COMBINATORICS OF HYPERGEOMETRIC FUNCTIONS ASSOCIATED WITH POSITIVE ROOTS

Israel M. Gelfand

Department of Mathematics
Rutgers University
New Brunswick, NJ 08903, U.S.A.
E-mail: igelfand@math.rutgers.edu

Mark I. Graev

Department of Mathematics
Research Institute for System Studies RAS
23 Avtozavodskaya St, Moscow 109280, Russia
E-mail: Graev@systud.msk.su

Alexander Postnikov

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139, U.S.A.
E-mail: apost@math.mit.edu

January 18, 1995

Abstract. In this paper we study the hypergeometric system on unipotent matrices. This system gives a holonomic D-module. We find the number of independent solutions of this system at a generic point. This number is equal to the famous Catalan number. An explicit basis of Γ-series in solution space of this system is constructed in the paper. We also consider restriction of this system to certain strata. We introduce several combinatorial constructions with trees, polyhedra, and triangulations related to this subject.

Contents

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX

1. General Hypergeometric Systems

In this paper we use the following notation: $[a,b] := \{a, a{+}1, \dots, b\}$ and $[n] := [1,n]$.

Recall several definitions and facts from the theory of general hypergeometric functions (see [GGZ, GZK, GGR2]).

Consider the following action of the complex n-dimensional torus $T = (\mathbb{C}^*)^n$ with coordinates $t = (t_1, t_2, \dots, t_n)$ on the space $\mathbb{C}^N$

$$x = (x_1, x_2, \dots, x_N) \longmapsto x \cdot t = (x_1 t^{a_1}, \dots, x_N t^{a_N}), \tag{1.1}$$

where $a_j = (a_{1j}, \dots, a_{nj}) \in \mathbb{Z}^n$, $j = 1, 2, \dots, N$ and t^{a_j} denotes $t_1^{a_{1j}} \dots t_n^{a_{nj}}$.

Definition 1.1. The *General Hypergeometric System* associated with action of torus (1.1) is the following system of differential equations on $\mathbb{C}^N$

$$\sum_{j=1}^{N} a_{ij} x_j \frac{\partial f}{\partial x_j} = \alpha_i f, \qquad i = 1, 2, \dots, n; \tag{1.2}$$

$$\prod_{j\colon l_j > 0} \left(\frac{\partial}{\partial x_j}\right)^{l_j} f = \prod_{j\colon l_j < 0} \left(\frac{\partial}{\partial x_j}\right)^{-l_j} f, \tag{1.3}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n$ and $l = (l_1, l_2, \dots, l_N)$ ranges over the lattice L of integer vectors such that $l_1 a_1 + l_2 a_2 + \dots + l_N a_N = 0$.

Solutions of the system (1.2), (1.3) are called *hypergeometric functions* on $\mathbb{C}^N$ associated with action of torus (1.1). The numbers α_i are called *exponents.*

Remark 1.2. Equations (1.2) are equivalent to the following homogeneous conditions

$$f(x \cdot t) = t^{\alpha} f(x), \tag{1.4}$$

where $t = (t_1, t_2, \dots, t_n) \in T$ and $t^{\alpha} = {t_1}^{\alpha_1} {t_2}^{\alpha_2} \dots {t_n}^{\alpha_n}$.

Remark 1.3. For generic α system (1.2), (1.3) is equivalent to the subsystem, where L ranges over any set of generators for the lattice L.

By A denote the set of integer vectors $a_1, a_2, \dots, a_N$. Let H_A be the sublattice in $\mathbb{Z}^n$ generated by $a_1, a_2, \dots, a_N$ and $m = \dim H_A$ be the dimension of H_A. Let P_A denote the convex hull of the origin 0 and $a_1, a_2, \dots, a_N$. Then P_A is a polyhedron with vertices in the lattice H_A.

