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Smoothness of Schubert varieties via patterns in root subsystems

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Abstract

The aim of this article is to present a smoothness criterion for Schubert varieties in generalized flag manifolds G/B in terms of patterns in root systems. We generalize Lakshmibai–Sandhya's well-known result that says that a Schubert variety in $SL(n)/B$ is smooth if and only if the corresponding permutation avoids the patterns 3412 and 4231. Our criterion is formulated uniformly in general Lie theoretic terms. We define a notion of pattern in Weyl group elements and show that a Schubert variety is smooth (or rationally smooth) if and only if the corresponding element of the Weyl group avoids a certain finite list of patterns. These forbidden patterns live only in root subsystems with star-shaped Dynkin diagrams. In the simply-laced case the list of forbidden patterns is especially simple: besides two patterns of type A_3 that appear in Lakshmibai–Sandhya's criterion we only need one additional forbidden pattern of type D_4. In terms of these patterns, the only difference between smoothness and rational smoothness is a single pattern in type B_2. Remarkably, several other important classes of elements in Weyl groups can also be described in terms of forbidden patterns. For example, the fully commutative elements in Weyl groups have such a characterization. In order to prove our criterion we used several known results for the classical types. For the exceptional types, our proof is based on computer verifications. In order to conduct such a verification for the computationally challenging type E_8, we derived several general results on Poincaré polynomials of

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cohomology rings of Schubert varieties based on parabolic decomposition, which have an independent interest.

Keywords: Schubert varieties; Singular locus; Root system; Patterns

1. Introduction

Let G be a semisimple simply-connected complex Lie group and B be a Borel subgroup. The generalized flag manifold G/B decomposes into a disjoint union of *Schubert cells* BwB/B, labeled by elements w of the corresponding Weyl group W. The *Schubert varieties* $X_w = \overline{BwB/B}$ are the closures of the Schubert cells. A classical question of Schubert calculus is: *For which elements w in the Weyl group W, is the Schubert variety X_w smooth*?

This question has a particularly nice answer for $G = SL(n)$. In this case the Weyl group is the symmetric group $W = S_n$ of permutations of n letters. For a permutation $w = w_1 w_2 \cdots w_n$ in S_n and another permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ in S_k, with $k \leqslant n$, we say that w *contains the pattern* σ if there is a sequence $1 \leqslant p_1 < \cdots < p_k \leqslant n$ such that $w_{p_i} > w_{p_j}$ if and only if $\sigma_i > \sigma_j$ for all $1 \leqslant i < j \leqslant k$. In other words, w contains the pattern σ if there is a subsequence in w of size k with the same relative order of elements as in σ. If w does not contain the pattern σ, then we say that w *avoids the pattern* σ.

Theorem 1.1 (Lakshmibai–Sandhya [19]). *For a permutation $w \in S_n$, the Schubert variety X_w in $SL(n)/B$ is smooth if and only if w avoids the patterns* 3412 *and* 4231.

There are several general approaches to determining smoothness of Schubert varieties. See Billey and Lakshmibai [6] for a survey of known results. Kazhdan and Lusztig defined a weaker condition called *rational smoothness*. Rational smoothness can be interpreted in terms of Kazhdan–Lusztig polynomials [15,16]. A Schubert variety is rationally smooth whenever certain Kazhdan–Lusztig polynomials are trivial. Kumar [18] presented smoothness and rational smoothness criteria in terms of the nil Hecke ring, defined in [17]. There are many other results due to Carrell, Peterson, and other authors related to (rational) smoothness of the Schubert varieties. For example, according to a result of D. Peterson, see Carrell and Kuttler [9], smoothness of Schubert varieties is equivalent to rational smoothness in the case of a simply-laced root system. Nevertheless none of these general criteria give a simple efficient nonrecursive method (such as the Lakshmibai–Sandhya criterion) for determining if a given Schubert variety is smooth or not. Recently, Billey [1] presented analogues of Lakshmibai–Sandhya's theorem, for all classical types B_n, C_n, and D_n. However, these constructions, including the definitions of patterns, depend on a particular way to represent elements in classical Weyl groups as signed permutations.

The main goals of this paper are to present a uniform approach to pattern avoidance in general terms of root systems and to extend the Lakshmibai–Sandhya criterion to the case of an arbitrary semisimple Lie group G. This approach using root subsystems will be described in the next section. Theorem 2.2 gives a polynomial time algorithm for determining

smoothness and rational smoothness of Schubert varieties in G/B in terms of root subsystems. As a consequence of the main theorem, we get two additional criteria for (rational) smoothness in terms of root systems embeddings and double parabolic factorizations (see Theorems 3.1 and 6.2).

Based on the ideas of root subsystems presented in this work, Braden and the first author [3] refined this notion and gave a lower bound for the Kazhdan–Lusztig polynomials evaluated at $q = 1$ in terms of patterns. They also introduce a geometrical construction which identifies "pattern Schubert varieties" as torus fixed point components inside larger Schubert varieties. This can be used to give another proof of one direction of our main theorem. However, due to a delay in publication, those results will appear first.

In Section 2, we formulate our smoothness criterion and describe the minimal lists of patterns needed to identify singular (rationally singular) Schubert varieties. In Section 3, we present a computational improvement using root system embeddings that reduces the minimal lists to just 4 patterns (3 patterns) for (rational) smoothness test. The difference between smoothness and rational smoothness is exhibited in the presence or absence of rank 2 patterns. The connection to fully commutative elements is described in Section 4. In Section 5, we recall several known characterizations of smoothness and rational smoothness from the literature which we will use in the proof of the main theorem. In Section 6, we reformulate our main result in terms of parabolic subgroups. Then we prove two statements on parabolic decomposition which will be used in the proof of Theorem 2.2, including Theorem 6.4 which gives a criterion for factoring Poincaré polynomials of Schubert varieties. In Section 7, we give the details of the proof of the main theorem.

2. Root subsystems and the main results

As before, let G be a semisimple simply-connected complex Lie group with a fixed Borel subgroup B. Let $\mathfrak{h}$ be the Cartan subalgebra corresponding to a maximal torus contained in B. Let $\Phi \in \mathfrak{h}^*$ be the corresponding root system, and let $W = W_\Phi$ be its Weyl group. The choice of B determines the subset $\Phi_+ \subset \Phi$ of positive roots. The fact that a Schubert variety X_w, $w \in W$, in G/B is smooth (or rationally smooth) depends only on the pair (Φ_+, w). We call such a pair (*rationally*) *smooth* whenever the corresponding Schubert variety is (rationally) smooth. The *inversion set* of an element w in the Weyl group W_Φ is defined by

$$I_\Phi(w) = \Phi_+ \cap w(\Phi_-),$$

where $\Phi_- = \{-\alpha \mid \alpha \in \Phi_+\}$ is the set of negative roots.

The following properties of inversion sets are well known, see [7, §1, no. 6].

Lemma 2.1. *The inversion set $I_\Phi(w)$ uniquely determines the Weyl group element $w \in W_\Phi$. Furthermore, a subset $I \subseteq \Phi_+$ in the set of positive roots is the inversion set $I_\Phi(w)$ for some w if and only if there exist a linear form h on the vector space $\mathfrak{h}^*$ such that $I = \{\alpha \in \Phi_+ \mid h(\alpha) > 0\}$.*

A *root subsystem* of Φ is a subset of roots $\Delta \subset \Phi$ which is equal to the intersection of Φ with a vector subspace. Clearly, a root subsystem Δ is a root system itself in the

subspace spanned by Δ, see [7, §1, no. 1]. It comes with the natural choice of positive roots $\Delta_+ = \Delta \cap \Phi_+$.

