PERMUTOHEDRA, ASSOCIAHEDRA, AND BEYOND

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Abstract. The volume and the number of lattice points of the permutohedron P_n are given by certain multivariate polynomials that have remarkable combinatorial properties. We give several different formulas for these polynomials. We also study a more general class of polytopes that includes the permutohedron, the associahedron, the cyclohedron, the Pitman-Stanley polytope, and various generalized associahedra related to wonderful compactifications of De Concini-Procesi. These polytopes are constructed as Minkowski sums of simplices. We calculate their volumes and describe their combinatorial structure. The coefficients of monomials in $\mathrm{Vol}\, P_n$ are certain positive integer numbers, which we call the mixed Eulerian numbers. These numbers are equal to the mixed volumes of hypersimplices. Various specializations of these numbers give the usual Eulerian numbers, the Catalan numbers, the numbers $(n+1)^{n-1}$ of trees, the binomial coefficients, etc. We calculate the mixed Eulerian numbers using certain binary trees. Many results are extended to an arbitrary Weyl group.

1. Introduction

The *permutohedron* $P_n(x_1, \ldots, x_n)$ is the convex hull of the $n!$ points obtained from $(x_1, \ldots, x_n)$ by permutations of the coordinates. Permutohedra appear in representation theory as *weight polytopes* of irreducible representations of GL_n and in geometry as *moment polytopes*.

In this paper we calculate volumes of permutohedra and numbers of their integer lattice points. Let us give a couple of examples. It was known before that the volume of the regular permutohedron $P_n(n, n-1, \ldots, 1)$ equals the number n^{n-2} of *trees* on n labeled vertices and the number of lattice points of this polytope equals the number of *forests* on n labeled vertices. Another example is the *hypersimplex* $\Delta_{k,n} = P_n(1, \ldots, 1, 0, \ldots, 0,)$ (with k ones). It is well-know that the volume of $\Delta_{k,n}$ is the Eulerian number, that is the number of permutations of size $n-1$ with $k-1$ descents, divided by $(n-1)!$. This calculation dates back to Laplace [Lap]. These examples are just a tip of an iceberg. They indicate at a rich combinatorial

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structure. Both the volume and the number of lattice points of the permutohedron $P_n(x_1, \ldots, x_n)$ are given by multivariate polynomials in $x_1, \ldots, x_n$ that have remarkable properties.

We present three different combinatorial interpretations of these polynomials using three different approaches. Our first approach is based of Brion's formula that expresses the sum of exponents over lattice points of a polytope as a rational function. From this we deduce a formula for the volume of the permutohedron as a sum on $n!$ polynomials. Then we deduce a combinatorial formula for the coefficients in terms permutations with given descent sets. We extend the formula for the volume to weight polytopes for any Lie type. There are some similarities between this formula and the Weyl's character formula.

Our second approach is based on a way to represent permutohedra as a weighted Minkowski sum $\sum y_I \Delta_I$ of the coordinate simplices. We extend our results to a larger class of polytopes that we call *generalized permutohedra*. These polytopes are obtained from usual permutohedra by parallel translations of their faces.

We discuss combinatorial structure of generalized permutohedra. This class includes many interesting polytopes: associahedra, cyclohedra, various generalized associahedra related to De Concini-Procesi's wonderful compactifications, graph associahedra, Pitman-Stanley polytopes, graphical zonotopes, etc. We describe the combinatorial structure for a class of generalized permutohedra in terms of *nested families.* This description leads to a generalization of the Catalan numbers.

We calculate volumes of generalized permutohedra by first calculating *mixed volumes* of various coordinate simplices using *Bernstein's theorem* on systems of algebraic equations. More generally, we calculate the *Ehrhart polynomial* of generalized permutohedra, i.e., the polynomial that expresses their number of lattice points. Interestingly, the formula for the number of lattice points is obtained from the formula for the volume by replacing usual powers in monomials with raising powers. We also found an interesting new *duality* for generalized permutohedra that preserves the number of lattice points.

We introduce and study *root polytopes* and their triangulations. These are convex hulls of the origin and end-points of several positive roots for a type A root system. In particular, this class of polytopes includes direct products of two simplices. We apply the *Cayley trick* to show that the volume of a root polytope is related to the number of lattice points in a certain associated generalized permutohedron. Each triangulation of a root polytope leads to a bijection between lattice points of the associated generalized permutohedron and its dual generalized permutohedron.

As an application of these techniques we solve a problem about combinatorial description of diagonal vectors of shifted Young tableaux of the triangular shape.

Our third approach is based on a way to represent permutohedra as a Minkowski sum of the hypersimplices $\sum u_k \Delta_{k,n}$. We express volumes of permutohedra in terms of mixed volumes of the hypersimplices. We call these mixed volume the *mixed Eulerian numbers.* Various specializations of these numbers lead to the usual Eulerian numbers, the Catalan numbers, the binomial coefficients, the factorials, the number $(n+1)^{n-1}$ of trees, and many other combinatorial sequences. We prove several identities for the mixed Eulerian number and give their combinatorial interpretation in terms of weighted binary trees. We also extend this approach and generalize mixed Eulerian numbers to an arbitrary root system.

A brief overview of the paper follows. In Section 2, we define permutohedra, give their several known properties, and discuss their relationship with zonotopes. In Section 3, we give a formula for volumes of permutohedra (Theorem 3.1) based on Brion's formula and derive another formula for volumes (Theorem 3.2) that involves numbers of permutations with given descents sets. In Section 4, we give a formula for volumes and lattice points enumerators of weight polytopes for any Lie type (Theorems 4.2 and 4.3). In Section 5, we give a formula for volume of permutohedra (Theorem 5.1) based on our second approach. In Section 6, we discuss generalized permutohedra and several ways to parametrize this class of polytopes. In Section 7, we discuss combinatorial structure for a class of generalized permutohedra in terms of nested families (Theorem 7.4). In Section 8, we apply this description to several special cases of generalized permutohedra. In Section 9, we extend Theorem 5.1 to generalized permutohedra and calculate their volumes (Theorem 9.3) using Bernstein's theorem. In Section 10, we give alternative formulas for volumes (Theorems 10.1 and 10.2) based on our first approach. In Section 11, we state a formula for the Ehrhart polynomial of generalized permutohedra (Theorem 11.3) and derive the duality theorem (Corollary 11.8). In Section 12, we discuss root polytopes and their triangulations for bipartite graphs. In Section 13, we treat the case of non-bipartite graphs. In Section 14, we show how triangulations of roots polytopes are related to lattice points of generalized permutohedra. We also finish the proof of Theorem 11.3. In Section 15, we describe diagonals of shifted Young tableaux. In Section 16, we discuss our third approach based on the mixed Eulerian numbers. We prove several properties of these numbers (Theorems 16.3 and 16.4). In Section 17, we give the third combinatorial formula for volumes of permutohedra (Theorem 17.1) and give a combinatorial interpretation for the mixed Eulerian numbers (Theorem 17.7). Finally, in Section 18 we extend our third approach to weight polytopes for an arbitrary root system (Theorems 18.3 and 18.5). In Appendix 19, we review and give short proofs of needed general results on enumeration of lattice points in polytopes.

Let us give a notational remark about our use of various coordinate systems. We use the x-coordinates to parametrize permutohedra expressed in the standard form as convex hulls of S_n-orbits of $(x_1, \dots, x_n)$. We use the z-coordinates to parametrize (generalized) permutohedra expressed by linear inequalities as $\{t \mid f_i(t) \geq z_i\}$, i.e., the z-coordinates correspond to the facets of these polytopes. We use the y-coordinates to parametrize (generalized) permutohedra written as weighted Minkowski sums $\sum y_I \Delta_I$ of the coordinate simplices. Finally, we use the u-coordinates to parametrize permutohedra written as weighted Minkowski sums $\sum u_k \, \Delta_{n,k}$ of the hypersimplices. For all other purposes we use the t-coordinates. Throughout the paper, we use the notation $[n] := \{1, 2 \dots, n\}$ and $[m, n] := \{m, m+1, \dots, n\}$.

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2. Permutohedra and zonotopes

Definition 2.1. For $x_1, \dots, x_n \in \mathbb{R}$, the *permutohedron* $P_n(x_1, \dots, x_n)$ is the convex polytope in $\mathbb{R}^n$ defined as the convex hull of all vectors obtained from

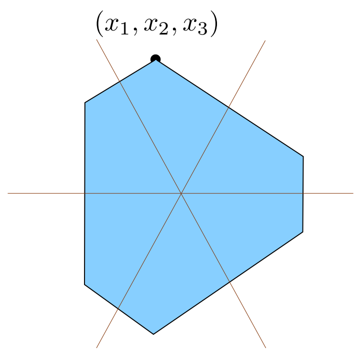
$(x_1, \ldots, x_n)$ by permutations of the coordinates:

$$P_n(x_1, \ldots, x_n) := \mathrm{ConvexHull}((x_{w(1)}, \ldots, x_{w(n)}) \mid w \in S_n),$$

where S_n is the symmetric group. This polytope lies in the hyperplane $H_c = \{(t_1, \ldots, t_n) \mid t_1 + \cdots + t_n = c\} \subset \mathbb{R}^n$, where $c = x_1 + \cdots + x_n$. Thus $P_n(x_1, \ldots, x_n)$ has the dimension at most $n - 1$.

For example, for $n = 3$ and distinct x_1, x_2, x_3, the permutohedron $P_3(x_1, x_2, x_3)$ is the hexagon shown below. If some of the numbers x_1, x_2, x_3 are equal to each other then the permutohedron degenerates into a triangle, or even a single point.

$P_2(x_1, x_2, x_3) =$



For a polytope $P \in H_c$, define its volume $\mathrm{Vol}\, P$ as the usual $(n-1)$-dimensional volume of the polytope $p(P) \in \mathbb{R}^{n-1}$, where p is the projection $p : (t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_{n-1})$. If $c \in \mathbb{Z}$, then the volume of any parallelepiped formed by generators of the integer lattice $\mathbb{Z}^n \cap H_c$ is 1.

In this paper, we calculate the volume

$$V_n(x_1, \ldots, x_n) := \mathrm{Vol}\, P_n(x_1, \ldots, x_n)$$

of the permutohedron. Also, for integer $x_1, \ldots, x_n$, its number of lattice points

$$N_n(x_1, \ldots, x_n) := P_n(x_1, \ldots, x_n) \cap \mathbb{Z}^n.$$

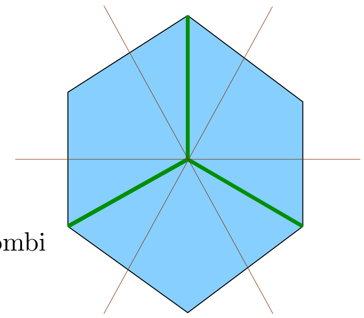
We will see that both $V_n(x_1, \ldots, x_n)$ and $N_n(x_1, \ldots, x_n)$ are polynomials of degree $n - 1$ in the variables $x_1, \ldots, x_n$. The polynomial V_n is the top homogeneous part of N_n. The *Ehrhart polynomial* of the permutohedron is $E_{P_n}(t) = N_n(tx_1, \ldots, tx_n)$. We will give 3 totally different formulas for these polynomials.

The special permutohedron for $(x_1, \ldots, x_n) = (n - 1, n - 2, \ldots, 0)$,

$$P_n(n - 1, \ldots, 0) = \mathrm{ConvexHull}((w(1) - 1, ..., w(n) - 1) \mid w \in S_n)$$

is the most symmetric permutohedron. It is invariant under the action of the symmetric group S_n. For example, for $n = 3$, it is the regular hexagon:

regular hexagon subdivided into 3 rombi



We will call this special permutohedron $P_n(n-1,\dots,0)$ the *regular permutohedron.* The volume of the regular permutohedron and its Ehrhart polynomial can be easily calculated using the general result on graphical zonotopes given below.

Recall that the *Minkowski sum* of several subsets $A, \dots, B$ in a linear space is the locus of sums of vectors that belong to these subsets $A+\cdots+B := \{a+\cdots+b \mid a \in A,\dots,b \in B\}$. If $A,\dots,B$ are convex polytopes then so is their Minkowski sum. The Newton polytope $\mathrm{Newton}(f)$ for a polynomial $f = \sum_{a\in\mathbb{Z}^n} \beta_a\, t_1^{a_1}\cdots t_n^{a_n}$ is the convex hull of integer points $a \in \mathbb{Z}^n$ such that $\beta_a \neq 0$. Then $\mathrm{Newton}(f\cdot g)$ is the Minkowski sum $\mathrm{Newton}(f)+\mathrm{Newton}(g)$. A *zonotope* is a Minkowski sum of several line intervals.

Definition 2.2. For a graph Γ on the vertex set $[n] := \{1,\dots,n\}$, the *graphical zonotope* Z_Γ is defined as the Minkowski sum of the line intervals:

$$Z_\Gamma := \sum_{(i,j)\in\Gamma} [e_i, e_j] = \mathrm{Newton}\left(\prod_{(i,j)\in\Gamma} (t_i - t_j)\right),$$

where the Minkowski sum and the product are over edges (i,j), $i<j$, of the graph Γ, and $e_1,\dots,e_n$ are the coordinate vectors in $\mathbb{R}^n$. The zonotope Z_Γ lies in the hyperplane H_c, where c is the number of edges of Γ. The polytope Z_Γ was first introduced by Zaslavsky (unpublished).

The following two claims express well-know properties of graphical zonotopes and permutohedra.

Proposition 2.3. *The regular permutohedron $P_n(n-1,\dots,0)$ is the graphical zonotope Z_{K_n} for the complete graph K_n.*

Proof. The permutohedron $P_n(n-1,\dots,0)$ is the Newton polytope of the Vandermonde determinant $\det(t_i^{j-1})_{1\le i,j\le n}$. On the other hand, the Vandermonde determinant is equal to the product $\prod_{1\le i<j\le n}(t_j - t_i)$, whose Newton polytope is the zonotope Z_{K_n}. $\square$

The following claim is given in Stanley [St2, Exer. 4.32].

Proposition 2.4. *For a connected graph Γ on n vertices, the volume* $\mathrm{Vol}\, Z_\Gamma$ *of the graphical zonotope Z_Γ equals the number of spanning trees of the graph Γ. The number of lattice points of Z_Γ equals to the number of forests in the graph Γ.*

In particular, the volume of the regular permutohedron is $\mathrm{Vol}\, P_n(n-1,\dots,0) = n^{n-2}$ *and its number of lattice points equals the number of forests on n labeled vertices.*

The zonotope Z_Γ can be subdivided into unit parallelepipeds associated with spanning trees of Γ, which implies the first claim.

In general, for arbitrary $x_1, \dots, x_n$, the permutohedron $P_n(x_1, \dots, x_n)$ is not a zonotope. We cannot easily calculate its volume by subdividing it into parallelepipeds.

One can alternatively describe the permutohedron $P_n(x_1, \dots, x_n)$ in terms of linear inequalities.

Proposition 2.5. Rado [Rad] *Let us assume that $x_1 \geq \cdots \geq x_n$. Then a point $(t_1, \dots, t_n) \in \mathbb{R}^n$ belongs to the permutohedron $P_n(x_1, \dots, x_n)$ if and only if*

$$t_1 + \cdots + t_n = x_1 + \cdots + x_n$$

and, for any nonempty subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, we have

$$t_{i_1} + \cdots + t_{i_k} \leq x_1 + \cdots + x_k.$$

The combinatorial structure of the permutohedron $P_n(x_1, \dots, x_n)$ does not depend on $x_1, \dots, x_n$ as long as all these numbers are distinct. More precisely, we have the following well-know statement.

Proposition 2.6. *Let us assume that $x_1 > \cdots > x_n$. The d-dimensional faces of $P_n(x_1, \dots, x_n)$ are in one-to-one correspondence with disjoint subdivisions of the set $\{1, \dots, n\}$ into nonempty ordered blocks $B_1 \cup \cdots \cup B_{n-d} = \{1, \dots, n\}$. The face corresponding to the subdivision into blocks $B_1, \dots, B_{n-d}$ is given by the $n-d$ linear equations*

$$\sum_{i \in B_1 \cup \cdots \cup B_k} t_i = x_1 + \cdots + x_{|B_1 \cup \cdots \cup B_k|}, \quad \text{for } k = 1, \dots, n-d.$$

In particular, two vertices $(x_{u(1)}, \dots, x_{u(n)})$ and $(x_{w(1)}, \dots, x_{w(n)})$, $u, w \in S_{n+1}$, are connected by an edge if and only if $w = u\, s_i$, for some adjacent transposition $s_i = (i, i+1)$.

3. Descents and divided symmetrization

Theorem 3.1. *Let us fix distinct numbers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. The volume of the permutohedron $P_n = P_n(x_1, \dots, x_n)$ is equal to*

$$\operatorname{Vol} P_n = \frac{1}{(n-1)!} \sum_{w \in S_n} \frac{(\lambda_{w(1)} x_1 + \cdots + \lambda_{w(n)} x_n)^{n-1}}{(\lambda_{w(1)} - \lambda_{w(2)})(\lambda_{w(2)} - \lambda_{w(3)}) \cdots (\lambda_{w(n-1)} - \lambda_{w(n)})}.$$

Notice that all λ_i's in the right-hand side cancel each other after the symmetrization. Theorem 4.2 below gives a similar formula for any Weyl group. Its proof is based on the Brion's formula [Bri]; see Appendix 19.

Theorem 3.1 gives an efficient way to calculate the polynomials $V_n = \operatorname{Vol} P_n$. However this theorem does not explain the combinatorial significance of the coefficients in these polynomials. The next theorem gives a combinatorial interpretation for the coefficients.

Given a sequence of nonnegative integers $(c_1, \dots, c_n)$ such that $c_1 + \cdots + c_n = n - 1$, let us construct the sequence $(\epsilon_1, \dots, \epsilon_{2n-2}) \in \{1, -1\}^{2n-2}$ by replacing each entry 'c_i' with '$1, \dots, 1, -1$' (c_i '1's followed by one '-1'), for $i = 1, \dots, n$, and then removing the last '-1'. For example, the sequence $(2, 0, 1, 1, 0, 1)$ gives $(1, 1, -1, -1, 1, -1, 1, -1, -1, 1)$. This map is actually a bijection between the sets

$\{(c_1,\dots,c_n)\in\mathbb{Z}_{\geq 0}^n \mid c_1+\cdots+c_n=n-1\}$ and $\{(\epsilon_1,\dots,\epsilon_{2n-2})\in\{1,-1\}^{2n-2} \mid \epsilon_1+\cdots+\epsilon_{2n-2}=0\}$. Let us define the set $I_{c_1,\dots,c_n}$ by

$$I_{c_1,\dots,c_n} := \{i\in\{1,\dots,n-1\} \mid \epsilon_1+\cdots+\epsilon_{2i-1}<0\}.$$

The *descent set* of a permutation $w\in S_n$ is $I(w)=\{i\in\{1,\dots,n-1\} \mid w_i>w_{i+1}\}$. Let $D_n(I)$ be the number of permutations in S_n with the descent set $I(w)=I$.

Theorem 3.2. *The volume of the permutohedron $P_n=P_n(x_1,\dots,x_n)$ is equal to*

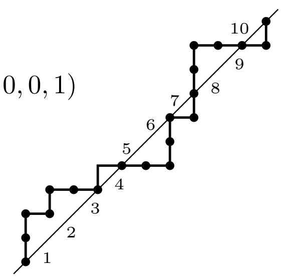
$$\mathrm{Vol}\, P_n = \sum (-1)^{|I_{c_1,\dots,c_n}|}\, D_n(I_{c_1,\dots,c_n})\,\frac{x_1^{c_1}}{c_1!}\cdots\frac{x_n^{c_n}}{c_n!}\,,$$

where the sum is over sequences of nonnegative integers $c_1,\dots,c_n$ such that $c_1+\cdots+c_n=n-1$.

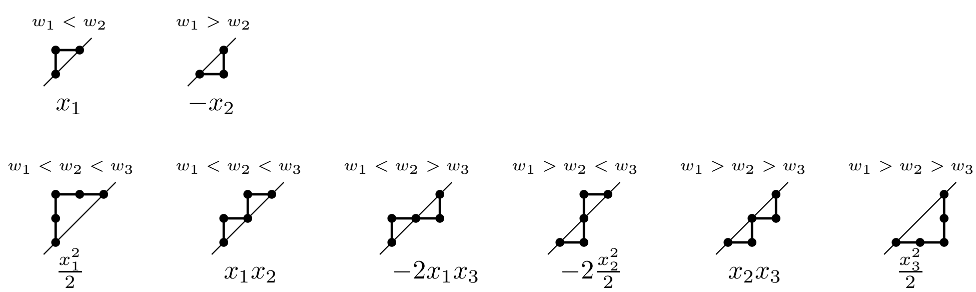
We can graphically describe the set $I_{c_1,\dots,c_n}$, as follows. Let us construct the lattice path P on $\mathbb{Z}^2$ from $(0,0)$ to $(n-1,n-1)$ with steps of the two types $(0,1)$ "up" and $(1,0)$ "right" such that P has exactly c_i up steps in the $(i-1)$-st column, for $i=1,\dots,n$. Notice that the $(2i-1)$-th and $2i$-th steps in the path P are either both above the $x=y$ axis or both below it. The set $I_{c_1,\dots,c_n}$ is the set of indices i such that the $(2i-1)$-th and $2i$-th steps in P are below the $x=y$ axis.

$(c_1,\dots,c_{10})=(2,1,0,1,0,0,2,3,0,0,1)$

$I_{c_1,\dots,c_{10}}=\{5,6,7,10\}$



Example 3.3. We have $V_2=x_1-x_2$ and $V_3=\frac{x_1^2}{2}+x_1x_2-2x_1x_3-2\frac{x_2^2}{2}+x_2x_3+\frac{x_3^2}{2}$. The following figure shows the paths corresponding to all terms in V_2 and V_3.



For example, $I_{1,0,1}=\{2\}$ and there are 2 permutations $132, 231\in S_3$ with the descent set $\{2\}$. Thus the coefficient of x_1x_3 in V_3 is -2.

For a polynomial $f(\lambda_1,\dots,\lambda_n)$, define its *divided symmetrization* by

$$\langle f\rangle := \sum_{w\in S_n} w\left(\frac{f(\lambda_1,\dots,\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)\cdots(\lambda_{n-1}-\lambda_n)}\right),$$

where the symmetric group S_n acts by permuting the variables λ_i.

Proposition 3.4. *Let f be a polynomial of degree $n-1$ in the variables $\lambda_1, \dots, \lambda_n$. Then its divided symmetrization $\langle f\rangle$ is a constant. If $\deg f < n-1$, then $\langle f\rangle = 0$.*

Proof. We can write $\langle f\rangle = g/\Delta$, where $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)$ is the common denominator of all terms in $\langle f\rangle$ and g is a certain polynomial of degree $\deg\Delta = \binom{n}{2}$. Since $\langle f\rangle$ is a symmetric rational function, g should be an anti-symmetric polynomial and thus it is divisible by Δ. Since g and Δ have the same degree, their quotient is a constant. If $\deg f < n-1$, then $\deg g < \deg \Delta$ and, thus, $g = 0$. $\square$

Proposition 3.5. *We have $\langle \lambda_1^{c_1} \cdots \lambda_n^{c_n}\rangle = (-1)^{|I|} D_n(I)$, where $c_1, \dots, c_n$ are nonnegative integers with $c_1 + \cdots + c_n = n-1$ and $I = I_{c_1,\dots,c_n}$.*

Proof. We can expand the expression $\frac{1}{\lambda_i - \lambda_j}$, $i<j$ as the Laurent series that converges in the region $\lambda_1 > \cdots > \lambda_n > 0$:

$$\frac{1}{\lambda_i - \lambda_j} = \lambda_i^{-1}\frac{1}{1-\lambda_j/\lambda_i} = \sum_{k\geq 0} \lambda_i^{-k-1}\,\lambda_j^k.$$

Let us use this formula to expand each term $w\left(\frac{\lambda_1^{c_1}\cdots\lambda_n^{c_n}}{(\lambda_1-\lambda_2)\cdots(\lambda_{n-1}-\lambda_n)}\right)$ as a Laurent series f_w that converges in this region. Let CT_w be the constant term of the series f_w. Then, according to Proposition 3.4, we have $\langle \lambda_1^{c_1}\cdots\lambda_n^{c_n}\rangle = \sum_{w\in S_n} CT_w$. Equivalently, the number CT_w is the constant term in the series $w^{-1}(f_w)$, i.e., the Laurent series obtained by the expansion of each term $\frac{1}{\lambda_i - \lambda_{i+1}}$ in $\frac{\lambda_1^{c_1}\cdots\lambda_n^{c_n}}{(\lambda_1-\lambda_2)\cdots(\lambda_{n-1}-\lambda_n)}$ as

$$\frac{1}{\lambda_i - \lambda_{i+1}} = \begin{cases} \displaystyle\sum_{k\geq 0}\lambda_i^{-k-1}\,\lambda_{i+1}^k, & \text{for } w(i) < w(i+1), \\ \displaystyle -\sum_{k\geq 0}\lambda_i^{k}\,\lambda_{i+1}^{-k-1}, & \text{for } w(i) > w(i+1). \end{cases}$$

Let $I = I(w)$ be the descent set of the permutation w. Then CT_w equals $(-1)^{|I|}$ times the number of nonnegative integer sequences $(k_1, \dots, k_{n-1})$ such that we have $(c_1, \dots, c_n) = v_1 + \cdots + v_{n-1}$, where

$$v_i = \begin{cases} (k_i+1)\,e_i - k_i\,e_{i+1}, & \text{for } i \notin I \quad (w_i < w_{i+1}), \\ -k_i\,e_i + (k_i+1)\,e_{i+1}, & \text{for } i \in I \quad (w_i > w_{i+1}), \end{cases}$$

and the e_i are the coordinate vectors. Notice that, for a fixed permutation w, there is at most 1 sequence $(k_1,\dots,k_{n-1})$ that produces $(c_1,\dots,c_n)$, as above. Thus $CT_w \in \{1,-1,0\}$.

Let P be the lattice path from $(1,1)$ to (n,n) constructed from the sequence $(c_1,\dots,c_n)$ as shown after Theorem 3.2. In other words, P is the continuous piecewise-linear path obtained by joining the points

$$(0,0)-(0,c_1)-(1,c_1)-(1,c_1+c_2)-(2,c_1+c_2)-(2,c_1+c_2+c_3)-\cdots-(n-1,n-1)$$

by the straight lines.

Let r be the maximal index such that $w(1) < w(2) < \cdots < w(r)$. Then we have $c_1 = k_1 + 1$, $c_2 = k_2 + 1 - k_1$, $\dots$, $c_{r-1} = k_{r-1} + 1 - k_{r-2}$, $c_r = -k_r - k_{r-1}$. Thus $k_i = c_1 + \cdots + c_i - i \geq 0$, for $i = 1, \dots, r-2$, $k_{r-1} = c_1 + \cdots + c_{r-1} - (r-1) = 0$ and $k_r = c_r = 0$. This means that the path P stays weakly above the $x=y$ axis as it goes from the point $(0,0)$ to the point $(r-1,r-1)$, then it passes through the point $(r-1,r-1)$, and goes strictly below the $x=y$ axis (if $r<n+1$). For

$i = 1, \ldots, r-1$, the number k_i is exactly the distance between the lowest point of the path P on the line $x = i$ and the point (i,i).

Let r' be the maximal index such $w(r) > w(r+1) > \cdots > w(r')$. Then we have $c_r = -k_r = 0$, $c_{r+1} = k_r + 1 - k_{r+1}$, $\ldots$, $c_{r'-1} = k_{r'-2} + 1 - k_{r'-1}$, and $c_{r'} = (k_{r'-1}+1)+(k_r+1)$. Thus $k_i = i - r - c_r - \cdots - c_i = i - 1 - c_1 - \cdots - c_i \geq 0$, for $i = r, \ldots, r'-1$, and $k_{r'} = c_r + \cdots + c_{r'} - r \geq 0$. This means that the path P stays weakly below the $x = y$ axis as it goes from the point $(r-1, r-1)$ to the point $(r'-1, r'-1)$, then it passes through the point $(r'-1, r'-1)$ and goes strictly above the $x = y$ axis (if $r' < n+1$). For $i = r, \ldots, r'-1$, the number k_i is the distance between the highest point of the path P on the line $x = i-1$ and the point $(i-1, i-1)$.

