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Affine Weyl groups in K-theory and representation theory

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Abstract. We present a simple combinatorial model for the characters of the irreducible representations of complex semisimple Lie groups and, more generally, for Demazure characters. On the other hand, we give an explicit combinatorial Chevalley-type formula for the T-equivariant K-theory of generalized flag manifolds G/B. The construction is given in terms of alcove paths, which correspond to decompositions of affine Weyl group elements, and saturated chains in the Bruhat order on the (nonaffine) Weyl group. A key ingredient is a certain R-matrix, that is, a collection of operators satisfying the Yang-Baxter equation. Our model has several advantages over the Littelmann path model and the LS-galleries of Gaussent and Littelmann. The relationship between our model and the latter ones is yet to be explored.

Résumé. Nous présentons un modèle combinatoire simple pour les caractères des représentations d'un groupe de Lie complexe semisimple et, en général, pour les caractères de Demazure. D'autre part, nous présentons une généralisation combinatoire de la formule de Chevelley pour la K-théorie équivariante des variétés de drapeaux G/B. Notre construction est en termes de chemins sur les alcôves déterminées par le groupe de Weyl affine (qui correspondent aux décompositions réduites dans ce groupe) et de chemins saturés sur le groupe de Weyl (nonaffine). Un ingrédient important est une certaine R-matrice, c'est-à-dire une collection des opératoires qui vérifient l'équation de Yang-Baxter. Notre modèle a plusieurs avantages par comparaison avec le modèle de chemins de Littelmann et les galeries LS de Gaussent et Littelmann. La relation entre notre modèle et les deux autres n'a pas encore été étudiée.

1. Introduction

Littelmann paths give a model for characters of irreducible representations V_λ of a semisimple Lie group G, and, more generally, for a complex symmetrizable Kac-Moody algebra. The theory extends to the characters of *Demazure modules* $V_{\lambda,w}$, which are B-modules. Littelmann [**Li1, Li2**] showed that the mentioned characters can be described by counting certain continuous paths in $\mathfrak{h}^*_\mathbb{R}$. These paths are constructed recursively, using certain operators acting on them, known as *root operators*. A special case of Littelmann paths are the *Lakshmibai-Seshadri paths* (L-S paths), which have been introduced before, in the context of *standard monomial theory* [**LS1**]. L-S paths also have a nonrecursive characterization.

A geometric application of Littelmann paths was given by Pittie and Ram [**PR**], who used them to derive a Chevalley-type multiplication formula in the T-equivariant K-theory of the generalized flag variety G/B. Let $K_T(G/B)$ be the Grothendieck ring of T-equivariant coherent sheaves on G/B. According to Kostant

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and Kumar [**KK**], the ring $K_T(G/B)$ is a free module over the representation ring $R(T)$ of the maximal torus, with basis given by the classes $[\mathcal{O}_w]$ of structure sheaves of Schubert varieties. Pittie and Ram showed that the basis expansion of the product of $[\mathcal{O}_w]$ with the class $[\mathcal{L}_\lambda]$ of a line bundle can be expressed as a sum over certain L-S paths. The Pittie-Ram formula extends the classical *Chevalley formula* [**Chev**] for the cohomology ring $H^*(G/B)$, and its special case for the cohomology of the classical flag variety SL_n/B, known as *Monk's rule*.

Let us also mention some important results related to the Pittie-Ram formula. The fact that the product in this formula expands as a nonnegative combination was also explained by Brion [**Bri**] and Mathieu [**Mat**]. Brion [**Bri**] noted that the special case of the Pittie-Ram formula corresponding to a fundamental weight is closely related to the multiplication of $[\mathcal{O}_w]$ with the class of the structure sheaf of a codimension 1 Schubert variety (that is, to the hyperplane section of a Schubert variety in equivariant K-theory). The coefficients in the Pittie-Ram formula were identified as certain characters by Lakshmibai and Littelmann [**LL**] using geometry. Finally, Littelmann and Seshadri [**LS2**] showed that the Pittie-Ram formula is a consequence of standard monomial theory [**LS1, Li3**], and, furthermore, that it is almost equivalent to standard monomial theory.

When it comes to explicit calculations, it is often quite difficult to use the Littelmann path model, for the following reasons.

- The recursive process of constructing Littelmann paths via root operators is quite complex. On the other hand, there is no nonrecursive characterization of Littelmann paths in general, with the exception of L-S paths (see the next remark).
- L-S paths are not purely combinatorial objects, since their characterization involves rational numbers. Furthermore, their complexity is reflected in the fact that some applications (the Pittie-Ram formula, standard monomial theory [**LLM**]) require, in the case of nonregular weights λ, Deodhar's lift operators $W/W_\lambda \to W$ from cosets modulo parabolic subgroups; these operators are defined by a nontrivial recursive procedure. The picture becomes even more complicated when, beside W_λ, there is another parabolic subgroup involved; this siuation appears, for instance, in standard monomial theory [**LLM**].
- The recently defined *LS-galleries* [**GL**], which are closely related to the path model, are given by complicated conditions.
- L-S paths did not seem to allow an extension of the Pittie-Ram formula to the case of arbitrary weights λ.
- It is difficult to use L-S paths to compute hyperplane sections of Schubert varieties via Brion's result mentioned above, because the Pittie-Ram formula would have to be applied a large number of times. Essentially, this means that the Pittie-Ram formula is hard to "invert".

In this paper, we present an alternative model for both Demazure characters and Chevalley-type formulas in $K_T(G/B)$. This model has the following nice features.

- It is simple, nonrecursive, and purely combinatorial (no rational numbers are involved). The related computations are very explicit and straightforward, since they only involve enumerating certain saturated chains in Bruhat order.
- Deodhar's lifts from cosets modulo parabolic subgroups are not needed.
- The corresponding Chevalley-type formula is equally simple for any weight, regular or nonregular, dominant or nondominant.
- This formula is straightforward to "invert", in order to compute hyperplane sections of Schubert varieties in T-equivariant K-theory.

Our model is based on enumerating certain *saturated chains* in the Bruhat order on the corresponding Weyl group. This enumeration is determined by an *alcove path*, which is a sequence of adjacent alcoves for the affine Weyl group of the Langland's dual group $G^\vee$. Alcove paths correspond to representations of elements in the affine Weyl group as products of generators.

Our Chevalley-type formula in $K_T(G/B)$ can be conveniently formulated in terms of a certain *R-matrix*, that is, in terms of a collection of operators satisfying the *Yang-Baxter equation.* We express the operator E^λ of multiplication by the class of a line bundle $[\mathcal{L}_\lambda] \in K_T(G/B)$ as a composition $R^{[\lambda]}$ of elements of the R-matrix given by an alcove path. In order to prove the formula, we simply verify that the operators $R^{[\lambda]}$ satisfy the same commutation relations with the elementary Demazure operators T_i as the operators E^λ.

Currently, we are working on clarifying the relationship between the Littelmann path model and LS-galleries on the one hand, and our construction on the other hand. We are planning to describe root operators and give an explicit Littlewood-Richardson rule in terms of our model in forthcoming publications. Generalizing our construction to Kac-Moody groups is also a joint project.