By Lemma 2.1, for any $w \in W_\Phi$ and any root subsystem $\Delta \subset \Phi$, the set of roots $I_\Phi(w) \cap \Delta$ is the inversion set $I_\Delta(\sigma)$ for a unique element $\sigma \in W_\Delta$ in the Weyl group of Δ. Let us define the *flattening map* $f_\Delta : W_\Phi \to W_\Delta$ by setting $f_\Delta(w) = \sigma$ where σ is determined by its inversion set $I_\Delta(\sigma) = I_\Phi(w) \cap \Delta$.

Recall that a graph is called a *star* if it is connected and it contains a vertex incident with all edges. Let us say that a root system Δ is *stellar* if its Dynkin diagram is a star and Δ is not of type A_1 or A_2. For example, B_3 is stellar but F_4 is not. Our first analogue of the Lakshmibai–Sandhya criterion can be formulated as follows. See also Theorems 2.3, 2.4, 3.1 and 6.2.

Theorem 2.2. *Let G be any semisimple simply-connected Lie group, B be any Borel subgroup, with corresponding root system Φ and Weyl group $W = W_\Phi$. For $w \in W$, the Schubert variety $X_w \subset G/B$ is smooth* (*rationally smooth*) *if and only if, for every stellar root subsystem Δ in Φ, the pair $(\Delta_+, f_\Delta(w))$ is smooth* (*rationally smooth*).

The proof of Theorem 2.2 appears in Section 7. If Δ is a root subsystem in Φ and $\sigma = f_\Delta(w)$, then we say that the element w in W_Φ *contains the pattern* (Δ_+, σ). It follows from Theorem 2.2 that an element in W_Φ containing a non-smooth (non-rationally-smooth) pattern is also non-smooth (non-rationally-smooth). Another explanation of this fact for rational smoothness based on intersection homology can be found in the work of Billey and Braden [3] mentioned above.

Let us say that an element w *avoids the pattern* (Δ_+, σ) if w does not contain a pattern isomorphic to (Δ_+, σ). Clearly, Theorem 2.2 implies that the set of (rationally) smooth elements $w \in W_\Phi$ can be described as the set of all elements w that avoid patterns of several types. Since there are finitely many types of stellar root systems, the list of forbidden patterns is also finite.

Actually, the list of stellar root systems is relatively small: B_2, G_2, A_3, B_3, C_3, and D_4. Figure 1 shows their Dynkin diagrams labeled according to standard conventions from [7]. In order to use Theorem 2.2 as a (rational) smoothness test we need to know all non-smooth and non-rationally-smooth elements in the Weyl groups with stellar root systems. The following table gives the numbers of such elements.

stellar type	B_2	G_2	A_3	B_3	C_3	D_4
non-smooth elements	1	5	2	20	20	49
non-rationally-smooth elements	0	0	2	14	14	49

There are several things to notice about the table. In the simply-laced cases A_3 and D_4 the numbers of non-smooth and non-rationally-smooth elements coincide. The rationally smooth elements in B_n are exactly the same as the rationally smooth elements in C_n. This explains why the number 14 appears in both B_3 and C_3 cases. Note that in general the number of non-smooth elements in B_n is not equal to the number of non-smooth elements in C_n. For examples, we have 268 non-smooth elements in the B_4 case and 270 non-smooth elements in the C_4 case.

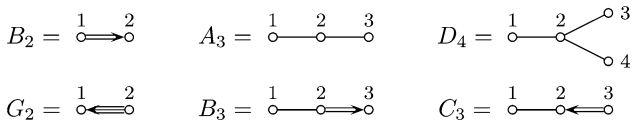


Fig. 1. Dynkin diagrams of stellar root systems.

There are exactly two non-smooth elements of type A_3—they correspond to the two forbidden patterns that appear Lakshmibai–Sandhya's criterion. Although there are 49 non-smooth elements of type D_4, only one (!) of these 49 elements contains no forbidden A_3 patterns. These three patterns (two of type A_3 and one of type D_4) are all patterns that are needed in the case of a simply-laced root system (A-D-E case).

For all stellar types, let $s_1, s_2, \ldots$ be the simple reflections generating the corresponding Weyl groups labeled as shown on Fig. 1. Thus in both A_3 and D_4 cases the reflection s_2 corresponds to the central node of the corresponding Dynkin diagram. We will write elements of corresponding Weyl groups as products of the generators s_i.

Theorem 2.3. *Suppose that Φ is a simply-laced root system. Then the Schubert variety X_w, $w \in W_\Phi$, is smooth if and only if w avoids the following three patterns*: *two patterns of type A_3 given by the elements $s_2s_1s_3s_2$ and $s_1s_2s_3s_2s_1$ and one pattern of type D_4 given by the element $s_2s_1s_3s_4s_2$.*

Remark that, D. Peterson has shown (unpublished, see Carrell and Kuttler [9]) that in the simply-laced case a Schubert variety is smooth if and only if it is rationally smooth. Thus in the previous claim we can replace the word "smooth" by the phrase "rationally smooth."

For the case of arbitrary root systems (including non-simply-laced ones), we need to list forbidden patterns of types B_2, G_2, B_3, and C_3. The only non-smooth element of B_2 is $s_2s_1s_2$. The non-smooth elements of type G_2 are the 5 elements in the interval in the Bruhat order $[s_1s_2s_1, w_o[$ (w_o is excluded), where w_o is the longest Weyl group element for type G_2. There are also 6 non-smooth elements of type B_3 and 6 non-smooth elements of type C_3 that contain no forbidden B_2 patterns. The following theorem summarizes this data and gives the minimal list of patterns for the smoothness test.

We will write $[a, b, \ldots, c]$ to denote the collection of words $1, a, b, \ldots, c$. We concatenate this collection with another word as follows: $[a, b, c]\,d$ is a shorthand for the four words d, ad, bd, cd.

Theorem 2.4. *Let Φ be an arbitrary root system. The Schubert variety X_w, $w \in W_\Phi$, is smooth if and only if w avoids the patterns listed in the following table*:

stellar type	*forbidden patterns*	*# patterns*
B_2	$s_2s_1s_2$	1
G_2	$[s_2]s_1s_2s_1[s_2]$, $s_1s_2s_1s_2s_1$	5
A_3	$s_2s_1s_3s_2$, $s_1s_2s_3s_2s_1$	2
B_3	$s_2s_1s_3s_2$, $s_1s_2s_3s_2s_1[s_3, s_3s_2, s_2s_3, s_2s_3s_2]$	6
C_3	$[s_3]s_2s_1s_3s_2[s_3]$, $s_3s_2s_1s_2s_3$, $s_1s_2s_3s_2s_1s_3s_2s_3$	6
D_4	$s_2s_1s_3s_4s_2$	1

All Weyl group elements for the types B_2 and G_2 are rationally smooth. Thus we can ignore all root subsystems of these types in rational smoothness test. Rational smoothness can be defined in terms of Kazhdan–Lusztig polynomials that depend only on the Weyl group. The Weyl groups of types B_3 and C_3 are isomorphic. Thus the lists of non-rationally-smooth elements are identical in these two cases. The following theorem presents these lists.