We can continue working with maximal monotone intervals in the permutation w in this fashion. Let r'' be the maximal index such that $w(r') < \cdots < w(r'')$. Similarly to the above argument, we obtain that that path P' stays weakly above the $x = y$ axis until it crosses it at the point $(r''-1, r''-1)$, etc.

We deduce that the indices $r, r', r'', \ldots$ characterizing the descent set of w correspond to the points where the path P crosses the $x = y$ axis. Thus the descent set of w is uniquely reconstructed from the sequence $(c_1, \ldots, c_n)$ as $I = I_{c_1,\ldots,c_n}$. Moreover, for any permutation w with such descent set, the nonnegative integer sequence $(k_1, \ldots, k_{n-1})$ is uniquely reconstructed from the sequence $(c_1, \ldots, c_n)$ as

$$k_i = \begin{cases} \min\{y - i \mid (i,y) \in P\} & \text{if } i \notin I, \\ \min\{i - 1 - y \mid (i-1,y) \in P\} & \text{if } i \in I, \end{cases}$$

and, thus, $CT_w = (-1)^{|I|}$. This shows that only permutations with the descent set $I = I_{c_1,\ldots,c_n}$ make a contribution to $\langle \lambda_1^{c_1} \cdots \lambda_n^{c_n} \rangle$, and the contribution of any such permutation is $(-1)^{|I|}$. This finishes the proof. $\square$

Proof of Theorem 3.2. According to Theorem 3.1, the volume of the permutohedron can be written as the divided symmetrization of the power of a linear form:

$$V_n = \frac{1}{(n-1)!} \left\langle (x_1\lambda_1 + \cdots + x_n\lambda_n)^{n-1} \right\rangle = \sum_{c_1+\cdots+c_n=n-1} \langle \lambda_1^{c_1} \cdots \lambda_n^{c_n} \rangle \, \frac{x_1^{c_1}}{c_1!} \cdots \frac{x_n^{c_n}}{c_n!} \,.$$

Now apply Proposition 3.5. $\square$

4. Weight polytopes

Theorem 3.1 can be extended to any Weyl group, as follows. Let Φ be a root system of rank r. Let Λ be the associated integer *weight lattice* and $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ be the weight space. The roots in Φ span the root lattice $L \subseteq \Lambda$. The associated *Weyl group* W acts on the weight space $\Lambda_{\mathbb{R}}$. Let (x, y) be a nondegenerate W-invariant inner product on $\Lambda_{\mathbb{R}}$.

Definition 4.1. For $x \in \Lambda_{\mathbb{R}}$, we can define the *weight polytope* $P_W(x)$ as the convex hull of a Weyl group orbit:

$$P_W(x) := \mathrm{ConvexHull}(w(x) \mid w \in W) \subset \Lambda_{\mathbb{R}}.$$

For the Lie type A_r, the weight polytope $P_W(x)$ is the permutohedron $P_{r+1}(x)$.

Let us fix a choice of *simple roots* $\alpha_1, \ldots, \alpha_r$ in Φ. Let Vol be the volume form on $\Lambda_{\mathbb{R}}$ normalized so that the volume of the parallelepiped generated by the simple roots α_i is 1. Recall that a weight $\lambda \in \Lambda_{\mathbb{R}}$ is called *regular* if $(\lambda, \alpha) \neq 0$ for any root $\alpha \in \Phi$. A weight λ is called *dominant* if $(\lambda, \alpha_i) \geq 0$, for $i = 1, \dots, r$.

Theorem 4.2. *Let $\lambda \in \Lambda_{\mathbb{R}}$ be a regular weight. The volume of the weight polytope is equal to*

$$\operatorname{Vol} P_W(x) = \frac{1}{r!} \sum_{w \in W} \frac{(\lambda, w(x))^r}{(\lambda, w(\alpha_1)) \cdots (\lambda, w(\alpha_r))}.$$

For type A_r, $W = S_{r+1}$ and Theorem 4.2 specializes to Theorem 3.1.

Let G be a Lie group with the root system Φ. For a dominant weight λ, let V_λ be the irreducible representation of G with the highest weight λ. The character of V_λ is a certain nonnegative linear combination $ch(V_\lambda)$ of the formal exponents e^μ, $\mu \in \Lambda$. (These formal exponents are subject to the relation $e^\mu \cdot e^\nu = e^{\mu+\nu}$.) The weights that occur in the representation V_λ with nonzero multiplicities, i.e., the weights μ such that e^μ has a nonzero coefficient in $ch(V_\lambda)$, are exactly the points of the weight polytope $P_W(\lambda)$ in the lattice $L + \lambda$ (the root lattice shifted by λ). Let

$$S(P_W(\lambda)) := \sum_{\mu \in P_W(\lambda) \cap (L+\lambda)} e^\mu$$

be the sum of formal exponents over these lattice points. In other words, $S(P_W(\lambda))$ is obtained from the character $ch(V_\lambda)$ by replacing all nonzero coefficients with 1. For example, in the type A, the expression $S(P_n(\lambda))$ is obtained from the Schur polynomial by erasing the coefficients of all monomials.

We have the following identity in the field of rational expressions in the formal exponents.

Theorem 4.3. *For a dominant weight λ, the sum of exponents over lattice points of the weight polytope $P_W(\lambda)$ equals*

$$S(P_W(\lambda)) = \sum_{w \in W} \frac{e^{w(\lambda)}}{(1 - e^{-w(\alpha_1)}) \cdots (1 - e^{-w(\alpha_r)})}.$$

Notice that if we replace the product over simple roots α_i in the right-hand side of Theorem 4.3 by a similar product over *all* positive roots, we obtain exactly Weyl's character formula for $ch(V_\lambda)$.

Theorems 3.1, 4.2, and 4.3 follow from Brion's formula [Bri] on summation over lattice points in a rational polytope. In Appendix 19, we give a brief overview of this result and related results of Khovanskii-Pukhlikov [KP1, KP2] and Brion-Vergne [BV1, BV2]. The following proof assumes reader's familiarity with the Appendix.

Proof of Theorems 3.1, 4.2, 4.3. Let us identify the lattice $L + \lambda$ embedded into $\Lambda_{\mathbb{R}}$ with $\mathbb{Z}^r \subset \mathbb{R}^r$. Then (for a regular weight λ) the polytope $P_W(\lambda)$ is a Delzant polytope, i.e., for any vertex of $P_W(\lambda)$, the cone at this vertex is generated by an integer basis of the lattice $\mathbb{Z}^r$; see Appendix 19. Indeed, the generators of the cone at the vertex λ are $-\alpha_1, \dots, -\alpha_r$. Thus the generators of the cone at the vertex $w(\lambda)$, for $w \in W$, are $g_{i,w(\lambda)} = -w(\alpha_i)$, $i = 1, \dots, r$. Now Theorem 4.3 is obtained from Brion's formula given in Theorem 19.2(2). As we mention in the proof of Theorem 19.3(1), this claim remains true for non-regular weights λ when some of the vertices $w(\lambda)$ may accidentally merge. Similarly, Theorems 3.1 and 4.2, are obtained from Theorem 19.2(4). $\square$

In a sense, Theorems 4.2 and 3.1 are deduced from Theorem 4.3 in the same way as Weyl's dimension formula is deduced from Weyl's character formula, cf. Appendix 19.

5. Dragon marriage condition

In this section we give a different combinatorial formula for the volume of the permutohedron.

Let us use the coordinates $y_1, \dots, y_n$ related to $x_1, \dots, x_n$ by

$$\begin{cases} y_1 = -x_1 \\ y_2 = -x_2 + x_1 \\ y_3 = -x_3 + 2x_2 - x_1 \\ \vdots \\ y_n = -\binom{n-1}{0} x_n + \binom{n-1}{1} x_{n-1} - \cdots \pm \binom{n-1}{n-1} x_1 \end{cases}$$

Write $V_n = \operatorname{Vol} P_n(x_1, \dots, x_n)$ as a polynomial in the variables $y_1, \dots, y_n$.

Theorem 5.1. *We have*

$$\operatorname{Vol} P_n = \frac{1}{(n-1)!} \sum_{(J_1, \dots, J_{n-1})} y_{|J_1|} \cdots y_{|J_{n-1}|},$$

where the sum is over ordered collections of subsets $J_1, \dots, J_{n-1} \subseteq [n]$ such that, for any distinct $i_1, \dots, i_k$, we have $|J_{i_1} \cup \cdots \cup J_{i_k}| \geq k+1$.

We will extend and prove Theorem 5.1 for a larger class of polytopes called generalized permutohedra; see Theorem 9.3. Theorem 5.1 implies that $(n-1)!\, V_n$ is a polynomial in $y_2, \dots, y_n$ with *positive* integer coefficients.

Example 5.2. We have $V_2 = \operatorname{Vol}([(x_1, x_2), (x_2, x_1)]) = x_1 - x_2 = y_2$ and $2V_3 = x_1^2 + 2x_1x_2 - 4x_1x_3 - 2x_2^2 + 2x_2x_3 + x_3^2 = 6\,y_2^2 + 6\,y_2\,y_3 + y_3^2$.

Remark 5.3. The condition on subsets $J_1, \dots, J_{n-1}$ in Theorem 5.1 is similar to the condition in Hall's marriage theorem [Hal]. One just needs to replace the inequality $\geq k+1$ with $\geq k$ to obtain Hall's marriage condition.

Let us give an analogue of the marriage problem and Hall's theorem.

Dragon marriage problem. *There are n brides, $n-1$ grooms living in a medieval town, and 1 dragon who likes to visit the town occasionally. Suppose we know all possible pairs of brides and grooms who do no mind to marry each other. A dragon comes to the village and takes one of the brides. When will it be possible to match the remaining brides and grooms no matter what the choice of the dragon was?*

Proposition 5.4. *Let $J_1, \dots, J_{n-1} \subseteq [n]$. The following three conditions are equivalent:*

(1) *For any distinct $i_1, \dots, i_k$, we have $|J_{i_1} \cup \cdots \cup J_{i_k}| \geq k+1$.*
(2) *For any $j \in [n]$, there is a system of distinct representatives in $J_1, \dots, J_{n-1}$ that avoids j. (This is a reformulation of the dragon marriage problem.)*
(3) *There is a system of 2-element representatives $\{a_i, b_i\} \subseteq J_i$, $i = 1, \dots, n-1$, such that $(a_1, b_1), \dots, (a_{n-1}, b_{n-1})$ are edges of a spanning tree in K_n.*

Proof. It is clear that (2) implies (1). On the other hand, (1) implies (2) according to usual Hall's theorem. We leave it as an exercise for the reader to check that either of these two conditions is equivalent to (3). $\square$

We will refer to the three equivalent conditions in Proposition 5.4 as the *dragon marriage condition.*

Example 5.5. Let M_n be the number of sequences of subsets $J_1, \dots, J_{n-1} \subseteq [n]$ satisfying the dragon marriage condition. Equivalently, M_n is the number of bipartite subgraphs $G \subseteq K_{n-1,n}$ such that for any vertex j in the second part there is a matching in G covering the remaining vertices. According to Theorem 5.1 with $y_1 = \cdots = y_n = 1$, we have $M_n = (n-1)!\,\mathrm{Vol}\, P_n(-1, -2, -4, \dots, -2^{n-1})$. Let us calculate a few numbers M_n using Theorem 3.1.

n	2	3	4	5	6	7	8
M_n	1	13	1009	354161	496376001	2632501072321	52080136110870785

6. Generalized permutohedra

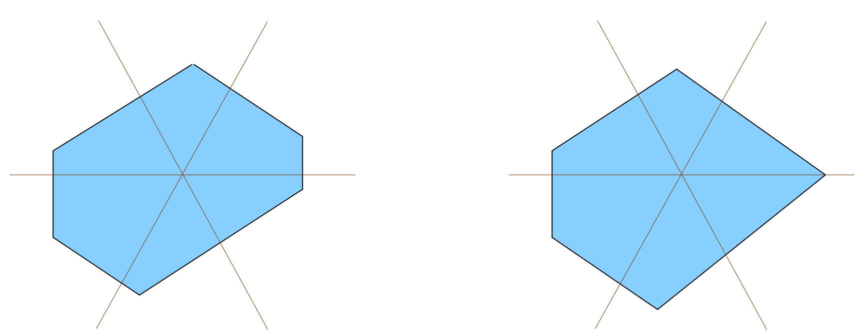
Definition 6.1. Let us define *generalized permutohedra* as deformations of the usual permutohedron, i.e., as polytopes obtained by moving vertices of the usual permutohedron so that directions of all edges are preserved (and some of the edges may accidentally degenerate into a singe point); see Appendix 19. In other words, a generalized permutohedron is the convex hull of $n!$ points $v_w \in \mathbb{R}^n$ labeled by permutations $w \in S_n$ such that, for any $w \in S_n$ and any adjacent transposition $s_i = (i, i{+}1)$, we have $v_w - v_{w\, s_i} = k_{w,i}(e_{w(i)} - e_{w(i+1)})$, for some nonnegative number $k_{w,i} \in \mathbb{R}_{\geq 0}$, where $e_1, \dots, e_n$ are the coordinate vectors in $\mathbb{R}^n$, cf. Proposition 2.6.

Each generalized permutohedron is obtained by parallel translation of the facets of a usual permutohedron. Recall that these facets are given by Rado's theorem (Proposition 2.5). Thus generalized permutohedra are parametrized by collections $\{z_I\}$ of the $2^n - 1$ coordinates z_I, for nonempty subsets $I \subseteq [n]$, that belongs to a certain deformation cone $\mathcal{D}_n$. Each generalized permutohedron has the form

$$P_n^z(\{z_I\}) = \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} t_i = z_{[n]},\ \sum_{i \in I} t_i \geq z_I,\ \text{for subsets } I \right\},$$

for $\{z_I\} \in \mathcal{D}_n$. If $z_I = z_J$ whenever $|I| = |J|$, then $P_n(\{z_I\})$ is a usual permutohedron.

The following figure shows examples of generalized permutohedra:



According to Theorem 19.3, we have the following statement.

Proposition 6.2. *The volume of the generalized permutohedron $P_n(\{z_I\})$ is a polynomial function of the z_I's defined on the deformation cone $\mathcal{D}_n$. The number of* *lattice points $P_n(\{z_I\}) \cap \mathbb{Z}^n$ in the generalized permutohedron is a polynomial function of the z_I's defined on the lattice points $\mathcal{D}_n \cap \mathbb{Z}^{2^n-1}$ of the deformation cone.*

Let us call the multivariate polynomial that expresses the number of lattice points in $P_n(\{z_I\})$ the *generalized Ehrhart polynomial* of the permutohedron.

Let us give a different construction for a class of generalized permutohedra. Let $\Delta_{[n]} = \mathrm{ConvexHull}(e_1, \dots, e_n)$ be the standard coordinate simplex in $\mathbb{R}^n$. For a subset $I \subset [n]$, let $\Delta_I = \mathrm{ConvexHull}(e_i \mid i \in I)$ denote the face of the coordinate simplex $\Delta_{[n]}$:

$$\Delta_I = \mathrm{ConvexHull}(e_i \mid i \in I).$$

Let $\{y_I\}$ be a collection of nonnegative parameters $y_I \geq 0$, for all nonempty subsets $I \subset [n]$. Let us define the polytope $P_n^y(\{y_I\})$ as the Minkowski sum of the simplices Δ_I scaled by the factors y_I:

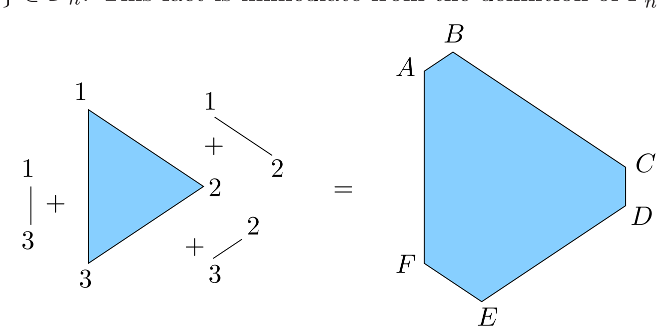
$$P_n^y(\{y_I\}) := \sum_{I \subset [n+1]} y_I \cdot \Delta_I.$$

Proposition 6.3. *Let $\{y_I\}$ be a collection of nonnegative real numbers for all nonempty subsets $I \subseteq [n]$, and let $\{z_I\}$ be the collection of numbers given by*

$$z_I = \sum_{J \subseteq I} y_J, \quad \text{for all nonempty } I \subseteq [n].$$

Then $P_n^y(\{y_I\} = P_n^z(\{z_I\}$.

Proof. Let us first pick a nonempty subset $I_0 \subseteq [n]$ and set $y_I = \delta(I, I_0)$ (Kronecker's delta). Then $P_n^y(\{y_I\}) = \Delta_{I_0}$, because the Minkowski contains only 1 nonzero term. In this case, we have $z_I = 1$, if $I \supseteq I_0$, and $z_I = 0$, otherwise. The inequalities describing the polytope $P_n^z(\{z_I\})$ give the same coordinate simplex Δ_{I_0}. The general case follows from the fact that the Minkowski sum of two generalized permutohedra $P_n^z(\{z_I\})$ and $P_n^z(\{z_I'\})$, for $\{z_I\}, \{z_I'\} \in \mathcal{D}_n$, is exactly the generalized permutohedron $P_n^z(\{z_I + z_I'\})$ parametrized by the coordinatewise sum $\{z_I + z_I'\} \in \mathcal{D}_n$. This fact is immediate from the definition of $P_n^z(\{z_I\})$. $\square$



Remark 6.4. Not every generalized permutohedron $P_n^z(\{z_I\})$ can be written as a Minkowski sum $P_n^y(\{y_I\})$ of the coordinate simplices. For example, for $n = 3$, the polytope $P_3^y(\{y_I\})$ (usually a hexagon) is the Minkowski sum of the coordinate triangle $\Delta_{[3]}$ and 3 line intervals $\Delta_{\{1,2\}}$, $\Delta_{\{1,3\}}$, $\Delta_{\{2,3\}}$ parallel to its edges (scaled by some factors); see the figure above. For this hexagon we always have $|AB| \leq |DE|$. On the other hand, any hexagon with edges parallel to the edges of $\Delta_{[3]}$ is a certain generalized permutohedron $P_3^z(\{z_I\})$.

The points $\{z_I\}$ of the deformation cone $\mathcal{D}_n$ that can be expressed as $z_I = \sum_{J\subseteq I} y_J$ through nonnegative parameters y_I form a certain region $\mathcal{D}'_n$ of top dimension in the deformation cone $\mathcal{D}_n$. Since the volume and generalized Ehrhart polynomial are polynomial functions on $\mathcal{D}_n$, it is enough to calculate them for the class of polytopes $P_n^y(\{y_I\})$ and then extend from $\mathcal{D}'_n$ to $\mathcal{D}_n$ by the polynomiality.

In what follows will refer to the polytopes $P_n^y(\{y_I\})$ as generalized permutohedra, keeping in mind that they form a special class of polytopes $P_n^z(\{z_I\})$.

7. Nested complex

The combinatorial structure of the generalized permutohedron $P_n^y = P_n^y(\{y_I\})$ depends only on the set $B \subset 2^{[n]}$ of nonempty subsets $I \subseteq [n]$ such that $y_I > 0$. In this section, we describe the combinatorial structure of P_n^y when the set B satisfies some additional conditions.

Definition 7.1. Let us say that a set B of nonempty subsets in S is a *building set* on S if it satisfies the conditions:

(B1) If $I, J \in B$ and $I \cap J \neq \emptyset$, then $I \cup J \in B$.
(B2) B contains all singletons $\{i\}$, for $i \in S$.

Condition (B1) is a certain "connectivity condition" for building sets. Note that condition (B2) does not impose any additional restrictions on the structure of generalized permutohedra and was added only for convenience. Indeed, the Minkowski sum of a polytope with $\Delta_{\{i\}}$, which is a single point, is just a parallel translation of the polytope.

Let $B_{\max} \subset B$ be the subset of maximal by inclusion elements in B. Let us say that a building set B is *connected* if it has a unique maximal by inclusion element S. According to (B1) all elements of $B_{\max}$ are pairwise disjoint. Thus each building set B is a union of pairwise disjoint connected building sets, called the *connected components* of B, that correspond to elements of $B_{\max}$.

For a subset $C \subset S$, define the *induced building set* as $B|_C = \{I \in B \mid I \subseteq C\}$.

Example 7.2. Let Γ be a graph on the set of vertices S. Define the *graphical building* $B(\Gamma)$ as the set of all nonempty subsets $C \subseteq S$ of vertices such that the induced graph $\Gamma|_C$ is connected. Clearly, it satisfies conditions (B1) and (B2). The building set $B(\Gamma)$ is connected if and only if the graph Γ is connected. The connected components of $B(\Gamma)$ correspond to connected components of the graph Γ. The induced building set is the building set for the induced graph: $B(\Gamma)|_C = B(\Gamma|_C)$.

Definition 7.3. A subset N in the building set B is called a *nested set* if it satisfies the following conditions:

(N1) For any $I, J \in N$, we have either $I \subseteq J$, or $J \subseteq I$, or I and J are disjoint.
(N2) For any collection of $k \geq 2$ disjoint subsets $J_1, \dots, J_k \in N$, their union $J_1 \cup \cdots \cup J_k$ is not in B.
(N3) N contains all elements of $B_{\max}$.

The *nested complex* $\mathcal{N}(B)$ is defined as the poset of all nested families in B ordered by inclusion.

Clearly, the collection of all nested sets in B (with elements of $B_{\max}$ removed) is a simplicial complex.

Theorem 7.4. *Let us assume that the set B associated with a generalized permutohedron P_n^y is a building set on $[n]$. Then the poset of faces of P_n^y ordered by reverse inclusion is isomorphic to the nested complex $\mathcal{N}(B)$.*

This claim was independently discovered by E. M. Feichtner and B. Sturmfels [FS, Theorem 3.14]. They also defined objects similar to B-forests discussed below; see [FS, Proposition 3.17].

Proof. Each face of an arbitrary polytope can be described as the set of points of the polytope that minimize a linear function f. Moreover, the face of a Minkowski sum $Q_1 + \cdots + Q_m$ that minimizes f is exactly the Minkowski sum of the faces of Q_i's that minimize f.

Let us pick a linear function $f(t_1, \dots, t_n) = a_1 t_1 + \cdots + a_n t_n$ on $\mathbb{R}^n$. It gives an ordered set partition of $[n]$ into a disjoint union of nonempty blocks $[n] = A_1 \cup \cdots \cup A_s$ such that $a_i = a_j$, whenever i and j are in the same block A_s, and $a_i < a_j$, whenever $i \in A_s$ and $j \in A_t$, for $s < t$. The face of a coordinate simplex Δ_I that minimizes the linear function f is the simplex $\Delta_{\widehat{I}}$, where $\widehat{I} := I \cap A_{j(I)}$ and $j = j(I)$ is the minimal index such that the intersection $I \cap A_j$ is nonempty. We deduce that the face of P_n^y minimizing f is the Minkowski sum $\sum_{I\in B} y_I\, \Delta_{\widehat{I}}$.

We always have $j(I) \geq j(J)$, for $I \subset J$. Let $N \subseteq B$ be the collection of elements $I \in B$ such that $j(I) \gneq j(J)$, for any $J \supsetneq I$, $J \in B$. We can also recursively construct the subset $N \subseteq B$, as follows. First, all maximal by inclusion elements of B should be in N. According to (B1), all other elements of B should belong to one of the maximal elements I_m. For each maximal element $I_m \in B$, all elements $I \subsetneq I_m$ such that $j(I) = j(I_m)$, i.e., the elements I that have a nonempty intersection with $\widehat{I}_m$, do not belong to N. The remaining elements $I \subsetneq I_m$ are exactly the elements of the induced building set $B|_{I_m \setminus \widehat{I}_m}$. Let us repeat the above procedure for each of the induced building sets. In other words, find all maximal by inclusion elements $I_{m'}$ in $B|_{I_m \setminus \widehat{I}_m}$. These maximal elements should be in N. Then, for each maximal element $I_{m'}$, construct the induced building set $B|_{I_{m'} \setminus \widehat{I}_{m'}}$, etc. Let us keep on doing this branching procedure until we arrive to building sets that consist of singletons, all of which should be in N.

It follows from this branching construction that N is a nested set in B. It is immediate that N satisfies conditions (N1) and (N3). If $J_1, \dots, J_k \in N$ are disjoint subsets and $J_1 \cup \cdots \cup J_k \in B$, $k \geq 2$, then we should have included $J_1 \cup \cdots \cup J_k$ in N in the recursive construction, and then the J_i cannot all belong to N. This implies condition (N2). It is also clear that, given N, we can uniquely reconstruct the subset $\widehat{I} \subseteq I$, for each $I \in B$. Indeed, find the minimal by inclusion element $J \in N$ such that $J \supseteq I$. Then $\widehat{J} = J \setminus \bigcup_{K\subsetneq J, K\in N} K$ and $\widehat{I}$ is the intersection of the last set with I. Thus the nested set N uniquely determines the face $\sum_{I\in B} y_I\, \Delta_{\widehat{I}}$ of P_n^y that minimizes f.

Let us show that, for any nested set $N \in \mathcal{N}(B)$, there exists a face of P_n^y associated with N. Indeed, let $A_I = I \setminus \bigcup_{J\subsetneq I, J\in N} J$, for any $I \in N$. Then $\bigcup_{I\in N} A_I$ is a disjoint decomposition of $[n]$ into nonempty blocks. Let us pick any linear order of $A_1 < \cdots < A_s$ of the blocks A_I such that $A_I < A_J$, for $I \subsetneq J$, and any linear function f on $\mathbb{R}^n$ that gives this set partition, for example, $f(t_1, \dots, t_n) = \sum_{i, j\in A_i} i\, t_j$. Then the function f minimizes a certain face F_N of P_n^y and if we apply the above procedure to F_N we will recover the nested set N. We also see from this construction that the face F_N contains the face $F_{N'}$ if and only if $N \subseteq N'$. $\square$

We can express the generalized permutohedron $P_n^y(\{y_I\})$ as $P_n^z(\{z_I\})$, where $z_I = \sum_{i: I_i \subseteq I} y_i$; see Section 6. Let us give an explicit description of its faces.

Proposition 7.5. *As before, let us assume that B is a building set. The face P_N of $P_n^y(\{y_I\}) = P_n^z(\{z_I\})$ associated with a nested set $N \in \mathcal{N}(B)$ is given by*

$$P_N = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid \sum_{i \in I} t_i = z_I, \text{ for } I \in N; \ \sum_{i \in J} t_i \geq z_J, \text{ for } J \in B\}.$$

The dimension of the face P_N equals $n - |N|$. In particular, the dimension of $P_n^y(\{y_I\})$ is $n - |B_{\max}|$.

Proof. According to the proof of Theorem 7.4, for a nested set $N \in \mathcal{N}(B)$, we have the disjoint decomposition $[n] = \bigcup_{I \in N} A_I$ into nonempty blocks, and the corresponding face of P_n^y is given by

$$P_N = \sum_{I \in N,\, J \in B,\, J \cap A_I \neq \emptyset} y_J \, \Delta_{J \cap A_I}.$$

This Minkowski sum involves the terms Δ_{A_I}, among others. Thus $\dim P_N \geq \dim(\sum_{I \in N} \Delta_{A_I}) = n - |N|$. It also follows from the construction that $J \cap A_I \neq \emptyset$ implies that $J \subseteq I$. Thus we have the equality $\sum_{i \in I} t_i = z_I$, for $I \in N$ and any point $(t_1, \dots, t_n) \in P_N$. It follows that the codimension of P_N in $\mathbb{R}^n$ is at least $|N|$. Together with the inequality for the dimension, this implies that $\dim P_N = n - |N|$ and the face P_N is described by the above $|N|$ linear equations, as needed. $\square$

Theorem 7.4 implies that vertices of P_n^y are in a bijective correspondence with maximal by inclusion elements of the nested complex $\mathcal{N}(B)$. We will call these elements *maximal nested families.* The following proposition gives their description.

