COUNTING MORSE CURVES AND LINKS

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Abstract. We investigate the connected components of the space of Morse curves. This leads to new interesting combinatorial sequences related to many classical objects such as alternating permutations, Catalan numbers, knots, braid groups, meanders, and continued fractions. We prove some enumerative and arithmetic results about these sequences and formulate several conjectures.

1. Morse curves

Let $\mathcal{M}$ be a real differentiable manifold, and let $h : \mathcal{M} \to \mathbb{R}$ be a smooth function without critical points called the *height function.* A *simple curve* is a smooth closed curve $f : S^1 \to \mathcal{M}$ without self-intersections. A *link* of simple curves is a collection of pairwise nonintersecting simple curves.

Definition 1.1. For a pair $(\mathcal{M}, h)$, a *Morse curve* is a simple curve $f : S^1 \to \mathcal{M}$ such that the composition $h \circ f$ is a Morse function on S^1, i.e., $h \circ f$ has a finite number of isolated nondegenerate critical points with all distinct critical values. A *Morse k-link* is a link of k Morse curves such that all critical values of all curves are distinct. The *combinatorial type* of a Morse curve/link is its connected component in the space of all Morse curves/links.

There are several invariants of Morse curves that depend only on the combinatorial type. Recall that a permutation σ is called *alternating* if $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots$. For a Morse curve f, let us order the critical values $v_1, v_2, \ldots, v_m$ of $h \circ f$ as we go around the curve so that v_m is the maximal critical value. Then the critical values alternate: $v_m > v_1 < v_2 > v_3 < \cdots > v_{m-1} < v_m$. In particular, the total number of critical values $m = 2n$ is even, a half of them are local maximums and the other half are local minimums. Let σ be the permutation of size $2n-1$ such that $v_{\sigma^{-1}(1)} < v_{\sigma^{-1}(2)} < \cdots < v_{\sigma^{-1}(2n-1)}$. Then σ is an alternating permutation.

Another invariant of a Morse curve f is its *knot* that can be defined as the connected component of f in the space of all simple curves. There is only one (trivial) knot for $\mathcal{M} = \mathbb{R}^d$, $d \geq 4$, two knots (clockwise and counterclockwise) for $\mathcal{M} = \mathbb{R}^2$, and four knots for the cylinder $\mathcal{M} = S^1 \times \mathbb{R}$. It is hard to classify all knots for $\mathcal{M} = \mathbb{R}^3$.

It would be interesting to describe all possible combinatorial types of Morse curves with given knot and alternating permutation. It is not hard to see that, for $\mathcal{M} = \mathbb{R}^d$, $d \geq 4$, with the standard height function $h : (x_1, \ldots, x_d) \mapsto x_d$, the type of Morse curve is uniquely determined by its alternating permutation. However, in general it is not true that the knot and the alternating permutation uniquely

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determine the type of a Morse curve. Figure 1 shows two Morse curves in $\mathbb{R}^3$ of different types (with respect to the standard height function). They have the same (trivial) knot and the same alternating permutation.

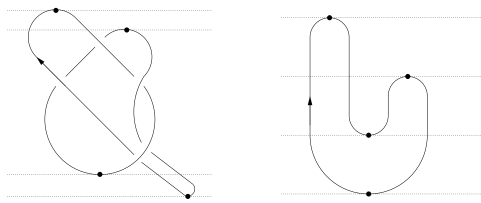

FIGURE 1. Two Morse curves in $\mathbb{R}^3$ of different types with the same knot and the same alternating permutation

In this article, we will concentrate on 2-dimensional manifolds $\mathcal{M}$.

2. Plane Morse curves and links

Definition 2.1. A *plane Morse curve* is a Morse curve on $\mathcal{M} = \mathbb{R}^2$ oriented clockwise with the height function $h : (x, y) \mapsto y$. A *plane Morse k-link* as a Morse link of k such curves (all oriented clockwise). The *order* of a curve/link is half the total number of its critical points.