Theorem 2.5. *Let Φ be an arbitrary root system. The Schubert variety X_w, $w \in W_\Phi$, is rationally smooth if and only if w avoids the patterns listed in the following table*:

stellar type	*forbidden patterns*	*# patterns*
A_3	$s_2s_1s_3s_2$, $s_1s_2s_3s_2s_1$	2
$B_3 = C_3$	$[s_3]s_2s_1s_3s_2[s_3]$, $[s_2]s_3s_2s_1s_2s_3[s_2]$, $s_1s_2s_3s_2s_1[s_3, s_2s_3, s_3s_2, s_2s_3s_2, s_3s_2s_3]$	14
D_4	$s_2s_1s_3s_4s_2$	1

A quick glance on the tables in Theorems 2.4 and 2.5 reveals that the lists of forbidden patterns for non-simply-laced cases are longer than the list of three simply-laced forbidden patterns. In Section 3 we show how to reduce the list patterns above to just the forbidden patterns of types B_2, A_3, D_4 using embeddings of root systems.

3. Root system embeddings

In this section, we present an alternative notion of pattern avoidance in terms of embedded root systems. Again we can characterize smoothness and rational smoothness of Schubert varieties. The key advantage of this approach is that we reduce the minimal number of patterns to just 3 for rational smoothness and 4 for smoothness. While we believe this approach is useful for computational purposes, we suspect root subsystems are better for geometrical considerations.

Let Φ and Δ be two root systems in the vectors spaces U and V, respectively. An *embedding* of Δ into Φ is a map $e : \Delta \to \Phi$ that extends to an injective linear map $U \to V$. For example, any three positive roots $\alpha, \beta, \gamma \in \Phi_+$ define an A_3-embedding whenever $\alpha + \beta$, $\beta + \gamma$ and $\alpha + \beta + \gamma$ are all in Φ_+.

Note that inner products are not necessarily preserved by embeddings as they are with root subsystems. Also note that every root subsystem Δ in Φ gives an embedding of Δ into Φ, but it is not true that all embeddings come from root subsystems. It is possible that Δ embeds into Φ but the linear span of $e(\Delta)$ in Φ contains some additional roots. Nevertheless, in the simply-laced case this can never happen. For simply-laced root systems, the notions of root subsystems and embeddings are essentially equivalent.

We will say that a k-tuple of positive roots $(\beta_1, \ldots, \beta_k)$ in Φ, gives a *B_2-embedding*, *A_3-embedding*, or *D_4-embedding* if these vectors are the images of the simple roots in Δ for an embedding $\Delta \to \Phi$ with Δ of type B_2, A_3, or D_4, respectively. For example, B_2-embeddings are given by pairs of positive roots (β_1, β_2) such that both vectors $\beta_1 + \beta_2$ and $\beta_1 + 2\beta_2$ belong to Φ_+. Also *A_3-embeddings* are given by triples of positive roots $(\beta_1, \beta_2, \beta_3)$ such that all vectors $\beta_1 + \beta_2$, $\beta_2 + \beta_3$, and $\beta_1 + \beta_2 + \beta_3$ are roots in Φ_+.

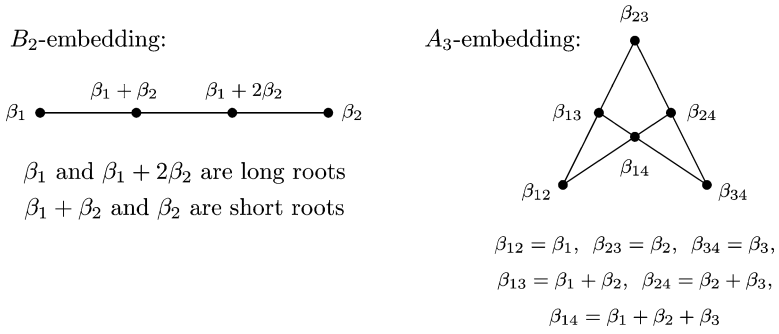


Fig. 2. B_2- and A_3-embeddings.

Figure 2 illustrates B_2- and A_3-embeddings. The vertices on the figure correspond to the positive roots in the image of the embedding. Here we used a $(k-1)$-dimensional picture in order to represent collections of k-dimensional vectors. The vertices on the figure are the intersections of the lines generated by the roots with a certain affine hyperplane. Therefore, inversion sets are determined by half planes in these pictures. A similar 3-dimensional figure can be constructed for D_4. Egon Schulte pointed out [22] that the figure can be obtained by projecting 12 vertices of the regular 24-cell onto a tetrahedron spanned by 4 vertices of the 24-cell. In fact, it can be viewed as a model in projective 3-space, and then it is actually related to the half-24-cell.

The set of positive roots Φ_+ in Φ and the embedding e determines the set of positive roots $\Delta_+ = e^{-1}(\Phi_+)$ in Δ. We can extend the definition of the flattening map to embeddings of root systems. For an embedding $e:\Delta\to\Phi$, let us define the *flattening map* $f_e: W_\Phi \to W_\Delta$ by setting $f_e(w)=\sigma$, if the inversion set of w pulls back to the inversion set of σ, i.e., $I_\Delta(\sigma) = e^{-1}(I_\Phi(w))$. According to Lemma 2.1, the element σ is uniquely defined.

There are five A_3-embeddings into a root system Φ of type B_3 and seven into Φ of type C_3. Among these twelve embeddings, five are necessary to classify rationally singular Schubert varieties and three of these embeddings lead to false positive classifications of rationally smooth elements in W_{C_3}. Therefore, we introduce the following definition in order to eliminate the false conditions. For an embedding $e:\Delta\to\Phi$, let $\bar{\Delta}\subset\Phi$ be the root subsystem in Φ spanned by the image $e(\Delta)$. We say that an embedding $e:\Delta\to\Phi$ is *proper* if either $\bar{\Delta}$ is not of type B_3, C_3 or $\bar{\Delta}$ is of type B_3, C_3 and there exists a B_2-embedding $\varepsilon: B_2\to\bar{\Delta}$ such that

(1) If B_2 has basis β_1,β_2, then $\varepsilon(\beta_1+\beta_2)=e(\alpha_i)$ for some simple root $\alpha_i\in\Delta$.
(2) We have $\varepsilon^{-1}(e(\Delta)) = I_{B_2}(s_2s_1s_2) = \{\beta_1+\beta_2, \beta_1+2\beta_2, \beta_2\}$. In words, the image of the B_2-embedding intersects the image of the Δ-embedding in exactly three roots which correspond to the inversion set of the unique singular Schubert variety $X(s_2s_1s_2)$ of type B_2.

The root systems B_3 and C_3 each have three B_2-embeddings and each of these embeddings corresponds to exactly one proper A_3-embedding.

For an element w in the Weyl group W_Φ, we say that w contains an *embedded pattern* of type B_2, A_3, or D_4 if there is a proper embedding $e : \Delta \to \Phi$ such that

- $\boldsymbol{B_2}$: Δ is of type B_2 and $f_e(w) = s_2 s_1 s_2$;
- $\boldsymbol{A_3}$: Δ is of type A_3 and $f_e(w) = s_1 s_2 s_3 s_2 s_1$ or $f_e(w) = s_2 s_1 s_3 s_2$;
- $\boldsymbol{D_4}$: Δ is of type D_4 and $f_e(w) = s_2 s_1 s_3 s_4 s_2$.