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2. Notation

Let G be a connected, simply connected, simple complex Lie group. Fix a Borel subgroup B and a maximal torus T such that $G \supset B \supset T$. Let $\mathfrak{h}$ be the corresponding Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G. Let r be the rank of Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system.* Let $\mathfrak{h}^*_\mathbb{R} \subset \mathfrak{h}^*$ be the real span of the roots. Let $\Phi^+ \subset \Phi$ be the set of positive roots corresponding to our choice of B. Then Φ is the disjoint union of Φ^+ and $\Phi^- = -\Phi^+$. Let $\alpha_1, \dots, \alpha_r \in \Phi^+$ be the corresponding set of *simple roots*, which form a basis of $\mathfrak{h}^*_\mathbb{R}$. Let (λ, μ) denote the scalar product on $\mathfrak{h}^*_\mathbb{R}$ induced by the Killing form. Given a root α, the corresponding *coroot* is $\alpha^\vee := 2\alpha/(\alpha,\alpha)$. The collection of coroots $\Phi^\vee := \{\alpha^\vee : \alpha \in \Phi\}$ forms the *dual root system.*

The *Weyl group* $W \subset \mathrm{Aut}(\mathfrak{h}^*_\mathbb{R})$ of the Lie group G is generated by the reflections $s_\alpha : \mathfrak{h}^*_\mathbb{R} \to \mathfrak{h}^*_\mathbb{R}$, for $\alpha \in \Phi$, given by $s_\alpha : \lambda \mapsto \lambda - (\lambda, \alpha^\vee)\,\alpha$. In fact, the Weyl group W is generated by *simple reflections* $s_1, \dots, s_r$ corresponding to the simple roots $s_i := s_{\alpha_i}$. An expression of a Weyl group element w as a product of generators $w = s_{i_1} \cdots s_{i_l}$ which has minimal length is called a *reduced decomposition* for w; its length $\ell(w) = l$ is called the *length* of w. The Weyl group contains a unique *longest element* $w_\circ$ with maximal length $\ell(w_\circ) = |\Phi^+|$. For $u, w \in W$, we say that u *covers* w, and write $u \gtrdot w$, if $w = us_\beta$, for some $\beta \in \Phi^+$, and $\ell(u) = \ell(w) + 1$. The transitive closure of the relation $\gtrdot$ is called the *Bruhat order* on W.

The *weight lattice* Λ is given by $\Lambda := \{\lambda \in \mathfrak{h}^*_\mathbb{R} \ : \ (\lambda, \alpha^\vee) \in \mathbb{Z}$ for any $\alpha \in \Phi\}$. The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \dots, \omega_r$, which are defined as the elements of the dual basis to the basis of simple coroots, i.e., $(\omega_i, \alpha_j^\vee) = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by $\Lambda^+ := \{\lambda \in \Lambda :$ $(\lambda, \alpha^\vee) \geq 0$ for any $\alpha \in \Phi^+\}$.

Let $\rho := \omega_1 + \cdots + \omega_r = \frac{1}{2}\sum_{\beta \in \Phi^+} \beta$. The *height* of a coroot $\alpha^\vee \in \Phi^\vee$ is $(\rho, \alpha^\vee) = c_1 + \cdots + c_r$ if $\alpha^\vee = c_1\alpha_1^\vee + \cdots + c_r\alpha_r^\vee$. Since we assumed that Φ is irreducible, there is a unique *highest coroot* $\theta^\vee \in \Phi^\vee$ that has maximal height. The *dual Coxeter number* is $h^\vee := (\rho, \theta^\vee) + 1$.

3. The K-theory of Generalized Flag Varieties

The *generalized flag variety* G/B is a smooth projective variety. It decomposes into a disjoint union of *Schubert cells* $X_w^\circ := BwB/B$ indexed by elements $w \in W$ of the Weyl group. The closures of Schubert cells $X_w := \overline{X_w^\circ}$ are called *Schubert varieties.* Let $\mathcal{O}_w := \mathcal{O}_{X_w}$ be the structure sheaves of Schubert varieties X_w.

The group of characters $X = X(T)$ of the maximal torus T is isomorphic to the weight lattice Λ. Its group algebra $\mathbb{Z}[X] = R(T)$ is the representation ring of T. This is generated by formal exponents $\{x^\lambda \ : \ \lambda \in \Lambda\}$ with multiplication $x^\lambda \cdot x^\mu := x^{\lambda+\mu}$, i.e., $\mathbb{Z}[X] = \mathbb{Z}[x^{\pm\omega_1}, \cdots, x^{\pm\omega_r}]$ is the algebra of Laurent polynomials in r variables. Let $\mathcal{L}_\lambda := G \times_B \mathbb{C}_\lambda$ be the line bundle over G/B associated with the weight λ.

Denote by $K_T(G/B)$ the *Grothendieck ring* of coherent T-equivariant sheaves on G/B. According to Kostant and Kumar [**KK**], the Grothendieck ring $K_T(G/B)$ is a free $\mathbb{Z}[X]$-module. The classes $[\mathcal{O}_w] \in$

$K_T(G/B)$ of the structure sheaves $\mathcal{O}_w$ form a $\mathbb{Z}[X]$-basis of $K_T(G/B)$. The classes $[\mathcal{L}_\lambda] \in K_T(G/B)$ of the line bundles $\mathcal{L}_\lambda$ span the Grothendieck ring (as a $\mathbb{Z}[X]$-module). The product $[\mathcal{L}_\lambda]\cdot[\mathcal{O}_u]$ in the Grothendieck ring $K_T(G/B)$ can be written as a finite sum

$$[\mathcal{L}_\lambda]\cdot[\mathcal{O}_u] = \sum_{w\in W,\, \mu\in\Lambda} c_{u,w}^{\lambda,\mu}\; x^\mu\,[\mathcal{O}_w], \tag{3.1}$$

where $c_{u,w}^{\lambda,\mu}$ are some integer coefficients. It makes sense to call these coefficients K_T*-Chevalley coefficients*; indeed, they are related to the coefficients in Chevalley's formula via applying the *Chern character* map to both sides of (3.1). In this paper, we present an explicit combinatorial formula for $c_{u,w}^{\lambda,\mu}$, see Theorems 5.1 and 6.2. We will see that $c_{u,w}^{\lambda,\mu} = 0$ unless $w \le u$ in the Bruhat order, and that $c_{u,u}^{\lambda,\mu} = \delta_{\lambda,\mu}$.

If λ is a dominant weight, then we will see that all coefficients $c_{u,w}^{\lambda,\mu}$ are nonnegative. In this case, Pittie and Ram [**PR**] showed that $c_{u,w}^{\lambda,\mu}$ count certain L-S paths, cf. also Lakshmibai-Littelmann [**LL**] and Littelmann-Seshadri [**LS2**].

4. Affine Weyl Groups

Let W_{aff} be the *affine Weyl group* for the Langland's dual group $G^\vee$. The affine Weyl group W_{aff} is generated by the affine reflections $s_{\alpha,k} : \mathfrak{h}^*_\mathbb{R} \to \mathfrak{h}^*_\mathbb{R}$, for $\alpha\in\Phi$ and $k\in\mathbb{Z}$, that reflect the space $\mathfrak{h}^*_\mathbb{R}$ with respect to the affine hyperplanes

$$H_{\alpha,k} := \{\lambda \in \mathfrak{h}^*_\mathbb{R} \;:\; (\lambda,\alpha^\vee) = k\}. \tag{4.1}$$

The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_\mathbb{R}$ into open regions, called *alcoves*. The following important property can be found, e.g., in [**Hum**, Chapter 4].