Figure 2 gives an example of plane Morse curve of order 11. There is one com-

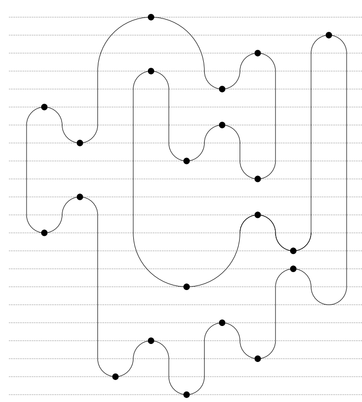

FIGURE 2. A plane Morse curve of order 11

binatorial type of plane Morse curves of order 1 and four types of plane Morse curves of order 2, see Figure 3. An example of Morse 6-link of order 10 is shown on Figure 4. There are 6 combinatorial types of 2-links of order 2, see Figure 5.

Let F_n be the number of connected components of the space of plane Morse curves of order n. Also let $L_n^{(k)}$ be the the number of connected components of the space of plane Morse k-links of order n and $L_n = \sum_k L_n^{(k)}$ be the total number of components of the space of links of order n.

The first few numbers F_n and L_n are given in the following table.[1]

[1]The author wrote a C++ program for calculation the numbers L_n.

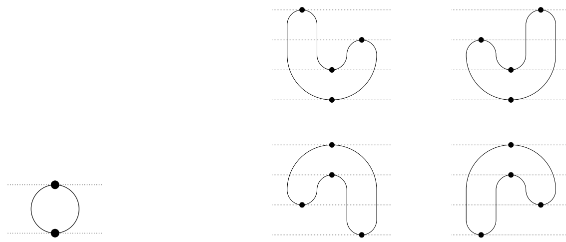

FIGURE 3. All plane Morse curves of orders 1 and 2

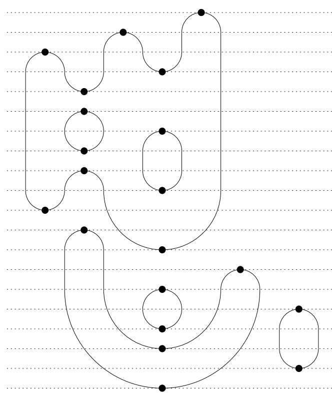

FIGURE 4. A plane Morse 6-link of order 10

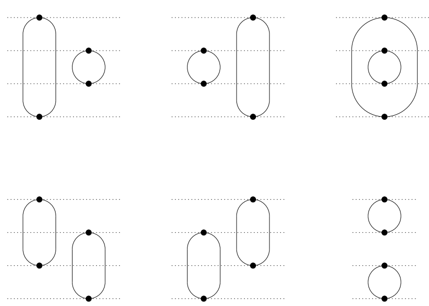

FIGURE 5. All plane Morse 2-links of order 2

n	0	1	2	3	4	5	6	7
F_n	0	1	4	70	2752	191888	20860608	?
L_n	1	1	10	325	22150	2586250	461242900	116651486125

n	8	9	10
L_n	39713286199150	17511670912894750	9709015945443877900

Define the n-th *link polynomial* $L_n(q)$ as the generating function for k-link numbers of order n:

$$L_n(q) = \sum_k L_n^{(k)}\, q^k$$

It is clear that F_n is the coefficient of q in $L_n(q)$ and $L_n = L_n(1)$. The first few polynomials $L_n(q)$ are the following

$$\begin{aligned}
L_0(q) &= 1\,, \\
L_1(q) &= q\,, \\
L_2(q) &= 4\,q + 6\,q^2\,, \\
L_3(q) &= 70\,q + 160\,q^2 + 95\,q^3\,, \\
L_4(q) &= 2752\,q + 8198\,q^2 + 8316\,q^3 + 2884\,q^4\,, \\
L_5(q) &= 191888\,q + 695312\,q^2 + 958698\,q^3 + 597792\,q^4 + 142560\,q^5\,, \\
L_6(q) &= 20860608\,q + 88459928\,q^2 + 151680336\,q^3 + 131749186\,q^4 + \\
&\qquad + 58080440\,q^5 + 10412402\,q^6\,.
\end{aligned}$$