Recall here that the Coxeter generators s_i of Weyl groups of types B_2, A_3, and D_4 are labeled as shown on Fig. 1. Note, that the reduced expressions above are all of the form: central node conjugated by its neighbors or neighbors conjugated by the central node.

Let $\Phi^\vee$ be the root system dual to Φ. Its Weyl group $W_{\Phi^\vee}$ is naturally isomorphic to W_Φ. For an element $w \in W_\Phi$, we say that w contains a *dual embedded pattern* whenever the corresponding element in $W_{\Phi^\vee} \simeq W_\Phi$ contains an embedded pattern given by a proper embedding $e : \Delta \to \Phi^\vee$.

Theorem 3.1. *Let G be any semisimple simply-connected Lie group, B be any Borel subgroup, with corresponding root system Φ and Weyl group $W = W_\Phi$.*

(1) *For $w \in W$, the Schubert variety X_w is rationally smooth if and only if w has no embedded patterns or dual embedded patterns of types A_3 or D_4.*
(2) *For $w \in W_\Phi$, the Schubert variety X_w is smooth if and only if w has neither embedded patterns of types B_2, A_3, or D_4, nor dual embedded patterns of types A_3 or D_4.*

Note that, the element w, corresponding to a smooth Schubert variety X_w, may contain dual embedded patterns of type B_2. Thus smoothness of Schubert varieties, unlike rational smoothness, is not invariant with respect to duality of root system.

Proof. Any B_2, A_3, or D_4-embedding spans a root subsystem whose rank must be at most 4. Therefore, this theorem follows directly from Theorem 2.2 by checking all root systems of rank at most 4. □

We mention one more computational simplification in applying Theorem 3.1. For any $w \in W_\Phi$, there exists a hyperplane that separates the sets of roots $I_\Phi(w)$ and $\Phi_+ \setminus I_\Phi(w)$. Figure 3 illustrates embedded patterns of types B_2 and A_3. It is easy to see that each of these inversion sets is determined by a half plane. The black vertices "●" correspond to the roots in the inversion set $I_\Phi(w)$ and the white vertices "○" correspond to the roots outside the inversion set $I_\Phi(w)$.

Therefore, in order to search for embedded patterns for w of types B_2, A_3 and D_4 we only need to look for pairs, triple or quadruples of the following forms:

$\boldsymbol{B_2}$: A pair of positive roots (β_1, β_2) which forms the basis of a B_2-embedding such that $\beta_1 \notin I_\Phi(w)$ and $\beta_1 + \beta_2 \in I_\Phi(w)$.
$\boldsymbol{A_3}$: A triple of positive roots $(\beta_1, \beta_2, \beta_3)$ which forms the basis of a *proper* A_3-embedding such that (1) $\beta_{12}, \beta_{34} \notin I_\Phi(w)$ and $\beta_{14} \in I_\Phi(w)$; or (2) $\beta_{23} \notin I_\Phi(w)$ and $\beta_{13}, \beta_{24} \in I_\Phi(w)$;

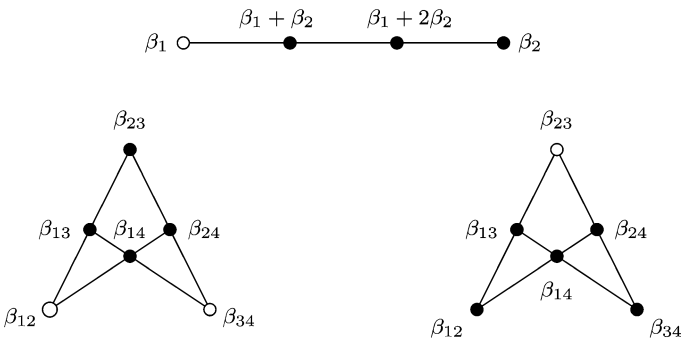


Fig. 3. Forbidden embedded patterns of types B_2 and A_3.

***D*$_4$**: A 4-tuple of positive roots $(\beta_1, \beta_2, \beta_3, \beta_4)$ which forms the basis of a D_4-embedding such that $\beta_1 + 2\beta_2 + \beta_3 + \beta_4 \in I_\Phi(w)$ and $\beta_1 + \beta_2 + \beta_3, \beta_1 + \beta_2 + \beta_4, \beta_2 + \beta_3 + \beta_4 \notin I_\Phi(w)$.

4. Other elements characterized by pattern avoidance

In a series of papers (see [11,23,24] and reference wherein), Fan and Stembridge have developed a theory of *fully commutative elements* in arbitrary Coxeter groups. By definition, an element in a Coxeter group is fully commutative if all its reduced decompositions can be obtained from each other by using only the Coxeter relations that involve commuting generators.

According to [5] the fully commutative elements in type A are exactly the permutations avoiding the pattern 321. In types B and D, Stembridge has shown that the fully commutative elements can again be characterized by pattern avoidance [23, Theorems 5.1 and 10.1].

We note here that fully commutative elements are easily characterized by root subsystems as well. The following is an unpublished theorem originally due to Stembridge [25].

Proposition 4.1. *Let W be any Weyl group with corresponding root system Φ. Then $w \in W$ is fully commutative if and only if for every root subsystem Δ of type A_2, B_2, or G_2 we have $f_\Delta(w) \neq w_0^\Delta$ where w_0^Δ is the unique longest element of W_Δ. In other words, w is fully commutative if and only if w avoids the patterns given by the longest elements in rank* 2 *irreducible root systems.*

Remark 4.2. Fan, Stembridge and Kostant also investigated *abelian elements* in Weyl groups. An element $w \in W$ is abelian if its inversion set $I(w)$ contain no three roots α, β, and $\alpha + \beta$. Equivalently, $w \in W$ is abelian if the Lie algebra $\mathfrak{b} \cap w(\mathfrak{b}_-)$ is abelian, where $\mathfrak{b}$ is Borel and $\mathfrak{b}_-$ is opposite Borel algebras. For simply-laced root systems, the set of abelian elements coincides with the set of fully commutative elements. The set of abelian elements has a simple characterization in terms of embedded patterns. Indeed, by definition, $w \in W_\Phi$

is abelian if and only if there is no A_2-embedding $e : \Delta \to \Phi$ such that the flattening $f_e(w)$ is the longest element w_0^Δ of W_Δ.

5. Criteria for smoothness and rational smoothness

In this section, we summarize the three criteria for smoothness and rational smoothness we rely on for the proof of Theorem 2.2.

Let $\alpha_1, \ldots, \alpha_n$ be the simple roots in Φ and let $\mathbb{Z}[\mathfrak{h}]$ denote the symmetric algebra generated by $\alpha_1, \ldots, \alpha_n$. For any $w, v \in W$ such that $w \leqslant v$, let us define $K_{w,v} \in \mathbb{Z}[\mathfrak{h}]$ by the recurrence

$$K_{w,w} = \prod_{\alpha \in I_\Phi(w)} \alpha \quad \text{for } w = v,$$

$$K_{w,v} = \begin{cases} K_{ws_i,v} & \text{if } v < vs_i, \\ K_{ws_i,v} + (ws_i\alpha_i)K_{ws_i,vs_i} & \text{if } v > vs_i \end{cases}$$

for $v \leqslant w$ and any simple reflection s_i such that $ws_i < w$. Then $K_{w,v}$ is a polynomial of degree $\ell(v)$ in the simple roots with non-negative integer coefficients. These polynomials first appeared in the work of Kostant and Kumar [17] on the nil Hecke ring, see [2] for the recurrence.