Lemma 4.1. *The affine Weyl group W_{aff} acts simply transitively on the collection of all alcoves.*

The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \{\lambda\in\mathfrak{h}^*_\mathbb{R} \;:\; 0 < (\lambda,\alpha^\vee) < 1 \text{ for all } \alpha\in\Phi^+\}.$$

Lemma 4.1 implies that, for any alcove A, there exists a unique element v_A of the affine Weyl group W_{aff} such that $v_A(A_\circ) = A$. Hence the map $A \mapsto v_A$ is a one-to-one correspondence between alcoves and elements of the affine Weyl group.

Recall that $\theta^\vee \in \Phi^\vee$ is the highest coroot. Let $\theta\in\Phi^+$ be the corresponding root, and let $\alpha_0 := -\theta$. The fundamental alcove $A_\circ$ is, in fact, the simplex given by

$$A_\circ = \{\lambda\in\mathfrak{h}^*_\mathbb{R} \;:\; 0 < (\lambda,\alpha_i^\vee) \text{ for } i = 1,\dots,r, \text{ and } (\lambda,\theta^\vee) < 1\}, \tag{4.2}$$

Lemma 4.1 also implies that the affine Weyl group is generated by the set of reflections $s_0, s_1, \dots, s_k$ with respect to the walls of the fundamental alcove $A_\circ$, where $s_0 := s_{\alpha_0,-1}$ and $s_1,\dots,s_r\in W$ are the simple reflections $s_i = s_{\alpha_i,0}$. As before, a decomposition $v = s_{i_1}\cdots s_{i_l}\in W_{\mathrm{aff}}$ is called *reduced* if it has minimal length; its length $\ell(v) = l$ is called the length of v.

We say that two alcoves A and B are *adjacent* if B is obtained by an affine reflection of A with respect to one of its walls. In other words, two alcoves are adjacent if they are distinct and have a common wall. For a pair of adjacent alcoves, let us write $A \xrightarrow{\beta} B$ if the common wall of A and B is of the form $H_{\beta,k}$ and the root $\beta\in\Phi$ points in the direction from A to B. By definition, all alcoves that are adjacent to the fundamental alcove $A_\circ$ are obtained from $A_\circ$ by the reflections $s_0,\cdots,s_r$, and $A_\circ \xrightarrow{-\alpha_i} s_i(A_\circ)$.

Definition 4.2. An *alcove path* is a sequence of alcoves $(A_0, A_1, \dots, A_l)$ such that A_{j-1} and A_j are adjacent, for $j = 1,\dots,l$. Let us say that an alcove path is *reduced* if it has minimal length l among all alcove paths from A_0 to A_l.

Let $v \mapsto \bar{v}$ be the homomorphism $W_{\text{aff}} \to W$ defined by ignoring the affine translation. In other words, $\bar{s}_{\alpha,k} = s_\alpha \in W$.

The following Lemma, which is essentially well-known, summarizes some properties of decompositions in affine Weyl groups, cf. [**Hum**].

LEMMA 4.3. *Let v be any element of W_{aff}, and let $A = v(A_\circ)$ be the corresponding alcove. Then the decompositions $v = s_{i_1} \cdots s_{i_l}$ of v (reduced or not) as a product of generators in W_{aff} are in one-to-one correspondence with alcove paths $A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_l} A_l$ from the fundamental alcove $A_0 = A_\circ$ to $A_l = A$. This correspondence is explicitly given by $A_j = s_{i_1} \cdots s_{i_j}(A_\circ)$, for $j = 0, \ldots, l$; and the roots $\beta_1, \ldots, \beta_l$ are given by*

$$\beta_1 = \alpha_{i_1},\ \beta_2 = \bar{s}_{i_1}(\alpha_{i_2}),\ \beta_3 = \bar{s}_{i_1}\bar{s}_{i_2}(\alpha_{i_3}), \ldots,\ \beta_l = \bar{s}_{i_1} \cdots \bar{s}_{i_{l-1}}(\alpha_{i_l}). \tag{4.3}$$

Let $r_j \in W_{\text{aff}}$ denote the affine reflection with respect to the common wall of the alcoves A_{j-1} and A_j, for $j = 1, \ldots, l$. Then the affine reflections $r_1, \ldots, r_l$ are given by

$$r_1 = s_{i_1},\ r_2 = s_{i_1}s_{i_2}s_{i_1},\ r_3 = s_{i_1}s_{i_2}s_{i_3}s_{i_2}s_{i_1},\ \ldots,\ r_l = s_{i_1} \cdots s_{i_r} \cdots s_{i_1}. \tag{4.4}$$

We have $\bar{r}_i = s_{\beta_i}$ and $v = s_{i_1} \cdots s_{i_l} = r_l \cdots r_1$.

The affine translations by weights preserve the set of affine hyperplanes $H_{\alpha,k}$, and map alcoves to alcoves. For $\lambda \in \Lambda$, let $A_\lambda = A_\circ + \lambda$ be the alcove obtained by the affine translation of the fundamental alcove $A_\circ$ by the vector λ. Let $v_\lambda = v_{A_\lambda}$ be the corresponding element of W_{aff}, i.e,. v_λ is defined by $v_\lambda(A_\circ) = A_\lambda$. Note that v_λ may not be an affine translation, although it translates the alcove $A_\circ$.

DEFINITION 4.4. Let λ be a weight, and let $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$ be any decomposition, reduced or not, of $v_{-\lambda}$ as a product of generators of W_{aff}. Let $r_1, \ldots, r_l \in W_{\text{aff}}$ be the affine reflections given by (4.4), and let $\beta_1, \ldots, \beta_l$ be the roots given by (4.3). Thus $\bar{r}_i = s_{\beta_i}$. We say that the sequence $(r_1, \ldots, r_l)$ is the λ-*chain of reflections* and the sequence $(\beta_1, \ldots, \beta_l)$ is the λ-*chain of roots* associated with the decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l}$.

Equivalently, a sequence of roots $(\beta_1, \ldots, \beta_l)$ is a λ-chain of roots if there is an alcove path $A_0 \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l$. By Lemma 4.3, the elements of the corresponding λ-chain of reflections are the affine reflections r_j with respect to the common walls of the alcoves A_{j-1} and A_j, for $j = 1, \ldots, l$.

Finally, we say that a λ-chain is *reduced* if it is associated with a reduced decomposition of $v_{-\lambda}$.

5. The K_T-Chevalley Formula

We can formulate our main result as follows.

THEOREM 5.1. *Fix any weight λ. Let $(r_1, \ldots, r_l)$ and $(\beta_1, \ldots, \beta_l)$ be the λ-chain of reflections and the λ-chain of roots associated with a decomposition $v_{-\lambda} = s_{i_1} \cdots s_{i_l} \in W_{\text{aff}}$, which may or may not be reduced. Let $u, w \in W$, and $\mu \in \Lambda$. Then the K_T-Chevalley coefficient $c_{u,w}^{\lambda,\mu}$, i.e., the coefficient of $x^\mu\,[\mathcal{O}_w]$ in the expansion of the product $[\mathcal{L}_\lambda] \cdot [\mathcal{O}_u]$, can be expressed as follows:*

$$c_{u,w}^{\lambda,\mu} = \sum_J (-1)^{n(J)}\,; \tag{5.1}$$

the summation ranges over all subsets $J = \{j_1 < \cdots < j_s\}$ of $\{1, \ldots, l\}$ satisfying the following conditions:

(a) *$u \gtrdot u\,\bar{r}_{j_1} \gtrdot u\,\bar{r}_{j_1}\bar{r}_{j_2} \gtrdot \cdots \gtrdot u\,\bar{r}_{j_1}\bar{r}_{j_2} \cdots \bar{r}_{j_s} = w$ is a saturated decreasing chain from u to w in the Bruhat order on the Weyl group W;*

(b) *$-\mu = u\,r_{j_1} \cdots r_{j_s}(-\lambda)$,*

where $n(J)$ is the number of negative roots in $\{\beta_{j_1}, \ldots, \beta_{j_s}\}$.