Proposition 7.6. *A nested set $N \in \mathcal{N}(B)$ is maximal if and only if, for each $I \in N$, we have $|A_I| = 1$, where $A_I = I \setminus \bigcup_{J \subsetneq I, J \in N} J$. For a maximal nested set N, the map $I \mapsto i_I$, where $\{i_I\} = A_I$, is a bijection between N and $[n]$.*

Proof. According to the proof of Theorem 7.4 and Proposition 7.5, a nested set $N \in \mathcal{N}(B)$ is maximal (and F_N is a point) if and only if $\dim(\sum_{I \in N} \Delta_{A_I}) = \sum_{I \in N}(|A_I| - 1) = 0$, i.e., all A_I should be one elements sets. The map $I \mapsto i_I$ is clearly an injection. On the other hand, for any $i \in [n]$ and the minimal by inclusion element I of N that contains i, we have $I \mapsto i$. $\square$

For a maximal nested set $N \in \mathcal{N}(B)$, let us partially order the set $[n]$ by $i \geq_N j$ whenever $I \supseteq J$. The Hasse diagram of the order "$\geq_N$" is a rooted forest, i.e., a forest with a chosen root in each connected component and edges directed away from the roots. The set of such forests can be described, as follows.

For two nodes i and j in a rooted forest, we say that i is a *descendant* of j if the node j belongs to the shortest chain connecting i and the root of its connected component. In particular, each node is a descendant of itself. Let us say that two nodes i and j are *incomparable* if neither i is a descendant of j, nor j is a descendant of i.

Definition 7.7. For a rooted forest F and a node i, let $\mathrm{desc}(i, F)$ be the set of all descendants of the node i in F (including the node i itself). Define a *B-forest* as a rooted forest F on the vertex set $[n]$ such that

(F1) For any $i \in [n]$, we have $\mathrm{desc}(i, F) \in B$.
(F2) There are no $k \geq 2$ distinct incomparable nodes $i_1, \ldots, i_k$ in F such that $\bigcup_{j=1}^{k} \mathrm{desc}(i_j, F) \in B$.
(F3) The sets $\mathrm{desc}(i, F)$, for all roots i of F, are exactly the maximal elements of the building set B.

Condition (F3) implies that the number of connected components in a B-forest equals the number of connected components of the building set B. We will call such graphs *B-trees* in the case when B is connected.

Proposition 7.8. *The map $N \mapsto F_N$ is a bijection between maximal nested families $N \in \mathcal{N}(B)$ and B-forests.*

Proof. The claim is immediate from the above discussion. Indeed, note that each maximal nested set $N \in \mathcal{N}(B)$ can be reconstructed from the forest $F = F_N$ as $N = \{\mathrm{desc}(1, F), \ldots, \mathrm{desc}(n, F)\}$. □

Let us describe the vertices of the generalized permutohedron in the coordinates.

Proposition 7.9. *The vertex $v_F = (t_1, \ldots, t_n)$ of the generalized permutohedron P_n^y associated with a B-forest F is given by $t_i = \sum_{J \in B:\, i \in J \subseteq \mathrm{desc}(i,F)} y_J$, for $i = 1, \ldots, n$.*

Proof. Let N be the maximal nested set associated with the B-forest F. By Proposition 7.5, the associated vertex $v_F = (t_1, \ldots, t_n)$ is given by the n linear equations $\sum_{i \in I} t_i = z_I$, for each $I \in N$. Notice that, for each $J \in B$, there exists a unique $i \in J$ such that $i \in J \subseteq \mathrm{desc}(i, F)$. Indeed, $\mathrm{desc}(i, F)$ should be the minimal element of N containing J. Thus, for the numbers t_i defined as in Proposition 7.9 and any $I \in N$, we have

$$\sum_{i \in I} t_i = \sum_{i \in I} \sum_{J \in B:\, i \in J \subseteq \mathrm{desc}(i,F)} y_J = \sum_{J \subseteq I} y_J = z_I,$$

as needed. □

Proposition 7.10. *Let F be a B-forest and let v_F be the associated vertex of the generalized permutohedron P_n^y. For each nonrooted node i of F, define the n-vector $g_{i,F} = e_i - e_j$, where the node j is the parent of the node i in F. (Here $e_1, \ldots, e_n$ are the coordinate vectors in $\mathbb{R}^n$.) Then the integer vectors $g_{i,F}$ generate the local cone of the polytope P_n^y at the vertex v_F. In particular, the generalized permutohedron P_n^y is a simple Delzant polytope; see Appendix 19.*

Proof. Let N be the maximal nested set associated with the forest F. Then each edge of P_n^y incident to v_F correspond to a nested sets obtained from N by removing an element $I \in N \setminus B_{\max}$. There are $n - |B_{\max}|$ such edges and Proposition 7.5 implies that they are generated by the vectors $g_{i,F}$. □

Let $f_B(q)$ be the *f-polynomial* of the generalized permutohedron P_n^y. According to Theorem 7.4 is is given by

$$f_B(q) = \sum_{i=0}^{n-1} f_i\, q^i = \sum_{N \in \mathcal{N}(B)} q^{n-|N|},$$

where f_i is the number of i-dimensional faces of P_n^y. The recursive construction of nested families implies the following recurrence relations fort the f-vector.

Theorem 7.11. *The f-polynomial $f_B(q)$ is determined by the following recurrence relations:*

(1) *If B consists of a single singleton, then $f_B(q) = 1$.*
(2) *If B has connected components $B_1, \ldots, B_k$, then*

$$f_B(q) = f_{B_1}(q) \cdots f_{B_k}(q).$$

(3) *If B is a connected building on S, then*

$$f_B(q) = \sum_{C \subsetneq S} q^{|S|-|C|-1} f_{B|_C}(q).$$

Definition 7.12. We define the *generalized Catalan number*, for a building set B, as the number $C(B) = f_B(0)$ of vertices of the generalized permutohedron P_n^y, or, equivalently, the number of maximal nesting families in $\mathcal{N}(B)$, or, equivalently, the number of B-forests.

The reason for this name will become apparent from examples in the next section. The generalized Catalan numbers $C(B)$ are determined by the recurrence relations similar to the ones in Theorem 7.11, where in (3) we sum only over subsets $C \subset S$ of cardinality $|S| - 1$.

In the following section we show that the associahedron is a special case of generalized permutohedra. Thus we can also call this class of polytopes *generalized associahedra*. However this name is already reserved for a different generalization of the associahedron studied by Chapoton, Fomin, and Zelevinsky [CFZ].

Even though Chapoton-Fomin-Zelevinsky's generalized associahedra are different from our "generalized associahedra," there are some similarities between these two families of polytopes. In [Zel] Zelevinsky gives an alternative construction for generalized permutohedra associated with building sets which is parallel to the construction from [CFZ]. He first constructs the dual fan for the nested complex $\mathcal{N}(B)$ and then shows that it has a polytopal realization.

A natural question to ask is how to find a common generalization of Chapoton-Fomin-Zelevinsky's generalized associahedra and generalized permutohedra discussed in this section.

8. Examples of generalized permutohedra

8.1. **Permutohedron.** Let us assume that building set $B = B_{all} = 2^{[n]} \setminus \{\emptyset\}$ is the set of all nonempty subsets in $[n]$. Then P_n^y is combinatorially equivalent to the usual permutohedron, say, $P_n(n, n-1, \ldots, 1)$. This is the generic case of generalized permutohedra. In this case, nested families are flags of subsets $J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_s = [n]$. Indeed, two disjoint subsets I and J cannot belong to a nested set because their union $I \cup J$ is in B. The maximal nested families are complete flags on n subsets. Clearly, there are $n!$ such flags, which correspond to the $n!$ vertices of the permutohedron. In this case, B_{all}-trees are directed chains of the form $(w_1, w_2), (w_2, w_3), \ldots, (w_{n-1}, w_n)$, where $w_1, \ldots, w_n$ is a permutation in S_n. The generalized Catalan number in this case is $C(B_{all}) = n!$.

8.2. **Associahedron.** Assume that the building set $B = B_{int} = \{[i, j] \mid 1 \le i \le j \le n\}$ is the set of all continuous intervals in $[n]$. In this case, the generalized permutohedron is combinatorially equivalent to the *associahedron,* also known as the *Stasheff polytope,* which first appeared in the work of Stasheff [Sta].

A nested set $N \subseteq B_{int}$ is a collection of intervals such that, for any $I, J \in N$, we either have $I \subseteq J$, $J \subseteq I$, or I and J are disjoint *non-adjacent* intervals, i.e., $I \cup J$ is not an interval. Let us describe B_{int}-trees.

Recall that a *plane binary tree* is an tree such that each node has at most 1 left child and at most one right child. (If a node has only one child, we specify if it is the left or the right child.) It is well known that there are the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ of plane binary trees on n unlabeled nodes.

For a node v in such a tree, let L_v be the left branch and R_v be the right branch at this node, both of which are smaller plane binary trees. If v has no left child, then L_v is the empty graph, and similarly for R_v. For any plane binary tree on n nodes, there is a unique way to label the nodes by the numbers $1, \dots, n$ so that, for any node v, all labels in L_v are less than the label of v and all labels in R_v are greater than the label of v. Indeed, label each node v by the number $|L_v| + 1$.

We can also describe this labeling using the *depth-first search*. This is the walk on the nodes of a tree that starts at the root and is determined by the rules: (1) if we are at a some node and have never visited its left child, then go to the left child; (2) otherwise, if we have never visited its right child, then go to the right child; (3) otherwise, if the node has the parent, then go to the parent; (4) otherwise stop. Let us mark the nodes by the integers $1, \dots, n$ in the order of their appearance in this walk, as follows. Each time when we visit an unmarked vertex and *do not* apply rule (1), we mark this node. The labeling of nodes defined by any of these equivalent ways is called the *binary search labeling*. It was described by Knuth in [Knu, 6.2.1]. Example 8.3 below shows a plane binary tree with the binary search labeling.

Proposition 8.1. *The B_{int}-trees are exactly plane binary trees on n nodes with the binary search labeling.*

Proof. Let N be a maximal nested set. Suppose that the maximal element $[n] \in N$ corresponds to $i = i_{[n]}$ under the bijection in Proposition 7.6. Then $N \setminus [n]$ is the union of two maximal nested families on $[1, i-1]$ and on $[i+1, n]$. Equivalently, each B_{int}-tree is a rooted tree with root labeled i and two branches which are B_{int}-trees on the vertex sets $[1, i-1]$ and $[i+1, n]$. This implies the claim. $\square$

Thus in this case the generalized permutohedron has the Catalan number C_n vertices associated with plane binary trees. Proposition 7.9 implies the following description of the vertices of $P_n^y(\{y_{ij}\})$, where $y_{ij} = y_{[i,j]}$ for each interval $[i, j] \subseteq [n]$. For a plane binary trees T with binary search labeling, let $\mathrm{desc}(k, T) = [l_k, r_k]$, for $k = 1, \dots, n$. Then the left branch of a vertex k is $L_k = [l_k, k-1]$ and the right branch is $R_k = [k+1, r_k]$.

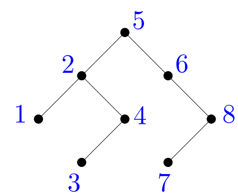
Corollary 8.2. *The vertex $v_T = (t_1, \dots, t_n)$ associated with a plane binary tree T is given by $t_k = \sum_{l_k \le i \le k \le j \le r_k} y_{ij}$. In particular, in the case when $y_{ij} = 1$, for any $1 \le i \le j \le n$, we have*

$$v_T = ((|L_1| + 1)(|R_1| + 1), \cdots, ((|L_n| + 1)(|R_n| + 1)).$$

The polytope Ass_n with the C_n vertices given by the second part of Corollary 8.2 is exactly the realization of the *associahedron* described by Loday [Lod]. The will refer to this particular geometric realization of the associahedron as the *Loday realization.* This polytope can be equivalently defined as the Newton polytope $\mathrm{Ass}_n := \mathrm{Newton}\left(\prod_{1 \le i \le j \le n}(t_i + t_{i+1} + \cdots + t_j)\right)$. We will calculate volumes and numbers of lattice points in Ass_n, for $n = 1, \dots, 8$, in Examples 10.3 and 15.3.

We can also describe the Loday realization, as follows. There are C_n subdivisions of the triangular Young diagram of the shape $(n, n-1, \dots, 1)$ into a disjoint union of n rectangles; see Thomas [Tho, Theorem 1.1] and Stanley's Catalan addendum [St3, Problem 6.19(u^5)]. These subdivisions are in simple a bijective correspondence with plane binary trees on n nodes. The i-th rectangle in such a subdivision is the rectangle that contains the i-th corner of the triangular shape. Then, for a vertex $v_T = (t_1, \dots, t_n)$ of the associahedron in the Loday realization, the i-the coordinate t_i equals the number of boxes in the i-th rectangle of the associated subdivision; see Example 8.3 below.

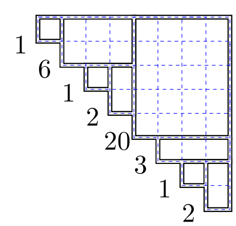
Example 8.3. Here is an example of a plane binary tree T with the binary search labeling:



This tree is associated with the maximal nested set

$$N = \{\mathrm{desc}(1, T), \dots, \mathrm{desc}(8, T)\} = \{[1, 1], [1, 4], [3, 3], [3, 4], [1, 8], [6, 8], [7, 7], [7, 8]\}.$$

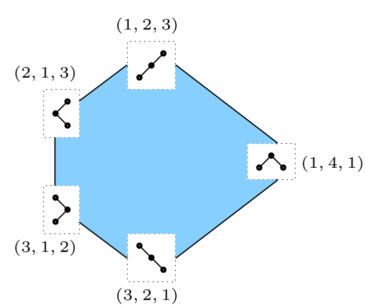
This tree corresponds to the following subdivision of the triangular shape into rectangles. (Here we used shifted Young diagram notation for a future application; see Section 15.)



The corresponding vertex of the associahedron in the Loday realization is

$$(1 \cdot 1,\ 2 \cdot 3,\ 1 \cdot 1,\ 2 \cdot 1,\ 5 \cdot 4,\ 1 \cdot 3,\ 1 \cdot 1,\ 2 \cdot 1).$$

Example 8.4. The next figure shows the Loday realization of the associahedron for $n = 3$:



8.3. **Cyclohedron.** Let $B = B_{cyc}$ be the set of all *cyclic intervals* in $[n]$, i.e., subsets of the form $[i, j]$ and $[1, i] \cup [j, n]$, for $1 \le i \le j \le n$. In this case, the generalized permutohedron is the *cyclohedron* that was also introduced by Stasheff [Sta]. If we restrict the building set B_{cyc} to $[n] \setminus \{i\}$, then we obtain the building set isomorphic to the set B_{int} of usual intervals in $[n-1]$. Thus we obtain the following description of B_{cyc}-trees.

Proposition 8.5. *The set of B_{cyc}-trees is exactly the set of trees that have a root at some vertex i attached to a plane binary tree on $n-1$ nodes with the binary search labeling by integers in $[n] \setminus \{i\}$ with respect to the order $i+1 < i+2 < \cdots < n < 1 < \cdots < i-1$.*

The generalized Catalan number in this case is $C(B_{cyc}) = n \cdot C_{n-1} = \binom{2n-2}{n-1}$.

8.4. **Graph associahedra.** Let Γ be a graph on the vertex set $[n]$. Let us assume that $B = B(\Gamma)$ is is the set of subsets $I \subseteq [n]$ such that the induced graph $\Gamma|_I$ is connected; see Example 7.2. In this case, the generalized permutohedron P_n^y is called the *graph associahedron.* The above examples are special cases of graph associahedra. If $\Gamma = A_n$ is the chain with n nodes, i.e., the type A_n Dynkin diagram, then we obtain the usual associahedron discussed above. In the case of the complete graph $\Gamma = K_n$ we obtain the usual permutohedron. If Γ is the n-cycle, then we obtain the cyclohedron.

Various graph associahedra, especially those graph associahedra that correspond to Dynkin diagrams and extended Dynkin diagrams, came up earlier in the work of De Concini and Procesi [DP] on wonderful models of subspace arrangements and then in the work on Davis-Januszkiewitz-Scott [DJS]. The class of graph associahedra was independently discovered by Carr and Devadoss in [CD]. They constructed these polytopes using blow-ups, cf. [DJS]. These polytopes also recently appeared in the paper by Toledano-Laredo [Tol] under the name De Concini-Procesi associahedra. We borrowed the term graph associahedra from [CD]. Since they are special cases of our generalized permutohedra, we can also call them *graph permutohedra.*

In the case of graph associahedra it is enough to require condition (N2) of Definition 7.3 and condition (F2) of Definition 7.7 only for $k = 2$. Indeed, if we have several disjoint subsets $I_1, \dots, I_k \in B(\Gamma)$ such that $\Gamma|_{I_1 \cup \cdots \cup I_k}$ is connected, then $\Gamma|_{I_i \cup I_j}$ is connected for some pair i and j.

Definition 8.6. For a graph Γ, let us define the Γ-*Catalan number* as $C(\Gamma) = C(B(\Gamma))$, i.e., it the number of vertices of the graph associahedron, or, equivalently, the number of $B(\Gamma)$-trees; see Definition 7.12.

For the n-chain $\Gamma = A_n$, i.e., the Dynkin diagram of the type A_n, the A_n-Catalan number is the usual Catalan number $C(A_n) = C_n$. For the complete graph, we have $C(K_n) = n!$. Let us calculate several other G-Catalan numbers.

Let $T_{n_1,\dots,n_r}$ be the star graph that has a central node with r attached chains with $n_1, \dots, n_r$ nodes. For example, $T_{1,1,1}$ is the Dynkin diagram of the type D_4.

Proposition 8.7. *For a positive integer r, the generating function $\tilde{C}(x_1, \dots, x_r)$ for the $T_{n_1,\dots,n_r}$-Catalan numbers is given by*

$$\sum_{n_1,\dots,n_r \geq 0} C(T_{n_1,\dots,n_r})\, x_1^{n_1} \dots x_r^{n_r} = \frac{C(x_1) \cdots C(x_r)}{1 - x_1\, C(x_1) - \cdots - x_r C(x_r)},$$

where $C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the usual Catalan numbers.

Proof. According to the recurrence relation in Theorem 7.11, we have

$$C(T_{n_1,\dots,n_r}) = C_{n_1}\cdots C_{n_r} + \sum_{k=1}^{r}\sum_{i=1}^{n_k} C(T_{n_1,\dots,n_{k-1},n_k-i,n_{k+1},\dots,n_r})\cdot C_{i-1}. \tag{8.1}$$

Indeed, the first term corresponds to removing the central node and splitting the graph $T_{n_1,\dots,n_k}$ into r chains. The remaining terms correspond to removing a node in one of the chains and splitting the graph into two connected components. This relation can be written in terms of generating functions as

$$\tilde{C}(x_1,\dots,x_r) = C(x_1)\dots C(x_r) + \sum_{k=1}^{r} x_k\cdot \tilde{C}(x_1,\dots,x_r)\cdot C(x_k),$$

which is equivalent to the claim. □

Let us calculate Γ-Catalan numbers for a class of graphs which includes all Dynkin diagrams. Let $D_n = T_{1,1,n-3}$, $\hat{A}_n$ be the $(n+1)$-cycle, $E_n = T_{1,2,n-4}$.

Proposition 8.8. *The Γ-Catalan numbers for these graphs are given by*

$$\begin{aligned}
&C(A_n) = C_n = \tfrac{1}{n+1}\tbinom{2n}{n},\ \textit{for } n\geq 1,\\
&C(\hat{A}_n) = (n+1)C_n = \tbinom{2n}{n},\ \textit{for } n\geq 3,\\
&C(D_n) = 2\,C_n - 2\,C_{n-1} - C_{n-2},\ \textit{for } n\geq 3,\\
&C(E_n) = 3\,C_n - 4\,C_{n-1} - 3\,C_{n-2} - 2\,C_{n-3},\ \textit{for } n\geq 4.
\end{aligned}$$

Proof. We already proved that $C(A_n) = C_n$. Using Theorem 7.11, we deduce that $C(\hat{A}_n) = (n+1)C(A_n)$. According Theorem 7.11 or (8.1), we deduce that the numbers $C(D_n)$ can be calculated using the recurrence relations $C(D_n) = C_{n-3} + 2C_{n-1} + \sum_{i=1}^{n-3} C(D_{n-i})\,C_{i-1}$, for $n\geq 4$, and $C(D_3) = 5$. In order to prove that $C(D_n) = 2C_n - 2C_{n-1} - C_{n-2}$ it is enough to check that the right-hand side satisfy the recurrence this relation and that $2C_3 - 2C_2 - C_1 = 5$. We can easily do this using the recurrence relation for the Catalan numbers $C_n = \sum_{i=1}^{n} C_{n-i}C_{i-1}$, for $n\geq 1$. Similarly, the numbers $C(E_n)$ are given by the recurrence relation $C(E_n) = C_{n-1} + C(D_{n-1}) + C_{n-2} + 2\,C_{n-4} + \sum_{i=1}^{n-4} C(E_{n-i})\,C_{i-1} = 3\,C_{n-1} - C_{n-2} - C_{n-3} + 2\,C_{n-4} + \sum_{i=1}^{n-4} C(E_{n-i})\,C_{i-1}$, for $n\geq 5$, and $C(E_4) = 14$. Again, we can easily check that the right hand side of $C(E_n) = 3\,C_n - 4\,C_{n-1} - 3\,C_{n-2} - 2\,C_{n-3}$ satisfies this relation, and that $3\,C_4 - 4\,C_3 - 3\,C_2 - 2\,C_1 = 14$. □

Similarly, for any fixed $n_1,\dots,n_{k-1}$, the number $f_n = C(T_{n_1,n_2,\dots,n_{k-1},n})$ can be expressed as a linear combination of several Catalan numbers.

Remark 8.9. One can define the generalized Catalan number for any Lie type. However this number does not depend on multiplicity of edges in the Dynkin diagram. The Catalan number for the Lie types B_n and C_n is the usual Catalan number C_n.

8.5. **Pitman-Stanley polytope.** All above examples are special cases of graph associahedra. Let us consider an example that does not belong to this class.

Let $B = B_{\mathit{flag}} = \{[1],[2],\dots,[n]\}$ be the complete flag of subsets in $[n]$, and let $z_i = \sum_{j=1}^{i} y_{[j]}$, for $i = 1,\dots,n$. According to Proposition 6.3, the generalized permutohedron is this case is the polytope given by the inequalities:

$$\{(t_1,\dots,t_n)\mid t_i\geq 0,\ t_1+\cdots+t_i\geq z_i,\ \text{for } i=1,\dots,n-1,\ t_1+\cdots+t_n = z_n\}$$

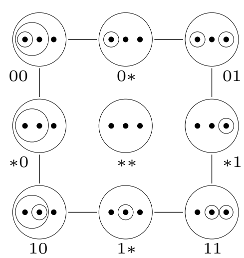
This is exactly the polytope studied by Pitman and Stanley [PiSt]. We will call it the *Pitman-Stanley polytope.*

Let $B^+_{\mathit{flag}} = B_{\mathit{flag}} \cup \{\{1\}, \dots, \{n\}\}$ be the set obtained from B_{flag} by adding all singletons. The generalized permutohedron for B^+_{flag} is just a parallel translation of the Pitman-Stanley polytope. The set B^+_{flag} is a building set. Nested families $N \in \mathcal{N}(B^+_{\mathit{flag}})$ are the subsets $N \subset B^+_{\mathit{flag}}$ such that (1) if $[i] \in N$ then $\{i+1\} \notin N$, and (2) $[n] \in N$. Let us encode a nested set N by a word $u_1, \dots, u_{n-1}$ in the alphabet $\{0, 1, *\}$ such that, for $i = 1, \dots, n-1$, if $[i] \in N$ then $u_i = 0$, if $\{i+1\} \in N$ then $u_i = 1$, otherwise $u_i = *$. This gives a bijection between nested families and 3^{n-1} words of length $n-1$ with these 3 letters. A nested set N contains a nested set N' whenever the word for N is obtained from the word for N' by replacing some $*$'s with 0's and/or 1's. In particular, a nested set is maximal if its words contains only 0's and 1's. Thus the nested complex $\mathcal{N}(B^+_{\mathit{flag}})$ is isomorphic to the face lattice of the $(n-1)$-dimensional hypercube.

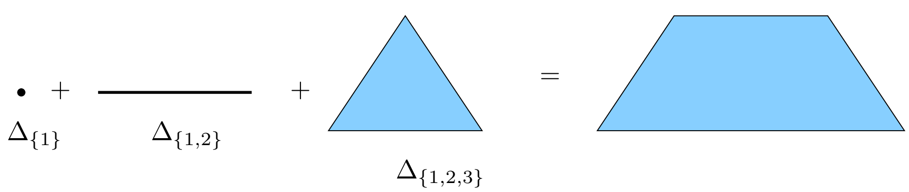
Proposition 8.10. *The Pitman-Stanley polytope has 2^{n-1} vertices and it is combinatorially equivalent to the $(n-1)$-dimensional hypercube.*

Thus the generalized Catalan number in this case is $C(B^+_{\mathit{flag}}) = 2^{n-1}$.

Example 8.11. The following figure shows the combinatorial structure of the Pitman-Stanley polytope for $n = 3$ in terms of nested families.



Note that, as a geometric polytope, the Pitman-Stanley polytope is a *non-regular* quadrilateral, as shown on the following figure.



8.6. **Graphical zonotope.** Let Γ be a graph on the vertex set $[n]$, and let B be the set of all pairs $\{i, j\} \subset [n]$ such that (i, j) is an edge of Γ. The set B *does not* satisfy the axioms of a building set; see Definition 7.1. The minimal building set that contains B is the graphical building set $B(\Gamma)$; see Example 7.2. The generalized permutohedron for the set B is the graphical zonotope Z_Γ; see Definition 2.2. In this case, we can not describe combinatorial structure of Z_Γ using nested families. However it is well-known that the vertices of Z_Γ correspond to acyclic orientations of the graph Γ. It is not hard to describe the faces of this polytope as well. Note that the polytope Z_Γ is dual to the graphic arrangement for the graph Γ.

9. Volume of generalized permutohedra via Bernstein's theorem

Let $G \subseteq K_{m,n}$ be a bipartite graph with no isolated vertices. (This graph should not be confused with graphs used in Section 8.) We will label the vertices G by $1, \dots, m, \bar{1}, \dots, \bar{n}$ and call $1, \dots, m$ the *left vertices* and $\bar{1}, \dots, \bar{n}$ the *right vertices.* Let us associate this graph with the collection $\mathcal{I}_G$ of subsets $I_1, \dots, I_m \subseteq [n]$ such that $j \in I_i$ if and only if $(i, \bar{j})$ is an edge of G. Let us define the polytope $P_G(y_1, \dots, y_m)$ is the Minkowski sum

$$P_G(y_1, \dots, y_m) = y_1 \Delta_{I_1} + \cdots + y_m \Delta_{I_m}.$$

The polytope $P_G(y_1, \dots, y_m)$ is exactly the generalized permutohedron $P_n^y(\{y_I\})$, where $y_I = \sum_{i \mid I_i = I} y_i$.

Remark 9.1. The class of polytopes $P_G(1, \dots, 1)$ is as general as $P_G(y_1, \dots, y_m)$ for arbitrary nonnegative integers $y_1, \dots, y_m$. Indeed, we can always replace a term $y_i \Delta_{I_i}$ with y_i terms Δ_{I_i}. We use the notation $P_G(y_1, \dots, y_m)$ in order to emphasize dependence of this class of polytopes on the parameters $y_1, \dots, y_m$.

Definition 9.2. Let us say that a sequence of nonnegative integers $(a_1, \dots, a_m)$ is a *G-draconian sequence* if $\sum a_i = n - 1$ and, for any subset $\{i_1 < \cdots < i_k\} \subseteq [m]$, we have $|I_{i_1} \cup \cdots \cup I_{i_k}| \geq a_{i_1} + \cdots + a_{i_k} + 1$. Equivalently, $(a_1, \dots, a_m)$ is an *G-draconian* sequence of integers if the sequence of subsets $I_1^{(a_1)}, \dots, I_m^{(a_m)}$, where $I^{(a)}$ means I repeated a times, satisfies the dragon marriage condition; see Proposition 5.4.