Theorem 2.2. *The numbers L_n are the coefficients of the expansion of the continued fraction*

$$\cfrac{1}{1 - \cfrac{1^2\,x}{1 - \cfrac{3^2\,x}{1 - \cfrac{5^2\,x}{1 - \cfrac{7^2\,x}{1 - \cdots}}}}} = 1 + x + 10\,x^2 + 325\,x^3 + \cdots = \sum_{n\geq 0} L_n\, x^n.$$

Recall that the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ equals the number of Catalan path, which are lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ (up step) and $(1,-1)$ (down step) that always stay in the upper halfplane $\{(x,y) \mid y \geq 0\}$. Let us say that an up step $(i,j-1) \to (i+1,j)$ and a down step or $(i,j) \to (i+1,j-1)$ in a Catalan path has *level* j. Let us fix two infinite sets of parameters $u_1, u_2, u_3, \ldots$ and $d_1, d_2, d_3, \ldots$ and weight each up step of level j by u_j and each down step of level j by d_j. Then define the weight $wt(P)$ of a Catalan number as the product of weights of all its steps. It is well-known the generating series of weighted Catalan paths is given by the following continued fraction; see [GJ]:

$$\sum_{n\geq 0} \sum_{P:(0,0)\to(2n,0)} wt(P)\, x^n = \cfrac{1}{1 - \cfrac{u_1 d_1\, x}{1 - \cfrac{u_2 d_2\, x}{1 - \cfrac{u_3 d_3\, x}{1 - \cdots}}}}.$$

Proof of Theorem 2.2. Let us extablish a bijection between plane Morse links of degree n and weighted Catalan paths with weights $u_j = d_j = 2j-1$. Let us totally order all $2n$ critical points of a Morse link so that their critical values increase. Let us constract a Catalan path whose steps correspond to the critical points of the link.

More presicely, up steps correspond local minimums and down steps correspond to maximums. For example, the link shown on Figure 4 gives the Catalan path with the following sequence of up and down steps:

$$(U, U, U, U, D, D, D, U, D, U, U, D, U, D, D, U, U, D, D, D).$$

A step of level j in the Catalan path correspond a critial point p in the link such that there are exactly $2j-1$ points in the link with the same height $h(p)$ as p. Let us count the number of combinatorial types of plane Morse links that produce a given Catalan path. As we construct such a link by inserting critical points one by one from the botton to the top, we have $2j-1$ options to insert a local minimum or local maximum associated with a step of level j in the Catalan path. This gives a correspondence between weighted Catalan paths and Morse links. □

Theorem 2.3. *The value $(-1)^n L_n(-1)$ is equal to the n-th Catalan number:*

$$(-1)^n L_n(-1) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Let us say that a critial point p in a plane Morse link is *left-movable* if the number of (non-critical) points on the link to the left of p with the same height $h(p)$ is odd. Also say that a critial point p in a plane Morse link is *right-movable* if it is not left-movable, i.e., there are even number of points on the link to the left of p with the same height $h(p)$, and there is at least one such point to the right of p. Let us use the *Involution Principle* to show that the alternating sum of plane Morse links with at least one left- or right-movable critical point is 0. Find the left- or right-movable critical point p with the minimal critical value. Then replace the local neighboringhood of the point p (together with its left or right neighboring strand) as shown on Figure 6. The left-hand side of Figure 6 corresponds to a left-movable point and the right-hand side corresponds to a right-movable point. Note that, if we apply this operation twice, then we obtain a link of the same combinatorial type as the original link. Also notice that this operation changes the number of curves in a link by 1. Thus all contributions to $L_n(-1)$ of links with movable critical points cancel each other.

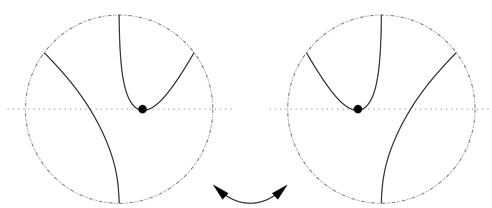

Figure 6. Involution on plane Morse links

If a link has no movable critial points then it is uniquely determined by the Catalan path constructed in the proof of Theorem 2.2. For any Catalan path with $2n$ steps there is such a link. This link consists of n disjoint curves. □

Let π be an *alternating* permutation of size $2n$. An index $k \in \{2, \ldots, 2n-1\}$ is called *interior* for π if there are two indices $i < k < j$ such that $\pi(i) < \pi(k) < \pi(j)$. Let $\mathrm{int}(\pi)$ the number of interior indices for π.

