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Note

A generalization of Sylvester's identity

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Abstract

We consider a new generalization of Euler's and Sylvester's identities for partitions. Our proof is based on an explicit bijection.

1. Main results

A *partition* λ of n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of positive integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l > 0$ and $\sum \lambda_i = n$. The numbers λ_i are called *parts* of λ. Denote by $l(\lambda)$ the number l of parts in λ.

One of the well-known facts in the theory of partitions is Euler's identity.

Theorem (Euler, 1748). *The number of partitions of n with odd parts is equal to the number of partitions of n with distinct parts.*

There exist several generalizations of Euler's identity (e.g. see [2, 5]). One of them is Sylvester's identity.

By $\mathscr{A}(n, k)$ denote the set of partitions of n into odd parts (repetitions allowed) with exactly k different parts. By $\mathscr{B}(n, k)$ denote the set of partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l)$ of n such that the sequence $(\lambda_1 - l, \lambda_2 - l + 1, \ldots, \lambda_l - 1)$ has exactly k different elements. Let $A(n, k) = \#\mathscr{A}(n, k)$ and $B(n, k) = \#\mathscr{B}(n, k)$.

Theorem (Sylvester, 1882). $A(n, k) = B(n, k)$.

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Example. Let $n = 13$, $k = 3$. Then $\mathscr{A}(13,3) = \{(9,3,1), (7,5,1), (7,3,1^3), (5,3^2,1^2), (5,3,1^5)\}$ and $\mathscr{B}(13,3) = \{(9,3,1), (8,4,1), (7,4,2), (7,5,1), (6,4,2,1)\}$. Hence $A(13,3) = B(13,3) = 5$.

We present a generalization of these identities.

By $\mathscr{A}(n,m,c)$ denote the set of all partitions of n with parts $\equiv c \bmod m$; $A(n,m,c) = \# \mathscr{A}(n,m,c)$.

We say that λ is a *partition of type* $(h_1, h_2, \ldots)$, if λ has exactly $h_1 \geqslant 1$ parts of maximal length, $h_2 \geqslant 1$ second by length parts, etc. Let $1 \leqslant c < m$. By $\mathscr{B}(n,m,c)$ denote the set of all partitions of n of type $(c, m-c, c, m-c, \ldots)$; $B(n,m,c) = \# \mathscr{B}(n,m,c)$.

Theorem 1. $A(n,m,c) = B(n,m,c)$.

By $\mathscr{A}(n,m,c,k)$ denote the set of partitions $\lambda \in \mathscr{A}(n,m,c)$ with exactly k different parts.

A *chain* in a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ is a subsequence $\lambda_p, \lambda_{p+1}, \ldots, \lambda_p$ such that $\lambda_i - \lambda_{i+1} \leqslant 1$ for $p \leqslant i < q$. Let $\mathscr{B}(n,m,c,k)$ denote the set of all partitions $\lambda \in \mathscr{B}(n,m,c)$ such that λ has exactly k maximal chains.

Let $A(n,m,c,k) = \# \mathscr{A}(n,m,c,k)$ and $B(n,m,c,k) = \# \mathscr{B}(n,m,c,k)$.

Theorem 2. $A(n,m,c,k) = B(n,m,c,k)$.

It is clear that for $m = 2$, $c = 1$, Theorem 1 is Euler's identity and Theorem 2 is Sylvester's identity. In Section 2 we present a bijective proof of Theorems 1 and 2. Our bijection generalizes the original Sylvester's bijection but the construction is quite different. We also generalize the following Fine's identity.

Theorem (Fine, 1954). *The number of partitions of n into distinct parts with largest part s is equal to the number of partitions of n into odd parts such that $2s + 1$ is equal to the largest part plus two times the number of parts.*

We call the number $d(\lambda, m, c) = \# \{i\colon \lambda_i \geqslant mi + c\}$ *(m,c)-depth* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, where $1 \leqslant c < m$.

Theorem 3. *The number of partitions $\lambda \in \mathscr{B}(n,m,c,k)$ with the largest part $\lambda_1 = s$ such that the number of parts $l(\lambda) = mr + m$ or $l(\lambda) = mr + c$ is equal to the number of partitions $\mu \in \mathscr{A}(n,m,c,k)$ such that $\mu_1 + ml(\mu) = ms + c$ and $d(\lambda, m, c) = r$.*

Obviously, for $m = 2$, $c = 1$ Theorem 3 generalizes Fine's identity.