Kumar has given very general criteria for smoothness and rational smoothness in terms of the nil-Hecke ring. Through a series of manipulations which were given in [6], one can obtain the following statement from Kumar's theorem for finite Weyl groups. Kumar's theorem in full generality applies to the Schubert varieties for any Kac–Moody group. However we would need to work with rational functions of the roots.

Theorem 5.1 [6,18]. *Given any $v, w \in W$ such that $v \leqslant w$, the Schubert variety X_{vw_0} is smooth at e_{ww_0} if and only if*

$$K_{w,v} = \prod_{\alpha \in Z(w,v)} \alpha, \tag{1}$$

where $Z(w, v) = \{\alpha \in \Phi_+ \colon v \not\leqslant s_\alpha w\}$.

We can simplify the computations in Theorem 5.1 by evaluating this identity at a well-chosen point. The modification reduces the problem from checking a polynomial identity to checking degrees plus a numerical identity. Checking the degrees can be done with a polynomial time algorithm since this only depends on the number of positive roots.

Lemma 5.2. *Let $r \in \mathfrak{h}$ be any regular dominant integral weight, i.e., $\alpha(r) \in \mathbb{N}_+$ for each $\alpha \in \Phi_+$. Given any $v, w \in W$ such that $v \leqslant w$, the Schubert variety X_{vw_0} is smooth at e_{ww_0} if and only if*

$$\big|Z(w, v)\big| = \ell(v) \quad \textit{and} \quad K_{w,v}(r) = \prod_{\alpha \in Z(w,v)} \alpha(r). \tag{2}$$

Proof. We just need to prove the equivalence of (1) and (2). Dyer [10] has shown that $K_{w,v}$ is divisible by $\prod_{\alpha\in Z(w,v)}\alpha$. Therefore, $K_{w,v}=\prod_{\alpha\in Z(w,v)}\alpha$ if and only if their quotient is 1. We can check that the quotient is 1 by checking the degrees are equal

$$\ell(v)=\deg(K_{w,v})=\deg\Bigg(\prod_{\alpha\in Z(w,v)}\alpha\Bigg)=\big|Z(w,v)\big|$$

(in which case the quotient is a constant) and that $K_{w,v}(r)=\prod_{\alpha\in Z(w,v)}\alpha(r)$. □

Remark 5.3. Note, by choosing r such that $\alpha(r)$ is always an integer, we do not have to consider potential round off errors when checking equality.

Remark 5.4. A Schubert variety X_w is smooth at every point if and only if it is smooth at e_{id}. Therefore, we only need to check $K_{w_0,ww_0}(r)=\prod_{\alpha\in Z(w_0,ww_0)}\alpha(r)$ when $|Z(w_0,ww_0)|=\ell(ww_0)$ or equivalently $|\{\alpha\in\Phi_+\colon s_\alpha\leqslant w\}|=\ell(w)$.

The next criterion due to Carrell–Peterson is for rational smoothness. The *Bruhat graph* $B(w)$ *for* $w\in W$ is the graph with vertices $\{x\in W\colon x\leqslant w\}$ and edges between x and y if $x=s_\alpha y$ for some $\alpha\in\Phi_+$ where s_α is the reflection corresponding to α,

$$s_\alpha v=v-\frac{(v,\alpha)}{2(\alpha,\alpha)}\alpha.$$

Note the Bruhat graph contains the Hasse diagram of the lower order ideal below w in Bruhat order plus some extra edges. Let

$$P_w(t)=\sum_{v\leqslant w}t^{\ell(v)},$$

then $P_w(t^2)$ is the *Poincaré polynomial* for the cohomology ring of the Schubert variety X_w.

Theorem 5.5 [8]. *The following are equivalent*:

(1) X_w *is rationally smooth at every point.*
(2) *The Poincaré polynomial* $P_w(t)=\sum_{v\leqslant w}t^{\ell(v)}$ *of* X_w *is symmetric* (*palindromic*).
(3) *The Bruhat graph* $B(w)$ *is regular of degree* $\ell(w)$*, i.e., every vertex in* $B(w)$ *is incident to* $\ell(w)$ *edges.*

We can relate this theorem to the inversion sets $I_\Phi(w)$ using the following simple lemma, see [7].

Lemma 5.6. *Fix a reduced expression* $s_{a_1}s_{a_2}\cdots s_{a_p}=w\in W$*. Let* $\beta_1,\ldots,\beta_n$ *be the simple roots in* Φ_+*. The following sets are all equal to the inversion set* $I_\Phi(w)$:

(1) $\Phi_+ \cap w(\Phi_-)$.
(2) $\{\alpha \in \Phi_+ \colon s_\alpha w < w\}$.
(3) $\{s_{a_1} s_{a_2} \cdots s_{a_{j-1}} \beta_{a_j} \colon 1 \leqslant j \leqslant p\}$.

Let us label an edge $(x, s_\alpha x)$ in $B(w)$ by the root $\alpha \in \Phi_+$. Then, by Lemma 5.6, the edges adjacent to w in the Bruhat graph $B(w)$ are labeled by the elements of $I_\Phi(w)$ so the degree $\deg(w)$ of the vertex w is $\ell(w)$. At any other vertex $x < w$ we know $\#\{\alpha \in \Phi_+ \colon s_\alpha x < x\} = \ell(x)$ so $\deg(x) = \ell(x) + \#\{\alpha \in \Phi_+ \colon x < s_\alpha x \leqslant w\}$. Therefore, we have the following lemma.

Lemma 5.7. *The Bruhat graph $B(w)$ is not regular if and only if there exists an $x < w$ such that*

$$\deg(x) > \deg(w) \quad \Leftrightarrow \quad \#\{\alpha \in \Phi_+ \mid x < s_\alpha x \leqslant w\} > \ell(w) - \ell(x).$$

6. Parabolic decomposition

In the first lemma below, we give an alternative characterization of pattern containment in terms of a parabolic factorization. This leads to an alternative characterization of smooth and rationally smooth elements in the Weyl group. We also give a method for factoring some Poincaré polynomials of Schubert varieties.

Fix a subset J of the simple roots. Let Φ^J be the root subsystem spanned by roots in J, and let $\Phi^J_+ = \Phi^J \cap \Phi_+$ be its set of positive roots. Let W_J be the parabolic subgroup generated by the simple reflections corresponding to J. Let W^J be the set of minimal length coset representatives for $W_J \backslash W$ (moding out on the left). In other words,

$$W^J = \left\{v \in W \mid v^{-1}(\alpha) \in \Phi_+ \text{ for any } \alpha \in \Phi^J_+\right\}. \tag{3}$$

Every $w \in W$ has a unique parabolic decomposition as the product $uv = w$ where $u \in W_J$, $v \in W^J$ and $\ell(w) = \ell(u) + \ell(v)$, and conversely, every product $u \in W_J$, $v \in W^J$ has $\ell(uv) = \ell(u) + \ell(v)$ [14, Proposition 1.10]. Equivalently, if $w = uv$ is the parabolic decomposition and $s_{a_1} s_{a_2} \cdots s_{a_p}$, $s_{b_1} s_{b_2} \cdots s_{b_q}$ are reduced expressions for u, v respectively then each $s_{a_i} \in W_J$ and $s_{a_1} s_{a_2} \cdots s_{a_p} s_{b_1} s_{b_2} \cdots s_{b_q}$ is a reduced expression for w.