Theorem 2.4. *We have*

$$L_n = \sum_{\pi} 2^{\mathrm{int}(\pi)},$$

where the sum is over all alternating permutations of length $2n$.

3. Binary arithmetics

The numbers L_n does not seem to have simple factorization. Nevertheless, the they have interesting arithmetic properties for modulo powers of primes. Note that Theorem 2.3 implies that the numbers $L_n = L_n(1)$ and $C_n = \pm\, L_n(-1)$ have the same parity. Actually, a stronger property seems to be true.

For an nonzero integer number m, let $d_2(m)$ denotes the maximal power of 2 that divides the number m.

Conjecture 3.1. For $n \geq 1$, the maximal power of 2 that divides the number L_n is equal to the maximal power of 2 that divides the Catalan number C_n:

$$d_2(L_n) = d_2(C_n).$$

We have verified this conjecture for $n \leq 100,000$ using a computer.

The maximal power $d_2(C_n)$ is given by the following Kummer's theorem that can be found in [D]. For an nonnegative integer number m, let $s(m)$ be the sum of digits $\epsilon_0 + \epsilon_1 + \cdots$ in the binary expansion $m = \sum_{i\geq 0} \epsilon_i\, 2^i$, $\epsilon_i \in \{0,1\}$.

Theorem 3.2. (Kummer) *The maximal power of 2 that divides the Catalan number C_n is equal to*

$$d_2(C_n) = s(n+1) - 1.$$

Thus the records of the sequence $d_2(C_n)$ are the numbers $n = 2^k - 2$.

Theorem 3.3. *The number $L_n - C_n$ is always divisible by 8. Thus $d_2(L_n) = d_2(C_n)$ provided $d_2(C_n) = 0,\ 1,$ or 2 (i.e., when the number $n+1$ has less than four 1's in the binary expansion).*

Proof. Use the fact that $a^2 - 1 = (a-1)(a+1)$ is always divisible by 8 for even a. □

Let $R_n = (L_n - C_n)/2$. For even n, the number R_n is equal to the number of links of order n with even number of curves. For odd n, the number R_n is equal to the number links of order n with odd number of curves. Conjecture 3.1 would follow from the next conjecture.

For an infinite sequence $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ of 0's and 1's, let $t_\alpha(m)$ denote the first index k where the sequence α disagree with the binary expansion $m = \sum_{i\geq 0} \epsilon_i\, 2^i$, $\epsilon_i \in \{0,1\}$, i.e., $\epsilon_0 = \alpha_0, \dots, \epsilon_{k-1} = \alpha_{k-1}$, and $\epsilon_k \neq \alpha_k$.

Conjecture 3.4. There is a unique infinite sequence of 0's and 1's

$$(\alpha_0, \alpha_1, \alpha_2, \dots) = (1,0,1,0,1,0,1,0,1,1,0,0,0,1,1,0,0,0,0,1,\dots)$$

such that for any number $n \geq 2$ we have

$$d_2(R_n) = s(n) + t_\alpha(n-2) + 1.$$

In particular, $d_2(R_n) = s(n) + d_2(n - 549719) + 1$ for $n \not\equiv 549719 \pmod{2^{20}}$.

We have verified this statement for $n \leq 100000$.

Conjecture 3.4 implies the following two statements. Recall that a *record* of a sequence A_n is an index r such that $A_r > A_n$ for all $n < r$.

Theorem 3.5. *Suppose that Conjecture 3.4 is true. Then there is an infinite increasing sequence*

$$a_1, a_2, a_3, \dots = 0, 2, 4, 6, 8, 9, 13, 14, 19, \dots$$

(a_i is the position of the i-th 1 in the sequence α) such that the records of the sequence $d_2(R_n)$, $n \geq 2$, are exactly the numbers

$$r_k = 2 + \sum_{i=1}^{k} 2^{a_i}.$$

There is a recursive procedure for finding the records r_k.

