Transversal Matroids and Strata on Grassmannians

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In the present paper we investigate strata on Grassmannians that are associated with transversal matroids and study the restrictions of hypergeometric functions on these strata (see [1–6]).

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1. Let Z_{kn} be the manifold of all nondegenerate complex $k \times n$-matrices, $k < n$, let G_{kn} be the Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$, and let $\pi\colon Z_{kn} \to G_{kn}$ be the natural projection ($\pi(z)$ for $z \in Z_{kn}$ is the k-dimensional subspace of $\mathbb{C}^n$ generated by the rows of z).

For a matrix $z \in Z_{kn}$ denote by $p_I(z) = p_{i_1 \dots i_k}(z)$ the minor of z composed by the columns with indices from the set $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. A fixed set B of k-element subsets of $\{1, \dots, n\}$ is called a *collection*. By the *stratum* $S = S_B \subset Z_{kn}$ associated with a collection B we mean the manifold of all $z \in Z_{kn}$ such that $p_I(z) \neq 0 \iff I \in B$. The image $s = s_B = \pi(S)$ of the stratum S_B is called a *stratum on the Grassmannian* G_{kn} (see [2]). Obviously, $G_{kn} = \bigcup_B s_B$.

If the stratum associated with a collection B is nonempty, then B satisfies the axioms for bases of a matroid (see [7, 8]). This matroid is said to be associated with the stratum.

For $U \subset \{1, \dots, k\} \times \{1, \dots, n\}$ denote by $Z(U)$ the submanifold of matrices $z = (z_{ij}) \in Z_{kn}$ such that $z_{ij} = 0$ whenever $(i, j) \notin U$.

Lemma. *If $Z(U) \neq \varnothing$, then there exists a unique stratum $S(U) \subset Z_{kn}$ such that the set $S(U) \cap Z(U)$ is dense in $Z(U)$.*

The subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ is a base of the matroid associated with the stratum $S(U)$ if and only if there exists a rearrangement $\sigma_1 \dots \sigma_k$ of the set $\{1, \dots, k\}$ such that $(\sigma_1, i_1), \dots, (\sigma_k, i_k) \in U$.

Strata of the form $S(U)$ and $s(U) := \pi(S(U)) \subset G_{kn}$ and associated matroids are said to be *transversal.* There is a vast literature devoted to transversal matroids (e.g., see [7–10]).

Note that transversal strata played a substantial role in [3] and [4], where they are called special or linearizable.

Let $\overline{s}$ denote the closure of a stratum $s \subset G_{kn}$.

Theorem 1. *For a stratum $s \subset G_{kn}$ there are transversal strata $s_1, \dots, s_l \subset G_{kn}$ such that $\overline{s} = \overline{s_1} \cap \dots \cap \overline{s_l}$.*

The proof is based on the results of [10].

2. There is a natural right action of the complex torus T^n (embedded in $GL(n)$ as the group of diagonal matrices) on G_{kn}. Let $\widetilde{G}_{kn} := G_{kn}/T^n$ and let $\tau\colon G_{kn} \to \widetilde{G}_{kn}$ be the natural projection. The map τ transfers the stratification to $\widetilde{G}_{kn}$.

We calculate the dimension of certain strata in $\widetilde{G}_{kn}$.

More precisely, let

$$U = \{(1,1), \dots, (k,k)\} \cup V, \quad \text{where } V \subset \{1, \dots, k\} \times \{k+1, \dots, n\}.$$

Strata of the form $S(U)$, $s(U)$, and $\tilde{s}(U) := \tau(s(U))$ are said to be *strict transversal.* Matroids associated with such strata appear in [9], where they are said to be simplicial, and in [10], where they are said to be fundamental transversal.

It is convenient to represent the set $V \subset \{1, \dots, k\} \times \{k+1, \dots, n\}$ in the form of a bipartite graph Γ_V. Namely, Γ_V is the graph on the set of vertices $\{1, \dots, n\}$ such that (i, j) is an edge of Γ_V if and only if $(i, j) \in V$.