Let $\Delta \subset \Phi$ be any root subsystem. It was shown in [3] that Δ is conjugate to Φ^J for some subset J of the simple roots, i.e., there exists a $v_1 \in W^J$ such that $v_1(\Delta) = \Phi^J$. Clearly, W_Δ and W_J are isomorphic subgroups since the Dynkin diagrams for Δ and Φ^J are isomorphic as graphs. If there exist multiple isomorphisms, any one will suffice.

Lemma 6.1. *Let $\Delta \subset \Phi$ be any root subsystem. Suppose that $v_1(\Delta) = \Phi^J$ for $v_1 \in W^J$ as above. Let $w \in W$. Let $u \in W_J$ and let $u' = v_1^{-1} u v_1$ be the corresponding element in W_Δ under the natural isomorphism. Then $\mathrm{fl}_\Delta(w) = u'$ if and only if there exits $v_2 \in W^J$ such that $w = v_1^{-1} u v_2$.*

Proof. The element $v_1 \in W^J$ gives a one-to-one correspondence $\alpha \mapsto v_1(\alpha)$ between *positive* roots in Δ and *positive* roots in Φ^J. Also for any $v_2 \in W^J$, v_2^{-1} maps positive roots of Φ^J to positive roots in $\Delta \subset \Phi$ and negative roots of Φ^J to negative roots of $\Delta \subset \Phi$.

Let $v_2 = u^{-1}v_1 w$. We claim $v_2 \in W^J$ since for any $\alpha \in \Phi_J^+$, $v_2^{-1}(\alpha) = w^{-1}v_1^{-1}u(\alpha) > 0$. Then we have a bijection from the inversions of u to the inversions of w in Δ:

$$\begin{aligned} \alpha \in \Delta_+ \cap I_\Phi(w) \quad &\Leftrightarrow \quad w^{-1}(\alpha) < 0 \quad \Leftrightarrow \quad v_2^{-1}u^{-1}v_1(\alpha) < 0 \\ &\Leftrightarrow \quad u^{-1}v_1(\alpha) \in \Phi_-^J \quad \Leftrightarrow \quad v_1(\alpha) \in I_\Phi(u). \qquad \square \end{aligned}$$

Theorem 6.2. *Let Φ be any root system and let $J_1, \ldots, J_s$ be a collection of subsets of simple roots such that all parabolic subsystems $\Phi^{J_1}, \ldots, \Phi^{J_s}$ are stellar and they include all possible stellar types present in the Dynkin diagram of Φ. Then $w \in W$ is* (*rationally*) *smooth if and only if it cannot be presented in the form $w = v_1^{-1}uv_2$, where $v_1, v_2 \in W^{J_i}$, $u \in W_{J_i}$, for $i \in \{1, \ldots, s\}$, and u is* (*rationally*) *singular element in W_{J_i}.*

Proof. Suppose that Δ is any stellar root subsystem in Φ. Then Δ is conjugate to Φ^J, where $J = J_i$ for some i. Now Lemma 6.1 shows that Theorem 6.2 is equivalent to Theorem 2.2. $\square$

Each stellar parabolic subset J_i in Theorem 6.2 consists of a node in the Dynkin diagram together with its neighbors. We need to pick all nonisomorphic such subsets. For example, $s = 1$, for Φ of type A_n; $s = 2$, for any other simply laced type; and $s = 3$ for Φ of type B_n or C_n with $n \geqslant 4$.

Theorem 6.2 implies the following statement.

Corollary 6.3. *Let us fix any subset of simple roots J. Suppose that $u \in W_J$ and $v_1, v_2 \in W^J$ are such that $v_1^{-1}uv_2$ is a* (*rationally*) *smooth element in W. Then u is a* (*rationally*) *smooth in element in W_J.*

It was shown in [4] that, for any $w \in W$ and a subset J of the simple roots, the parabolic subgroup W_J has a unique maximal element $m(w, J) \in W_J$ below w in the Bruhat order. The following theorem generalizes the factoring formulas for Poincaré polynomials found in [1,12] and [6, Theorem 11.23]. Using this theorem one can simplify the search for the palindromic Poincaré polynomials which appear in Theorem 5.5.

Theorem 6.4. *Let J be any subset of the simple roots. Assume $w \in W$ has the parabolic decomposition $w = u \cdot v$ with $u \in W_J$ and $v \in W^J$ and furthermore, $u = m(w, J)$. Then*

$$P_w(t) = P_u(t)\, P_v^{W^J}(t)$$

where $P_v^{W^J}(t) = \sum_{z \in W^J,\, z \leqslant v} t^{\ell(z)}$ is the Poincaré polynomial for v in the quotient.

Proof. Let $B(w) = \{x \in W \mid x \leqslant w\}$ and $B_{W^J}(v) = \{z \in W^J \mid z \leqslant v\}$. We will show there exists a rank preserving bijection $f : B(w) \to B(u) \times B_{W^J}(v)$.

Given any $x \leqslant w$, say x has parabolic decomposition $x = yz$ with respect to J, then $y \leqslant u = m(w, J)$ since $m(w, J)$ is the unique maximal element below w and in W_J. Furthermore, Proctor [20, Lemma 3.2] has shown that if $z \in W$ is also a minimal length element in the coset $W_J z$, then $z \leqslant w$ if and only if $z \leqslant v$. Therefore, we can define a map

$$f : B(w) \to B(u) \times B_{W^J}(v) \tag{4}$$

by mapping x to (y, z). Note that this map is injective and rank preserving since $\ell(x) = \ell(y) + \ell(z)$ by the properties of the parabolic decomposition.

Conversely, given any $y \in W$ such that $y \leqslant u$ and given any $z \in B_{W^J}(v)$, then actually $y \in W_J$ and we have $yz \leqslant w$ with $\ell(yz) = \ell(y) + \ell(z)$. Therefore, yz can be written as a subexpression of the reduced expression for w which is the concatenation of reduced expressions for u and v. Furthermore, $f(yz) = (y, z)$ since z is the unique minimal length coset representative in the coset containing yz. Hence f is surjective. □

Gasharov [12] and Billey [1] have shown that, for the classical types, the Poincaré polynomials of rationally smooth Schubert varieties have very nice factorizations. Gasharov and Reiner [13], Ryan [21], and Wolper [26] have in fact shown that smooth Schubert varieties can be described as iterated fiber bundles over Grassmannians. We see a similar phenomena for the exceptional types.

Corollary 6.5. *Every Poincaré polynomial of a rationally smooth Schubert variety in types A_n, B_n, C_n, D_n, E_6, E_7, E_8, G_2, F_4 factors into a product of symmetric factors each of which are Poincaré polynomials indexed by elements in a maximal parabolic quotient W/W_J.*

We only use the following lemma in the proof of Theorem 2.2 while proving our criteria for rational smoothness.

Lemma 6.6. *Let $w = uv$ be the parabolic decomposition of $w \in W$ with $u \in W_J$ and $v \in W^J$.*

(1) *If X_u is not rationally smooth then X_w is not rationally smooth.*
(2) *For any root subsystem $\Delta \subset \Phi^J$, we have $I_\Phi(u) \cap \Delta = I_\Phi(w) \cap \Delta$. Therefore, if u has a pattern (Δ_+, σ) then so does w.*

Note, it is not true that if X_v is singular in the quotient G/P_J then X_w is necessarily singular in G/B. Also, it is possible for both X_u to be smooth in G/B, X_v smooth in the quotient and yet X_w to be singular in G/B if $u \neq m(w, J)$.