Theorem 5.1 can be extended to generalized permutohedra, as follows.

Theorem 9.3. *The volume of the generalized permutohedron $P_G(y_1, \dots, y_m)$ equals*

$$\operatorname{Vol} P_G(y_1, \dots, y_m) = \sum_{(a_1, \dots, a_m)} \frac{y_1^{a_1}}{a_1!} \cdots \frac{y_m^{a_m}}{a_m!},$$

where the sum is over all G-draconian sequences $(a_1, \dots, a_m)$.

We can also reformulate Theorem 9.3, as follows.

Corollary 9.4. *The volume of the generalized permutohedron $P_n^y(\{y_I\})$ is given by*

$$\operatorname{Vol} P_n^y(\{y_I\}) = \frac{1}{(n-1)!} \sum_{(J_1, \dots, J_{n-1})} y_{J_1} \cdots y_{J_{n-1}},$$

where the sum is over ordered collections of nonempty subsets $J_1, \dots, J_{n-1} \subset [n]$ such that, for any distinct $i_1, \dots, i_k$, we have $|J_{i_1} \cup \cdots \cup J_{i_k}| \geq k + 1$.

Proof. Assume in Theorem 9.3 that G is the bipartite graph associated with the collection $I_1, \dots, I_m$, $m = 2^n - 1$, of all nonempty subsets in $[n]$. Then replace the summation over G-draconian sequences $(a_1, \dots, a_m)$ by the summation over $\binom{n-1}{a_1, \dots, a_m}$ rearrangements $(J_1, \dots, J_{n-1})$ of the sequence $(I_1^{(a_1)}, \dots, I_m^{(a_m)})$. $\square$

Example 9.5. Suppose that $I_1, \dots, I_m$, $m = \binom{n}{2}$, is the collection of all 2-element subsets in $[n]$ and $G \subset K_{m,n}$ is the associated bipartite graph. Then $P_G(1, \dots, 1)$ is the regular permutohedron $P_{n-1}(n-1, n-2, \dots, 0)$. In this case, there are n^{n-2} G-draconian sequences $(a_1, \dots, a_m)$, which are in a bijective correspondence with trees on n vertices. For a tree $T \subset K_n$, the a_i's corresponding to the edges of T are equal to 1 and the remaining a_i' are zero, cf. Proposition 5.4. Thus we recover the result that $\operatorname{Vol} P_n(n-1, n-2, \dots, 0) = n^{n-2}$.

Definition 9.6. A sequence of positive integers $(b_1, \dots, b_m)$ is called a *parking function* if its increasing rearrangement $c_1 \leq c_2 \leq \cdots \leq c_m$ satisfies $c_i \leq i$, for $i = 1, \dots, m$.

Recall that there are $(m+1)^{m-1}$ parking functions of the length m.

Example 9.7. Suppose that $I_i = [n+1-i]$, for $i = 1, \dots, m$, where $m = n-1$. In this case, the generalized permutohedron $P_G(y_1, \dots, y_m)$ is the Pitman-Stanley polytope; see Subsection 8.5. A G-draconian sequence is a nonnegative integer sequence $(a_1, \dots, a_m)$ such that $a_1 + \cdots + a_i \geq i$, for $i = 1, \dots, m$, and $a_1 + \cdots + a_m = m$. There are the Catalan number $C_m = \frac{1}{m+1}\binom{2m}{m}$ such sequences. Let us call them *Catalan sequences.* A collection of intervals $I_{b_1}, \dots, I_{b_m}$ satisfies the dragon marriage condition if and only if $(b_1, \dots, b_m)$ is a parking function. We recover the following two formulas for the volume of the Pitman-Stanley polytope proved in [PiSt]:

$$\mathrm{Vol}\, P_G(y_1, \dots, y_m) = \sum_{(a_1,\dots,a_m)} \frac{y_1^{a_1}}{a_1!} \cdots \frac{y_m^{a_m}}{a_m!} = \frac{1}{m!} \sum_{(b_1,\dots,b_m)} y_{b_1} \cdots y_{b_m},$$

where the first sum is over Catalan sequences $(a_1, \dots, a_m)$ and the second sum is over parking functions $(b_1, \dots, b_m)$. In particular, $\mathrm{Vol}\, P_G(1, \dots, 1) = \frac{(m+1)^{m-1}}{m!} = \frac{n^{n-2}}{(n-1)!}$.

The proof of Theorem 9.3 relies on Bernstein's theorem on systems of polynomial equations. Let us first recall the definition of the *mixed volume* $\mathrm{Vol}\,(Q_1, \dots, Q_n)$ of n polytopes $Q_1, \dots, Q_n \subset \mathbb{R}^n$. It is based on the following proposition.

Proposition 9.8. *There exists a unique function* $\mathrm{Vol}\,(Q_1, \dots, Q_n)$ *defined on* n*-tuples of polytopes in* $\mathbb{R}^n$ *such that, for any collection of* m *polytopes* $R_1, \dots, R_m \subset \mathbb{R}^n$*, the usual volume of the Minkowski sum* $y_1 R_1 + \cdots + y_m R_m$*, for nonnegative factors* y_i*, is the polynomial in* $y_1, \dots, y_m$ *given by*

$$\mathrm{Vol}\,(y_1 R_1 + \cdots + y_m R_m) = \sum_{(i_1,\dots,i_n)} \mathrm{Vol}\,(R_{i_1}, \dots, R_{i_n})\, y_{i_1} \cdots y_{i_n},$$

where the sum is over ordered sequences $(i_1, \dots, i_n) \in [m]^n$*.*

For a finite subset $A \subset \mathbb{Z}^n$, let $f_A(t_1, \dots, t_n) = \sum_{a \in A} \beta_a\, t_1^{a_1} \cdots t_n^{a_n}$ be a Laurent polynomial in $t_1, \dots, t_n$ with some complex coefficients β_a.

Theorem 9.9. Bernstein [Ber] *Fix* n *finite subsets* $A_1, \dots, A_n \subset \mathbb{Z}^n$*. Let* Q_i *be the convex hull of* A_i*, for* $i = 1, \dots, n$*. Then the system*

$$\begin{cases} f_{A_1}(t_1, \dots, t_n) = 0, \\ \qquad \vdots \\ f_{A_n}(t_1, \dots, t_n) = 0 \end{cases}$$

of n *polynomial equations in the* n *variables* $t_1, \dots, t_n$ *has exactly* $n!\, \mathrm{Vol}\,(Q_1, \dots, Q_n)$ *isolated solutions in* $(\mathbb{C} \setminus \{0\})^n$ *whenever the collection of all coefficients of the polynomials* f_{A_i} *belong to a certain Zariski open set in* $\mathbb{C}^{\sum |A_i|}$*.*

Bernstein's theorem is usually used for finding the number of solutions of a system of polynomial equations by calculating the mixed volume. We will apply Bernstein's theorem in the opposite direction. Namely, we will calculate the mixed volume by solving a system of polynomial equations. Actually, in our case we need to solve a system of linear equations.

Proof of Theorem 9.3. According to Proposition 9.8 and the definition of the polytope $P_G(y_1,\dots,y_m)$ as the Minkowski sum of simplices, we have

$$\mathrm{Vol}\, P_G(y_1,\dots,y_m) = \sum_{i_1,\dots,i_{n-1}} \mathrm{Vol}\,(\Delta_{I_{i_1}},\dots,\Delta_{I_{i_{n-1}}})\, y_{i_1}\cdots y_{i_{n-1}},$$

where the sum is over all $i_1,\dots,i_{n-1}\in[m]$. Here we can define $(n-1)$-dimensional (mixed) volumes of polytopes embedded into $\mathbb{R}^n$ as (mixed) volumes of their projections into, say, the first $n-1$ coordinates. It remains to show that the mixed volume of several coordinate simplices is equal to

$$\mathrm{Vol}\,(\Delta_{J_1},\dots,\Delta_{J_{n-1}}) = \begin{cases} \frac{1}{(n-1)!} & \text{if } J_1,\dots,J_{n-1} \text{ satisfy DMC,} \\ 0 & \text{otherwise,} \end{cases}$$

where "DMC" stands for the dragon marriage condition; see Proposition 5.4. Consider the following system of $n-1$ linear equations in the variables $t_1,\dots,t_{n-1}$:

$$\begin{cases} \sum_{j\in J_1} \beta_{1,j}\, t_j = 0, \\ \vdots \\ \sum_{j\in J_{n-1}} \beta_{n-1,j}\, t_j = 0, \end{cases}$$

where we assume that $t_n = 1$. According to Bernstein's theorem, this system has exactly $(n-1)!\,\mathrm{Vol}\,(\Delta_{J_1},\dots,\Delta_{J_{n-1}})$ isolated solutions in $(\mathbb{C}\setminus\{0\})^{n-1}$ for generic coefficients $\beta_{i,j}\in\mathbb{C}$, for $j\in J_i$.

Of course, we can always solve this linear system using Cramer's rule. Let $B=(\beta_{ij})$ be the $(n-1)\times n$-matrix formed by the coefficients of the system, where we assume that $\beta_{i,j}=0$, for $j\notin I_i$; and let $|B^{(i)}|$ be the i-th maximal minor of this matrix. The system in nondegenerate if and only if $|B^{(n)}|\neq 0$. In this case, its only solution is given by $t_i = (-1)^i|B^{(i)}|/|B^{(n)}|$, for $i=1,\dots,n-1$. Thus the system has a single isolated solution in $(\mathbb{C}\setminus\{0\})^{n-1}$ if and only if *all* n maximal minors of B are nonzero. Otherwise, the system has no isolated solutions in $(\mathbb{C}\setminus\{0\})^{n-1}$.

The matrix $B=(\beta_{i,j})$ is subject to the only constraint $\beta_{i,j}=0$, for $j\notin J_i$. For generic values of $\beta_{i,j}$, the k-th maximal minor of this matrix is nonzero if and only if there is a system of distinct representatives of $J_1,\dots,J_{n-1}$ that avoids k. According to Proposition 5.4, these conditions are equivalent to the needed condition. This finishes the proof. □

10. Volumes via Brion's formula

Let us give a couple of alternative formulas for volume of generalized permutohedra that extend results of Section 3. It is more convenient to expresses generalized permutohedra in the form $P_n^y(\{y_I\})$; see Section 6.

Theorem 10.1. *For any distinct $\lambda_1,\dots,\lambda_n$, we have*

$$\mathrm{Vol}\, P_n^y(\{y_I\}) = \frac{1}{(n-1)!}\sum_{w\in S_n} \frac{\left(\sum_{I\subseteq[n]} \lambda_{w(\min(I))} y_{w(I)}\right)^{n-1}}{(\lambda_{w(1)}-\lambda_{w(2)})\cdots(\lambda_{w(n-1)}-\lambda_{w(n)})}.$$

This theorem is deduced from Brion's formula (see Appendix 19) in exactly the same way as Theorems 4.2 and 3.1.

For example, we have

$$\operatorname{Vol} P_2^y(\{y_I\}) = \frac{\lambda_1 y_{\{1\}} + \lambda_2 y_{\{2\}} + \lambda_1 y_{\{1,2\}}}{\lambda_1 - \lambda_2} + \frac{\lambda_2 y_{\{2\}} + \lambda_1 y_{\{1\}} + \lambda_2 y_{\{2,1\}}}{\lambda_2 - \lambda_1} = y_{\{1,2\}}$$

Note the terms $\lambda_i y_{\{i\}}$ make a zero contribution. Thus in the summation in Theorem 10.1 we can skip singleton subsets I.

For a collection of subsets $J_1, \dots, J_{n-1} \subseteq [n]$, construct the integer vector $(a_1, \dots, a_n) = e_{\min(J_1)} + \cdots + e_{\min(J_{n-1})}$. Let $I(J_1, \dots, J_{n-1}) = I_{a_1,\dots,a_n}$, defined as in Section 3. Theorem 3.2 can be extended as follows.

Theorem 10.2. *We have*

$$\operatorname{Vol} P_n^y(\{y_I\}) = \sum_{J_1,\dots,J_{n-1}\in[n]} (-1)^{|I(J_1,\dots,J_{n-1})|} \sum_w y_{w(J_1)} \cdots y_{w(J_{n-1})},$$

where the second sum is over permutations $w \in S_n$ with the descent set $I(w) = I(J_1, \dots, J_{n-1})$.

This result is deduced from Theorem 10.1 using the same argument as in the proof of Theorem 3.2.

Theorem 10.1 is convenient for explicit calculations of volumes. Let us give a couple of examples obtained with some help of a computer.

Example 10.3. Let $A_n = (n-1)! \operatorname{Vol} \operatorname{Ass}_n$, where Ass_n is the associahedron in the Loday realization; see Subsection 8.2. According to Theorem 10.1 we have

$$A_n = \sum_{w\in S_n} \frac{\left(\sum_{1\le i\le j\le n} \lambda_{m(i,j,w)}\right)^{n-1}}{(\lambda_{w(1)} - \lambda_{w(2)}) \cdots (\lambda_{w(n-1)} - \lambda_{w(n)})},$$

where $m(i,j,w) = w(\min(w^{-1}([i,j]))) = \min\{k \mid w(k) \in [i,j]\}$. The numbers A_n, for $n = 1, \dots, 8$, are the following:

n	1	2	3	4	5	6	7	8
A_n	1	1	7	142	5895	417201	45046558	6891812712

Example 10.4. (cf. Example 5.5) Let us call a subgraph $G \subseteq K_{n,n}$ a *Hall graph* if it contains a perfect matching or, equivalently, satisfies the Hall marriage condition. Let H_n be the number of Hall subgraphs in $K_{n,n}$. According to Corollary 9.4, $\frac{1}{(n-1)!} H_{n-1}$ is the volume of the generalized permutohedron $P_n^y(\{y_I\})$ with $y_I = 1$, for subsets $I \subseteq [n]$ such that $n \in I$, and $y_I = 0$, otherwise. Using Theorem 10.1 we can calculate several numbers H_n.

n	1	2	3	4	5	6	7
H_n	1	7	247	37823	23191071	54812742655	494828369491583

11. Generalized Ehrhart polynomial

In this section we give a formula for the number of lattice points of generalized permutohedra.

Let us define the *Minkowski difference* of two polytopes $P, Q \subset \mathbb{R}^n$ as $P - Q = \{x \in \mathbb{R}^n \mid x + Q \subseteq P\}$. Its main property is the following.

Lemma 11.1. *For any two polytopes, we have $(P+Q)-Q=P$.*

Proof. We need to prove that, for a point x, we have $x+Q\subseteq P+Q$ if and only if $x\in P$. The "if" direction is trivial. Let us check the "only if" direction. It is enough to assume that $x=0$. We need to show that $Q\subseteq P+Q$ implies that $0\in P$. Suppose that $0\notin P$. Because of convexity of P we can find a linear form f such that $f(p)>0$, for any point $p\in P$ (and, of course, $f(0)=0$). Let $q_{\min}\in Q$ be the point of Q with minimal possible value of $f(q_{\min})$. Then for any point $p+q\in P+Q$, where $p\in P$ and $q\in Q$, we have $f(p+q)=f(p)+f(q)>f(q_{\min})$. Thus $q_{\min}\notin P+Q$. Contradiction. □

Definition 11.2. Let us define the *trimmed generalized permutohedron* as the Minkowski difference of $P_G(y_1,\dots,y_m)$ and the simplex $\Delta_{[n]}$:

$$P_G^-(y_1,\dots,y_m)=P_G(y_1,\dots,y_m)-\Delta_{[n]}=\{x\in\mathbb{R}^n\mid x+\Delta_{[n]}\subseteq P_G\}.$$

This is a slightly more general class of polytopes than generalized permutohedra P_G. Suppose that $I_1=[n]$, i.e., the vertex 1 in G is connected with all vertices in the right part. (If this is not the case, we can always add such a vertex to G.) According to Lemma 11.1, we have

$$P_G(y_1,\dots,y_m)=P_G^-(y_1+1,y_2,\dots,y_m).$$

In other words, if one of the summands in the Minkowski sum for P_G is $\Delta_{[n]}$ then the trimmed generalized permutohedron P_G^- equals the (untrimmed) generalized permutohedron given by a similar Minkowski sum without this summand. Also notice that the class of polytopes $P_G^-(1,\dots,1)$ is as general as $P_G^-(y_1,\dots,y_m)$ for arbitrary nonnegative integer $y_1,\dots,y_m$, cf. Remark 9.1.

Let us give a formula for the generalized Ehrhart polynomial of (trimmed) generalized permutohedra. Define raising powers as $(y)_a:=y(y+1)\cdots(y+a-1)$, for $a\geq 1$, and $(y)_0:=1$. Equivalently, $\frac{(y)_a}{a!}:=\binom{y+a-1}{a}$.

Theorem 11.3. *For nonnegative integers $y_1,\dots,y_m$, the number of lattice points in the trimmed generalized permutohedron $P_G^-(y_1,\dots,y_m)$ equals*

$$P_G^-(y_1,\dots,y_m)\cap\mathbb{Z}^n=\sum_{(a_1,\dots,a_m)}\frac{(y_1)_{a_1}}{a_1!}\cdots\frac{(y_m)_{a_m}}{a_m!},$$

where the sum is over all G-draconian sequences $(a_1,\dots,a_m)$. In particular, the number of lattice points in $P_G(y_1,\dots,y_m)$ equals the above expression with y_1 replaced by y_1+1, assuming that $I_1=[n]$.

This also implies that the number of lattice points in $P_G^-(1,\dots,1)$ equals the number of G-draconian sequences.

In other words, the formula for the number of lattice points in P_G^- is obtained from the formula for the volume of P_G by replacing usual powers in all terms by raising powers. We will prove this theorem in Section 14.

Example 11.4. Let $I_1=[n]$ and $I_2,\dots,I_m$, $m=\binom{n}{2}+1$, be all 2-element subsets in $[n]$, cf. Example 9.5. Then the polytope $P_G^-(1,\dots,1)$ is the regular permutohedron $P_n(n-1,\dots,0)$ and

$$P_G^-(0,1,\dots,1)=P_n(n-1,\dots,0)-\Delta_{[n]}=P_n(n-2,n-2,n-3,\dots,0).$$

In this case, G-draconian sequences are in a bijection with forests $F \subset K_n$. The G-draconian sequence $(a_1, \ldots, a_m)$ associated with a forest F with c connected components is given by $a_1 = c - 1$, $a_i = 1$ if I_i is an edge of F, and $a_i = 0$ otherwise, for $i = 2, \ldots, m$. Theorem 11.3 implies that the number of lattice points of lattice points in the regular permutohedron equals the number of labeled forests on n nodes. More generally, if we set some y_i's to zero, then we deduce that the number of lattice points in a graphical zonotope equals the number of forests in the corresponding graph; see Proposition 2.4.

Theorem 11.3 and Example 11.4 also imply the following statement.

Corollary 11.5. *Let Γ be a connected graph on the vertex set $[n]$. Let Z_Γ be the graphical zonotope, i.e., the Minkowski sum of intervals $[e_i, e_j]$, for edges (i, j) of Γ. Also consider the Minkowski difference $Z_\Gamma^- = Z_\Gamma - \Delta_{[n]}$. Then the volume of Z_Γ equals the number of lattice points in Z_Γ^-:*

$$\mathrm{Vol}\, Z_\Gamma = \#(Z_\Gamma^- \cap \mathbb{Z}^n),$$

and both these numbers are equal to the number of spanning trees in the graph Γ. In particular, the number of lattice points in the permutohedron $P_n(n-2, n-2, n-3, \ldots, 0)$ equals n^{n-2}.

Example 11.6. Suppose that $I_i = [n + 1 - 1]$, for $i = 1, \ldots, m$, where $m = n - 1$, as in Example 9.7. Theorem 11.3 implies the following expression for the number of lattice points in the Pitman-Stanley polytope proved in [PiSt]:

$$\#(P_G(y_1, \ldots, y_m) \cap \mathbb{Z}^n) = \sum_{(a_1,\ldots,a_m)} \frac{(y_1 + 1)_{a_1}}{a_1!} \cdots \frac{(y_m)_{a_m}}{a_m!},$$

where the sum is over Catalan sequences $(a_1, \ldots, a_m)$ as in Example 9.7. Thus the number of lattice points in $P_G^-(1, \ldots, 1) = P_G(0, 1, \ldots, 1) = \Delta_{[2]} + \cdots + \Delta_{[n-1]}$ equals the Catalan number $C_m = C_{n-1}$. Also the number of lattice points in $P_G(1, \ldots, 1) = \Delta_{[2]} + \cdots + \Delta_{[n]}$ equals the Catalan number $\sum_{(a_1,\ldots,a_m)} (a_1 + 1) = C_n$, where the sum is over Catalan sequences.

For a bipartite graph $G \subseteq K_{m,n}$, let $G^* \subseteq K_{n,m}$ be mirror image of G obtained by switching the left and write components. In other words, G^* is the same graph with the relabeled vertices $1, \ldots, m, \bar{1}, \ldots, \bar{n} \longrightarrow \bar{1}, \ldots, \bar{m}, 1, \ldots, n$.

Lemma 11.7. *The set of G-draconian sequences is exactly the set of lattice points of the polytope $P_{G^*}^-(1, \ldots, 1) \subset \mathbb{R}^m$.*

Proof. In order to prove the lemma we just need to check all definitions. Let $I_1^*, \ldots, I_n^* \subseteq [m]$ be the collection of subsets associated with the graph G^*, i.e., $j \in I_i^*$ whenever $(i, \bar{j}) \in G^*$, or, equivalently, $(j, \bar{i}) \in G$. Then $P_{G^*}(1, \ldots, 1) = \Delta_{I_1^*} + \cdots + \Delta_{I_n^*} \subseteq \mathbb{R}^m$. This is exactly the polytope $P_m^z(\{z_I\}$, where $z_I = \#\{i \mid I_i^* \subseteq I\}$, for nonempty $I \subseteq [m]$; see Proposition 6.3. According to Section 6, this polytope is given by the inequalities

$$P_{G^*}(1, \ldots, 1) = \{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid \sum_{i \in [m]} t_i = n,\ \sum_{i \in I} t_i \geq z_I,\ \text{for } I \subset [m]\}.$$

Thus the polytope $P^-_{G^*}(1,\dots,1)$, which is the Minkowski difference of the above polytope and $\Delta_{[m]}$, is given by

$$P^-_{G^*}(1,\dots,1) = \{(t_1,\dots,t_m)\in\mathbb{R}^m \mid \sum_{i\in[m]} t_i = n-1,\ \sum_{i\in I} t_i \geq z_I,\ \text{for } I\subset[m]\}.$$

We have $z_I = \#\{j\in[n] \mid i\in I, \text{ for any edge } (i,\bar{j})\in G\} = n - \left|\bigcup_{j\in J} I_j\right|$, for $I\subseteq[m]$ and $J=[m]\setminus I$. Thus we can rewrite the inequality $\sum_{i\in I} t_i \geq z_I$ as $\sum_{j\in J} t_j \leq \left|\bigcup_{j\in J} I_j\right| - 1$. These are exactly the inequalities from the definition of G-draconian sequence, which proves the claim. □

This shows that Theorem 11.3 gives a formula for the number of lattice points of the polytope $P^-_G(y_1,\dots,y_m)$ as a sum over the lattice points of $P^-_{G^*}(1,\dots,1)$, and vise versa. In particular, we obtain the following duality for trimmed generalized permutohedra.

Corollary 11.8. *The number of lattice points in the polytope $P^-_G(1,\dots,1)$ equals the number of lattice points in the polytope $P^-_{G^*}(1,\dots,1)$:*

$$\#(P^-_G(1,\dots,1)\cap\mathbb{Z}^n) = \#(P^-_{G^*}(1,\dots,1)\cap\mathbb{Z}^m).$$

Notice that the polytopes $P^-_G(1,\dots,1)$ and $P^-_{G^*}(1,\dots,1)$ have different dimensions and they might be very different. In Theorem 12.9 we will describe a class of bijections between lattice points of these polytopes.

Example 11.9. Let $G=K_{m,n}$ be the complete bipartite graph. Then $P^-_{K_{m,n}}$ is the $(n-1)$-dimensional simplex inflated $m-1$ times: $P^-_{K_{m,n}} = (m-1)\Delta_{[n]}$. The polytope for the mirror image of the graph is obtained by switching m and n: $P^-_{K^*_{m,n}} = (n-1)\Delta_{[m]}$. Corollary 11.8 says that these two polytopes have the same number of lattice points. This is a advanced way to say that $\binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}$.

Theorem 19.3(2) from Appendix B (Euler-MacLaurin formula for polytopes) gives the following alternative expression for the generalized Ehrhart polynomial, i.e, for the number of lattice points in $P^z_n(\{z_I\})$. Without loss of generality, we will assume that $z_{[n]}=0$. The volume $\operatorname{Vol} P^z_n(\{z_I\})$ is a homogeneous polynomial $\tilde{V}_n$ in the z_I, for all nonempty $I\subsetneq[n]$.

Proposition 11.10. *The number of lattice points in the generalized permutohedron $P^z_n(\{z_I\})$ is given by the polynomial obtained from the polynomial $\tilde{V}_n$ by applying the Todd operator* $\mathrm{Todd}_n = \prod_{I\subsetneq[n]} \mathrm{Todd}\left(-\frac{\partial}{\partial z_I}\right)$, *where* $\mathrm{Todd}(q) = q/(1-e^{-q}) = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \cdots$.

12. Root polytopes and their triangulations

Definition 12.1. For a graph G on the vertex set $[n]$, let $\tilde{Q}_G\subset\mathbb{R}^n$ be the convex hull of the origin 0 and the points $e_i - e_j$, for all edges (i,j), $i<j$, of G. We will call polytopes $\tilde{Q}_G$ *root polytopes.* In other words, a root polytope is the convex hull of the origin and some subset of end-points of positive roots for a root system of type A_{n-1}. Polytopes $\tilde{Q}_G$ belong to an $n-1$ dimensional hyperplane.

In the case of the complete graph $G = K_n$, the polytope $\tilde{Q}_{K_n}$ was studied in [GGP]. In particular, we constructed a triangulation of this polytope and proved that its $(n-1)$-dimensional volume equals $\frac{1}{(n-1)!}C_{n-1}$, where $C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}$ is the $(n-1)$-st Catalan number.

In this section we study root polytopes for a bipartite graphs $G \subseteq K_{m,n}$. It is convenient to introduce related polytopes

$$Q_G = \mathrm{ConvexHull}(e_i - e_{\bar{j}} \mid \text{for edges } (i, \bar{j}) \text{ of } G) \subset \mathbb{R}^{m+n},$$

where $e_1, \dots, e_m, e_{\bar{1}}, \dots, e_{\bar{n}}$ are the coordinate vectors in $\mathbb{R}^{m+n}$. Since G is a bipartite graph, the polytope Q_G belongs to an $(m+n-2)$-dimensional affine subspace. The polytope $\tilde{Q}_G$ is the pyramid with the base Q_G and the vertex 0. Thus $r!\,\mathrm{Vol}_r\,\tilde{Q}_G = (r-1)!\,\mathrm{Vol}_{r-1}\,Q_G$, where Vol_r stands for the r-dimensional volume. Slightly abusing notation, we will also refer to polytopes Q_G as *root polytopes*.

The polytope $Q_{K_{m,n}}$ for the complete bipartite graph $K_{m,n}$ is the direct product of two simplices $\Delta^{m-1} \times \Delta^{n-1}$ of dimensions $(m-1)$ and $(n-1)$. (Here $\Delta^{m-1} \simeq \Delta_{[m]}$.) For other bipartite graphs, the polytope Q_G is the convex hull of some subset of vertices of $\Delta^{m-1} \times \Delta^{n-1}$. These polytopes are intimately related to generalized permutohedra.

Let $I_1, \dots, I_m$ be the sequence of subsets associated with the graph G, i.e., $j \in I_i$ whenever $(i, \bar{j}) \in G$. Let $P_G = P_G(1, \dots, 1) = \Delta_{I_1} + \cdots + \Delta_{I_m}$ and $P_G^- = P_G - \Delta_{[n]}$.