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The *cyclomatic number* of a graph Γ is $c(\Gamma) := e - v + d$, where e is the number of edges, v is the number of vertices, and d is the number of connected components of Γ. The cyclomatic number is also the dimension of the first cohomology class of the graph.

Theorem 2. *Let $\tilde{s}$ be a strict transversal stratum in $\widetilde{G}_{kn}$, i.e., $\tilde{s} = \tilde{s}(U)$, where*

$$U = \{(1,1), \dots, (k,k)\} \cup V, \qquad V \subset \{1, \dots, k\} \times \{k+1, \dots, n\}.$$

Then the complex dimension of the stratum $\tilde{s}$ is equal to the cyclomatic number of the graph Γ_V.

Proof. Denote by $\widetilde{Z}(V)$ the manifold of all $k \times (n-k)$-matrices $A = (a_{ij})$ such that $a_{ij} = 0$ for $(i, j+k) \notin V$. Let $x \in \tilde{s}$. Then there exists a block matrix $z = (\mathbb{I}_k, A) \in (\tau \circ \pi)^{-1}(x)$, where $\mathbb{I}_k$ is the identity k-matrix and $A \in \widetilde{Z}(V)$. Moreover, for $z' = (\mathbb{I}_k, A')$ and $z'' = (\mathbb{I}_k, A'')$ we have $(\tau \circ \pi)(z') = (\tau \circ \pi)(z'')$ if and only if $A' = t_1 A'' t_2$, where $t_1 \in T^k$ and $t_2 \in T^{n-k}$. For almost all $A \in \widetilde{Z}(V)$ we have $(\tau \circ \pi)(\mathbb{I}_k, A) \in \tilde{s}$. Thus, $\dim \tilde{s} = \dim T^k \backslash \widetilde{Z}(V) / T^{n-k}$. On the other hand, it is clear that $\dim T^k \backslash \widetilde{Z}(V) / T^{n-k} = c(\Gamma_V)$. $\square$

3. Solutions of the following system of differential equations are called *general hypergeometric functions* on Z_{kn} (see [1–6]):

$$\begin{aligned}
&\frac{\partial^2 \Phi}{\partial z_{ij} \partial z_{i'j'}} = \frac{\partial^2 \Phi}{\partial z_{ij'} \partial z_{i'j}}, && i, i' \in \{1, \dots, k\},\ j, j' \in \{1, \dots, n\}, \\
&\sum_i z_{ij} \frac{\partial \Phi}{\partial z_{ij}} = \alpha_j \Phi, && j \in \{1, \dots, n\}, \\
&\sum_j z_{ij} \frac{\partial \Phi}{\partial z_{ij}} = -\Phi, && i \in \{1, \dots, k\}, \\
&\sum_j z_{ij} \frac{\partial \Phi}{\partial z_{i'j}} = 0, && i, i' \in \{1, \dots, k\},\ i \neq i',
\end{aligned}$$

where α_i are arbitrary complex numbers such that $\sum \alpha_j = -k$.

These equations imply the following homogeneity conditions:

$$\begin{aligned}
&\Phi(\alpha;\, z\delta) = \prod_j \delta_j^{\alpha_j} \Phi(\alpha;\, z), && \delta = \operatorname{diag}(\delta_1, \dots, \delta_n) \in T^n, \\
&\Phi(\alpha;\, gz) = (\det g)^{-1} \Phi(\alpha;\, z), && g \in GL(k).
\end{aligned}$$

Therefore, the restrictions $\Phi|_S$ to a stratum $S \subset Z_{kn}$ substantially depend on $\dim(\tau \circ \pi)(S)$ parameters.

We consider the restrictions of hypergeometric functions to a strict transversal stratum

$$S = S(U), \quad U = \{(1,1), \dots, (k,k)\} \cup V,\ V \subset \{1, \dots, k\} \times \{k+1, \dots, n\}.$$

Let $S_0 \subset S$ be the space of all block matrices $(\mathbb{I}_k, A)$, where $A = (a_{ij}) \in \widetilde{Z}(V)$. Clearly, $GL(k) \cdot S_0 \cdot T^n = S$ (see the proof of Theorem 2). Therefore, hypergeometric functions given on S_0 can be uniquely extended to S via homogeneity conditions.