Proof. To prove the first statement, assume X_u is not rationally smooth. Then there exits a vertex $x < u$ in the Bruhat graph for u where $\deg(x)$ is too large by Theorem 5.5 and Lemma 5.7, namely

$$\#\{\alpha \in \Phi_+ \colon x < s_\alpha x \leqslant u\} > \ell(u) - \ell(x).$$

We claim $\deg(xv)$ is too large for X_w. Note, $x \leqslant s_\alpha x \leqslant u$ implies $s_\alpha \in W_J$, and therefore $xv \leqslant s_\alpha xv \leqslant w$ by the properties of the parabolic decomposition. Therefore,

$$\begin{aligned}\#\{\alpha \in \Phi_+ \colon xv < s_\alpha xv \leqslant w\} &\geqslant \#\{\alpha \in \Phi_+ \colon x < s_\alpha x \leqslant u\} \\ &> \ell(u) - \ell(x) = \ell(w) - \ell(v) - \ell(x) = \ell(w) - \ell(xv).\end{aligned}$$

Hence, X_w is not rationally smooth, proving the first statement.

The second statement follows directly from Lemma 6.1. □

7. Proof of Theorem 2.2

The semisimple Lie groups come in 4 series: A_n, B_n, C_n, D_n and 5 exceptional types E_6, E_7, E_8, F_4, G_2. The proof of Theorem 2.2 for the four infinite series follows easily on the known characterization for smooth Schubert varieties in terms of pattern avoidance. The proof for the exceptional types was much more difficult. One might imagine that a routine verification would suffice for these finite Weyl groups. However, both verifying smoothness in F_4 and rational smoothness in E_8 directly would be impossible in our life time using previously known techniques. The exceptional types were proved with the aide of a large parallel computer after making several reductions in complexity. These reductions in complexity also give insight into the intricate geometry of the exceptional types.

Recall, classical pattern avoidance is defined in terms of the following function which *flattens* any subsequence into a signed permutation. Let $\mathcal{B}_n$ denote the signed permutation group. Elements in $\mathcal{B}_n$ can be written in *one-line notation* as an ordered list of the numbers $1, \ldots, n$ with a choice of sign for each entry. For example, $3\bar{2}1 = s_2 s_3 s_2 s_1 s_2 \in \mathcal{B}_n$ (barred numbers are negative). The group $\mathcal{B}_n$ is isomorphic to the Weyl group of type B_n and C_n. The Weyl group of type D_n is the subgroup of $\mathcal{B}_n$ in which all elements have an even number of negative entries; and the Weyl group of type A_{n-1} (the symmetric group S_n) is the subgroup in which all elements have no negative entries.

Definition 7.1. Given any sequence $a_1 a_2 \ldots a_k$ of distinct non-zero real numbers, define $\mathrm{fl}(a_1 a_2 \ldots a_k)$ to be the unique element $b = b_1 \ldots b_k$ in $\mathcal{B}_k$ such that

- both a_j and b_j have the same sign.
- for all i, j, we have $|b_i| < |b_j|$ if and only if $|a_i| < |a_j|$.

For example, $\mathrm{fl}(\bar{6}, 3, \bar{7}, 1) = \bar{3}2\bar{4}1$. Any sequence containing the subsequence $\bar{6}, 3, \bar{7}, 1$ does not avoid the pattern $\bar{3}2\bar{4}1$.

Theorem 7.2 [1,19]. *Let W be one of the groups $W_{A_{n-1}}$, W_{B_n}, W_{C_n} or W_{D_n} and let $w \in W$. Then X_w is (rationally) smooth if and only if for each subsequence $1 \leqslant i_1 < i_2 < i_3 < i_4 \leqslant n$, $\mathrm{fl}(w_{i_1} w_{i_2} w_{i_3} w_{i_4})$ corresponds to a (rationally) smooth Schubert variety.*

In order to prove Theorem 2.2, we claim $\mathrm{fl}(w_{i_1} w_{i_2} w_{i_3} w_{i_4}) = v$ if and only if $I_\Delta(v) = I_\Phi(w) \cap \Delta$ where Δ is the root subsystem of type B_4 in the span of $e_{|w_{i_1}|}$, $e_{|w_{i_2}|}$,

$e_{|w_{i_3}|}$, $e_{|w_{i_4}|}$. We will prove this claim in type B, the remaining cases are similar. Then verification of types A_3, B_4, C_4 and D_4 suffices to check the theorem in the classical case.

For type B_n, let us pick the linear basis $e_1, \dots, e_n$ in $\mathfrak{h}^*$ such that the simple roots are given by $e_1, e_2 - e_1, \dots, e_n - e_{n-1}$. Then $\Phi_+ = \{e_k \pm e_j \colon 1 \leqslant j < k < n\} \cup \{e_j \colon 1 \leqslant j \leqslant n\}$. A signed permutation w acts on $\mathbb{R}^n$ by

$$w(e_j) = \begin{cases} e_{w_j} & \text{if } w_j > 0, \\ -e_{|w_j|} & \text{if } w_j < 0. \end{cases}$$

Explicitly, $I_\Phi(w) = \Phi_+ \cap w\Phi_-$ is the union of the following three sets

$$\begin{gathered} \{w(-e_j) \colon w_j < 0\}, \\ \{w(e_j - e_k) \colon j < k,\ w_j > |w_k|\}, \\ \{w(\pm e_j - e_k) \colon j < k,\ w_k < 0 \text{ and } |w_j| < |w_k|\}. \end{gathered}$$

Therefore, deciding if $w(-e_j)$ or $w(\pm e_j - e_k) \in I_\Phi(w)$ depends only on the relative order and sign patterns on w_j and w_k. By definition of the classical flattening function $\mathrm{fl}(w_i w_{i_2} w_{i_3} w_{i_4}) = v \in W_{B_4}$ if $w_i w_{i_2} w_{i_3} w_{i_4}$ and $v_1 v_2 v_3 v_4$ have the same relative order and sign pattern. Hence, when Δ is the root subsystem of type B_4 determined by $e_{|w_{i_1}|}, e_{|w_{i_2}|}, e_{|w_{i_3}|}, e_{|w_{i_4}|}$, we have $I_\Phi(w) \cap \Delta = I_\Delta(v)$ if and only if $\mathrm{fl}(w_{i_1} w_{i_2} w_{i_3} w_{i_4}) = v$. This proves the claim and finishes the proof of Theorem 2.2 for the classical types.

Next consider the root systems of types G_2 and F_4. We can simply check Theorem 2.2 by computer using the modified version of Kumar's criterion for determining smoothness and the Carrell–Peterson criteria discussed in Section 5. In particular, for G_2, we use Remark 5.4 to find all singular elements Schubert varieties. They are $X_{s_1 s_2 s_1}$, $X_{s_1 s_2 s_1 s_2}$, $X_{s_2 s_1 s_2 s_1}$, $X_{s_1 s_2 s_1 s_2 s_1}$, $X_{s_2 s_1 s_2 s_1 s_2}$ (assuming α_1 is the short simple root). Pattern avoidance using root subsystems does not offer any simplification of this list. However, using root systems embeddings in Section 3, these singular Schubert varieties all follow from one B_2 pattern. All Schubert varieties of type G_2 are rationally smooth.