Remark. Sylvester's identity and bijection were first published in [9]. Fine's identity was found in [2]. Andrews was probably the first who realized that Fine's identity may be concluded from Sylvester's bijection (see [1]). There exists another Fine's theorem related to this subject (see [6, 1, 3]). In contrast with the first one it does not follow from Sylvester's bijection. It would be interesting to find its generalization in a spirit of the combinatorial proof given in [4].

2. Construction of bijection and proof of theorems

Proof of Theorems 1–3 is based on properties of the bijection $\psi:\mathscr{A}(n,m,c)\to\mathscr{B}(n,m,c)$ which we will construct later.

Recall several standard definitions from the theory of partitions (see [2, 8]).

With a partition $\lambda=(\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_l>0)$ we associate its *Young diagram* which is the set of pairs $(i,j)\in\mathbb{Z}^2$ such that $1\leqslant i\leqslant l$, $1\leqslant j\leqslant\lambda_i$. Pairs (i,j) are arranged on the plane $\mathbb{R}^2$ with i increasing downward and j increasing from left to right. The diagram will be presented in form of a set of 1×1-boxes centered at (i,j).

A partition $\lambda'=(\lambda'_1\geqslant\lambda'_2\geqslant\cdots\geqslant\lambda'_s>0)$ is called *conjugate* to a partition $\lambda=(\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_l>0)$ if their Young diagrams are symmetric to each other with respect to the principal diagonal. Note that $\lambda'_1=l(\lambda)$ is the number of parts of λ and $(\lambda')'=\lambda$.

The *sum* of partitions λ and μ is a partition $\lambda+\mu$ such that $(\lambda+\mu)_i=\lambda_i+\mu_i$. The *union* of partitions λ and μ is a partition $\lambda\cup\mu$ such that its parts are the union of parts of λ and μ arranged in nonincreasing order. It is easy to see that $(\lambda\cup\mu)'=\lambda'+\mu'$. Let $m\cdot\lambda=\lambda+\lambda+\cdots+\lambda$ (m times) and $\lambda/m=\mu$ if $\lambda=m\cdot\mu$.

The *Generalized Frobenius Representation* (α,β) of $\lambda\in\mathscr{A}(n,m,c)$ is defined as follows. Let $d=d(\lambda,m,c)$ be (m,c)-depth of λ. Let $\alpha_i=\lambda_i-mi+m-c$, $1\leqslant i\leqslant d$ be the number of boxes in the ith row of λ to the right of the point $(i,mi-m+c)$. Let $\beta_j=\lambda'_j-\lfloor(j-a)/m\rfloor)$, $1\leqslant j\leqslant mc+d$ be the number of boxes in the jth column of λ below the point $(\lfloor(j-a)/m\rfloor,j)$, (here $\lfloor x\rfloor$ denotes the maximal integer such that $\lfloor x\rfloor\leqslant x$).

Then $\alpha=(\alpha_1>\alpha_2>\cdots>\alpha_d>0)$, $\alpha_i\equiv 0\bmod m$; and $\beta=(\beta_1\geqslant\beta_2\geqslant\cdots)$.

Define a map ψ by $\psi(\lambda):=\beta+\gamma$, where $\gamma=(m\cdot(\alpha/m)')'$.

Prove that ψ is a bijection between $\mathscr{A}(n,m,c)$ and $\mathscr{B}(n,m,c)$. It is clear that β is a partition of type $(c,m,m,\ldots)$ and γ is of type $(m,m,m,\ldots)$. Hence, $\beta+\gamma$ is of type $(c,m-c,c,m-c,\ldots)$, i.e. $\psi(\lambda)\in\mathscr{B}(n,m,c)$. Conversely, for each $\mu\in\mathscr{B}(n,m,c)$ there is a unique decomposition $\mu=\delta+\nu$, where δ is a partition of type $(c,m,m,\ldots)$ and ν of type $(m,m,m,\ldots)$. Indeed, δ' consists of all parts of μ' congruent with $c\bmod m$ and ν' consists of all parts of μ' congruent with $0\bmod m$. Then $\mu=(\delta'\cup\nu')=\delta+\nu$. Therefore, ψ is a bijection between $\mathscr{A}(n,m,c)$ and $\mathscr{B}(n,m,c)$ and we have proved Theorem 1.

Example. Let $\lambda=(10,7^2,4^3,1)\in\mathscr{A}(37,3,1)$. Then $\alpha=(9,3)$; $\beta=(7,5^3,1^3)$; $\gamma=(3\cdot(\alpha/3)')'=(3\cdot(3,1)')'=(3\cdot(2,1^2))'=(6,3^2)'=(3^3,1^3)$; $\psi(\lambda)=\beta+\gamma=(10,8^2,6,2^2,1)\in\mathscr{B}(37,3,1)$ (See Fig. 1).