Theorem 12.2. *For any connected bipartite graph $G \subseteq K_{m,n}$, the $(m+n-2)$-dimensional volume of the root polytope Q_G is expressed in terms of the number of lattice points of the trimmed generalized permutohedron P_G^- as*

$$\mathrm{Vol}\,Q_G = \frac{\#(P_G^- \cap \mathbb{Z}^n)}{(m+n-2)!}.$$

We will prove this theorem by constructing a bijection between simplices in a triangulation of the polytope Q_G and lattice points of the polytope P_G^-; see Theorem 12.9.

For a bipartite graph $G \subseteq K_{m,n}$, let $G^+ \subseteq K_{m+1,n}$ be the bipartite graph obtained from G by adding a new vertex $m+1$ connected by the edges $(m+1, \bar{j})$, $j = 1, \dots, n$, with all vertices of the second part. Then $P_{G^+}^- = P_G$.

Corollary 12.3. *For any bipartite graph $G \subseteq K_{m,n}$ without isolated vertices, the $(m+n-1)$-dimensional volume of the polytope Q_{G^+} is related to the number of lattice points in the generalized permutohedron as*

$$\mathrm{Vol}\,Q_{G^+} = \frac{\#(P_G \cap \mathbb{Z}^n)}{(m+n-1)!}.$$

Definition 12.4. A *polyhedral subdivision* of a polytope Q is a subdivision of Q into a union of cells of the same dimension as P such that each cell is the convex hull of some subset of vertices of Q and any two cells intersect properly, i.e., the intersection of any two cells is their common face. Polyhedral subdivisions are partially ordered by refinement. Minimal elements of this partial order, i.e., unsubdividable polyhedral subdivisions, are called *triangulations*. In a triangulation each cell is a simplex.

Triangulations of the product $\Delta^{m-1} \times \Delta^{n-1}$ were first discussed by Gelfand-Kapranov-Zelevinsky [GKZ, 7.3.D] and then studied by several authors; e.g., see

Santos [San]. We will analyze triangulations of more general root polytopes Q_G. The following 3 lemmas were originally discovered circa 1992 by the author in collaboration with Zelevinsky and Kapranov in the context of triangulations of $\Delta^{m-1} \times \Delta^{n-1}$.

Assume that the graph $G \subseteq K_{m,n}$ is connected. First, let us describe the simplices inside the polytope Q_G.

Lemma 12.5. *For a subgraph $H \subseteq G$, the convex hull of the collection $\{e_i - e_{\bar{j}} \mid (i, \bar{j})$ is an edge of $H\}$ of vertices of Q_G is a simplex if and only if H is a forest in the graph G. Such a simplex has maximal dimension $m + n - 2$ if and only H is a spanning tree of G. All $(m + n - 2)$-dimensional simplices of this form have the same volume $\frac{1}{(m+n-2)!}$.*

Proof. If H contains a cycle $(i_1, \bar{j}_1), (\bar{j}_1, i_2), (i_2, \bar{j}_2), \ldots, (\bar{j}_k, i_1)$, then the vectors $e_{i_1} - e_{\bar{j}_1}, e_{\bar{j}_1} - e_{i_2}, \ldots, e_{\bar{j}_k} - e_{i_1}$ corresponding to the edges in this cycle are linearly dependent. (Their sum is zero.) Thus the end-points of these vectors cannot be vertices of a simplex. Conversely, for a forest, i.e., a graph without cycles, all vectors are linearly independent and, thus, form a simplex. □

For a forest $F \subseteq G$, we will denote the simplex from this lemma by

$$\Delta_F := \mathrm{ConvexHull}(e_i - e_{\bar{j}} \mid (i, \bar{j}) \text{ is an edge of } F).$$

A triangulation of Q_G as a collection of simplices $\{\Delta_{T_1}, \ldots, \Delta_{T_s}\}$, for some spanning trees $T_1, \ldots, T_s$ of G such that $Q_G = \cup \Delta_{T_i}$; and each intersection $\Delta_{T_i} \cap \Delta_{T_j}$ is the common face of these two simplices.

Let us now describe pairs of simplices that intersect properly. For two spanning trees T and T' of G, let $U(T, T')$ be the *directed* graph with the edge set $\{(i, \bar{j}) \mid (i, \bar{j}) \in T\} \cup \{(\bar{j}, i) \mid (i, \bar{j}) \in T'\}$, i.e., $U(T, T')$ is the union of edges T and T' with edges of T oriented from left to right and edges of T' oriented from right to left. An directed *cycle* is a sequence of directed edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), (i_k, i_1)$ such that all $i_1, \ldots, i_k$ are distinct.

Lemma 12.6. *For two trees T and T', the intersection $\Delta_T \cap \Delta_{T'}$ is a common face of the simplices Δ_T and $\Delta_{T'}$ if and only if the directed graph $U(T, T')$ has no directed cycles of length ≥ 4.*

Proof. Suppose that $U(T, T')$ has a directed cycle of length ≥ 4. Then the graphs T and T' have nonempty partial matching (i.e., subgraphs with disjoint edges) M and M' such that (1) M and M' have no common edges; and (2) M and M' are matching on the same vertex set. Then both M and M' should have $k \geq 2$ edges. Let $x = \frac{1}{k} \sum_{(i,\bar{j}) \in M} (e_i - e_{\bar{j}}) = \frac{1}{k} \sum_{(i,\bar{j}) \in M'} (e_i - e_{\bar{j}})$. Thus $x \in \Delta_T \cap \Delta_{T'}$. However, the minimal face of the simplex Δ_T that contains x is Δ_M and the minimal face of $\Delta_{T'}$ that contains x is $\Delta_{M'}$. Since $M \neq M'$, we have $\Delta_M \neq \Delta_{M'}$. Thus the intersection of the simplices Δ_T and $\Delta_{T'}$ *is not* their common face.

Conversely, assume that $U(T, T')$ has no directed cycles of length ≥ 4. Let $F = T \cap T'$ be the forest formed by the common edges of T and T'. Because $U(T, T')$ is acyclic, we can find a function $h : \{1, \ldots, m, \bar{1}, \ldots, \bar{n}\} \to \mathbb{R}$ such that (1) h is constant on connected components of the forest F; and (2) for any directed edge $(a, b) \in U(T, T')$ that joins two different connected components of F, we have $h(a) < h(b)$. The second condition says that if $(a, b) = (i, \bar{j})$ is an edge of T then $h(i) < h(\bar{j})$, and if $(a, b) = (\bar{j}, i)$ is an edge of T' then $h(i) > h(\bar{j})$.

The function h defines a linear form f_h on the space $\mathbb{R}^{m+n}$ with the coordinates $h(1), \dots, h(m), h(\bar{1}), \dots, h(\bar{n})$ in the standard basis. The above conditions imply that (1) for any vertex x in the common face Δ_F of Δ_T and $\Delta_{T'}$, we have $f_h(x) = 0$, (2) for any vertex $x \in \Delta_T \setminus \Delta_F$, we have $f_h(x) < 0$; and (3) for any vertex $x \in \Delta_{T'} \setminus \Delta_F$, we have $f_h(x) > 0$. In other words, the hyperplane $f_h(x) = 0$ intersects the simplices Δ_T and $\Delta_{T'}$ at their common face and separates the remaining vertices of these simplices. This implies that $\Delta_T \cap \Delta_{T'} = \Delta_F$, as needed. □

Definition 12.7. For a spanning tree $T \in K_{m,n}$, let us define the *left degree vector* $LD = (d_1, \dots, d_m)$ and the *right degree vector* $RD = (d_{\bar{1}}, \dots, d_{\bar{n}})$, where $d_i = \deg_i(T) - 1$ and $d_{\bar{j}} = \deg_{\bar{j}}(T) - 1$ are the degrees of the vertices i and $\bar{j}$ in T minus 1. Note that $LD(T)$ and $RD(T)$ are nonnegative integer vectors because all degrees of vertices in a tree are strictly positive.

Lemma 12.8. *Let $\{\Delta_{T_1}, \dots, \Delta_{T_s}\}$ be a triangulation of Q_G. Then, for $i \neq j$, T_i and T_j have different left degree vectors $LD(T_i) \neq LD(T_j)$ and different right degree vectors $RD(T_i) \neq RD(T_j)$.*

Proof. It is enough to prove that it is impossible to find two different spanning trees T and T' have have same degrees in, say, the left part $\deg_i(T) = \deg_i(T')$, for $i = 1, \dots, m$, and such that the directed graph $U(T, T')$ has no directed cycles of length ≥ 4. Suppose that we found two such trees. Let F be the forest formed by the common edges of T and T'. The directed graph $U(T, T')$ induces an acyclic directed graph on connected components of F. Because of the acyclicity of this graph, we can find a minimal connected component C of F such that no directed edge of $U(T, T')$ enters to any vertex of C from outside of this component. Since $T \neq T'$, the component C cannot include all vertices. Thus some vertex i of C should be joined by an edge $(i, \bar{j}) \in T \setminus F$ with a vertex in some other component. Since we assumed that $\deg_i(T) = \deg_i(T')$, there is an edge $(i, \bar{k}) \in T' \setminus F$. But this edge should be oriented as $(\bar{k}, i)$ in the graph $U(T, T')$, i.e., it enters the vertex i of C. Contradiction. □

An alternative proof of Lemma 12.8 follows from Lemma 14.9 below.

For a bipartite graph $G \in K_{m,n}$, let $G^* \in K_{n,m}$ be the same graph with the left and right components switched, i.e., G^* is the mirror image of G. Recall that the trimmed generalized permutohedron P_G^- is the Minkowski difference of the generalized permutohedron P_G and the simplex $\Delta_{[n]}$.

Theorem 12.9. *For any triangulation $\{\Delta_{T_1}, \dots, \Delta_{T_s}\}$ of the root polytope Q_G, the set of right degree vectors $\{RD(T_1), \dots, RD(T_s)\}$ is exactly the set of lattice points in the trimmed generalized permutohedron P_G^- (without repetitions). Similarly, the set of left degree vectors $\{LD(T_1), \dots, LD(T_s)\}$ is exactly the set of lattice points in the polytope $P_{G^*}^-$ for the mirror image of the graph G.*

We will prove this theorem in Section 14. This theorem says that each triangulation $\tau = \{\Delta_{T_1}, \dots, \Delta_{T_s}\}$ of the root polytope G_Q gives a bijection

$$\phi_\tau : \#(P_G^- \cap \mathbb{Z}^n) \to \#(P_{G^*}^- \cap \mathbb{Z}^m)$$

between lattice points of the polytope P_G^- and the lattice points of the polytope $P_{G^*}^-$ such that $\phi_\tau : RD(T_i) \mapsto LD(T_i)$, for $i = 1, \dots, s$.

It is interesting to investigate which properties of a triangulation τ can be recovered from the bijection ϕ_τ. Also it is interesting to intrinsically describe the class of bijections associated with triangulations of Q_G.

Example 12.10. Suppose that $G = K_{m,n}$. Theorem 12.9 says that each triangulation of the product $\Delta^{m-1} \times \Delta^{n-1}$ of two simplices gives a bijection between lattice points two inflated simplices $P^-_{K_{m,n}} = (m-1)\Delta^{n-1}$ and $P^-_{K_{n,m}} = (n-1)\Delta^{m-1}$; see Example 11.9.

Another instance of a similar phenomenon related to maximal minors of matrices was investigated by Bernstein-Zelevinsky [BZ].

13. Root polytopes for non-bipartite graphs

Let us show how to extend the above results to root polytopes $\tilde{Q}_G$ for a more general class of graphs G that may not be bipartite. Assume that G is a connected graph on the vertex set $[n]$ that satisfies the following condition:

For $i < j < k$, if (i,j) and (j,k) are edges of G, then (i,k) is also an edge of G.

The polytope $\tilde{Q}_G$ is has the dimension $(n-1)$. Let us say that a triangulation of the polytope $\tilde{Q}_G$ is *central* if any $(n-1)$-dimensional simplex in this triangulation contains the origin 0.

Definition 13.1. Let us say that a tree is *alternating* if there are no $i < j < k$ such that (i,j) and (j,k) are edges in T. Equivalently, labels in any path in an alternating tree T should alternate.

Alternating trees were first introduced in [GGP] in order to describe triangulations of $\tilde{Q}_{K_n}$. They also appeared in [Pos] and [PoSt].

For a spanning tree $T \subseteq G$, let $\tilde{\Delta}_T = \mathrm{ConvexHull}(0, e_i - e_j \mid (i,j) \in T, i < j)$.

Lemma 13.2. cf. [GGP] *A simplex $\tilde{\Delta}_T$ may appear in a central triangulation of $\tilde{Q}_G$ if and only if T is an alternating tree. All these simplices have the same volume $\frac{1}{(n-1)!}$.*

Proof. Suppose that a tree T is not alternating. Let us find a pair of edges (i,j) and (j,k) in T with $i < j < k$. Let T' be the tree obtained from T by replacing the edge (i,j) with (i,k) and T'' be the tree obtained for T by replacing the edge (j,k) with (i,k). Then two simplices $\tilde{\Delta}_{T'}$ and $\tilde{\Delta}_{T''}$ intersect at their common face. Their union $\tilde{\Delta}_{T'} \cup \tilde{\Delta}_{T''}$ properly contain the simplex $\tilde{\Delta}_T$. Moreover, for neighborhood B of the origin, $(\tilde{\Delta}_{T'} \cup \tilde{\Delta}_{T''}) \cap B = \tilde{\Delta}_T \cap B$. If the simplex $\tilde{\Delta}_T$ belongs to some central triangulation then with can replace it by the pair of simplices $\tilde{\Delta}_{T'}$ and $\tilde{\Delta}_{T'}$ and obtain a "bigger" triangulation, which is impossible. $\square$

For an alternating tree T, we say that a vertex $i \in [n]$ is a *left vertex* if, for any edge (i,j) in T, we have $i < j$. Otherwise, if, for any edge (i,j) in T, we have $i > j$, we say that i is a *right vertex.* For a disjoint decomposition $[n] = L \cup R$, let $G_{L,R}$ be the subgraph of G given by

$$G_{L,R} = \{(i,j) \in G \mid i \in L, j \in R, i < j\}.$$

The graph $G_{L,R}$ is a bipartite graph with the parts L and R. Spanning trees of the graph $G_{L,R}$ are exactly alternating trees of G with fixed sets L and R of left and right vertices. Note that in general there are 2^{n-2} possible choices of the subsets L

and R because we always have $1 \in L$ and $n \in R$ and for any other vertex we have 2 options. However, some of these choices may lead to disconnected graphs $G_{L,R}$ that contain no spanning trees.

Since each alternating tree in G belongs to one of the graphs $G_{L,R}$, we deduce that each simplex $\tilde{\Delta}_T$ in a central triangulation of $\tilde{Q}_G$ belongs to one of the polytopes $\tilde{Q}_{G_{L,R}}$. Thus we obtain the following claim.

Proposition 13.3. *The polytope $\tilde{Q}_G$ decomposes into the union of polytopes $\tilde{Q}_G = \bigcup_{L,R} \tilde{Q}_{G_{L,R}}$ over disjoint decompositions $[n] = L \cap R$ such that the graph $G_{L,R}$ is connected. Terms of this decompositions are in a bijection with the facets of $\tilde{Q}_G$ that do not contain the origin. Each such facet F has the form $F = Q_{G_{L,R}}$ and $\tilde{Q}_{G_{L,R}}$ is the pyramid with the base F. Each central triangulation of $\tilde{Q}_G$ is obtained by selecting a triangulation of each part $Q_{G_{L,R}}$.*

Since each graph $G_{L,R}$ is bipartite, we can apply the results of this section and relate the volume of $\tilde{Q}_{G_{L,R}}$ to the number of lattice points in a certain (trimmed) generalized permutohedron. By Proposition 13.3, we can express the volume of the root polytope $\tilde{Q}_G$ as a sum of numbers of lattice points in several trimmed generalized permutohedra.

Example 13.4. In [GGP] we constructed a triangulation of the polytope $\tilde{Q}_{K_n}$, for the complete graph $G = K_n$. This triangulation is formed by the simplices $\tilde{\Delta}_T$, for all *noncrossing* alternating trees T, i.e., alternating trees that contain no pair of crossing edges (i, k) and (j, l), for $i < j < k < l$. The number of such trees equals the $(n-1)$-st Catalan number C_{n-1}.

For a disjoint decomposition $[n] = L \cap R$, let $K_{L,R}$ be the bipartite graph with the edge set $\{(i, j) \mid i \in L, j \in R, i < j\}$. According to Proposition 13.3, we have $\tilde{Q}_{K_n} = \bigcup_{L,R} \tilde{Q}_{K_{L,R}}$, where different terms have no common interior points. The collection of simplices $\tilde{\Delta}_T$, for all noncrossing spanning trees T of the graph $K_{L,R}$, form a triangulation of the polytope $\tilde{Q}_{K_{L,R}}$.

This example and Theorem 12.2 imply the following statement.

Corollary 13.5. *For any disjoint decomposition $[n] = L \cap R$ such that $1 \in L$ and $n \in R$, the number of noncrossing spanning trees of the graph $K_{L,R}$ equals the number of lattice points in the trimmed generalized permutohedron $P^-_{K_{L,R}}$.*

For example, if $L = \{1, \dots, l\}$ and $R = \{l+1, \dots, n\}$, then $K_{L,R} = K_{l,n-l}$ is the complete bipartite graph. We deduce that the number of noncrossing trees in the complete bipartite graph $K_{l,n-l}$ equals the number of lattice points in the polytope $P^-_{K_{l,n-l}} = (l-1)\Delta^{n-l-1}$, which equals $\binom{n-2}{l-1}$.

14. Mixed subdivisions of generalized permutohedra

In this section we study mixed subdivisions of generalized permutohedra into parts isomorphic to direct products of simplices. For this we use the Cayley trick that relates mixed subdivisions of the Minkowski sum of several polytopes $P_1 + \cdots + P_m$ to all polyhedral subdivision of a certain polytope $\mathcal{C}(P_1, \dots, P_m)$ of higher dimension. The Cayley trick was first developed by Sturmfels [Stu] for coherent subdivisions and by Humber-Rambau-Santos [HRS] for arbitrary subdivisions. Santos [San] used this trick to study triangulations of the product of two simplices.

Definition 14.1. Let d be the dimension of the Minkowski sum $P_1 + \cdots + P_m$. A *Minkowski cell* in this Minkowski sum is a polytope $B_1 + \cdots + B_m$ of the top dimension d, where each B_i is a convex hull of some subset of vertices of P_i. A *mixed subdivision* of the Minkowski sum is its decomposition into a union of several Minkowski cells such that the intersection of any two cells is their common face. Mixed subdivisions form a poset with respect to refinement. A *fine mixed subdivision* is a minimal element in this poset.

Lemma 14.2. *A mixed subdivision is fine if and only if, for each mixed cell* $B = B_1 + \cdots + B_m$ *in this subdivision, all* B_i *are simplices and* $\sum \dim B_i = \dim B = d$.

Proof. We leave this claim as an exercise, or refer to [San, Proposition 2.3]. □

The mixed cells described in this lemma are called *fine mixed cells.* The lemma implies that each fine mixed cell $B_1 + \cdots + B_m$ is isomorphic to the direct product $B_1 \times \cdots \times B_m$ of simplices, i.e., the simplices B_i span independent affine subspaces. In order to emphasize this fact, we will use the direct product notation for fine cells.

Let $e_1, \dots, e_m, e_{\bar{1}}, \dots, e_{\bar{n}}$ be the standard basis of $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. Embed the space $\mathbb{R}^n$, where the polytopes $P_1, \dots, P_n$ live, into $\mathbb{R}^{m+n}$ as the subspace with the basis $e_{\bar{1}}, \dots, e_{\bar{n}}$.

Definition 14.3. Following Sturmfels [Stu] and Huber-Rambau-Santos [HRS], we define the *Cayley embedding* of $P_1, \dots, P_m$ as the polytope $\mathcal{C}(P_1, \dots, P_m)$ given by

$$\mathcal{C}(P_1, \dots, P_m) = \mathrm{ConvexHull}(e_i + P_i \mid i = 1, \dots, m).$$

Let $(y_1, \dots, y_m) \times \mathbb{R}^n$ denote the n-dimensional affine subspace in $\mathbb{R}^{m+n}$ such that the first m coordinates are equal to some fixed parameters $y_1, \dots, y_m$. (Here we think of the y_i not as coordinates but as fixed parameters.)

Lemma 14.4. [Stu, HRS] *For any choice of parameters* $y_1, \dots, y_m \geq 0$ *such that* $\sum y_i = 1$, *the intersection of* $\mathcal{C}(P_1, \dots, P_m)$ *with the affine subspace* $(y_1, \dots, y_m) \times \mathbb{R}^n$ *is exactly the weighted Minkowski sum* $y_1 P_1 + \cdots + y_m P_m$ *(shifted into this affine subspace).*

Proof. Indeed, by the definition, the polytope $\mathcal{C}(P_1, \dots, P_m)$ is the locus of points of the form $\sum_{i=1}^m \lambda_i (e_i + p_i)$, where $p_i \in P_i$, $\lambda_i \geq 0$ and $\sum \lambda_i = 1$. Intersecting a point of this form with $(y_1, \dots, y_n) \times \mathbb{R}^n$ means that we fix $\lambda_i = y_i$, for $i = 1, \dots, m$. This gives the needed Minkowski sum. □

The next proposition expresses the *Cayley trick.*

Proposition 14.5. [HRS] *Fix strictly positive parameters* $y_1, \dots, y_m > 0$ *such that* $\sum y_i = 1$. *For a polyhedral subdivision of* $\mathcal{C}(P_1, \dots, P_m)$, *intersecting its cells with* $(y_1, \dots, y_n) \times R^n$ *we obtain a mixed subdivision of* $y_1 P_1 + \cdots + y_m P_m$. *This gives a poset isomorphism between polyhedral subdivisions of* $\mathcal{C}(P_1, \dots, P_m)$ *and mixed subdivisions of* $y_1 P_1 + \cdots + y_m P_m$.

Proof. The first claim that a polyhedral subdivision of $\mathcal{C}(P_1, \dots, P_m)$ gives a mixed subdivision of $y_1 P_1 + \cdots + y_m P_m$ is immediate. On the other hand, we can recover a polyhedral subdivision of $\mathcal{C}(P_1, \dots, P_m)$ from a mixed subdivision of $y_1 P_1 + \cdots + y_m P_m$. We can always rescale cells of the mixed subdivision by changing values of $y_1, \dots, y_m$ and obtain a mixed subdivision of $y_1' P_1 + \cdots + y_m' P_m$, for any nonnegative $y_1', \dots, y_m'$. As we vary $y = (y_1, \dots, y_m)$ over all points of the simplex $y_1, \dots, y_m \geq$

0, $y_1 + \cdots + y_m = 1$, the unions $\cup_{y\in\Delta^{m-1}} yB$, for each mixed cell B, form cells of the polyhedral subdivision of $\mathcal{C}(P_1, \dots, P_m)$; see [HRS] for details. $\square$

Let $G \subseteq K_{m,n}$ be a connected bipartite graph. Let $I_1, \dots, I_m \subseteq [n]$ be the associated collection of nonempty subsets: $I_i = \{j \mid (i, \bar{j}) \in G\}$, for $i = 1, \dots, m$. Then the Cayley embedding of the simplices $\Delta_{I_1}, \dots, \Delta_{I_m}$ is exactly the root polytope Q_G from Section 12:

$$Q_G = \mathcal{C}(\Delta_{I_1}, \dots, \Delta_{I_m}).$$

Recall that the generalized permutohedron $P_G(y_1, \dots, y_m)$

$$P_G(y_1, \dots, y_m) = y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m},$$

for the nonnegative y_i. Proposition 14.5 specializes to the following claim.

Corollary 14.6. *For any strictly positive $y_1, \dots, y_m$, mixed subdivisions of the generalized permutohedron $P_G(y_1, \dots, y_m)$ are in one-to-one correspondence with polyhedral subdivisions of the root polytope Q_G. In particular, fine mixed subdivisions of $P_G(y_1, \dots, y_m)$ are in one-to-one correspondence with triangulations of Q_G. This correspondence is given by intersecting a polyhedral subdivision of Q_G with the subspace $(\frac{y_1}{s}, \dots, \frac{y_m}{s}) \times \mathbb{R}^n$, where $s = \sum y_i$, and then inflating the intersection by the factor s.*

In particular, this implies that the number of cells in a fine mixed subdivision of P_G equals $(m + n - 2)! \operatorname{Vol} Q_G$.

Let us describe fine mixed cells that appear in subdivisions of $P_G(y_1, \dots, y_m)$. For a sequence of nonempty subsets $\mathcal{J} = (J_1, \dots, J_m)$, let $G_{\mathcal{J}}$ be the graph with the edges $(i, \bar{j})$, for $j \in J_i$.

Lemma 14.7. *Each fine mixed cell in a mixed subdivision of $P_G(y_1, \dots, y_m)$ has the form $y_1\Delta_{J_1} \times \cdots \times y_m\Delta_{J_m}$, for some sequence of nonempty subsets $\mathcal{J} = (J_1, \dots, J_m)$ in $[n]$, such that $G_{\mathcal{J}}$ is a spanning tree of G.*

Proof. By Lemma 14.2, each fine cell has the form $y_1\Delta_{J_1} \times \cdots \times y_m\Delta_{J_m}$ where $J_i \subseteq I_i$, for $i = 1, \dots, m$, i.e., $G_{\mathcal{J}}$ is a subgraph of G, the simplices Δ_{J_i} span independent affine subspaces, and $\sum \Delta_{J_i} = \sum(|J_i| - 1) = n - 1$. This is equivalent to the condition that $G_{\mathcal{J}}$ is a tree. $\square$

Let us denote the fine cell associated with a spanning tree $T \subseteq G$, as described in the above lemma, by

$$\Pi_T := y_1\Delta_{J_1} \times \cdots \times y_m\Delta_{J_m},$$

where $J_i = \{j \mid (i, \bar{j}) \in T\}$, for $i = 1, \dots, m$. These fine cells Π_T are exactly the cells associated with the simplices $\Delta_T \subset Q_G$ from Section 12 via the Cayley trick:

$$\Pi_T = s\left(\Delta_T \cap \left(\frac{y_1}{s}, \dots, \frac{y_m}{s}\right) \times \mathbb{R}^n\right),$$

where $s = \sum y_i$. So it is not surprising that the fine cells Π_T are labeled by the same objects—spanning trees of G.

Let us explain the meaning of the left degree vector $LD(T) = (d_1, \dots, d_m)$ and the right degree vector $RD(T) = (d_{\bar{1}}, \dots, d_{\bar{n}})$ of a tree $T \subseteq G$ in terms of the fine cell Π_T.

Lemma 14.8. *Let $LD(T) = (d_1, \ldots, d_m)$ be the left degree vector of a tree T, then*

$$\operatorname{Vol}\Pi_T = \frac{y_1^{d_1}}{d_1!}\cdots\frac{y_m^{d_m}}{d_m!}.$$

Proof. Indeed, $d_{\bar{i}} = |J_i| - 1 = \dim \Delta_{J_i}$, for $i = 1, \ldots, m$. □

Lemma 14.9. *Let us specialize $y_1 = \cdots = y_m = 1$. For a spanning tree $T \subseteq G$, the fine cell Π_T contains the shift $(a_1, \ldots, a_n) + \Delta_{[n]}$ of the simplex $\Delta_{[n]}$ by an integer vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ if and only if $(a_1, \ldots, a_n)$ is the right degree vector $RD(T)$ of the tree T. Moreover, if $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ is not the right degree vector of T, then the shift $(a_1, \ldots, a_n) + \Delta_{[n]}$ has no common interior points with the cell Π_T.*

Proof. Notice that, for two subsets $I, J \subseteq [n]$ with a nonempty intersection, we have the following inclusion of Minkowski sums:

$$\Delta_I + \Delta_J \supseteq \Delta_{I\cup J} + \Delta_{I\cap J}.$$

Indeed, the polytope $\Delta_{I\cup J} + \Delta_{I\cap J}$ is the convex hull of all possible sums $e_i + e_j$, where e_i is a vertex of $\Delta_{I\cup J}$ and e_j a vertex of $\Delta_{I\cap J}$, i.e., $i \in I \cup J$ and $j \in I \cap J$. We have either ($i \in I$ and $j \in J$), or ($i \in J$ and $j \in I$), or both. In all cases, we have $e_i + e_j \in \Delta_I + \Delta_J$.