If λ is a dominant weight, then $c_{u,w}^{\lambda,\mu}$ equals the number of subsets $J \subseteq \{1, \ldots, l\}$ that satisfy conditions (a) *and* (b) *in Theorem* 5.1.

If λ is an anti-dominant weight, then $(-1)^{\ell(u)-\ell(w)}\, c_{u,w}^{\lambda,\mu}$ equals the number of subsets $J \subseteq \{1,\dots,l\}$ that satisfy conditions (a) *and* (b) *in Theorem* 5.1.

In the next section, we reformulate this Theorem in a compact form and then prove it, using a certain R-matrix. In Sections 7 and 8, we give several examples that illustrate this Theorem.

Given a dominant weight λ, let V_λ denote the finite dimensional irreducible representation of the Lie group G with highest weight λ. For $\lambda \in \Lambda^+$ and $w \in W$, the *Demazure module* $V_{\lambda,w}$ is the B-module that is dual to the space of global sections of the line bundle $\mathcal{L}_\lambda$ on the Schubert variety X_w, i.e., $V_{\lambda,w} = H^0(X_w, \mathcal{L}_\lambda)^*$. The formal characters of these modules, called *Demazure characters*, are given by $ch(V_{\lambda,w}) := \sum_{\mu\in\Lambda} m_{\lambda,w}(\mu)\, e^\mu \in \mathbb{Z}[\Lambda]$, where $m_{\lambda,w}(\mu)$ is the multiplicity of the weight μ in $V_{\lambda,w}$. The characters of irreducible representations of G are special cases, namely $ch(V_\lambda) = ch(V_{\lambda,w_\circ})$. The Demazure characters are given by Demazure's character formula [**Dem**].

Lemma 5.2. (cf. Lakshmibai-Littelmann [**LL**], Littelmann-Seshadri [**LS2**].) *For any $\lambda \in \Lambda^+$ and $u \in W$, the Demazure character $ch(V_{\lambda,u})$ can be expressed in terms of the K_T-Chevalley coefficients as follows: $ch(V_{\lambda,u}) = \sum_{w\in W,\, \mu\in\Lambda} c_{u,w}^{\lambda,\mu}\, e^\mu$.*

Theorem 5.1 implies the following combinatorial model for the Demazure characters $ch(V_{\lambda,u})$ and, in particular, for the characters $ch(V_\lambda)$ of the irreducible representations V_λ of the Lie group G.

Corollary 5.3. *Let λ be a dominant weight, let $u \in W$, and let $v_{-\lambda} = s_{i_1}\cdots s_{i_l} \in W_{\mathrm{aff}}$ be a reduced decomposition of $v_{-\lambda}$. Let $(r_1,\dots,r_l)$ be the corresponding λ-chain of reflections. Then the Demazure character $ch(V_{\lambda,u})$ is equal to the sum*

$$ch(V_{\lambda,u}) = \sum_J e^{-u\, r_{j_1}\cdots r_{j_s}(-\lambda)}$$

over all subsets $J = \{j_1 < \cdots j_s\} \subset \{1,\dots,l\}$ such that

$$u \gtrdot u\,\bar r_{j_1} \gtrdot u\,\bar r_{j_1}\bar r_{j_2} \gtrdot \cdots \gtrdot u\,\bar r_{j_1}\bar r_{j_2}\cdots \bar r_{j_s}$$

is a saturated decreasing chain in the Bruhat order on the Weyl group W.

We can slightly simplify the formula for the characters $ch(V_\lambda) = ch(V_{\lambda,w_\circ})$ of the irreducible representations of G, as follows.

Corollary 5.4. *Consider the setup in Corollary* 5.3. *We have*

$$ch(V_\lambda) = \sum_J e^{-r_{j_1}\cdots r_{j_s}(-\lambda)}\,,$$

where the summation is over all subsets $J = \{j_1 < \cdots j_s\} \subset \{1,\dots,l\}$ such that

$$1 \lessdot \bar r_{j_1} \lessdot \bar r_{j_1}\bar r_{j_2} \lessdot \cdots \lessdot \bar r_{j_1}\bar r_{j_2}\cdots \bar r_{j_s}$$

is a saturated increasing chain in the Bruhat order on the Weyl group W.

In order to make our formula completely combinatorial, we present one particular choice for the λ-chain of reflections, which is illustrated by Example 8.1. The construction depends on the choice of a total order on the simple roots in Φ. For simplicity, assume that λ is dominant. The set $\mathcal{R} = \mathcal{R}_\lambda \subset W_{\mathrm{aff}}$ of affine reflections with respect to the affine hyperplanes $H_{\alpha,k}$ that separate the alcoves $A_\circ$ and $A_{-\lambda}$ is given by

$$\mathcal{R} = \mathcal{R}_\lambda = \bigcup_{\alpha\in\Phi^+} \{s_{\alpha,k} \,:\, 0 \ge k > -(\lambda,\alpha^\vee)\}.$$

Let us choose a path $\pi : [0,1] \to \mathfrak{h}_{\mathbb{R}}^*$ that connects the alcoves $A_\circ$ and $A_{-\lambda}$; then let us totally order the set $\mathcal{R}$ according to the order in which the path π crosses the hyperplanes $H_{\alpha,k}$. If the path is given by $\pi = \pi_\varepsilon : t \mapsto -t\,\lambda + \varepsilon\,\omega_1 + \varepsilon^2\omega_2 + \cdots + \varepsilon^r\omega_r$, where ε is a sufficiently small positive constant, then the corresponding total order on $\mathcal{R}$ can be described as follows. Let $h : \mathcal{R} \to \mathbb{R}^{r+1}$ be the map given by

$$h : s_{\alpha,k} \mapsto (\lambda,\alpha^\vee)^{-1}\,(-k, (\omega_1,\alpha^\vee), \dots, (\omega_r,\alpha^\vee)), \tag{5.2}$$

for any $s_{\alpha,k} \in \mathcal{R}$ with $\alpha \in \Phi^+$. The map h is injective.

