G(V, V^*) \simeq \{G\text{-invariant symm. bilinear forms}\} \oplus \{G\text{-invariant skew-symm. bilinear forms}\}.$$

Suppose that $V \simeq V^*$. Then $\operatorname{Hom}_G(V, V^*) \simeq \operatorname{End}_G V \simeq \mathbb{C}$ by Schur's lemma, so there exists a unique nondegenerate G-invariant bilinear form B up to a scalar in $\mathbb{C}^\times$, and it is either symmetric or skew-symmetric. Since B is nondegenerate, the $\mathbb{C}$-linear functional $(-, w)$ equals $B(-, Jw)$ for a unique $Jw \in V$. Then $J := V \to V$ is $\mathbb{C}$-antilinear, and it is an isomorphism since $(\ ,\ )$ too is nondegenerate. Now J^2 is a $\mathbb{C}$-*linear* automorphism of the representation V, so by Schur's lemma, J^2 is multiplication-by-r for some $r \in \mathbb{C}^\times$. Also by Schur's lemma, every other $\mathbb{C}$-antilinear G-equivariant isomorphism is cJ for some $c \in \mathbb{C}$, and replacing J by cJ changes r to $c\bar{c}r$ (Proof: For $v \in V$, if $JJv = rv$, then $cJ(cJ(v)) = c\bar{c}J(J(v)) = c\bar{c}rv$).

- If B is symmetric, then for any choice of nonzero $v \in V$,

$$(Jv, Jv) = B(Jv, J^2v) = B(Jv, rv) = rB(Jv, v) = rB(v, Jv) = r(v, v)$$

  but $(\ ,\ )$ is positive definite, so r is a positive real number.
- If B is skew-symmetric, the same calculation shows that r is a negative real number.

Finally, the following are equivalent:

- V is realizable over $\mathbb{R}$
- We can choose $c \in \mathbb{C}^\times$ so that $(cJ)^2 = 1$.
- We can choose $c \in \mathbb{C}^\times$ so that $c\bar{c}r = 1$.
- r is positive.
- B is symmetric.

Frobenius–Schur indicator: We have

$$\begin{aligned}
\overline{\mathrm{FS}(V)} &= \frac{1}{\#G} \sum_g \chi_{V^*}(g^2) \\
&= \frac{1}{\#G} \sum_g \left(\chi_{(\mathrm{Sym}^2 V)^*}(g) - \chi_{(\bigwedge^2 V)^*}(g) \right) \quad \text{(by the formulas in Section 3)} \\
&= (\mathbb{C}, (\mathrm{Sym}^2 V)^*) - (\mathbb{C}, (\textstyle\bigwedge^2 V)^*) \\
&= \dim\{G\text{-invariant symm. bilinear forms}\} - \dim\{G\text{-invariant skew-symm. bilinear forms}\} \\
&= \begin{cases} 1 - 0 \\ 0 - 0 \\ 0 - 1 \end{cases} = \begin{cases} 1, & \text{if } D = \mathbb{R}; \\ 0, & \text{if } D = \mathbb{C}; \\ -1, & \text{if } D = \mathbb{H}. \end{cases}
\end{aligned}$$

$\square$

Proposition 5.1. *Every irreducible $\mathbb{C}$-representation V of G occurs in $W_{\mathbb{C}}$ for a unique irreducible $\mathbb{R}$-representation W of G.*

Proof. By Proposition 1.1(a), V occurs in $({}_{\mathbb{R}}V)_{\mathbb{C}}$, so V occurs in $W_{\mathbb{C}}$ for some irreducible $\mathbb{R}$-subrepresentation W of ${}_{\mathbb{R}}V$. If W is any irreducible $\mathbb{R}$-representation such that V occurs in $W_{\mathbb{C}}$, then the ${}_{\mathbb{R}}V$ column of the table shows that W equals the unique irreducible $\mathbb{R}$-subrepresentation of ${}_{\mathbb{R}}V$, so W is uniquely determined by V. □

Theorem 5.2 (Frobenius–Schur)**.** *We have*

$$\#\{g \in G : g^2 = 1\} \;=\; \sum_V (\dim V)\ \mathrm{FS}(V),$$

where V ranges over the irreducible $\mathbb{C}$-representations of G up to isomorphism.

Proof. The character of the regular representation $\mathbb{C}G$ is given by

$$\chi(g) = \begin{cases} \#G, & \text{if } g = 1; \\ 0, & \text{if } g \neq 1. \end{cases}$$

Thus

$$\begin{aligned} \#\{g \in G : g^2 = 1\} &= \frac{1}{\#G} \sum_g \chi(g^2) \\ &= \mathrm{FS}(\mathbb{C}G) \\ &= \sum_V (\dim V)\ \mathrm{FS}(V), \end{aligned}$$

since $\mathbb{C}G \simeq \bigoplus_V (\dim V) V$. □

Remark 5.3. Everything above for finite groups G holds also for *compact* groups G. The only changes required are:

- All representations should be given by *continuous* homomorphisms.
- Averages over G (such as in the definition of the Frobenius–Schur indicator) should be defined as integrals with respect to normalized Haar measure.

Remark 5.4. Let k be a field such that $\operatorname{char} k \nmid \#G$. Let $X_1, \ldots, X_r$ be the irreducible k-representations of G. Let $D_i = \operatorname{End}_G X_i$. Let n_i be the dimension of X_i as a right D_i-vector space. Then

$$\begin{aligned} kG &\simeq \prod_{i=1}^{r} \operatorname{End}_{D_i} X_i \\ &\simeq \prod_{i=1}^{r} \mathrm{M}_{n_i}(D_i). \end{aligned}$$

In particular,

$$\mathbb{R}G \simeq \prod \mathrm{M}_{d_i}(\mathbb{R}) \times \prod \mathrm{M}_{e_j}(\mathbb{C}) \times \prod \mathrm{M}_{f_k}(\mathbb{H})$$

for some positive integers d_i, e_j, f_k, and tensoring with $\mathbb{C}$ yields

$$\mathbb{C}G \simeq \prod \mathrm{M}_{d_i}(\mathbb{C}) \times \prod \left(\mathrm{M}_{e_j}(\mathbb{C}) \times \mathrm{M}_{e_j}(\mathbb{C})\right) \times \prod \mathrm{M}_{2f_k}(\mathbb{C}).$$

6. Some conclusions to remember

- Every irreducible $\mathbb{C}$-representation V of G occurs in $W_{\mathbb{C}}$ for a unique irreducible $\mathbb{R}$-representation of G.
- The representation V is said to be of real, complex, or quaternionic type according to whether $\operatorname{End}_G W$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.
- The type can be determined from the character χ_V by computing the Frobenius–Schur indicator.
- The representation V is realizable over $\mathbb{R}$ if and only if V is of real type, which happens if and only if there exists a nondegenerate G-invariant symmetric bilinear form $B \colon V \times V \to \mathbb{C}$.
- The representation V is of complex type if and only if $V \not\simeq V^*$; in this case, there does not exist any nondegenerate G-invariant bilinear form $B \colon V \times V \to \mathbb{C}$.
- The representation V is of quaternionic type if and only if there exists a nondegenerate G-invariant skew-symmetric bilinear form $B \colon V \times V \to \mathbb{C}$.
- If V is realizable over $\mathbb{R}$, then χ_V is real-valued. The converse is not true in general (it fails exactly in the quaternionic case).

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