For the fine cell $\Pi_T = \Delta_{J_1} \times \cdots \times \Delta_{J_m} = \Delta_{J_1} + \cdots + \Delta_{J_m}$, pick two summands Δ_{J_i} and Δ_{J_j} with a nonempty intersection $J_i \cap J_j$ (what should contain exactly one element k) and replace them by $\Delta_{J_i\cup J_j}$ and $\Delta_{J_i\cap J_j}$. We obtain another cell $\Pi_{T'} \subseteq \Pi_T$, where the tree T' is obtained from T by replacing all edges $(j, \bar{l}) \in T$, for $l \neq k$, with the edges $(i, \bar{l})$. Notice that the tree T' has exactly the same right degree vector $RD(T') = RD(T)$. Let us keep repeating this operation until we obtain a cell of the form $\Pi_{T''} = \Delta_{\{i_1\}} + \cdots + \Delta_{\{i_m\}} + \Delta_{[n]} \subseteq \Pi_T$, i.e., all summands are single vertices except for one simplex $\Delta_{[n]}$. Since the tree T'' has the same right degree vector $(d_{\bar{1}}, \ldots, d_{\bar{n}}) = RD(T'') = RD(T)$ as the tree T, we deduce that $\#\{j \mid i_j = i\} = d_{\bar{i}}$, for $i = 1, \ldots, n$. In other words, $\Pi_{T''} = (d_{\bar{1}}, \ldots, d_{\bar{n}}) + \Delta_{[n]} \subseteq \Pi_F$.

It remains to show that any other shift $(a_1, \ldots, a_n) + \Delta_{[n]}$, for an integer vector $(a_1, \ldots, a_n) \neq (d_{\bar{1}}, \ldots, d_{\bar{n}})$, has no common interior points with the cell Π_T. Suppose that there exists such a shift with a common interior point $b \in \Pi_T \cap ((a_1, \ldots, a_n) + \Delta_{[n]})$. Let $r = (d_{\bar{1}} - a_1, \ldots, d_{\bar{n}} - a_n) \in \mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$. Then the point $b + r$ is an interior point of $(d_{\bar{1}}, \ldots, d_{\bar{n}}) + \Delta_{[n]} \subseteq \Pi_T$. Thus the whole line interval $[b, b+r]$ belong to the interior of the fine cell $\Pi_F = \Delta_{J_1} \times \cdots \times \Delta_{J_m}$. Here $b \in \mathbb{R}^n$ and r is a nonzero integer vector. Thus at least one projection $[b', b' + r']$ of the interval $[b, b + r]$ to some component Δ_{J_i} of the direct product has a nonzero length. Here r' is should be a nonzero integer vector and $[b', b'+r']$ should belong to the interior of the simplex Δ_{J_i}. But this is impossible. No coordinate simplex can contain such an interval strictly in its interior. Indeed, the diameter of a coordinate simplex in the usual Euclidean metric on $\mathbb{R}^n$ is $2^{\frac{1}{n}}$. The only integer vectors that have smaller length are the coordinate vectors $\pm e_j$. If b' belongs to a coordinate simplex then $b' \pm e_j$ does not belong to it, because the vector $\pm e_j$ does not lie in the hyperplane where all coordinate simplices live. We obtain a contradiction. □

Let us now prove Theorems 12.9 and 11.3.

Proof of Theorem 12.9. It is enough to prove the statement about right degree vectors and deduce the statement about left degree vectors by symmetry. By Corollary 14.6, simplices in a triangulation $\{\Delta_{T_1}, \ldots, \Delta_{T_s}\}$ of the root polytope Q_G are in one-to-one correspondence with cells in the corresponding fine mixed subdivision $\{\Pi_{T_1}, \dots, \Pi_{T_s}\}$ of the generalized permutohedron P_G. By Lemma 14.9, each cell Π_{T_i} contains the shifted simplex $a + \Delta_{[n]}$, where $a = RD(T_i)$, and each integer shift $a + \Delta_{[n]} \subseteq P_G$ belongs to one of the cells Π_{T_i}. Notice that the set of integer vectors $a \in \mathbb{Z}^n$ such that $a + \Delta_{[n]} \subseteq P_G$ is exactly the set of lattice points of the trimmed generalized permutohedron P_G^-. This proves that the map $\Delta_{T_i} \mapsto RD(T_i)$ is a bijection between simplices in the triangulations and lattice points of P_G^-, as needed. □

Proof of Theorem 11.3. Let us fix a fine mixed subdivision $\{\Pi_{T_1}, \dots, \Pi_{T_m}\}$ of the polytope $P_G(y_1, \dots, y_m)$. According to Lemma 14.8, the volume of $P_G(y_1, \dots, y_m)$ can be written as

$$\operatorname{Vol} P_G(y_1, \dots, y_m) = \sum_{i=1}^{m} \frac{y_1^{d_1(T)}}{d_1!} \cdots \frac{y_m^{d_m(T)}}{d_m!}.$$

Let us compare this expression with the expression for $\operatorname{Vol} P_G(y_1, \dots, y_m)$ given by Theorem 9.3. We deduce that the map $\Pi(T_i) \mapsto LD(T_i)$ is a bijection between fine cells Π_{T_i} in this subdivision and G-draconian sequences. According to the Cayley trick and Theorem 12.9, the number of fine cells in this subdivision equals the number of simplices in a triangulation for Q_G equals the number of lattice points in $P_{G^-}(1, \dots, 1)$. We deduce that the number of G-draconian sequences equals the number of lattice points of $P_{G^-}(1, \dots, 1)$. This is exactly the claim of Theorem 11.3 in the case when $y_1 = \cdots = y_m = 1$.

The case of general $y_1, \dots, y_m$ follows from this special case. Indeed, we can write any weighted Minkowski sum $y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m}$, for nonnegative integers $y_1, \dots, y_m$, as the Minkowski sum of y_1 copies of Δ_{I_1}, y_2 copies of Δ_{I_2}, etc. When we do this transformation the right-hand sides of expressions given by Theorem 11.3 agree. For example, if we replace the term $y_1\Delta_{I_1}$ in the Minkowski sum with the sum $z_1\Delta_{I_1} + z_2\Delta_{I_1}$, where $y_1 = z_1 + z_2$, then the can correspondingly modify the right-hand side using the identity $\frac{(y_1)_{a_1}}{a_1!} = \binom{y_1+a_1-1}{a_1} = \sum_{b_1+b_2=a_1} \frac{(z_1)_{b_1}}{b_1!}\frac{(z_2)_{b_2}}{b_2!}$. □

Remark 14.10. We can also deduce that the number of G-draconian sequences equals $(m+n-2)! \operatorname{Vol} Q_G$, i.e., the number of simplices in a triangulation of Q_G, using integration. Let us calculate the volume $\operatorname{Vol} Q_G$ by integrating the volume of its slice $P_G(y_1, \dots, y_m)$ given by Theorem 9.3 over all points of the $(m-1)$-dimensional simplex $\Delta_{[m]}$:

$$\operatorname{Vol} Q_G = \int_{(y_1,\dots,y_m)\in\Delta_{[m]}} \operatorname{Vol} P_G(y_1, \dots, y_m) dy_1 \cdots dy_{m-1}.$$

Now we can use the fact that the integral of a monomial $\frac{y_1^{a_1}}{a_1!} \cdots \frac{y_m^{a_m}}{a_m!}$ over the simplex $\Delta_{[m]}$ equals $((m-1+\sum a_i)!)^{-1}$.

Also remark that the first part of the above proof and Theorem 12.9 gives an alternative proof of Lemma 11.7 saying that the set of G-draconian sequences is the set of lattice points in $P_{G^*}^-(1, \dots, 1)$.

Example 14.11. Let us assume that $I_1, \dots, I_m$, $m = 2^n - 1$, are all nonempty subsets of $[n]$ and G is the associated bipartite graph. The G-draconian sequences of integers are in one-to-one correspondence with all *unordered* collections of subsets in $[n]$ satisfying the dragon marriage condition. For a draconian sequence

$(a_1, \ldots, a_m)$ there are $\binom{n-1}{a_1,\ldots,a_m}$ associated ordered sequences of subsets. In this case, $P_G = P_n(2^{n-1}, 2^{n-2} \ldots, 2, 1)$ and $P_G^- = P_n(2^{n-1} - 1, 2^{n-2} \ldots, 2, 1)$ (both are usual permutohedra). The number of draconian sequences is exactly the number of lattice points in the permutohedron $P_n(2^{n-1} - 1, 2^{n-2} \ldots, 2, 1)$.

Another approach to counting lattice points in generalized permutohedra is based on constructing its fine mixed subdivision and paying a special attention to lower dimensional cells. Let us say that a *semi-polytope* is a bounded subset of points in a real vector space given by a finite collection of affine weak and strict equalities. Define coordinate *semi-simplices* as

$$\Delta_{I,j}^{semi} = \Delta_I \setminus \Delta_{I\setminus\{j\}} = \left\{ \sum_{i\in I} x_i\, e_i \mid \sum_{i\in I} x_i = 1;\ x_i \geq 0,\ \text{for } i \in I; \text{ and } x_j > 0 \right\},$$

for $j \in I \subseteq [n]$.

Alternative semiproof of Theorem 11.3. Let $P_G(y_1, \ldots, y_m) = y_1\Delta_{I_1} + \cdots + y_m\Delta_{I_m}$. Assume that $I_1 = [n]$. It seems feasible that there exists a *disjoint* decomposition of the polytope $P_G(y_1, \ldots, y_m)$ into semipolytopes of the form

$$P_G(y_1, \ldots, y_m) = \bigcup_{(J_1,\ldots,J_m)} y_1\Delta_{J_1} \times y_{I_2}\Delta_{J_2,j_2}^{semi} \cdots \times y_m\Delta_{J_m,j_m}^{semi}, \tag{14.1}$$

where the sum is over sequences of subsets $(J_1, \ldots, J_m)$ and $j_2, \ldots, j_m$ such that $j_i \in J_i \subseteq I_i$, and bipartite graphs associated with $(J_1, \ldots, J_m)$ are spanning trees T of G. In particular, the closure of each term is a fine mixed cell Π_T of top dimension.

Here is a not quite rigorous reason why this should be true. Let us start with the top dimensional simplex $y_1\Delta_{I_1}$, $I_1 = [n]$. When we add the simplex $y_2\Delta_{I_2}$, we create several new fine cells. Each of these cells is the direct product $y_1\Delta_{J_1} \times y_2\Delta_{J_2}$ of a face of $y_1\Delta_{I_1}$ and a face of $y_2\Delta_{I_2}$ glued to $y_1\Delta_{I_1}$ by one if its facets $y_1\Delta_{J_1} \times y_2\Delta_{J_2\setminus\{j_2\}}$. This is why we exclude elements of this facet. When we add $y_3\Delta_{I_3}$ we again create several new fine cells. Again each of these new cells is a direct product of one of the faces of the polytope created on the earlier stage and a face $y_3\Delta_{J_3}$ of $y_3\Delta_{I_3}$. Again each of these cells should be glued by a facet of $y_3\Delta_{J_3}$, etc.

Let us show that just an existence of a decomposition for the form (14.1) already implies Theorem 11.3. Indeed, the number of lattice points in one of the terms of this decomposition equals $\frac{(y_1+1)_{a_1}}{a_1!} \frac{(y_2)_{a_2}}{a_2!} \cdots \frac{(y_m)_{a_m}}{a_m!}$ and its volume is $\frac{(y_1+1)^{a_1}}{a_1!} \frac{y_2^{a_2}}{a_2!} \cdots \frac{y_m^{a_m}}{a_m!}$, where $a_i = \dim \Delta_{J_i} = |J_i| - 1$. Thus the formula for the number of lattice points in $P_G(y_1, \ldots, y_m)$ is obtained from the formula for the volume given by Theorem 9.3 by replacing usual powers with raising powers, as needed. □

In order to make this proof more rigorous, we need to carefully analyze all possible cases. Preferably one would like to have an explicit construction for a decomposition of the form (14.1).

In Section 15, we will need the following statement.

Proposition 14.12. *Any integer lattice point of the generalized permutohedron* $P_G = \Delta_{I_1} + \cdots + \Delta_{I_m}$ *has the form* $e_{j_1} + \cdots + e_{j_m}$*, where* $j_k \in I_k$*, for* $k = 1, \ldots, m$.

Remark 14.13. Proposition 14.12 says that any lattice point of the generalized permutohedron is the sum of vertices of its Minkowski summand. Note that of a similar clam is not true for an arbitrary Minkowski sum. For example, the Minkowski sum of two line intervals $[(0,1),(1,0)]$ and $[(0,0),(1,1)]$ contains the lattice point $(1,1)$ which cannot be presented as a sum of the vertices.

Proof of Proposition 14.12. Each lattice point of P_G belongs to a fine mixed cell in a fine mixed subdivision of P_G; see Section 14. According to Lemma 14.7, each fine mixed cell is a direct product $\Delta_{J_1} \times \cdots \times \Delta_{J_m}$ of simplices, where $J_i \subseteq I_i$, for $i = 1, \ldots, m$, and the graph $T = G_{(J_1,\ldots,J_m)} \subseteq K_{m,n}$ is a bipartite tree. Any lattice point $(b_1, \ldots, b_n)$ of $\Delta_{J_1} \times \cdots \times \Delta_{J_m}$ comes from a function $f : \{(i, \bar{j})\} \to \mathbb{R}_{\geq 0}$ defined on edges of the tree T such that (1) $f(i, \bar{j}) \geq 0$, (2) $\sum_j f(i, \bar{j}) = 1$, and (3) $\sum_i f(i, \bar{j}) = b_j$, for any $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Since T is a tree and the sum of values of f over edges at any node of T is integer, we deduce that f has all nonnegative integer values. (First, we prove this for leaves of T, then for leaves of the tree obtained by removing the leaves of T, etc.) Thus, for any $i = 1, \ldots, m$, we have $f(i, \bar{j}_i) = 1$, for some j_i, and $f(i, \bar{j}) = 0$, for $j \neq j_i$. Thus $(b_1, \ldots, b_n) = e_{j_1} + \cdots + e_{j_n}$, as needed.

□

15. Application: diagonals of shifted Young tableaux

A *standard shifted Young tableaux* of the triangular shape $(n, n-1, \ldots, 1)$ is a bijective map $T : \{(i,j) \mid 1 \leq i \leq j \leq n\} \to \{1, \ldots, \binom{n+1}{2}\}$ increasing in the rows and the columns, i.e., $T((i,j)) < T((i+1,j))$ and $T((i,j)) < T((i,j+1))$, whenever the entries are defined. Let us say that the *diagonal vector* of such a tableau T is the vector $\mathrm{diag}(T) = (d_1, \ldots, d_n) := (T(1,1), T(2,2), \ldots, T(n,n))$; see Example 15.5 below. Is clear, that $d_1 = 1$, $d_n = \binom{n+1}{2}$, and $d_{i+1} > d_i$. In this section we describe all possible diagonal vectors.

For a nonnegative integer $(n-1)$-vector $(a_1, \ldots, a_{n-1})$, let $N(a_1, \ldots, a_{n-1})$ be the number of standard shifted Young tableaux T of the triangular shape with the diagonal vector $\mathrm{diag}(T) = (1, a_1 + 2, a_1 + a_2 + 3, \ldots, a_1 + \cdots + a_{n-1} + n)$, or, equivalently, $a_i = d_{i+1} - d_i - 1$, for $i = 1, \ldots, n-1$.

Theorem 15.1. *We have the following identity:*

$$\sum_{a_1,\ldots,a_{n-1}\geq 0} N(a_1, \ldots, a_n)\, \frac{t_1^{a_1}}{a_1!} \cdots \frac{t_{n-1}^{a_{n-1}}}{a_{n-1}!} = \prod_{1 \leq i < j \leq n} \frac{t_i + t_{i+1} + \cdots + t_{j-1}}{j-i}.$$

Proof. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition. The *Gelfand-Tsetlin polytope* $GT(\lambda)$ is defined as the set of triangular arrays $(p_{ij})_{i,j\geq 1, i+j \leq n} \in \mathbb{R}^{\binom{n+1}{2}}$ such that the first row is $(p_{11}, p_{12}, \ldots, p_{1n}) = \lambda$ and entries in consecutive rows are interlaced $p_{i1} \geq p_{i+1\,1} \geq p_{i2} \geq p_{i+1\,2} \geq \cdots$, for $i = 1, \ldots, n-1$.

Let us calculate the volume of the polytope $GT(\lambda)$ in two different ways. First, recall that lattice points of $GT(\lambda)$ correspond to elements of the Gelfand-Tsetlin basis of the irreducible representation V_λ of $GL(n)$ with the highest weight λ. Thus the number of the lattice points is given by the Weyl dimension formula: $\#(GT(\lambda) \cap \mathbb{Z}^{\binom{n+1}{2}}) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$. We deduce that the volume of $GT(\lambda)$

is given by the top homogeneous component of this polynomial in $\lambda_1, \dots, \lambda_n$:

$$\operatorname{Vol} GT(\lambda) = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j}{j - i}.$$

On the other hand, note that the shape of an array $(p_{ij}) \in GT(\lambda)$ is equivalent to the shape of a shifted tableau. Let us subdivide $GT(\lambda)$ into parts by the hyperplanes $p_{ij} = p_{kl}$, for all i, j, k, l. A region of this subdivision of the Gelfand-Tsetlin polytopes $GT(\lambda)$ correspond to a choice of a total ordering of the p_{ij} compatible with all inequalities. Such ordering are in one-to-one correspondence with standard shifted Young tableaux of the triangular shape $(n, n-1, \dots, 1)$. For a tableau T with the diagonal vector $\operatorname{diag}(T) = (d_1, \dots, d_n)$, the associated region of $GT(\lambda)$ is isomorphic to $\{(y_1 < \cdots < y_{\binom{n+1}{2}}) \mid y_{d_i} = \lambda_i, \text{ for } i = 1, \dots, n\}$, that is, to the direct product of simplices $(\lambda_1 - \lambda_2)\Delta^{d_2 - d_1 - 1} \times \cdots \times (\lambda_{n-1} - \lambda_n)\Delta^{d_n - d_{n-1} - 1}$. The volume of this direct product equals

$$\prod_{i=1}^{n-1} \frac{(\lambda_i - \lambda_{i+1})^{d_{i+1} - d_i - 1}}{(d_{i+1} - d_i - 1)!}.$$

Thus the volume of $GT(\lambda)$ can be written as the sum of these expressions over standard shifted tableaux. Comparing these two expressions for $\operatorname{Vol} GT(\lambda)$ and writing them in the coordinates $t_i = \lambda_i - \lambda_{i+1}$, we obtain the needed identity. $\square$

Theorem 15.1 implies that $N(a_1, \dots, a_n)$ can be nonzero only if $(a_1, \dots, a_n)$ is a lattice point of the Newton polytope

$$\mathrm{Ass}_{n-1} := \mathrm{Newton}\left(\prod_{1 \le i < j \le n} (t_i + t_{i+1} + \cdots + t_{j-1})\right) = \sum_{1 \le i < j \le n} \Delta_{[i, j-1]}.$$

This Newton polytope is exactly the associahedron in the Loday realization, for $n-1$; see Subsection 8.2. Using Proposition 14.12, we obtain the following statement.

Corollary 15.2. *The number of different diagonal vectors in standard shifted Young tableaux of the shape $(n, n-1, \dots, 1)$ is exactly the number of integer lattice points in the associahedron Ass_{n-1}. More precisely, $N(a_1, \dots, a_{n-1})$ is nonzero if and only if $(a_1, \dots, a_{n-1})$ is an integer lattice point of Ass_{n-1}.*

It would be interesting to extend this claim to other shifted shapes.

Example 15.3. Let D_n be the number of different diagonal vectors, or, equivalently, the number integer lattice points in Ass_{n-1}, or, equivalently, the number of nonzero monomials in the expansion of the product $\prod_{1 \le i < j \le n} \sum_{k=i}^{j-1} t_k$. Several numbers D_n are given below.

n	1	2	3	4	5	6	7	8	9
D_n	1	1	2	8	55	567	7958	142396	3104160

Theorem 15.1 also implies that $N(a_1, \dots, a_n)$ equals $\prod_{i=1}^{n-1} (a_i)!/(1!2!\cdots(n-1)!)$ times the number of ways to write the point $(a_1, \dots, a_{n-1})$ as a sum of vertices of the simplices $\Delta_{[i, j-1]}$. In particular, if $(a_1, \dots, a_{n-1})$ is a vertex of the associahedron Ass_{n-1} then the second factor is 1.

Recall that vertices of Ass_{n-1} correspond to plane binary trees on $n-1$ nodes; see Subsection 8.2. For a plane binary tree on $n-1$ nodes, let L_i, R_i, $i = 1, \dots, n-1$, be the left and right branches of the nodes arranged in the binary search order; see Subsection 8.2. Also let $l_i = |L_i| + 1$ and $r_i = |R_i| + 1$.

Corollary 15.4. *The numbers of standard shifted Young tableaux with diagonal vectors corresponding to the vertices of the associahedron are given by*

$$T(l_1 \cdot r_1, \ldots, l_{n-1} \cdot r_{n-1}) = \frac{(l_1 \cdot r_1)! \cdots (l_{n-1} \cdot r_{n-1})!}{1!\,2! \cdots (n-1)!} = f_{l_1 \times r_1} \cdots f_{l_{n-1} \times r_{n-1}},$$

where $f_{k \times l}$ is the number of standard Young tableaux of the rectangular shape $k \times l$.

The second expression can be obtained from the first using the hook-length formula for the number of standard Young tableaux. We can also deduce it directly, as follows. Recall that binary trees on $n-1$ nodes are associated with subdivisions of the shifted shape $(n-1, n-2, \ldots, 1)$ into $n-1$ rectangles of sizes $l_1 \times r_1, \ldots, l_{n-1} \times r_{n-1}$; see Subsection 8.2. Each shifted tableaux with the diagonal vector $(d_1, \ldots, d_n) = (1,\ 2 + l_1 \cdot r_1,\ 3 + l_1 \cdot r_1 + l_2 \cdot r_2,\ \cdots)$ is obtained from such a subdivision by adding n diagonal boxes filled with the numbers $d_1, \ldots, d_n$ and filling the i-th rectangle $l_i \times r_i$ with the numbers $d_i + 1, d_i + 2, \ldots, d_{i+1} - 1$ so that they from a rectangular standard tableau, for $i = 1, \ldots, n-1$.

Example 15.5. The diagonal vector $(1, 3, 10, 12, 15, 36, 40, 43, 45)$ is associated with the plane binary tree and the subdivision into rectangles from Example 8.3. Here is a shifted tableau with the this diagonal vector obtained by filling the rectangles of this subdivision:

1	2	4	5	7	16	17	18	23
	3	6	8	9	19	20	21	24
		10	11	13	22	25	26	29
			12	14	27	28	31	34
				15	30	32	33	35
					36	37	38	39
						40	41	42
							43	44
								45

16. Mixed Eulerian numbers

Let us return to the usual permutohedron $P_{n+1} = P_{n+1}(x_1, \ldots, x_{n+1})$. Let us use the coordinates $u_1, \ldots, u_n$ related to $x_1, \ldots, x_{n+1}$ by

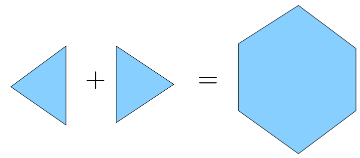
$$u_1 = x_1 - x_2,\ u_2 = x_2 - x_3,\ \cdots,\ u_n = x_n - x_{n+1}$$

This coordinate system is canonically defined for an arbitrary Weyl group as the coordinate system in the weight space given by the fundamental weights.

The permutohedron P_{n+1} can be written as the Minkowski sum

$$P_{n+1} = u_1\,\Delta_{1,n+1} + u_2\,\Delta_{2,n} + \cdots + u_n\,\Delta_{n,n+1}$$

of the *hypersimplices* $\Delta_{k,n+1} := P_{n+1}(1, \ldots, 1, 0, \ldots, 0)$ with k '1's. For example, the hexagon can be expressed as the Minkowski sum of the hypersimplices $\Delta_{1,3}$ and $\Delta_{2,3}$, which are two triangles with opposite orientations:



According to Proposition 9.8, the volume of P_{n+1} can be written as

$$\operatorname{Vol} P_{n+1} = \sum_{c_1,\dots,c_n} A_{c_1,\dots,c_n} \frac{u_1^{c_1}}{c_1!} \cdots \frac{u_n^{c_n}}{c_n!},$$

where the sum is over $c_1, \dots, c_n \geq 0$, $c_1 + \cdots + c_n = n$, and

$$A_{c_1,\dots,c_n} = n!\, V(\Delta_{1,n+1}^{c_1}, \dots, \Delta_{n,n+1}^{c_n}) \in \mathbb{Z}_{>0}$$

is the mixed volume of hypersimplices multiplied by $n!$. Here P^l means the polytope P repeated l times.

Definition 16.1. Let us call the integers $A_{c_1,\dots,c_n}$ the *mixed Eulerian numbers.*

The mixed Eulerian numbers are nonnegative integers because hypersimplices are integer polytopes. In particular, $n! \operatorname{Vol} P_{n+1}$ is a polynomial in $u_1, \dots, u_n$ with positive integer coefficients.

Example 16.2. We have

$$\begin{aligned}
&\operatorname{Vol} P_2 = \mathbf{1}\, u_1;\\
&\operatorname{Vol} P_3 = \mathbf{1}\, \tfrac{u_1^2}{2} + \mathbf{2}\, u_1 u_2 + \mathbf{1}\, \tfrac{u_2^2}{2};\\
&\operatorname{Vol} P_4 = \mathbf{1}\, \tfrac{u_1^3}{3!} + \mathbf{2}\, \tfrac{u_1^2}{2} u_2 + \mathbf{4}\, u_1 \tfrac{u_2^2}{2} + \mathbf{4}\, \tfrac{u_2^3}{3!} + \mathbf{3}\, \tfrac{u_1^2}{2} u_3 + \mathbf{6}\, u_1 u_2 u_3 +\\
&\qquad\qquad + \mathbf{4}\, \tfrac{u_2^2}{2} u_3 + \mathbf{3}\, u_1 \tfrac{u_3^2}{2} + \mathbf{2}\, u_2 \tfrac{u_3^2}{2} + \mathbf{1}\, \tfrac{u_3^3}{3!}.
\end{aligned}$$

Here the mixed Eulerian numbers are marked in bold.

Recall that the usual *Eulerian number* $A(n,k)$ is defined as the number of permutations in S_n with exactly $k-1$ descents. It is well-known that $n! \operatorname{Vol} \Delta_{k,n+1} = A(n,k)$; see Laplace [Lap, p. 257ff].