PROPOSITION 5.5. *Let $\mathcal{R} = \{r_1 < r_2 < \cdots < r_l\}$ be the total order on the set $\mathcal{R}$ such that $h(r_1) < h(r_2) < \cdots < h(r_l)$ in the lexicographic order on $\mathbb{R}^{r+1}$. Then $(r_1, \ldots, r_l)$ is a reduced λ-chain of reflections.*

6. K_T-Chevalley Formula: Operator Notation

Let us extend the ring of coefficients in $K_T(G/B)$, as follows. Let $\Lambda/h^\vee := \{\lambda/h^\vee \ : \ \lambda \in \Lambda\}$, where $h^\vee = (\rho, \theta^\vee) + 1$ is the dual Coxeter number. Let $\mathbb{Z}[\tilde{X}]$ be the group algebra of $\Lambda/h^\vee$ with formal exponents $x^{\lambda/h^\vee}$, for $\lambda \in \Lambda$. And let $\tilde{K}_T(G/B) := K_T(G/B) \otimes_{\mathbb{Z}[X]} \mathbb{Z}[\tilde{X}]$. For $\alpha \in \Phi^+$, define the $\mathbb{Z}[\tilde{X}]$-linear *Bruhat operators* B_α acting on $\tilde{K}_T(G/B)$ by

$$B_\alpha : [\mathcal{O}_w] \longmapsto \begin{cases} [\mathcal{O}_{ws_\alpha}] & \text{if } \ell(ws_\alpha) = \ell(w) - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Also define $B_\alpha := -B_{-\alpha}$, for negative roots α. The operators B_α move Weyl group elements one step down in the Bruhat order. For a weight λ, define the $\mathbb{Z}[\tilde{X}]$-linear operators X^λ acting on $\tilde{K}_T(G/B)$ by

$$X^\lambda : [\mathcal{O}_w] \mapsto x^{w(\lambda/h^\vee)}[\mathcal{O}_w]. \tag{6.2}$$

Let us define operators R_α by

$$R_\alpha := X^\alpha + X^{(\rho,\alpha^\vee)\,\alpha}\, B_\alpha = X^\rho\,(X^\alpha + B_\alpha)\,X^{-\rho}, \quad \text{for } \alpha \in \Phi. \tag{6.3}$$

The operators R_α generalize the operators considered in [**BFP**]. The following claim can be proved along the lines of [**BFP**].

THEOREM 6.1. *The family of operators R_α, $\alpha \in \Phi$, satisfies the Yang-Baxter equation (in the sense of Cherednik* [**Cher**, Definition 2.1a]*). In other words, $R_{-\alpha} = (R_\alpha)^{-1}$; the operators R_α and R_β commute whenever $(\alpha, \beta) = 0$; if α and β generate a root subsystem of type A_2, then*

$$R_\alpha R_{\alpha+\beta} R_\beta = R_\beta R_{\alpha+\beta} R_\alpha\,;$$

finally, there are similar relations for the other rank 2 root subsystems.

For $\lambda \in \Lambda$, let us define the operator $R^{[\lambda]}$ acting on $\tilde{K}_T(G/B)$ as

$$R^{[\lambda]} = R_{\beta_l} R_{\beta_{l-1}} \cdots R_{\beta_2} R_{\beta_1}, \tag{6.4}$$

where $(\beta_1, \ldots, \beta_l)$ is a λ-chain of roots and the R_α are given by (6.3). Theorem 6.1 implies that the operator $R^{[\lambda]}$ depends only on the weight λ and not on the choice of a λ-chain. The operator $R^{[\lambda]}$ preserves the space $K_T(G/B)$.

We can formulate the equivariant K-theory Chevalley formula using the operator notation, as follows.

THEOREM 6.2. *For any weight λ and any $u \in W$, we have*

$$[\mathcal{L}_\lambda] \cdot [\mathcal{O}_u] = R^{[\lambda]}([\mathcal{O}_u]),$$

i.e., the operator $R^{[\lambda]}$ acts on the space $K_T(G/B)$ as the operator of multiplication by the class $[\mathcal{L}_\lambda]$ of a line bundle.

If λ is a dominant weight, then all roots in a reduced λ-chain are positive; thus the operator $R^{[\lambda]}$ expands as a positive expression in the Bruhat operators B_α, $\alpha \in \Phi^+$, and the operators X^μ. In this case, Theorem 6.2 gives a positive formula for $[\mathcal{L}_\lambda] \cdot [\mathcal{O}_u]$.

7. Examples for Type A

Suppose that $G = SL_n$. Then the root system Φ is of type A_{n-1} and the Weyl group W is the symmetric group S_n. We can identify the space $\mathfrak{h}^*_{\mathbb{R}}$ with the quotient space $V := \mathbb{R}^n/\mathbb{R}(1,\dots,1)$, where $\mathbb{R}(1,\dots,1)$ denotes the subspace in $\mathbb{R}^n$ spanned by the vector $(1,\dots,1)$. The action of the symmetric group S_n on V is obtained from the (left) S_n-action on $\mathbb{R}^n$ by permutation of coordinates. Let $\varepsilon_1,\dots,\varepsilon_n \in V$ be the images of the coordinate vectors in $\mathbb{R}^n$. The root system Φ can be represented as $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \,:\, i \neq j,\ 1 \le i,j \le n\}$. The simple roots are $\alpha_i = \alpha_{i\,i+1}$, for $i = 1,\dots,n-1$. The longest coroot is $\theta^\vee = \alpha^\vee_{1n}$. The fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $i = 1,\dots,n-1$. We have $\rho = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + 2\varepsilon_{n-1} + \varepsilon_n$. The dual Coxeter number is $h^\vee = (\rho,\theta^\vee) + 1 = n$. The weight lattice is $\Lambda = \mathbb{Z}^n/\mathbb{Z}(1,\dots,1)$. We use the notation $[\lambda_1,\dots,\lambda_n]$ for a weight, as the coset of $(\lambda_1,\dots,\lambda_n)$ in $\mathbb{Z}^n$.

Let $Z \subset \Lambda$ be the set Z of central points of alcoves scaled by the factor $h^\vee = n$. The fundamental alcove corresponds to the point ρ in Z. Two alcoves are adjacent $A \xrightarrow{\alpha} B$, $\alpha \in \Phi$, if and only if the corresponding elements of Z are related by $\zeta_B - \zeta_A = \alpha$. In this case, we write $\zeta_A \xrightarrow{\alpha} \zeta_B$. Thus, we have the structure of a directed graph with labeled edges on the set Z. Alcove paths correspond to paths in this graph. The set Z can be explicitly described as

$$Z = \{[\mu_1,\dots,\mu_n] \in \Lambda \,:\, \mu_1,\dots,\mu_n \text{ have distinct residues modulo } n\}.$$

For an element $\mu = [\mu_1,\dots,\mu_n] \in Z$, there exists an edge $\mu \xrightarrow{\alpha_{ij}} (\mu + \alpha_{ij})$ if and only if $\mu_i + 1 \equiv \mu_j \bmod n$. Given a weight λ, the corresponding λ-chains are in one-to-one correspondence with directed paths in the graph Z from ρ to $\rho - n\lambda$.

Example 7.1. Suppose that $n = 4$ and $\lambda = \omega_2 = [1,1,0,0]$. The directed path

$$[4,3,2,1] \xrightarrow{-\alpha_{23}} [4,2,3,1] \xrightarrow{-\alpha_{13}} [3,2,4,1] \xrightarrow{-\alpha_{24}} [3,1,4,2] \xrightarrow{-\alpha_{14}} [2,1,4,3].$$

from $\rho = [4,3,2,1]$ to $\rho - n\,\omega_2 = [0,-1,2,1] = [2,1,4,3]$ produces the ω_2-chain $(\alpha_{23},\alpha_{13},\alpha_{24},\alpha_{14})$.