For F_4 we use the following algorithm to verify Theorem 2.2:

(1) Make a matrix of size $|W|^2$ containing the values $K_{w,v}(r)$ computed recursively using the formula

$$K_{w,v}(r) = \begin{cases} K_{ws_i,v}(r) & \text{if } v < vs_i, \\ K_{ws_i,v}(r) + (ws_i\alpha_i(r))K_{ws_i,vs_i}(r) & \text{if } v > vs_i \end{cases}$$

where s_i is any simple reflections such that $ws_i < w$.

(2) Identify all subsets $\{\gamma_1, \dots, \gamma_p\} \subset \Phi_+$ for $p = 2, 3, 4$ such that $\{\gamma_1, \dots, \gamma_p\}$ forms a basis for a root subsystem Δ of type B_2, A_3, B_3 or C_3 (no root subsystems of types G_2 or D_4 appear in F_4). Let $\mathcal{B}$ be the list of all such root subsystem bases.

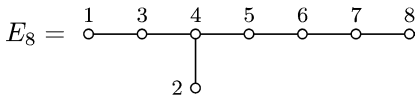
(3) For each such root subsystem Δ with basis $B \in \mathcal{B}$, find all singular Schubert varieties X_v using Remark 5.4 and add $I_\Delta(v)$ to a list called $\mathrm{BAD}(B)$. Note this list can be significantly simplified by removing all B_3, C_3 Schubert varieties which are classified as singular using a B_2 root subsystem.

(4) For each $w \in W_{F_4}$, check $\ell(w) = |\{\gamma\colon s_\gamma \leqslant w\}|$ and $K_{w_0,ww_0}(r) = \prod_{\gamma\in Z(w_0,ww_0)} \gamma(r)$ if and only if no $B \in \mathcal{B}$ exists such that $I_\Phi(w) \cap \operatorname{span}(B)$ is a member of $\operatorname{BAD}(B)$.

To verify the theorem for rational smoothness in F_4 we use the same algorithm except $\operatorname{BAD}(B)$ should contain all of the inversion sets for rationally singular Schubert varieties of types A_3, B_3, C_3, D_4 and in step (4) use the palindromic Poincaré polynomial criterion for rational smoothness from Theorem 5.5.

Finally, for E_6, E_7 and E_8 it suffices to check the theorem on E_8 since the corresponding root systems and Weyl groups are ordered by containment. Note, the Weyl group of type E_8 has 696,729,600 elements so creating the matrix as in step (1) above is not possible with the current technology. Therefore, a different method for verifying the main theorem was necessary. Recall, D. Peterson has shown, see [9], that smoothness and rational smoothness are equivalent for simply laced Lie groups, i.e., A_n, D_n and E_6, E_7, E_8. Unfortunately, computing $P_w(t)$ for all $w \in E_8$ and applying either of the Carrell–Peterson criteria in Theorem 5.5 is out of the question, however we made the following observations:

(1) In E_8, if $P_w(t)$ is not symmetric then approximately 99.989% of the time the only coefficients we need to check are of t^1 and $t^{\ell(w)-1}$. In fact, in all of E_8, all one ever needs to check is the first 6 coefficients (starting at t^1) equals the last 6 coefficients. Note, the coefficient of t^1 is just the number of distinct generators in any reduced expression for w and the coefficient of $t^{\ell(w)-1}$ is the number of $v = ws_\alpha$ for $\alpha \in \Phi_+$ such that $\ell(v) = \ell(w) - 1$ which can be efficiently computed.
(2) In E_8, if $P_w(t)$ is symmetric then there always exists a factorization according to Theorem 6.4 where J is a subset of all simple roots except one which corresponds to a leaf of the Dynkin diagram. This factored formula makes it easy to check the palindromic property recursively.
(3) Let J be the set of all simple roots in E_8 except α_1. Here we are labeling the simple roots according to the following Dynkin diagram:



By Lemma 6.6, we only need to test $w = uv$ where $v \in W/W_J$, $u \in W_J \approx W_{D_7}$ and X_u smooth. There are 9479 elements of W_{D_7} which correspond to smooth Schubert varieties and 2160 elements in W/W_J.

With these three observations, we can complete the verification of E_8 using the following algorithm:

(1) Identify all root subsystems of types A_3, D_4 and their bases as above (since E_8 is simply laced it has no root subsystems of the other types). Call the list of bases $\mathcal{B}$.
(2) For each such root subsystem Δ with basis $B \in \mathcal{B}$, find all singular (or equivalently rationally singular) Schubert varieties X_v using Remark 5.4 and add $I_\Delta(v)$ to a list

called BAD(B). Note this list can be significantly shortened by removing all D_4 Schubert varieties which are classified as singular using an A_3 root subsystem.

(3) Identify all 2160 minimal length coset representatives in the quotient $W_J \backslash W$ (moding out on the left). Call this set Q.

(4) Identify all 9479 elements in Smooth-$W_J \approx$ Smooth-$D_7 := \{u \in W_{D_7} \colon X_u \text{ is smooth}\}$ using classical pattern avoidance or root subsystems.

(5) For each $u \in$ Smooth-W_J and $v \in Q$, let $w = uv$ and check if $B \in \mathcal{B}$ exists such that $I_\Phi(w) \cap \text{span}(B) \in \text{BAD}(B)$.

   (a) If yes, check as many coefficients as necessary to show $P_w(t)$ is not symmetric. Here 5 coefficients sufficed for all u, v.

   (b) If no, attempt to factor $P_w(t)$ by taking J' to be all simple roots except one of the leaf nodes of the Dynkin diagram and if $w = m(w, J')v'$ or $w^{-1} = m(w^{-1}, J')v'$ in the corresponding parabolic decomposition then apply Theorem 6.4. For every w in this case, there exists some such J' so that

$$P_w(t) = P_{u'}(t) P_{v'}^{W^{J'}}(t)$$

   where $P_{v'}^{W^{J'}}(t)$ is symmetric and $P_{u'}(t)$ factored into symmetric factors recursively by peeling off one leaf node of the Dynkin diagram at a time.

This completes the proof of Theorem 2.2. Theorems 2.3, 2.4 and 2.5 follow directly.

Conjecture 7.3. *Let Φ be a simply laced root system of rank n. Say (Φ_+, w) is not a rationally smooth pair. We conjecture that one only need to compare the first n coefficients and the last n coefficients of $P_w(t)$ in order to find an asymmetry. Equivalently, the Kazhdan–Lusztig polynomial $P_{\text{id},w}(q)$ has a non-zero coefficient among the terms $q^1, q^2, \ldots, q^n$.*

We can show that for A_n one only needs to check $n-2$ coefficients, for D_5 and E_6 one needs to check 3 coefficients, and for E_8 one needs to check 5 coefficients. For F_4 (which is not simply laced) one needs to check 3 coefficients, and for B_5 one needs to check 6 coefficients.

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As with Appel and Haken's 1976 proof of the Four Color Theorem, our proof has met some criticism as to the merit of using a computer verification. In fact, it has been rejected by two journals based not on the importance or originality of the results, but on the method of proof. We believe quite to the contrary that every significant computer aided proof is a major accomplishment in expanding the role of computers in mathematics. It is like practicing to use induction 2000 years ago; it was a highly creative and influential

achievement. With years of practice, we have become quite proficient with the induction technique. However, computer aided proof is a fledgling technique that certainly will have a major impact on the future of mathematics. Therefore, we hope our method of proof will actually make a much broader impact on the future of mathematics than our main theorem.

Perhaps the only thing a complete human proof could add is an intuitive explanation for why stellar root subsystems contain all the bad patterns. This remains an open problem.

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