Theorem 3. *The space of restrictions of general hypergeometric functions to the submanifold S_0 coincides with the space of solutions of the following system of equations:*

$$\frac{\partial^l \Psi}{\partial a_{i_1 j_1} \partial a_{i_2 j_2} \cdots \partial a_{i_l j_l}} = \frac{\partial^l \Psi}{\partial a_{i_1 j_l} \partial a_{i_2 j_1} \cdots \partial a_{i_l j_{l-1}}} \tag{1}$$

for all circuits $((i_1, j_1+k), (j_1+k, i_2), (i_2, j_2+k), \dots, (i_l, j_l+k), (j_l+k, i_1))$ in the graph Γ_V, $i_1, \dots, i_l \in$

$\{1,\dots,k\}$, $j_1,\dots,j_l\in\{1,\dots,n-k\}$;

$$\sum_j a_{ij}\frac{\partial\Psi}{\partial a_{ij}} = -(\alpha_i+1)\Psi, \qquad i\in\{1,\dots,k\}; \tag{2}$$

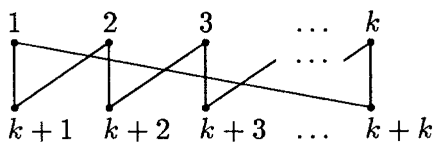
$$\sum_i a_{ij}\frac{\partial\Psi}{\partial a_{ij}} = \alpha_{j+k}\Psi, \qquad j\in\{1,\dots,n-k\}. \tag{3}$$

Remark. By [5], (1)–(3) is the hypergeometric system associated with the action of the torus $T^n = T^k\times T^{n-k}$ on $\widetilde{Z}(V)$. Theorem 3 means that the space of these hypergeometric functions coincides with the space of restrictions to S_0 of the hypergeometric functions on Z_{kn}. Note that this statement fails for arbitrary stratum. An example of the nonregular behavior of hypergeometric functions in a neighborhood of a nontransversal stratum is given in [2].

4. Example. If the graph Γ_V has no circuits, then $c(\Gamma_V)=0$ and, by Theorem 2, the space of hypergeometric functions on the corresponding stratum is trivial. The first meaningful example is the case in which Γ_V consists of a single circuit.

Note that the points of the stratum associated with the graph Γ_V that is formed by a cycle of length $2k$ are k-gons, which naturally appear in the study of cohomology of projective configurations and polylogarithms (see [11]).

Let $n=2k$ and let Γ_V be the graph with the edges $(1,k+1),(k+1,2),(2,k+2),\dots,(k,k+k)$, $(k+k,1)$:



The cyclomatic number $c(\Gamma_V)$ is equal to 1. Hence, by Theorem 2, in fact the solution Ψ to system (1)–(3) depends on a single parameter. Let $t=(a_{11}a_{22}\cdots a_{kk})(a_{21}a_{32}\cdots a_{1k})^{-1}$. Then

$$\Psi(a_{ij}) = a_{22}^{\gamma_1}a_{33}^{\gamma_2}\cdots a_{kk}^{\gamma_{k-1}}a_{21}^{\beta_1}a_{32}^{\beta_2}\cdots a_{1k}^{\beta_k}f(t),$$

where γ_i and β_j are determined by the system

$$\begin{aligned} \beta_k &= -\alpha_1-1, & \beta_1 &= \alpha_{k+1},\\ \gamma_1+\beta_1 &= -\alpha_2-1, & \gamma_1+\beta_2 &= \alpha_{k+2},\\ &\cdots\cdots\cdots & &\cdots\cdots\cdots\\ \gamma_{k-1}+\beta_{k-1} &= -\alpha_k-1, & \gamma_{k-1}+\beta_k &= \alpha_{k+k}. \end{aligned}$$

In this case differential equation (1) is equivalent to the following equation for $f(t)$:

$$\{D(D+\gamma_1-1)\cdots(D+\gamma_{k-1}-1) - t(D+\beta_1)\cdots(D+\beta_k)\}f(t)=0,$$

where $D:=t\,d/dt$.