Example 7.2. For an arbitrary n, we have $\omega_1 = \varepsilon_1 = [1,0,\dots,0]$. The path

$$\begin{aligned}&[n,n-1,\dots,1] \xrightarrow{-\alpha_{12}} [n-1,n,n-2,\dots,1] \xrightarrow{-\alpha_{13}} [n-2,n,n-1,n-3,\dots,1]\\ &\xrightarrow{-\alpha_{14}} [n-3,n,n-1,n-2,n-4,\dots,1] \xrightarrow{-\alpha_{15}} \cdots \xrightarrow{-\alpha_{1n}} [1,n,n-1,\dots,2].\end{aligned}$$

from ρ to $\rho - n\,\omega_1$ gives the ω_1-chain $(\alpha_{12},\alpha_{13},\alpha_{14},\dots,\alpha_{1n})$. In general, for any $k = 1,\dots,n$, we have the ε_k-chain

$$(\alpha_{k\,k+1},\alpha_{k\,k+2},\dots,\alpha_{k\,n},\alpha_{k\,1},\alpha_{k\,2},\dots,\alpha_{k\,k-1}) \tag{7.1}$$

given by the corresponding path from ρ to $\rho - n\varepsilon_k$.

Recall that $v_{-\lambda}$ is the unique element of W_{aff} such that $v_{-\lambda}(A_\circ) = A_{-\lambda}$. Equivalently, we can define $v_{-\lambda}$ in terms of central points of alcoves by the condition $v_{-\lambda}(\rho/h^\vee) = \rho/h^\vee - \lambda$.

Lemma 7.3. *Suppose that Φ is of type A_{n-1}. Then, for $k = 1,\dots,n-1$, the affine Weyl group element $v_{-\omega_k}$ belongs, in fact, to $S_n \subset W_{\mathrm{aff}}$. This permutation is given by*

$$v_{-\omega_k} = \begin{pmatrix} 1 & 2 & \cdots & n-k & n-k+1 & \cdots & n \\ k+1 & k+2 & \cdots & n & 1 & \cdots & k \end{pmatrix} \in S_n \subset W_{\mathrm{aff}}.$$

Let $R_{ij} := R_{\alpha_{ij}}$. Theorem 6.2 implies the following statement.

Corollary 7.4. *For $k = 1,\dots,n$, the operator of multiplication by $[\mathcal{L}_{\varepsilon_k}]$ in the Grothendieck ring $K_T(SL_n/B)$ is given by*

$$R^{[\varepsilon_k]} = R_{k\,k-1}R_{k\,k-2}\cdots R_{k\,1}R_{k\,n}R_{k\,n-1}\cdots R_{k\,k+1}.$$

For $k = 1, \dots, n-1$, the operator of multiplication by the line bundle $[\mathcal{L}_{\omega_k}]$ corresponding to the k-th fundamental weight ω_k is given by

$$R^{[\omega_k]} = R^{[\varepsilon_1]} \cdots R^{[\varepsilon_k]} = \overrightarrow{\prod_{i=1,\dots,k}} \; \overleftarrow{\prod_{j=k+1,\dots,n}} R_{ij}. \tag{7.2}$$

The combinatorial formula for multiplication by $[\mathcal{L}_{\omega_k}]_{x=1}$ in the Grothendieck ring $K(SL_n/B)$ that follows from formula (7.2) was originally found in [**Len**].

Example 7.5. For $n = 3$, Corollary 7.4 says that

$$R^{[\omega_1]} = R_{13}\, R_{12} \quad \text{and} \quad R^{[\omega_2]} = R_{13}\, R_{23}.$$

Example 7.6. Suppose that $n = 3$, $\lambda = \omega_1$, and $u = w_\circ = s_1 s_2 s_1 \in W$. Let us calculate the product $[\mathcal{L}_\lambda] \cdot [\mathcal{O}_u]$ in $K_T(SL_n/B)$ using Theorem 5.1. The ω_1-chain $(\beta_1, \beta_2) = (\alpha_{12}, \alpha_{13})$ is associated with the reduced decomposition $s_1 s_2 = v_{-\omega_1}$. The corresponding ω_1-chain of reflections is $(r_1, r_2) = (s_1, s_1 s_2 s_1) = (s_{\alpha_{12},0}, s_{\alpha_{13},0})$. Three out of four subsequences in (β_1, β_2) correspond to decreasing chains from $w_\circ$: (empty subsequence), (α_{12}), and $(\alpha_{12}, \alpha_{13})$. Thus we have

$$[\mathcal{L}_{\omega_1}] \cdot [\mathcal{O}_{w_\circ}] = x^{-w_\circ(-\omega_1)}[\mathcal{O}_{w_\circ}] + x^{-w_\circ r_1(-\omega_1)}[\mathcal{O}_{s_1 s_2}] + x^{-w_\circ r_1 r_2(-\omega_1)}[\mathcal{O}_{s_2}].$$

We can write this expression as

$$[\mathcal{L}_{[1,0,0]}] \cdot [\mathcal{O}_{w_\circ}] = x^{[0,0,1]}[\mathcal{O}_{w_\circ}] + x^{[0,1,0]}[\mathcal{O}_{s_1 s_2}] + x^{[1,0,0]}[\mathcal{O}_{s_2}].$$

This gives the character of the irreducible representation V_{ω_1}:

$$ch(V_{\omega_1}) = e^{[0,0,1]} + e^{[0,1,0]} + e^{[1,0,0]}.$$

Let us give a less trivial example.

Example 7.7. Suppose $n = 3$ and $\lambda = 2\,\omega_1 + \omega_2 = [3,1,0]$. The path

$$\begin{aligned} [3,2,1] \xrightarrow{-\alpha_{12}} [2,3,1] \xrightarrow{-\alpha_{13}} [1,3,2] \xrightarrow{-\alpha_{23}} [1,2,3] \\ \xrightarrow{-\alpha_{13}} [0,2,4] \xrightarrow{-\alpha_{12}} [-1,3,4] \xrightarrow{-\alpha_{13}} [-2,3,5] \end{aligned}$$

from $\rho = [3,2,1]$ to $\rho - n\lambda = [-2,3,5]$ gives the λ-chain

$$(\beta_1, \dots, \beta_6) = (\alpha_{12},\ \alpha_{13},\ \alpha_{23},\ \alpha_{13},\ \alpha_{12},\ \alpha_{13}),$$

which is associated with the reduced decomposition $v_{-\lambda} = s_1 s_2 s_1 s_0 s_1 s_2$ in the affine Weyl group. We have

$$R^{[\lambda]} = R_{\beta_6} \cdots R_{\beta_1} = R_{13}\, R_{12}\, R_{13}\, R_{23}\, R_{13}\, R_{12} = R^{[\omega_1]}\, R^{[\omega_2]}\, R^{[\omega_1]}.$$

The corresponding λ-chain of reflections is

$$(r_1, \dots, r_6) = (s_{\alpha_{12},0},\ s_{\alpha_{13},0},\ s_{\alpha_{23},0},\ s_{\alpha_{13},-1},\ s_{\alpha_{12},-1},\ s_{\alpha_{13},-2}).$$

Theorem 5.1 says that that the coefficient of $[\mathcal{O}_w]$ in the product $[\mathcal{L}_\lambda] \cdot [\mathcal{O}_u]$ in $K_T(SL_n/B)$ is given by the sum over subsequences in the λ-chain $(\beta_1, \dots, \beta_6)$ that correspond to saturated decreasing chains $u \gtrdot \cdots \gtrdot w$ in the Bruhat order on $W = S_3$.