Theorem 3.6. *Suppose that Conjecture 3.4 is true. If $n = r_k$ is a record of the sequence $d_2(R_n)$, then the next record is $r_{k+1} = n + 2^{a_{k+1}}$, where $a_{k+1} = d_2(L_n) - s(n) - 1$.*

We have verified that the first records of the sequence $d_2(R_n)$, $n \geq 2$, are given by this procedure. The records are 2, $3 = 2 + 1_2$, $7 = 2 + 101_2$, $23 = 2 + 10101_2$, $87 = 2 + 1010101_2$, $343 = 2 + 101010101_2$, $855 = 2 + 1101010101_2$, $9047 = 2 + 10001101010101_2$, and $25431 = 2 + 110001101010101_2$. We have $d_2(R_2) = 2$, $d_2(R_3) = 5$, $d_2(R_7) = 8$, $d_2(R_{23}) = 11$, $d_2(R_{87}) = 14$, $d_2(R_{343}) = 16$, $d_2(R_{855}) = 21$, $d_2(R_{9047}) = 23$, and $d_2(R_{25431}) = 29$. If Conjecture 3.4 is true the next record is $549719 = 2 + 10000110001101010101_2$.

4. Arithmetics modulo 3^k and 5^k

Conjecture 4.1. For any n, $d_3(L_n + 2) = 1$, $d_3(L_n + 5) = 1$, $d_3(L_n + 8) = 2$, $d_3(L_n + 11) = 1$. For any even $n \geq 2$, $d_3(L_n - 1) = 2$.

Theorem 4.2. *1. For any odd m, the sequence $L_n \pmod{m}$ is periodic.*
2. The sequence $L_n \equiv 1 \pmod{3}$ is periodic with period 1. The sequence $L_n \equiv 1 \pmod{3^2}$ is periodic with period 1. The sequence $L_n \pmod{3^3}$ is periodic with period 2. The sequence $L_n \pmod{3^4}$ is periodic with period 6. The sequence $L_n \pmod{3^5}$ is periodic with period 18. The sequence $L_n \pmod{3^6}$ is periodic with period 54.

Conjecture 4.3. 1. The sequence $L_n \pmod{3^{3+r}}$ is periodic with period $2 \cdot 3^r$.
2. The sequence $L_n \pmod{7}$ is periodic with period 12.
3. The sequence $L_n \pmod{11}$ is periodic with period 55.

For an nonzero integer number m, let $d_5(m)$ denotes the maximal power of 5 that divides the number m. Also let $k(n)$ be the number $k \geq 0$ such that in the 5-ary expansion $n = \sum_{i \geq 0} \epsilon_i 5^i$, $0 \leq \epsilon_i \leq 4$, we have $\epsilon_0 = \dots = \epsilon_{k-1} = 1$, and $\epsilon_k \neq 1$.

Conjecture 4.4. For $n \geq 4$, we have

$$d_5(L_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ k(n) \cdot (n-4) + 3 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 4.5. *If Conjecture 4.4 is true, then the records of the sequence $d_5(L_n)$, $n \geq 5$, are the numbers $n = 4 + (5^{2k} - 1)/4$, $k = 0, 1, \dots$.*

We have verified Conjecture 4.5 for the numbers L_n with $n \leq 300$. The first records are $5 = 4 + 1_5$, $35 = 4 + 111_5$, $785 = 4 + 11111_5$, $4 + 1111111_5$, etc.

5. Cylindric Morse curves

In this section we study Morse curves on the cylinder $\mathcal{M} = S^1 \times \mathbb{R}$. The height function $h : S^1 \times \mathbb{R} \to \mathbb{R}$ is given by the projection to the second coordinate, $h : (x, y) \to y$.

Theorem 5.1. *The number CL_n of links of order n on the cylinder is equal to $2^n \cdot \mathrm{UD}_{2n}$.*

Theorem 5.2. *The value $(-1)^n CL_n(-1)$ is equal to 2^n.*

6. Meanders

Let M_n be the *menadric number.*

Theorem 6.1. *We have $M_n < F_n < M_{???}$.*

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