Theorem 16.3. *The mixed Eulerian numbers have the following properties:*

1. *The numbers $A_{c_1,\dots,c_n}$ are positive integers defined for $c_1, \dots, c_n \geq 0$, $c_1 + \cdots + c_n = n$.*
2. *We have $A_{c_1,\dots,c_n} = A_{c_n,\dots,c_1}$.*
3. *For $1 \leq k \leq n$, the number $A_{0^{k-1},n,0^{n-k}}$ is the usual Eulerian number $A(n,k)$. Here and below 0^l denotes the sequence of l zeros.*
4. *We have $\sum \frac{1}{c_1! \cdots c_n!} A_{c_1,\dots,c_n} = (n+1)^{n-1}$, where the sum is over $c_1, \dots, c_n \geq 0$ with $c_1 + \cdots + c_n = n$.*
5. *We have $\sum A_{c_1,\dots,c_n} = n!\, C_n$, where again the sum is over all $c_1, \dots, c_n \geq 0$ with $c_1 + \cdots + c_n = n$ and $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number.*
6. *For $1 \leq k \leq n$ and $i = 0, \dots, n$, the number $A_{0^{k-1},n-i,i,0^{n-k-1}}$ is equal to the number of permutations $w \in S_{n+1}$ with k descents and $w(n+1) = i+1$.*
7. *We have $A_{1,\dots,1} = n!$.*
8. *We have $A_{k,0,\dots,0,n-k} = \binom{n}{k}$.*
9. *We have $A_{c_1,\dots,c_n} = 1^{c_1} 2^{c_2} \cdots n^{c_n}$ if $c_1 + \cdots + c_i \geq i$, for $i = 1, \dots, n-1$, and $c_1 + \cdots + c_n = n$. There are exactly C_n such sequences $(c_1, \dots, c_n)$.*

Proof. Properties (1) and (2) follow from the definition of the mixed Eulerian numbers. Property (3) follows from the fact that $n!\,\mathrm{Vol}\,\Delta_{k,n+1} = A(n,k)$. Property (4) follows from the fact that the volume of the regular permutohedron $P_{n+1}(n, n-1, \dots, 0)$, which corresponds to $u_1 = \cdots = u_n = 1$, equals $(n+1)^{n-1}$; see Proposition 2.4. Property (5) follows from Theorem 16.4 below. It was conjectured by R. Stanley. Property (6) is equivalent to the result by Ehrenborg, Readdy, and Steingrímsson [ERS, Theorem 1] about mixed volumes of two adjacent hypersimplices. Property (7) is a special case of Property (9).

(8) According to Theorem 3.2, we have

$$\mathrm{Vol}\,P_{n+1}(x_1, 0, \dots, 0, x_{n+1}) = \sum_{k=0}^{n} (-1)^{n-k}\, D_{n+1}([k+1,n])\, \frac{x_1^k}{k!}\, \frac{x_{n+1}^{n-k}}{(n-k)!},$$

where $D_{n+1}([k+1,n]) = \binom{n}{k}$ is the number of permutations $w \in S_{n+1}$ such that $w_1 < \cdots < w_{k+1} > w_{k+2} > \cdots > w_{n+1}$. This permutohedron corresponds to $u_1 = x_1$, $u_2 = \cdots u_{n-1} = 0$, $u_n = -x_{n+1}$, which implies that $A_{k,0,\dots,0,n-k} = \binom{n}{k}$.

(9) Let us use Theorem 5.1. The y-variables are related to the u-variables as

$$\begin{cases} y_2 = u_1, \\ y_3 = u_2 - u_1, \\ y_4 = u_3 - 2u_2 + u_1, \\ \vdots \\ y_{n+1} = \displaystyle\sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} u_{n-i} \end{cases}$$

Using these relations, we can express any coefficient $[u_n^{c_1} \cdots u_1^{c_n}]\, V_{n+1}$ of the polynomial $V_{n+1} = \mathrm{Vol}\,P_{n+1}$ written in the u-coordinates as a combination of coefficients $[y_{n+1}^{c_1'} \cdots y_2^{c_n'}]\, V_{n+1}$ of this polynomial written in the y-coordinates. Let us assume that $(c_1, \dots, c_n)$ satisfies $c_1 + \cdots + c_i \geq i$, for $i = 1, \dots, n-1$, and $c_1 + \cdots + c_n = n$. Then any sequence $(c_1', \dots, c_n')$ that appears in this expression satisfies the same conditions. For such a sequence, we have

$$[y_{n+1}^{c_1'} \cdots y_2^{c_n'}]\, V_{n+1} = \frac{1}{c_1'! \cdots c_n'!} \binom{n+1}{n+1}^{c_1'} \binom{n+1}{n}^{c_2'} \cdots \binom{n+1}{2}^{c_n'}.$$

Indeed, any collection of subsets $J_1, \dots, J_n \subseteq [n+1]$ such that c_i' of them have the cardinality $n+2-i$, for $i = 1, \dots, n$, automatically satisfies the dragon marriage condition; see Theorem 5.1. Thus we have

$$\begin{aligned} A_{c_1,\dots,c_n} &= \left(\tfrac{\partial}{\partial u_n}\right)^{c_1} \cdots \left(\tfrac{\partial}{\partial u_1}\right)^{c_n} V_{n+1} = \left(\left(\tfrac{\partial}{\partial y_{n+1}}\right)^{c_1} \left(\tfrac{\partial}{\partial y_n} - \tbinom{n-1}{1}\tfrac{\partial}{\partial y_{n+1}}\right)^{c_2} \times \right. \\ &\left. \times \left(\tfrac{\partial}{\partial y_{n-1}} - \tbinom{n-2}{1}\tfrac{\partial}{\partial y_n} + \tbinom{n-1}{2}\tfrac{\partial}{\partial y_{n+1}}\right)^{c_3} \cdots \right) V_{n+1} = \\ &= \tbinom{n+1}{n+1}^{c_1} \left(\tbinom{n+1}{n} - \tbinom{n-1}{1}\tbinom{n+1}{n+1}\right)^{c_2} \left(\tbinom{n+1}{n-1} - \tbinom{n-2}{1}\tbinom{n+1}{n} + \tbinom{n-1}{2}\tbinom{n+1}{n+1}\right)^{c_3} \cdots = \\ &= 1^{c_1} 2^{c_2} \cdots n^{c_n}. \end{aligned}$$

In the last equality we used the binomial identity

$$\sum_{i=0}^{k-1} (-1)^i \binom{n-k+i}{i} \binom{n+1}{n+2-k+i} = k, \quad \text{for } 1 \leq k \leq n,$$

which we leave as an exercise. $\square$

Let "$\sim$" be the equivalence relation of the set of nonnegative integer sequences $(c_1, \ldots, c_n)$ with $c_1 + \cdots + c_n = n$ given by $(c_1, \ldots, c_n) \sim (c'_1, \ldots, c'_n)$ whenever $(c_1, \ldots, c_n, 0)$ is a cyclic shift of $(c'_1, \ldots, c'_n, 0)$.

Theorem 16.4. *For a fixed $(c_1, \ldots, c_n)$, we have*

$$\sum_{(c'_1,\ldots,c'_n)\sim(c_1,\ldots,c_n)} A_{c'_1,\ldots,c'_n} = n!$$

In other words, the sum of mixed Eulerian numbers in each equivalence class is $n!$.

There are exactly the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ equivalence classes.

This claim was conjectured by R. Stanley. For example, it says that $A_{1,\ldots,1} = n!$ and that $A_{n,0,\ldots,0} + A_{0,n,0,\ldots,0} + A_{0,0,n,\ldots,0} + \cdots + A_{0,\ldots,0,n} = n!$, i.e., the sum of usual Eulerian numbers $\sum_k A(n,k)$ is $n!$.

Remark 16.5. The claim that there are C_n equivalence classes is well-known. Every equivalence class contains exactly one sequence $(c_1, \ldots, c_n)$ such that $c_1 + \cdots + c_i \geq i$, for $i = 1, \ldots, n$. For this special sequence, the mixed Eulerian number is given by the simple product $A_{c_1,\ldots,c_n} = 1^{c_1} \cdots n^{c_n}$; see Theorem 16.3.(9).

Theorem 16.4 follows from the following claim.

Proposition 16.6. *Let us write* $\mathrm{Vol}\, P_{n+1}$ *as a polynomial* $\hat{V}_{n+1}(u_1, \ldots, u_{n+1})$ *in* $u_1, \ldots, u_{n+1}$. *(This polynomial does not depend on u_{n+1}.) Then the sum of cyclic shifts of this polynomial equals*

$$\hat{V}_{n+1}(u_1, \ldots, u_{n+1}) + \hat{V}_{n+1}(u_{n+1}, u_1, \ldots, u_n) + \cdots + \hat{V}_{n+1}(u_2, \ldots, u_{n+1}, u_1) = \\ = (u_1 + \cdots + u_{n+1})^n$$

This claim has a simple geometric explanation in terms of alcoves of the affine Weyl group. Cyclic shifts come from symmetries of the type A_n extended Dynkin diagram.

Proof. Let $W = S_{n+1}$ be the type A_n Weyl group. The associated *affine Coxeter arrangement* is the hyperplane arrangement in the vector space $\mathbb{R}^{n+1}/(1, \ldots, 1)\mathbb{R} \simeq \mathbb{R}^n$ given by $t_i - t_j = k$, for $1 \leq i < j \leq n+1$ and $k \in \mathbb{Z}$. Here and below in this proof the coordinates $t_1, \ldots, t_{n+1}$ in $\mathbb{R}^{n+1}$ are understood modulo $(1, \ldots, 1)\mathbb{R}$. These hyperplanes subdivide the vector space into simplices, which are called the *alcoves.* The reflections with respect to these hyperplanes generate the *affine Weyl group* W_{aff} that acts simply transitively on the alcoves.

The *fundamental alcove* $A_\circ$ is given by the inequalities $t_1 > t_2 > \cdots > t_{n+1} > t_1 - 1$. It is the n-dimensional simplex with the vertices $v_0 = (0, \ldots, 0)$, $v_1 = (1, 0, \ldots, 0)$, $v_2 = (1, 1, 0, \ldots, 0)$, $\ldots$, $v_n = (1, \ldots, 1, 0)$. For $i = 1, \ldots, n$, the map

$$\phi_i : (t_1, \ldots, t_{n+1}) \mapsto (t_{i+1}, \ldots, t_{n+1}, t_1 - 1, \ldots, t_i - 1)$$

preserves the fundamental alcove and sends the vertex v_i to the origin v_0. We have $\mathrm{Vol}\, A_\circ = \frac{1}{|W|} = \frac{1}{(n+1)!}$, assuming that we normalize the volume as in Section 4.

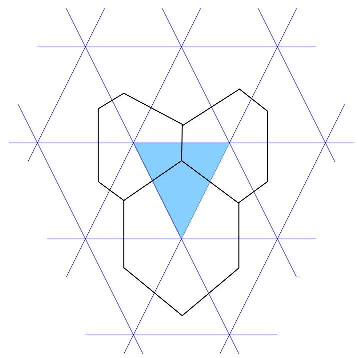
Let up pick a point $x = (x_1, \ldots, x_{n+1})$ in $A_\circ$. The W_{aff}-orbit of x has a unique representative in each alcove. For any vertex v of the affine Coxeter arrangement, i.e., for a 0-dimensional intersection of its hyperplanes, the convex hull of elements the orbit $W_{\mathrm{aff}} \cdot x$ contained in the alcoves adjacent to v is a (parallel translation) of a permutohedron. This collection of permutohedra associated with vertices of the arrangement forms a subdivision of the linear space.

For the origin $v = v_0$, we obtain the permutohedron $P_{(0)} = P_{n+1}(x_1, \dots, x_{n+1})$, and, for the vertex v_i, $i = 1, \dots, n$, we obtain the permutohedron

$$P_{(i)} = \phi_i^{-1} P_{n+1}(\phi_i(x)) = \phi_i^{-1} P_{n+1}(x_{i+1}, \dots, x_{n+1}, x_1 - 1, \dots, x_i - 1).$$

Note that, for $i = 0, \dots, n$, we have $\mathrm{Vol}\, P_{(i)} \cap A_\circ = \frac{1}{|W|} \mathrm{Vol}\, P_{(i)}$. Indeed, each permutohedron $P_{(i)}$ is composed of $|W|$ isomorphic parts obtained by reflections of $\mathrm{Vol}\, P_{(i)} \cap A_\circ$.

Thus the volume of the fundamental alcove times $|W|$ equals the sum of volumes of $n+1$ adjacent permutohedra, For example, the 6 areas of the blue triangle on the following picture is the sum of the areas of three hexagons.



In other words, we have $1 = |W| \cdot \mathrm{Vol}\, A_\circ = \sum_{i=0}^{n} \mathrm{Vol}\, P_{(i)}$. The last expression can be written in the u-coordinates as

$$\hat{V}_{n+1}(u_1, \dots, u_{n+1}) + \hat{V}_{n+1}(u_2, \dots, u_{n+1}, u_1) + \cdots + \hat{V}_{n+1}(u_{n+1}, u_1, \dots, u_n),$$

assuming that $u_1 + \cdots + u_n = 1$. The case of arbitrary $u_1, \dots, u_n$ is obtained by multiplying all u_i's by the same factor α which corresponds to multiplying the volume by α^n. □

Proof of Theorem 16.4. We obtain the required equality when we extract the coefficient of $u_1^{c_1} \cdots u_n^{c_n} u_{n+1}^0$ in the both sides of the identity in Proposition 16.6. □

Proposition 16.6 together with Theorem 3.1 implies the following identity. It would be interesting to find a direct proof of this claim.

Corollary 16.7. *The symmetrization of the expression*

$$\frac{1}{n!} \frac{(\lambda_1 u_1 + (\lambda_1 + \lambda_2) u_2 + \cdots (\lambda_1 + \cdots + \lambda_{n+1}) u_{n+1})^n}{(\lambda_1 - \lambda_2) \cdots (\lambda_n - \lambda_{n+1})}$$

with respect to $(n+1)!$ *permutations of* $\lambda_1, \dots, \lambda_{n+1}$ *and* $(n+1)$ *cyclic permutations of* $u_1, \dots u_{n+1}$ *equals* $(u_1 + \cdots + u_{n+1})^n$.

17. Weighted binary trees

Let us give a combinatorial interpretation for the mixed Eulerian numbers based on plane binary trees.

Let T be a plane binary tree on $[n]$ with the binary search labeling of the nodes; see Subsection 8.2. There are the Catalan number C_n of such trees. For any node $i = 1, \dots, n$, the set $\mathrm{desc}(i, T)$ of descendants of i (including the node i itself) is a consecutive interval $\mathrm{desc}(i, T) = [l_i, r_i]$ of integers. In particular, we have

$l_i \leq i \leq r_i$. For a pair nodes i and j in T such that $i \in \mathrm{desc}(j,T)$, i.e., $l_j \leq i \leq r_j$, define the weight

$$wt(i,j) = \min\left(\frac{i-l_j+1}{j-l_j+1}, \frac{r_j-i+1}{r_j-j+1}\right) = \begin{cases} \frac{i-l_j+1}{j-l_j+1} & \text{if } i \leq j, \\ \frac{r-i+1}{r_j-j+1} & \text{if } i > j. \end{cases} \tag{17.1}$$

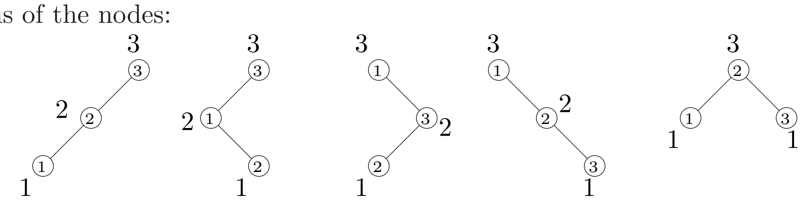
Let $h(j,T) := |\mathrm{desc}(j,T)|$ be the "hook-length" of a node j in a rooted tree T.

Theorem 17.1. *The volume of the permutohedron P_{n+1} is given by the following polynomial in the variables $u_1, \dots, u_n$:*

$$\mathrm{Vol}\, P_{n+1} = \sum_T \frac{n!}{\prod_{j=1}^n h(j,T)} \prod_{j=1}^n \left(\sum_{i \in \mathrm{desc}(j,T)} wt(i,j)\, u_i \right),$$

where the sum is over C_n plane binary trees T with n nodes.

Example 17.2. For $n = 3$, we have the following five binary trees, where we indicated the binary search labeling inside the nodes and also indicated the hook-lengths of the nodes:



hook-lengths of binary trees

Theorem 17.1 says that

$$\begin{aligned} \mathrm{Vol}\, P_4 &= (u_1)(\tfrac{1}{2}\, u_1 + u_2)(\tfrac{1}{3}\, u_1 + \tfrac{2}{3}\, u_2 + u_3) + (u_1 + \tfrac{1}{2}\, u_2)(u_2)(\tfrac{1}{3}\, u_1 + \tfrac{2}{3}\, u_2 + u_3) \\ &\quad + (u_1 + \tfrac{2}{3}\, u_2 + \tfrac{1}{3}\, u_3)(u_2)(\tfrac{1}{2}u_2 + u_3) + (u_1 + \tfrac{2}{3}\, u_2 + \tfrac{1}{3}\, u_3)(u_2 + \tfrac{1}{2}u_3)(u_3) \\ &\quad + 2 \cdot (u_1)(\tfrac{1}{2}\, u_1 + u_2 + \tfrac{1}{2}\, u_3)(u_3). \end{aligned}$$

Corollary 17.3. *We have*

$$(n+1)^{n-1} = \sum_T \frac{n!}{2^n} \prod_{j \in T} \left(1 + \frac{1}{h(j,T)}\right),$$

where is sum is over C_n plane binary trees T with n nodes.

For $n = 3$, the corollary says that $(3+1)^2 = 3 + 3 + 3 + 3 + 4$; see figure in Example 17.2.

Proof. Let us specialize Theorem 17.1 for $u_1 = \cdots = u_n = 1$. In this case, P_{n+1} is the regular permutohedron with volume $(n+1)^{n-1}$, see Proposition 2.4. Easy calculation shows that $\sum_{i \in \mathrm{desc}(j,T)} wt(i,j) = \frac{h(j,T)+1}{2}$. Thus the right-hand side of Theorem 17.1 gives the needed expression. $\square$

Various combinatorial proofs and generalizations of Corollary 17.3 were given by Seo [Seo], Du-Liu [DL], and Chen-Yang [CY].

An *increasing labeling* of nodes in a rooted tree T on $[n]$ is a permutation $v \in S_n$ such that, whenever $i \in \mathrm{desc}(j,T)$, i.e., the node i is a descendant of the node j, we have $v(i) \geq v(j)$. It is well-known that the number of increasing labelings is given by the following "hook-length formula;" see Knuth [Knu, Exer. 5.1.4.(20)] and Stanley [St1, Prop. 22.1]. It can be easily proved by induction.

Lemma 17.4. *The number of increasing labeling of a tree T equals $\frac{n!}{\prod_{j=1}^{n} h(j,T)}$.*

Let us say that an *increasing binary tree* (T, v) is a plane binary tree T with the binary search labeling as above and a choice of an increasing labeling v of its nodes. It is well-known that there are $n!$ increasing binary trees. The map $(T, v) \mapsto v$ is a bijection between increasing binary trees and permutations $v \in S_n$; cf. [St2, 1.3.13].

Let $\mathbf{i} = (i_1, \dots, i_n) \in [n]^n$ be a sequence of integers. Let us say that an increasing binary tree (T, v) is $\mathbf{i}$-*compatible* if $i_{v(j)} \in [l_j, r_j]$, for $j = 1, \dots, n$. Define the $\mathbf{i}$-*weight* of an $\mathbf{i}$-compatible increasing binary tree (T, v) as

$$wt(\mathbf{i}, T, v) = \prod_{j=1}^{n} wt(i_{v(j)}, j).$$

where $wt(i_{v(j)}, j)$ is given by (17.1). The number $n!\, wt(\mathbf{i}, T, v)$ is always a positive integer. The following lemma can be easily proved by induction, cf. Lemma 17.4. We leave it as an exercise.

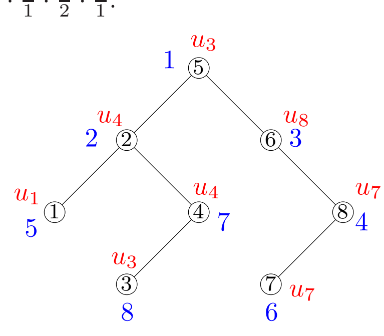
Lemma 17.5. *We have, $n!$ divided by all denominators in $wt(\mathbf{i}, T, v)$ equals the number labelings of the nodes of T by permutations $w \in S_n$ such that, for any node j, for which we pick the first (respectively, second) case in the definition of $wt(i_{v(j)}, j)$, the label $w(j)$ is less than labels $w(k)$ of all nodes k in the left (respectively, right) branch of the node j.*

Example 17.6. The following figure shows an $\mathbf{i}$-compatible increasing binary tree, for $\mathbf{i} = (3, 4, 8, 7, 1, 7, 4, 3)$. The labels for the binary search labeling are shown inside the nodes. The increasing labeling is $v = 5, 2, 8, 7, 1, 3, 6, 4$ (shown in blue color). The intervals $[l_j, r_j]$ are $[1, 1]$, $[1, 2]$, $[3, 3]$, $[3, 4]$, $[1, 8]$, $[6, 8]$, $[7, 7]$, $[7, 8]$. We also marked each node j by the variable $u_{i_{v(j)}}$ (shown in red color). The $\mathbf{i}$-weight of this tree is $wt(\mathbf{i}, T, v) = \frac{3}{5} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{2}{2} \cdot \frac{1}{1}$.

an $\mathbf{i}$-compatible binary tree

$T =$

$\mathbf{i} = (3, 4, 8, 7, 1, 7, 4, 3)$



Let us give a combinatorial interpretation for the mixed Eulerian numbers.

Theorem 17.7. *Let $(i_1, \dots, i_n)$ be any sequence such that $u_{i_1} \cdots u_{i_n} = u_1^{c_1} \cdots u_n^{c_n}$. Then*

$$A_{c_1,\dots,c_n} = \sum_{(T,v)} n!\, wt(\mathbf{i}, T, v),$$

where the sum is over $\mathbf{i}$-compatible increasing binary trees (T, v) with n nodes.

Note that all terms $n!\, wt(\mathbf{i}, T, v)$ in this formula are positive integers. Actually, this theorem gives not just one but $\binom{n}{c_1,\dots,c_n}$ different combinatorial interpretations of the mixed Eulerian numbers $A_{c_1,\dots,c_n}$ for each way to write $u_1^{c_1}\cdots u_n^{c_n}$ as $u_{i_1}\cdots u_{i_n}$. We will extend and prove Theorem 17.1 in Section 18. Let us now derive Theorem 17.7 from it.

Proof of Theorem 17.7. The volume of the permutohedron is obtained by multiplying the right-hand side of Theorem 17.1 by $\frac{1}{n!}\, u_{i_1}\cdots u_{i_n}$ and summing over all sequences $\mathbf{i} = (i_1, \dots, i_n) \in [n]^n$:

$$\mathrm{Vol}\, P_{n+1} = \sum_{\mathbf{i}\in[n]^n} u_{i_1}\cdots u_{i_n} \sum_{(T,v)} wt(\mathbf{i}, T, v),$$

where the second sum is over $\mathbf{i}$-compatible increasing binary trees (T, v) with n nodes. This formula together with Lemma 17.4 implies the needed expression. □

18. Volumes of weight polytopes via Φ-trees

In this section we extend the results of the previous section to weight polytopes for an arbitrary root system.

Let Φ be an irreducible root system of rank n with a choice of simple roots $\alpha_1, \dots, \alpha_n$, and let W be the associated Weyl group. Let (x, y) be a W-invariant inner product. Let $\omega_1, \dots, \omega_n$ be the fundamental weights. They form the dual basis to the basis of simple coroots $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$. Let $P_W(x)$ be the associated weight polytope, where $x = u_1\omega_1 + \cdots + u_n\omega_n$; see Definition 4.1. Its volume it a homogeneous polynomial V_Φ of degree n in the variables $u_1, \dots, u_n$:

$$V_\Phi(u_1, \dots, u_n) := \mathrm{Vol}\, P_W(u_1\omega_1 + \cdots + u_n\omega_n).$$

Recall the definition of $B(\Gamma)$-trees; cf. Definition 7.7 and Subsection 8.4.

Definition 18.1. For a connected graph Γ, a $B(\Gamma)$*-tree* is a rooted tree T on the same vertex set such that

(T1) For any node i and the set $I = \mathrm{desc}(i, T)$ of all descendants of i in T, the induced graph $\Gamma|_I$ is connected.

(T2) There are no two nodes $i \neq j$ such that the sets $I = \mathrm{desc}(i, T)$ and $J = \mathrm{desc}(j, T)$ are disjoint and the induced graph $\Gamma|_{I\cup J}$ is connected.

An *increasing* $B(\Gamma)$*-tree* (T, v) is $B(\Gamma)$-tree T together with an increasing labeling v of its nodes, defined as in Section 17. In the case when Γ is the Dynkin diagram of the root system Φ, we will call these objects Φ*-trees* and *increasing* Φ*-trees*.

The next proposition extends the well-known claim that there are $n!$ increasing binary trees on n nodes.

Proposition 18.2. *For any connected graph Γ on n nodes, the number of increasing $B(\Gamma)$-trees equals $n!$.*

Proof. The map $(T, v) \mapsto v$ is a bijection between increasing $B(\Gamma)$-trees and permutations $v \in S_n$. □

For a subset $I \subseteq [n]$, let Φ_I be the root system with simple roots $\{\alpha_i \mid i \in I\}$, and let $W_I \subset W$ be the associated parabolic subgroup. Let ω_i^I, $i \in I$ be the fundamental weights for the root system Φ_I. For $j \in I \subseteq [n]$, let us define the linear form $f_{I,j}(u) := \frac{1}{|I|}\sum_{i\in I} u_i\, (\omega_i^I, \omega_j^I)$ in the variables u_i.

Theorem 18.3. *The volume of the weight polytope $P_W(x)$ is given by*

$$V_\Phi(u_1,\dots,u_n) = \frac{2^n\cdot|W|}{\prod_{i=1}^n(\alpha_i,\alpha_i)}\sum_T\prod_{j=1}^n f_{\mathrm{desc}(j,T),j}(u),$$

where the sum is over all Φ-trees T.

Definition 18.4. The *mixed Φ-Eulerian numbers* $A^\Phi_{c_1,\dots,c_n}$, for $c_1,\dots,c_n\geq 0$, $c_1+\cdots+c_n=n$, are defined as the coefficients of the polynomial expressing the volume of the weight polytope:

$$V_\Phi(u_1,\dots,u_n)=\sum_{c_1,\dots,c_n}A^\Phi_{c_1,\dots,c_n}\,\frac{u_1^{c_1}}{c_1!}\cdots\frac{u_n^{c_n}}{c_n!}.$$

Equivalently, the mixed Φ-Eulerian numbers are the mixed volumes of the Φ-*hypersimplices,* which are the weight polytopes for the fundamental weights.

For a sequence $\mathbf{i}=(i_1,\dots,i_n)\in[n]^n$, let us say that an increasing Φ-tree (T,v) is $\mathbf{i}$-*compatible* if $i_{v(j)}\in\mathrm{desc}(j,T)$, for $j=1,\dots,n$.

Theorem 18.5. *Let $(i_1,\dots,i_n)$ be any sequence such that $u_{i_1}\cdots u_{i_n}=u_1^{c_1}\cdots u_n^{c_n}$. Then*

$$A^\Phi_{c_1,\dots,c_n}=\frac{2^n\cdot|W|}{\prod_{i=1}^n(\alpha_i,\alpha_i)}\sum_{(T,v)}\prod_{j=1}^n\left(\omega_{i_{v(j)}}^{\mathrm{desc}(j,T)},\omega_j^{\mathrm{desc}(j,T)}\right),$$

where the sum is over $\mathbf{i}$-compatible increasing Φ-trees (T,v).

The proof of these results is based on the following recurrence relation for volumes of weight polytopes. Let $\Phi_{(j)}:=\Phi_{[n]\setminus\{j\}}$ be the root system whose Dynkin diagram is obtained by removing the jth node, and let $W_{(j)}:=W_{[n]\setminus\{j\}}$ be the corresponding Weyl group, for $j=1,\dots,n$.

Proposition 18.6. *For $i=1,\dots,n$, we have*

$$\frac{\partial}{\partial u_i}V_\Phi(u_1,\dots,u_n)=\sum_{j=1}^n\frac{|W|}{|W_{(j)}|}\,\frac{(\omega_i,\omega_j)}{(\alpha_j,\omega_j)}\,V_{\Phi_{(j)}}(u_1,\dots,u_{j-1},u_{j+1},\dots,u_n).$$

Note that $(\alpha_j,\omega_j)=\frac{1}{2}(\alpha_j,\alpha_j)(\alpha_j^\vee,\omega_j)=\frac{1}{2}(\alpha_j,\alpha_j)$.