Suppose that $u = s_2 s_1$. There are five saturated chains in Bruhat order descending from u: (empty chain), $(u \gtrdot u s_{\alpha_{12}} = s_2)$, $(u \gtrdot u s_{\alpha_{13}} = s_1)$, $(u \gtrdot u s_{\alpha_{12}} \gtrdot u s_{\alpha_{12}} s_{\alpha_{23}} = 1)$, $(u \gtrdot u s_{\alpha_{13}} \gtrdot u s_{\alpha_{13}} s_{\alpha_{12}} = 1)$. Thus the expansion of $[\mathcal{L}_\lambda] \cdot [\mathcal{O}_u]$ is given by the sum over the following subsequences in the λ-chain $(\beta_1, \dots, \beta_6)$:

(empty subsequence), (α_{12}), $(\alpha_{13}), (\alpha_{12}, \alpha_{23})$, $(\alpha_{13}, \alpha_{12})$.

The sequence $(\beta_1, \ldots, \beta_6)$ contains one empty subsequence, two subsequences of the form (α_{12}), three subsequences of the form (α_{13}), one subsequence of the form $(\alpha_{12}, \alpha_{23})$, and two subsequence of the form $(\alpha_{13}, \alpha_{12})$. Hence, we have

$$\begin{aligned}[\mathcal{L}_\lambda] \cdot [\mathcal{O}_{s_2 s_1}] &= x^{-u(-\lambda)}\,[\mathcal{O}_{s_2 s_1}] + \left(x^{-ur_1(-\lambda)} + x^{-ur_5(-\lambda)}\right)\,[\mathcal{O}_{s_2}] + \\ &+\ \left(x^{-ur_2(-\lambda)} + x^{-ur_4(-\lambda)} + x^{-ur_6(-\lambda)}\right)\,[\mathcal{O}_{s_1}] + \\ &+ x^{-ur_1 r_3(-\lambda)}\,[\mathcal{O}_1] + \left(x^{-ur_2 r_5(-\lambda)} + x^{-ur_4 r_5(-\lambda)}\right)\,[\mathcal{O}_1].\end{aligned}$$

We can explicitly write this expression as

$$\begin{aligned}[\mathcal{L}_{[3,1,0]}] \cdot [\mathcal{O}_{s_2 s_1}] &= x^{[1,0,3]}\,[\mathcal{O}_{s_2 s_1}] + \left(x^{[3,0,1]} + x^{[2,0,2]}\right)\,[\mathcal{O}_{s_2}] + \\ &+\ \left(x^{[1,3,0]} + x^{[1,2,1]} + x^{[1,1,2]}\right)\,[\mathcal{O}_{s_1}] + x^{[3,1,0]}\,[\mathcal{O}_1] + \left(x^{[2,2,0]} + x^{[2,1,1]}\right)\,[\mathcal{O}_1].\end{aligned}$$

The Demazure character $ch(V_{\lambda,u})$ is obtained from the right-hand side of this expression by replacing each term $x^\mu[\mathcal{O}_w]$ with e^μ:

$$\begin{aligned}&ch(V_{[3,1,0],s_2 s_1}) = \\ &\quad e^{[1,0,3]} + e^{[3,0,1]} + e^{[2,0,2]} + e^{[1,3,0]} + e^{[1,2,1]} + e^{[1,1,2]} + e^{[3,1,0]} + e^{[2,2,0]} + e^{[2,1,1]}.\end{aligned}$$

8. Examples for Other Types

For root systems of other types, we can use the explicit construction of the λ-chain of reflections $(r_1, \ldots, r_l)$ given by Proposition 5.5.

Example 8.1. Suppose that the root system Φ is of type G_2. Let us find λ-chains for $\lambda = \omega_1$ and $\lambda = \omega_2$ using Proposition 5.5. The positive roots are $\gamma_1 = \alpha_1,\ \gamma_2 = 3\alpha_1 + \alpha_2,\ \gamma_3 = 2\alpha_1 + \alpha_2,\ \gamma_4 = 3\alpha_1 + 2\alpha_2,\ \gamma_5 = \alpha_1 + \alpha_2,\ \gamma_6 = \alpha_2$. The corresponding coroots are $\gamma_1^\vee = \alpha_1^\vee,\ \gamma_2^\vee = \alpha_1^\vee + \alpha_2^\vee,\ \gamma_3^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee,\ \gamma_4^\vee = \alpha_1^\vee + 2\alpha_2^\vee,\ \gamma_5^\vee = \alpha_1^\vee + 3\alpha_2^\vee,\ \gamma_6^\vee = \alpha_2^\vee$.

Suppose that $\lambda = \omega_1$. The set $\mathcal{R}_{\omega_1}$ of affine reflections with respect to the hyperplanes separating the alcoves $A_\circ$ and $A_{-\omega_1}$ is

$$\mathcal{R}_{\omega_1} = \{s_{\gamma_1,0},\ s_{\gamma_2,0},\ s_{\gamma_3,0},\ s_{\gamma_3,-1},\ s_{\gamma_4,0},\ s_{\gamma_5,0}\}.$$

The map $h : \mathcal{R}_{\omega_1} \to \mathbb{R}^{r+1}$ given by (5.2) sends these affine reflections to the vectors

$$(0,1,0),\ (0,1,1),\ (0,1,\tfrac{3}{2}),\ (\tfrac{1}{2},1,\tfrac{3}{2}),\ (0,1,2),\ (0,1,3),$$

respectively. The lexicographic order on vectors in $\mathbb{R}^3$ induces the following total order on the set $\mathcal{R}_{\omega_1}$:

$$s_{\gamma_1,0} < s_{\gamma_2,0} < s_{\gamma_3,0} < s_{\gamma_4,0} < s_{\gamma_5,0} < s_{\gamma_3,-1}\,.$$

Suppose now that $\lambda = \omega_2$. The set $\mathcal{R}_{\omega_2}$ of affine reflections with respect to the hyperplanes separating $A_\circ$ and $A_{-\omega_2}$ is

$$\mathcal{R}_{\omega_2} = \{s_{\gamma_2,0},\ s_{\gamma_3,0},\ s_{\gamma_3,-1},\ s_{\gamma_3,-2},\ s_{\gamma_4,0},\ s_{\gamma_4,-1},\ s_{\gamma_5,0},\ s_{\gamma_5,-1},\ s_{\gamma_5,-2}, s_{\gamma_6,0}\}.$$

The map $h : \mathcal{R}_{\omega_2} \to \mathbb{R}^{r+1}$ sends these affine reflections to the vectors

$$\begin{aligned}&(0,1,1),\ (0,\tfrac{2}{3},1),\ (\tfrac{1}{3},\tfrac{2}{3},1),\ (\tfrac{2}{3},\tfrac{2}{3},1),\ (0,\tfrac{1}{2},1),\ (\tfrac{1}{2},\tfrac{1}{2},1),\\ &\qquad\qquad\qquad (0,\tfrac{1}{3},1),\ (\tfrac{1}{3},\tfrac{1}{3},1),\ (\tfrac{2}{3},\tfrac{1}{3},1),\ (0,0,1),\end{aligned}$$

respectively. The lexicographic order on vectors in $\mathbb{R}^3$ induces the following total order on $\mathcal{R}_{\omega_2}$:

$$s_{\gamma_6,0} < s_{\gamma_5,0} < s_{\gamma_4,0} < s_{\gamma_3,0} < s_{\gamma_2,0} < s_{\gamma_5,-1} < s_{\gamma_3,-1} < s_{\gamma_4,-1} < s_{\gamma_5,-2} < s_{\gamma_3,-2}\,.$$