A solution of this equation is given by the Pochhammer series (see [12]):

$$f(t) = {}_kF_{k-1}\left(\begin{matrix}\beta_1,\dots,\beta_k\\ \gamma_1,\dots,\gamma_{k-1}\end{matrix};t\right) := \sum_{n\ge 0}\frac{(\beta_1)_n\cdots(\beta_k)_n}{(\gamma_1)_n\cdots(\gamma_{k-1})_n}\frac{t^n}{n!},$$

where $(a)_n := a(a+1)\cdots(a+n-1)$.

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Multiple Mixing and Local Rank of Dynamical Systems

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The Rokhlin multiple mixing problem is as follows: If an automorphism T of a Lebesgue space $(X, \mathcal{B}, \mu)$ (where $\mu(X) = 1$ and $\mathcal{B}$ is the algebra of μ-measurable subsets of X) has the mixing property of order 1, does it have the mixing property of order $k \geq 2$?

Recall that the automorphism T has the *k-fold mixing property* if for any $A_0, \dots, A_k \in \mathcal{B}$ we have

$$\mu(T^{z_0}A_0 \cap T^{z_1}A_1 \cap \dots \cap T^{z_k}A_k) \to \mu(A_0)\mu(A_1)\dots\mu(A_k) \tag{1}$$

as $|z_p - z_q| \to \infty$, $0 \leq p < q \leq k$.

Let $\mathbf{T} = \{T^z : z \in \mathbb{Z}^n, T^{z_1}T^{z_2} = T^{z_1+z_2}\ \forall z_1, z_2 \in \mathbb{Z}^n\}$ be a measure-preserving $\mathbb{Z}^n$-action on $(X, \mathcal{B}, \mu)$. We say that $\mathbf{T}$ has the *k-fold* mixing property if (1) holds. Ledrappier [3] produced examples of mixing $\mathbb{Z}^2$-actions without multiple mixing property. Thus, the following question is of particular interest: *which invariants of mixing dynamical systems imply the multiple mixing property*?

Kalikow [1] proved that the 1-fold mixing property is equivalent to the 2-fold mixing property for the $\mathbb{Z}$-actions of rank 1 (in our terms, for the $\mathbb{Z}$-actions of local rank 1). In [5] the author generalized Kalikow's result to the $\mathbb{Z}$-actions with D-approximation. The class of such systems contains finite rank actions (see [5]).

In this note we give a modification of the D-approximation. This also gives an invariant that leads to the multiple mixing property. Mixing $\mathbb{Z}^n$-actions of local rank b have D-approximation for $b > 2^{-n}$. King [2] proved that the mixing property of order 3 implies the mixing property of all orders for the $\mathbb{Z}^n$-actions of local rank $b = 1 - K(n)$, where $K(n) \to 0$ as $n \to 0$. Thus, we generalize results of [1, 2].

The mixing flows ($\mathbb{R}^n$-actions) of positive local rank have the mixing property of all orders. But for $\mathbb{Z}^n$-actions the corresponding question is open. We define the $(1+\varepsilon)$-mixing property that guarantees the multiple mixing property for systems of positive local rank.

1. Local rank and D-approximation of $\mathbb{Z}^n$-actions. Let $[0, h]$ denote the set $\{0, 1, \dots, h\}$ and let Q be the cube $[0, h]^n$. Let $\xi = \{\xi^z\}_{z\in Q}$ be a measurable partition of a set $U \subset X$, i.e., $U = \bigcup_{z\in Q} \xi^z$ and $\xi^v \cap \xi^w = \varnothing$ for $v \neq w$. If such a partition ξ with the configuration Q satisfies the condition

$$\forall z \in Q \qquad T^z\xi^0 = \xi^z,$$

where 0 is the zero vector in $\mathbb{Z}^n$, then ξ is called a *tower*.

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