Proof. The derivative $\partial V_\Phi/\partial u_i$ is the rate of change of the volume of the weight polytope as we move its generating vertex x in the direction of the ith fundamental weight ω_i. It can be written as the sum of $(n-1)$-dimensional volumes of facets of $P_W(x)$ scaled by some factors, which tell how fast the facets move. Facets of $P_W(x)$ have the form $w(P_{W_{(j)}}(x))$, where $j\in[n]$ and $w\in W/W_{(j)}$. In other words, $P_W(x)$ has $\frac{|W|}{|W_{(j)}|}$ facets isomorphic to $P_{W_{(j)}}(x)$.

The facet $P_{W_{(j)}}(x)$ is perpendicular to the fundamental weight ω_j. Note that this facet $P_{W_{(j)}}(x)$ is a parallel translate of $P_{W_{(j)}}(x')$, where $x'=u_1\omega_1^{(j)}+\cdots+u_{j-1}\omega_{j-1}^{(j)}+u_{j+1}\omega_{j+1}^{(k)}+\cdots+u_n\omega_n^{(j)}$ and $\omega_i^{(j)}:=\omega_i^{[n]\setminus\{j\}}$. Indeed, the fundamental weights $\omega_i^{(j)}$ for the root system $\Phi_{(j)}$ are projections of the fundamental weights ω_i, $i\neq j$, for Φ to the hyperplane perpendicular to ω_j. Thus the $(n-1)$-dimensional volume of this facet is $\mathrm{Vol}\,P_{W_{(j)}}(x)=V_{\Phi_{(j)}}(u_1,\dots,u_{j-1},u_{j+1},\dots,u_n)$.

If we move x in the direction of a vector v, then the facet $P_{W_{(j)}}(x)$ moves with the velocity proportional to (v,ω_j). Recall that we normalize the volume so that the volume of the parallelepiped generated by the simple roots $\alpha_1, \dots, \alpha_n$ is 1; see Section 4. Thus the scaling factor for $v = \alpha_j$ is 1, and, in general, the scaling factor is $\frac{(v,\omega_j)}{(\alpha_j,\omega_j)}$. In particular, for $v = \omega_i$, we obtain the needed factor $\frac{(\omega_i,\omega_j)}{(\alpha_j,\omega_j)}$. By symmetry, all facets $w(P_{W_{(j)}}(x)$ come with the same factors. $\square$

Proof of Theorem 18.5. Fix a sequence $\mathbf{i} = (i_1, \dots, i_n)$ such that $u_{i_1} \cdots u_{i_n} = u_1^{c_1} \cdots u_n^{c_n}$. Then, by the definition,

$$A^{\Phi}_{c_1,\dots,c_n} = \frac{\partial}{\partial\, u_{i_n}} \cdots \frac{\partial}{\partial\, u_{i_1}} \cdot V_{\Phi}(u_1, \dots, u_n).$$

Applying Proposition 18.6 repeatedly, we deduce that $A^{\Phi}_{c_1,\dots,c_n}$ equals the weighted sum over $\mathbf{i}$-compatible increasing Φ-trees (T, v), where each tree comes with the weight

$$\prod_{k=1}^{n} \left(\frac{|W_{I_k}|}{\prod_l |W_{I_{k,l}}|} \cdot \frac{2}{(\alpha_{j_k}, \alpha_{j_k})} \, (\omega_{i_k}^{I_k}, \omega_{j_k}^{I_k}) \right),$$

where $j_1, \dots, j_n$ is the inverse permutation to v, $I_k = \mathrm{desc}(j_k, T)$, and $I_{k,l}$, $l = 1, 2, \dots$, are the vertex sets of the branches of the vertex j_k in T. Note that all terms in the first quotient, except the term $|W|$, cancel each other. Thus we obtain the expression in the right-hand side of Theorem 18.5. $\square$

Proof of Theorem 18.3. The volume $V_\Phi(u_1, \dots, u_n)$ is obtained by multiplying the right-hand side of Theorem 18.5 by $\frac{1}{n!}\, u_{i_1} \cdots u_{i_n}$ and summing over all sequences $(i_1, \dots, i_n) \in [n]^n$. Thus we obtain

$$V_\Phi(u_1, \dots, u_n) = \frac{2^n \cdot |W|}{n! \cdot \prod_{i=1}^n (\alpha_i, \alpha_i)} \sum_T \mathrm{incr}(T) \prod_{j=1}^{n} (|\mathrm{desc}(j,T)| \cdot f_{\mathrm{desc}(j,T),j}(u)),$$

where the sum is over all Φ-trees T and $\mathrm{incr}(T)$ is the number of increasing labeling of T. Using Lemma 17.4, which says that $\mathrm{incr}(T) = n!/\prod |\mathrm{desc}(j,T)|$, we derive the needed statement. $\square$

For the Lie type A_n, Proposition 18.6 specializes to the following claim. Let us write $\mathrm{Vol}\, P_{n+1}$ as a polynomial $V_{n+1}(u_1, \dots, u_n)$ in $u_1, \dots, u_n$.

Proposition 18.7. *For any $i = 1, \dots, n$, we have* $\frac{\partial}{\partial u_i} V_{n+1}(u_1, \dots, u_n) =$

$$\sum_{j=1}^{n} \binom{n+1}{j} \frac{j\,(n+1-j)}{n+1}\, wt_{i,j,n}\, V_j(u_1, \dots, u_{j-1})\, V_{n-j+1}(u_{j+1}, \dots, u_n),$$

where $wt_{i,j,n} = \min(\frac{i}{j}, \frac{n+1-i}{n+1-j})$.

Proof. In this case, we have $W = S_{n+1}$, $V_W = V_{n+1}(u_1, \dots, u_n)$, $W_{(j)} = S_j \times S_{n+1-j}$, $P_{W_j} = P_j \times P_{n+1-j}$, and $V_{W_{(j)}} = V_j(u_1, \dots, u_{j-1})\, V_{n-j+1}(u_{j+1}, \dots, u_n)$. Thus $\frac{|W|}{|W_{(j)}|} = \binom{n+1}{j}$. The root system lives in the space $\{(t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1} \mid t_1 + \cdots t_{n+1} = 0\}$ with the inner product induced from $\mathbb{R}^{n+1}$. In this space, the simple roots are $\alpha_i = e_i - e_{i+1}$ and the fundamental weights are $\omega_i = e_1 + \cdots + e_i - \frac{i}{n+1}(1, \dots, 1)$, for $i = 1, \dots, n$. We have $(\alpha_j, \alpha_j) = 2$ and $(\alpha_j, \omega_j) = 1$. Thus $\frac{(\omega_i,\omega_j)}{(\alpha_j,\omega_j)} = (\omega_i, \omega_j) = \min(i,j) - \frac{i \cdot j}{n+1} = \frac{j\,(n+1-j)}{n+1}\, wt_{i,j,n}$. $\square$

Proof of Theorems 17.1 and 17.7. By Theorem 18.5 and proof of Proposition 18.7, the mixed Eulerian number $A_{c_1,\dots,c_n}$ equals the weighted sum over $\mathbf{i}$-compatible increasing binary trees, where each tree (T, v) comes with the weight

$$(n+1)! \cdot \prod_{j=1}^{n} \frac{(j-l_j+1)\,(h_j+1-j)}{h_j+1} \cdot \min\left(\frac{i_{v(j)}-l_j+1}{j-l_j+1}, \frac{r_j-i_{v(j)}+1}{r_j-j+1}\right),$$

where $l_j \le r_j$ are defined as in Section 17 and $h_j = |\mathrm{desc}(j,T)| = r_j - l_j + 1$. All terms in the first quotient, except the term $\frac{1}{n+1}$, cancel each other. Note that the product $\prod_{j=1}^{n} \min(\frac{i_{v(j)}-l_j+1}{j-l_j+1}, \frac{r_j-i_{v(j)}+1}{r_j-j+1})$ is exactly $wt(\mathbf{i}, T, v)$. Thus the total weight of (T, v) equals $(n+1)!\,\frac{1}{n+1}\,wt(\mathbf{i}, T, v)$, as needed. $\square$

19. Appendix: Lattice points and Euler-MacLaurin formula

In this section, we review some results of Brion [Bri], Khovanskii-Pukhlikov [KP1, KP2], Guillemin [Gui], and Brion-Vergne [BV1, BV2] related to counting lattice points and volumes of polytopes. For the completeness sake, we included short proofs of these results.

Instead of calculating the volume or counting the number of lattice points in a polytope, let us sum monomials over the lattice points in the polytope. We can work with unbounded polyhedra, as well.

Recall that a polytope in $\mathbb{R}^n$ is a convex hull of a finite set of vertices. A *rational polyhedron* in $\mathbb{R}^n$ is an intersection of a finite set of half-spaces with rational (equivalently, integer) coordinates. In particular, rational polyhedra include polytopes with rational vertices and *rational cones*, i.e., cones with a rational vertex and integer generating vectors.

Let $\chi_P : \mathbb{Z}^n \to \mathbb{Q}$ be the *characteristic function* (restricted to the integer lattice) of a polyhedron P given by $\chi_P(x) = 1$, if $x \in P$, and $\chi_P(x) = 0$, if $x \notin P$. The *algebra of rational polyhedra* A is the linear space of functions $\mathbb{Z}^n \to \mathbb{R}$ spanned by the characteristic functions χ_P of rational polyhedra. The space A is closed under multiplications of functions, because $\chi_P \cdot \chi_Q = \chi_{P\cap Q}$. The algebra A is generated by the *Heaviside functions* $H_{h,c} = \chi_{\{x \mid h(x)\ge c\}}$, where h is an integer linear form and $c \in \mathbb{Z}$.

The group algebra of the integer lattice $\mathbb{Z}^n$ is the algebra of Laurent polynomials $\mathbb{Q}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Let $\mathbb{Q}(t_1, \dots, t_n)$ be the *field of rational functions*, which is the field of fractions of the group algebra. For a vector $a \in \mathbb{Z}^n$, let $t^a := t_1^{a_1} \cdots t_n^{a_n}$.

Theorem 19.1. Khovanskii-Pukhlikov [KP1] *There exists a unique linear map $S : A \to \mathbb{Q}(t_1, \dots, t_n)$ such that*

(a) *$S(\delta) = 1$, where $\delta = \chi_{\{0\}}$ is the delta-function.*
(b) *For any $\nu \in A$ and $a \in \mathbb{Z}^n$, we have $S(\nu(x - a)) = t^a\, S(\nu)$.*

The map S has the following properties:

(1) *For a function ν on $\mathbb{Z}^n$ with a finite support, we have $S(\nu) = \sum_a \nu(a)\, t^a$. In particular, for a polytope P, we have $S(\chi_P) = \sum_{a\in P\cap\mathbb{Z}^n} t^a$.*
(2) *If $\nu \in A$ is a b-periodic function for some nonzero vector $b \in \mathbb{Z}^n$, i.e., $\nu(x) \equiv \nu(x - b)$, then $S(\nu) = 0$. Thus, for a rational polyhedron P that contains a line, we have $S(\chi_P) = 0$.*

(3) *For a simple rational cone* $C = v + \mathbb{R}_{\geq 0} g_1 + \cdots + \mathbb{R}_{\geq 0} g_m$, *where* $v \in \mathbb{Q}^n$ *and* $g_1, \ldots, g_m \in \mathbb{Z}^n$ *are linearly independent, we have*

$$S(\chi_C) = \left(\sum_{a \in \Pi \cap \mathbb{Z}^n} t^a \right) \prod_{i=1}^{m} (1 - t^{g_i})^{-1},$$

where Π *is the parallelepiped* $\{v + c_1 g_1 + \cdots + c_m g_m \mid 0 \leq c_i < 1\}$.

Proof. Let us first check that conditions (a) and (b) imply properties (1), (2), and (3). We have $S(\nu) = S(\sum_a \nu(a)\delta(x-a)) = \sum_a \nu(a)\, t^a$, for a function ν with a finite support. For a b-periodic function $\nu \in A$, we have $S(\nu) = t^b S(\nu)$ by (b), and, thus, $S(\nu) = 0$. Let us write, using the inclusion-exclusion principle, $\chi_\Pi = \chi_C - \sum_i \chi_{C+g_i} + \sum_{i<j} \chi_{C+g_i+g_j} - \cdots$. Thus by (b), we have $S(\chi_\Pi) = S(\chi_C) - (\sum_i t^{v_i}) S(\chi_C) + (\sum_{i<j} t^{g_i+g_j}) S(\chi_C) - \cdots = S(\chi_C) \prod_i (1 - t^{g_i})$, which is equivalent to (3).

Let us now prove the existence and uniqueness of the map S. We can subdivide any rational polyhedron P into rational simplices and simple rational cones. Furthermore, we can present the characteristic function of a simplex as an alternating sum of characteristic functions of simple rational cones. Thus we can write χ_P as a linear combination of characteristic functions of simple rational cones. Since conditions (a) and (b) imply expression (3) for $S(\chi_C)$ for each simple rational cone, the expression $S(\chi_P)$ is uniquely determined by linearity.

Let us verify that this construction for S is consistent. In other words, we need to check that, for any linear dependence $b_1 \chi_{C_1} + \cdots + b_N \chi_{C_N} = 0$ of characteristic functions of simple rational cones, we have $b_1 S(\chi_{C_1}) + \cdots + b_N S(\chi_{C_N}) = 0$, where each term $S(\chi_{C_i}) = f_i \cdot \prod_j (1 - t^{v_{ij}})^{-1}$ is given by expression (3). Here f_i are certain Laurent polynomials. Let us assume that $b_1 \chi_{C_1} + \cdots + b_N \chi_{C_N} = 0$ and $b_1 S(\chi_{C_1}) + \cdots + b_N S(\chi_{C_N}) = f/D$, where f is a nonzero Laurent polynomial and $D = \prod_{ij} (1 - t^{v_{ij}})$ is the common denominator of the terms $S(\chi_{C_i})$. Let us select a norm on $\mathbb{Z}^n$, for example, $|a| := \sqrt{a_1^2 + \cdots + a_n^2}$. Let R be a sufficiently large number such that $R > |a|$ for any monomial t^a that occurs in f or D with a nonzero coefficient. We can write each term as $S(\chi_{C_i}) = \sum_{|a| \leq 3R} \chi_{C_i}(a)\, t^a + \tilde{f}_i \cdot \prod_j (1 - t^{v_{ij}})^{-1}$, where, for any monomial t^a that occurs in $\tilde{f}_i$, we have $|a| > 2R$. Let us sum the right-hand sides of these expressions with the coefficients b_i. Then the first terms cancel and we obtain $b_1 S(\chi_{C_1}) + \cdots + b_N S(\chi_{C_N}) = \sum_i \tilde{f}_i \prod_j (1 - t^{v_{ij}})^{-1} = f/D$. We deduce that f is a linear combination of monomials t^a with $|a| > R$, which contradicts to our choice of R. This proves the existence and uniqueness of the map S. $\square$

Let A' be the subspace in the algebra of rational polyhedra A spanned by characteristic functions χ_P of rational polyhedra P that contain lines. According to Theorem 19.1, we have $S(f) = 0$, for any $f \in A'$. Thus we obtain a well-defined linear map $S : A/A' \to B$.

For a rational polyhedron P and a point $u \in \mathbb{Z}^n$, let $C_{P,u}$ denote the rational cone with the vertex at u such that $P \cap B = C_{P,u} \cap B$ for a sufficiently small open neighborhood B of u. Notice that $\chi_{C_{P,u}} \notin A'$ if and only if u is a vertex of the polyhedron P.

For an analytic function $f(t)$ defined in a neighborhood of 0, let $[t^n]\, f(t)$ denote the coefficient of t^n in its Taylor expansion. Notice that $\frac{t}{1-e^{-t}} = 1 + \frac{t}{2} +$

$\sum_{k=1}^{\infty}(-1)^{k-1} B_k \frac{t^{2k}}{(2k)!}$, is an analytic function at $t=0$, where B_k are the Bernoulli numbers.

Theorem 19.2. Brion [Bri], Khovanskii-Pukhlikov [KP1]

(1) *For any rational polyhedron P, we have $\chi_P \equiv \sum_{v\in V} \chi_{C_{P,v}}$ modulo the subspace A', where the sum is over the vertex set V of P.*

(2) *We have $S(P) = \sum_{v\in V} S(C_{P,v})$. In particular, for a simple rational polyhedron P, we have*

$$S(P) = \sum_{v\in V} \frac{\sum_{a\in\Pi_v\cap\mathbb{Z}^n} z^a}{\prod_{i=1}^n (1-z^{g_{i,v}})},$$

where the sum is over vertices v of P, $g_{1,v},\dots,g_{n,v}\in\mathbb{Z}^n$ are the integer generators of the cone $C_{P,v}$, and $\Pi_v = \{v + c_1 g_{1,v} + \cdots + c_n g_{n,v} \mid 0\le c_i < 1\}$.

(3) *For a simple rational polytope P, the number of lattice points in P equals*

$$\#\{P\cap\mathbb{Z}^n\} = [t^n]\left\{\sum_{v\in V}\left(\sum_{a\in\Pi_v\cap\mathbb{Z}^n} e^{t\cdot h(a)}\right)\prod_{i=1}^n \frac{t}{1-e^{t\cdot h(g_{i,v})}}\right\}.$$

where $h\in(\mathbb{R}^n)^*$ is any linear form such that $h(g_{i,v})\neq 0$, for all vectors $g_{i,v}$.

(4) *The volume of a simple rational polytope P equals*

$$\operatorname{Vol} P = \frac{1}{n!}\sum_{v\in V} \frac{|\det(g_{1,v},\dots,g_{n,v})|\, h(v)^n}{(-1)^n \prod_{i=1}^n h(g_{i,v})},$$

where $\det(g_{1,v},\dots,g_{n,v})$ is the determinant of the $n\times n$-matrix with the row vectors $g_{i,v}$ and $h\in(\mathbb{R}^n)^*$ is any linear form such that $h(g_{i,v})\neq 0$, for all vectors $g_{i,v}$.

The formula for the sum of exponents $S(P)$ was first obtained by M. Brion [Bri]. The formula for $\operatorname{Vol} P$ was given by Khovanskii-Pukhlikov [KP2] (in case of Delzant polytopes) and by Brion-Vergne [BV1] in general.

Proof. (1) As we have mentioned in the proof of Theorem 19.1, we can write the characteristic function of a rational polyhedron as a finite linear combination of characteristic functions of rational cones: $\chi_P = \sum_i b_i\,\chi_{C_i}$. Let $U\supseteq V$ be the set of vertices of all cones C_i. For $u\in U$, let I_u be the collection of indices i such that the cone C_i has the vertex u. Then $\sum_{i\in I_u} b_i\,\chi_{C_i} \equiv \chi_{C_{P,u}} \pmod{A'}$. Also $\chi_{C_{P,u}}\in A'$, for $u\in U\setminus V$. This proves the claim.

(2) This claim follows from (1) and Theorem 19.1.

(3) Let us pick a linear form h that does not annihilate any of the vectors $g_{i,v}$. Let B be the subalgebra of $\mathbb{Q}(t_1,\dots,t_n)$ generated by the z^a and $\frac{1}{1-z^b}$, for $a,b\in\mathbb{Z}^n$ such that $h(b)\neq 0$. Let $e_h : B\to\mathbb{R}((q))$ be the homomorphism from B to the ring of formal Laurent series in one variable q given by $z^a\mapsto e^{q\cdot h(a)}$ and $\frac{1}{1-z^b}\mapsto\frac{1}{1-e^{q\cdot h(b)}}$. Let us apply the homomorphism e_h to the expression for $S(P)$ given by (2). Then the number of lattice points $\#\{P\cap\mathbb{Z}^n\}$ is the constant coefficient of the resulting Laurent series. This is exactly the need claim.

(4) The volume of a polytope P can be calculated by counting the number of lattice points in the inflated polytope kP for large k. Explicitly, $\operatorname{Vol} P = \lim_{k\to\infty}\#\{kP\cap\mathbb{Z}^n\}/k^n$. The vertices of the inflated polytope kP are the vectors $k\,v$, for $v\in V$, and the generators of the cone $C_{kP,kv}$ are exactly the same vectors $g_{i,v}$ as for the original polytope P. We may assume that the limit is

taken over k's such that all vectors $k\,v$ are integer. Each term in the expression for $\#\{kP\cap\mathbb{Z}^n\}$ given by (3) has the form $[t^n]\left\{e^{t\cdot h(kv+a')}\prod_{i=1}^n \frac{t}{1-e^{t\cdot h(g_{i,v})}}\right\} = [t^n]\left\{e^{t\cdot h(kv+a')}\prod_{i=1}^n(-\frac{1}{h(g_{i,v})}+O(t))\right\}$, where $a'\in(\Pi_v - v)\cap\mathbb{Z}^n$. Since k appears only in the first exponent, this expression is a polynomial in k of degree n with the top term $k^n\left(\frac{1}{n!}h(v)^n(-1)^n\prod_{i=1}^n\frac{1}{h(g_{i,v})}\right)$. There are $|\det(g_{1,v},\dots,g_{n,v})| = |\Pi_v|$ choices for a'. Thus summing these expressions over all v and a' we obtain the needed expression for $\mathrm{Vol}\,P$. □

For a polytope P with the vertices $v_1,\dots,v_M$, we say that a *deformation* of P is a polytope of the form $P' = \mathrm{ConvexHull}(v_1',\dots,v_M')\in\mathbb{R}^n$ such that $v_i'-v_j' = k_{ij}(v_i-v_j)$, for some nonnegative $k_{ij}\in\mathbb{R}_{\geq 0}$, whenever $[v_i,v_j]$ is a 1-dimensional edge of P. A generic deformation of P has the same combinatorial structure as P. However in degenerate cases some of the vertices v_i' may merge with each other.

Deformations of P are obtained by parallel translations of its facets. Suppose that the polytope P has N facets and is given by the linear inequalities $P=\{x\in\mathbb{R}^n \mid h_i(x)\leq c_i,\ i=1,\dots,N\}$, for some $h_i\in(\mathbb{R}^n)^*$ and $c_i\in\mathbb{R}$. Then any deformation $P'=\mathrm{ConvexHull}(v_1',\dots,v_M')$ has the form

$$P(z_1,\dots,z_n) := \{x\in\mathbb{R}^n \mid h_i(x)\leq z_i,\ i=1,\dots,N\},\ \text{for some } z_1,\dots,z_N\in\mathbb{R},$$

where $h_i(v_j') = z_i$ whenever i-th facet of P contains the j-th vertex v_j. For this polytope we will write $v_i(z_1,\dots,z_N) = v_i'$. Let $\mathcal{D}_P\subset\mathbb{R}^N$ be the set of N-tuples $(z_1,\dots,z_N)$ corresponding to deformations of P. Then $\mathcal{D}_P$ is a certain polyhedral cone in $\mathbb{R}^N$ that we call the *deformation cone*. If P is a simple polytope then $\mathcal{D}_P$ has dimension N, because any sufficiently small parallel translations of the facets of P give a deformation of P. Deformations $P(z_1,\dots,z_n)$ for interior points $(z_1,\dots,z_n)\in\mathcal{D}_P\setminus\partial\mathcal{D}_P$ of the cone $\mathcal{D}_P$ are exactly the polytopes whose associated fan coincides with the fan of P.

A simple integer polytope P is called a *Delzant polytope* if, for each vertex v of P, the cone $C_{P,v}$ is generated by an integer basis of the lattice $\mathbb{Z}^n$. Such polytopes are associated with smooth toric varieties. Formulas in Theorem 19.2 are especially simple for Delzant polytopes. Indeed, in this case $\Pi_v\cap\mathbb{Z}^n$ consists of a single element v and $|\det(g_{1,v},\dots,g_{n,v})| = 1$. For Delzant polytopes, we assume that we pick the linear forms h_i corresponding to the facets of P so that h_i are integer and are not divisible by a nontrivial integer factor.

Let $I_P(z_1,\dots,z_N) = \#\{P(z_1,\dots,z_N)\cap\mathbb{Z}^n\}$ be the number of lattice points and $V_P(z_1,\dots,z_N) = \mathrm{Vol}\,P(z_1,\dots,z_N)$ be the volume of a deformation of P.

Let $\mathrm{Todd}(q) = \frac{q}{1-e^{-q}}$. Since $\mathrm{Todd}(q)$ expands as a Taylor series at $q=0$, we have the well-defined operators $\mathrm{Todd}\left(\frac{\partial}{\partial z_i}\right)$ acting on polynomials in $z_1,\dots,z_N$.

Theorem 19.3. (1) *For an integer polytope P, and $(z_1,\dots,z_N)\in\mathcal{D}_P\cap\mathbb{Z}^N$, the number of lattice points $I_P(z_1,\dots,z_N)$ and the volume $V_P(z_1,\dots,z_N)$ are given by polynomials in $z_1,\dots,z_N$ of degree n. The polynomial $V_P(z_1,\dots,z_N)$ is the top homogeneous component of the polynomial $I_P(z_1,\dots,z_N)$.*

(2) *If P is a Delzant polytope then we have*

$$I_P(z_1,\dots,z_N) = \left(\prod_{i=1}^N \mathrm{Todd}\left(\frac{\partial}{\partial z_i}\right)\right) V_P(z_1,\dots,z_N).$$

We will call the polynomial $I_P(z_1,\dots,z_N)$ the *generalized Ehrhart polynomial* of the polytope P.

Proof. (1) Assume P is a simple polytope. The vertices $v_i(z_1,\dots,z_N)$ of the deformation $P(z_1,\dots,z_N)$ linearly depend on $z_1,\dots,z_N$. According to formulas (3) and (4) in Theorem 19.2, $I_P(z_1,\dots,z_N)$ and $V_P(z_1,\dots,z_N)$ are polynomials in $z_1,\dots,z_N$, because each term in these formulas for $P(z_1,\dots,z_N)$ polynomially depend on v. This remains true for degenerate deformations $P(z_1,\dots,z_N)$ when some of the vertices $v_i(z_1,\dots,z_N)$ merge. Indeed, all claims of Theorem 19.2 remain valid (and proofs are exactly the same) if, instead of summation over actual vertices of $P(z_1,\dots,z_N)$, we sum over $v_i(z_1,\dots,z_N)$. If P is not simple then a generic small parallel translation of its facets results in a simple polytope. Thus P can be thought of as a degenerate deformation of a simple polytope and the above argument works.

(2) For a simple polytope P, we have

$$\frac{\partial}{\partial z_i} v_j(z_1,\dots,z_N) = \begin{cases} -\alpha_{ij}\, g_{k,v_j} & \text{if } v_j \text{ belongs to the } i\text{-th facet,} \\ 0 & \text{otherwise,} \end{cases}$$

for some positive constants α_{ij}, where g_{k,v_j} is the only generator of the cone C_{P,v_j} that is not contained in the i-th facet. Indeed, a small parallel translation of the i-th facet, moves each vertex v_j in this facet in the direction opposite to the generator g_{k,v_j} and does not change all other vertices. If P is a Delzant polytope then all constants α_{ij} are equal to 1. In this case, by Theorem 19.2(4), we have

$$V_P(z_1,\dots,z_N) = \frac{1}{n!}\sum_{j=1}^{M} \frac{h(v_j(z_1,\dots,z_N))^n}{(-1)^n\prod_{i=1}^n h(g_{i,v_j})} = [t^n]\left\{\sum_{j=1}^{M} \frac{e^{t\cdot h(v_j(z_1,\dots,z_N))}}{(-1)^n\prod_{i=1}^n h(g_{i,v_j})}\right\}$$

The only term in this expression that involves z_i's is the exponent $e^{t\cdot h(v_j(z_1,\dots,z_N))}$. For an analytic function $f(q)$, the operator $f\left(\frac{\partial}{\partial z_i}\right)$ maps this exponent to

$$e^{t\cdot h(v_j(z_1,\dots,z_N))} \mapsto \begin{cases} e^{t\cdot h(v_j(z_1,\dots,z_N))}\, f(-t\,h(g_{k,v_j})) & \text{if } v_j \text{ lies in the } i\text{-th facet,} \\ e^{t\cdot h(v_j(z_1,\dots,z_N))}\, f(0) & \text{otherwise,} \end{cases}$$

where k is the same as above. Using this for Todd operators, we obtain the expression for $I_P(z_1,\dots,z_N)$ given by Theorem 19.2(3). $\square$

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