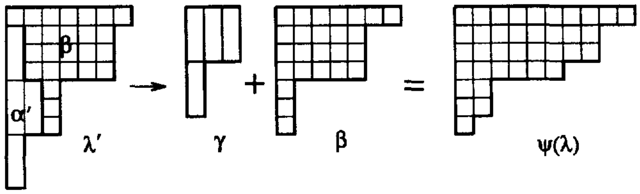

Fig. 1.

Note that $\lambda \in \mathscr{A}(37,3,1,4)$, because λ has exactly four different parts $11,7,4,1$; and $\psi(\lambda) \in \mathscr{B}(37,3,1,4)$, because $\psi(\lambda)$ has exactly four maximal chains (10), $(8,8)$, (6), $(2,2,1)$. Note also that $d(\lambda,3,1) = 2$.

In order to prove Theorems 2 and 3 we shall verify the corresponding properties of the bijection ψ.

Lemma 1. *If $\lambda \in \mathscr{A}(n,m,c,k)$ then $\psi(\lambda) \in \mathscr{B}(n,m,c,k)$.*

Proof. Let $d = d(\lambda,m,c)$. Note that the number of parts $l(\psi(\lambda))$ of $\psi(\lambda)$ is equal to $\max(l(\beta), l(\gamma)) = md + m$ if $l(\beta) < l(\gamma)$ and $l(\psi(\lambda)) = md + c$ if $l(\beta) > l(\gamma)$. Let

$$Q = \{1 \leqslant q \leqslant l(\psi(\lambda))\colon \psi(\lambda)_q - \psi(\lambda)_{q+1} > 1\};$$

$$Q_1 = \{1 \leqslant q \leqslant l(\beta)\colon \beta_q - \beta_{q+1} > 1\};$$

$$Q_2 = \{1 \leqslant q \leqslant l(\gamma)\colon \gamma_q - \gamma_{q+1} > 1\}.$$

Clearly, if $q \in Q_1$ (or $q \in Q_2$) then $q \equiv c$ (or $q \equiv 0$) $\bmod m$. Hence, $Q_1 \cup Q_2 = Q$, $Q_1 \cap Q_2 = \emptyset$, and $\#Q = \#Q_1 + \#Q_2$.

Then $\#Q = k - 1$, i.e. $\psi(\lambda)$ has exactly k maximal chains. Therefore, $\psi(\lambda) \in \mathscr{B}(n,m,c,k)$.

Theorem 2 immediately follows from Lemma 1.

Lemma 2. *If $\lambda \in \mathscr{A}(n,m,c,k)$ and $\mu = \psi(\lambda) \in \mathscr{B}(n,m,c,k)$ then $\lambda_1 + ml(\lambda) = m\mu_1 + c$.*

Proof. Clearly $\mu_1 = \psi(\lambda)_1 = \beta_1 + \gamma_1 = l(\lambda) + \alpha_1/m = l(\lambda) + (\lambda_1 - c)/m$. Hence, $\lambda_1 + ml(\lambda) = m\mu_1 + c$.

This lemma completes the proof of Theorem 3. □

3. Conclusion

Proposition 1. *If $m = 2$ and $c = 1$ then the bijection $\psi : \mathscr{A}(n,2,1) \to \mathscr{B}(n,2,1)$ coincides with Sylvester's bijection.*

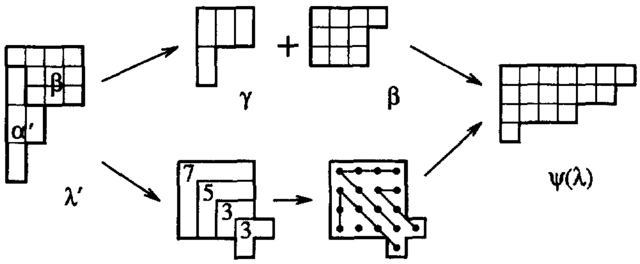

Fig. 2.

We do not present here the original Sylvester's construction (see [9]). There exists a simple inductive proof of this proposition.

Example. Let $\lambda = (7,5,3^2)$. Then $\alpha = (6,2)$; $\beta = (4,3^2)$; $\gamma = (3^2,2)$; and $\psi(\lambda) = (7,6,4,1)$. The idea of Sylvester's bijection is clear from Fig. 2.

Note also that for $m = 1$ and $c = 0$ the Generalized Frobenius Representation is exactly Frobenius Representation (see [8]) and $(1,0)$-depth is the size of *Durfee's Square* (see [2]).

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