The total orders on $\mathcal{R}_{\omega_1}$ and $\mathcal{R}_{\omega_2}$ correspond to the ω_1-chain $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_3)$ and the ω_2-chain $(\gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_2, \gamma_5, \gamma_3, \gamma_4, \gamma_5, \gamma_3)$. Thus the operators of multiplication by the classes $[\mathcal{L}_{\omega_1}]$ and $[\mathcal{L}_{\omega_2}]$ in $K_T(G/B)$ are given by

$$R^{[\omega_1]} = R_{\gamma_3}\, R_{\gamma_5}\, R_{\gamma_4}\, R_{\gamma_3}\, R_{\gamma_2}\, R_{\gamma_1},$$
$$R^{[\omega_2]} = R_{\gamma_3}\, R_{\gamma_5}\, R_{\gamma_4}\, R_{\gamma_3}\, R_{\gamma_5}\, R_{\gamma_2}\, R_{\gamma_3}\, R_{\gamma_4}\, R_{\gamma_5}\, R_{\gamma_6}.$$

By Lemma 7.3, we have $v_{-\omega_k} \in W$ for all fundamental weights ω_k in type A. In fact, similar a phenomenon occurs for minuscule fundamental weights in other types as well. The last two examples concern minuscule weights in types B and C. Recall that the element $v_{-\lambda}$ is defined by the condition $v_{-\lambda}(\rho/h^\vee) = \rho/h^\vee - \lambda$.

Example 8.2. Suppose that Φ is a root system of type C_r. This can be embedded into $\mathbb{R}^r$ as follows: $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm 2\varepsilon_i \ :\ i \neq j\}$, where $\varepsilon_1, \dots, \varepsilon_r$ are the coordinate vectors in $\mathbb{R}^r$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\dots \alpha_{r-1} = \varepsilon_{r-1} - \varepsilon_r$, $\alpha_r = 2\varepsilon_r$. The Weyl group W is the semidirect product of S_r and $(\mathbb{Z}/2\mathbb{Z})^r$. It acts on $\mathbb{R}^r$ by permuting the coordinates and changing their signs. The fundamental weights are $\omega_k = \varepsilon_1 + \cdots + \varepsilon_k$, $k = 1, \dots, r$; and $\rho = (r, \dots, 1) \in \mathbb{R}^r$. The dual Coxeter number is $h^\vee = (\rho, \theta^\vee) + 1 = 2r$.

Suppose that $\lambda = \omega_1$. Then $\rho - h^\vee\omega_1 = (-r, r-1, r-2, \dots, 1) \in \mathbb{R}^r$. This weight is obtained from ρ by applying the Weyl group element $s_{2\varepsilon_1}$ that changes the sign of the first coordinate. Thus $v_{-\omega_1} = s_{2\varepsilon_1} \in W \subset W_{\mathrm{aff}}$. The only reduced decomposition of this element is $v_{-\omega_1} = s_1 \cdots s_{r-1}\, s_r\, s_{r-1} \cdots s_1$, so $\ell(v_{-\omega_1}) = 2r-1$. This reduced decomposition corresponds to the ω_1-chain

$$(\alpha_1,\ s_1(\alpha_2),\ s_1 s_2(\alpha_3),\ \dots,\ s_1 \dots s_{r-1}(\alpha_r),\ \dots,\ s_1 \dots s_r \dots s_2(\alpha_1)) =$$
$$(\varepsilon_1 - \varepsilon_2,\ \varepsilon_1 - \varepsilon_3,\ \cdots,\ \varepsilon_1 - \varepsilon_r,\ 2\varepsilon_1,\ \varepsilon_1 + \varepsilon_r,\ \cdots,\ \varepsilon_1 + \varepsilon_3,\ \varepsilon_1 + \varepsilon_2),$$

cf. Definition 4.4. The operator $R^{[\omega_1]}$ is given by

$$R^{[\omega_1]} = R_{\varepsilon_1+\varepsilon_2} R_{\varepsilon_1+\varepsilon_3} \cdots R_{\varepsilon_1+\varepsilon_r} R_{2\varepsilon_1} R_{\varepsilon_1-\varepsilon_r} \cdots R_{\varepsilon_1-\varepsilon_3} R_{\varepsilon_1-\varepsilon_2}\,.$$

Example 8.3. Suppose that the root system Φ is of type B_r. This can be embedded into $\mathbb{R}^r$ as follows: $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i \ :\ i \neq j\}$, where $\varepsilon_1, \dots, \varepsilon_r$ are the coordinate vectors in $\mathbb{R}^r$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, $\dots \alpha_{r-1} = \varepsilon_{r-1} - \varepsilon_r$, $\alpha_r = \varepsilon_r$. The Weyl group W and its action on $\mathbb{R}^r$ are the same as in type C_r. The fundamental weights are $\omega_k = \varepsilon_1 + \cdots + \varepsilon_k$, $k = 1, \dots, r-1$, and $\omega_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$; on the other hand, $\rho = (r - \frac{1}{2}, \dots, 1 - \frac{1}{2}) \in \mathbb{R}^r$. The dual Coxeter number is $h^\vee = (\rho, \theta^\vee) + 1 = 2r$.

Suppose that $\lambda = \omega_r$ is the last fundamental weight. Then $\rho - h^\vee\omega_r = (-\frac{1}{2}, -1-\frac{1}{2}, -2-\frac{1}{2}, \dots, -r+\frac{1}{2}) \in \mathbb{R}^r$. This weight is obtained from ρ by applying the Weyl group element $v_{-\omega_r} \in W \subset W_{\mathrm{aff}}$ that reverses the order of all coordinates and changes their signs. The element $v_{-\omega_r} \in W$ has length $\ell(v_{-\omega_r}) = r(r+1)/2$. One of the reduced decompositions for this element is given by

$$v_{-\omega_r} = (s_r)(s_{r-1}\, s_r)(s_{r-2}\, s_{r-1}\, s_r) \cdots (s_2 \cdots s_r)(s_1 \cdots s_r).$$

The associated ω_r-chain is $(\alpha_r,\ s_r(\alpha_{r-1}),\ s_r s_{r-1}(\alpha_r),\ s_r s_{r-1} s_r(\alpha_{r-2}),\ \dots)$. We can explicitly find the roots in this ω_r-chain and write the operator $R^{[\omega_r]}$ as

$$R^{[\omega_r]} = (R_{\varepsilon_1}\, R_{\varepsilon_1+\varepsilon_2}\, R_{\varepsilon_1+\varepsilon_3} \cdots R_{\varepsilon_1+\varepsilon_r})(R_{\varepsilon_2}\, R_{\varepsilon_2+\varepsilon_3}\, R_{\varepsilon_2+\varepsilon_4} \cdots R_{\varepsilon_2+\varepsilon_r}) \cdots$$
$$\cdots (R_{\varepsilon_{r-2}}\, R_{\varepsilon_{r-2}+\varepsilon_{r-1}}\, R_{\varepsilon_{r-2}+\varepsilon_r})(R_{\varepsilon_{r-1}}\, R_{\varepsilon_{r-1}+\varepsilon_r})(R_{\varepsilon_r}).$$

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