Let Vol_{H_A} be the form of volume on the space $H_A \otimes_{\mathbb{Z}} \mathbb{R}$ such that volume of the identity cube is equal to 1. The volume of a polyhedron with vertices in the lattice H_A times $m!$ is an integer number. In particular, $m! \,\mathrm{Vol}_{H_A} P_A$ is integer.

Theorem 1.4. *The general hypergeometric system (1.2), (1.3) gives a holonomic D-module. The number of linearly independent solutions of this system in a neighborhood of a generic point is equal to* $m! \,\mathrm{Vol}_{H_A} P_A$.

If there exist an integer covector h such that

$$h(a_j) = 1 \qquad \text{for all} \qquad j = 1, 2, \dots, N \tag{1.5}$$

then we call the corresponding system (1.2), (1.3) *flat* or *nonconfluent.*

Theorem 1.4 in nonconfluent case was proved in [GZK]. Very close results were found by Adolphson in [Ad], but his technique is quite different from ours.

In this paper we study one special case of systems (1.2), (1.3) when condition (1.5) does not hold. We define these systems in the following section.

2. Hypergeometric System on Unipotent Matrices

Let $R \subset \mathbb{Z}^n$ be a *root system* and $R^+ \subset R$ be the set of *positive roots* (see [Bo]). Then we can define the hypergeometric system (1.2), (1.3) associated with the set of integer vectors $A = R^+$.

We consider the case of the root system A_n in more details.

Let $\epsilon_0, \epsilon_1, \dots, \epsilon_n$ be the standard basis in the lattice $\mathbb{Z}^{n+1}$. The root system A_n is the set of all vectors (roots) $e_{ij} = \epsilon_i - \epsilon_j$, $i \neq j$. Let $A = A_n^+$ be the set of all positive roots $A = \{e_{ij} \in A_n : 0{\le}i{<}j{\le}\, n\}$.

It is clear that positive roots generate the n-dimensional lattice $H_A \simeq \mathbb{Z}^n$ of all vectors $v = v_0\epsilon_0 + v_1\epsilon_1 + \dots + v_n\epsilon_n$, $v_i \in \mathbb{Z}$ such that $v_0 + v_1 + \dots + v_n = 0$.

By Z_n denote the group of unipotent matrices of order $n{+}1$, i.e. the group of upper triangular matrices $z = (z_{ij})$, $0{\le}i{\le}j{\le}n$ with 1's on the diagonal $z_{ii} = 1$.

The n-dimensional torus T presented as the group of diagonal matrices $t = \operatorname{diag}(t_0, t_1, \dots, t_n)$, $t_0 \cdot t_1 \dots t_n = 1$ acts on Z_n by conjugation $z \in Z_n \to tzt^{-1}$, or in coordinates

$$z = \{z_{ij}\} \longmapsto \{z_{ij} t_i t_j^{-1}\}. \tag{2.1}$$

Clearly, action of torus (1.1) associated with the set of vectors $A = A_n^+$ is the same as action (2.1). Here $N = \binom{n+1}{2}$ and z_{ij}, $0{\le}i{<}j{\le}n$ are coordinates in $\mathbb{C}^N$.

The main object of this paper is the hypergeometric system associated with action (2.1). Write down this system explicitly.

Definition 2.1. The *Hypergeometric System on the Group of Unipotent Matrices* is the following system of differential equation on the space $Z_n \simeq \mathbb{C}^N$ with coordinates z_{ij}, $0{\le}i{<}j{\le}n$

$$-\sum_{i=0}^{j-1} z_{ij}\,\frac{\partial f}{\partial z_{ij}} + \sum_{k=j+1}^{n} z_{jk}\,\frac{\partial f}{\partial z_{ij}} = \alpha_j f, \qquad j = 0, 1, \dots, n; \tag{2.2}$$

$$\frac{\partial f}{\partial z_{ik}} = \frac{\partial^2 f}{\partial z_{ij}\,\partial z_{jk}}, \qquad 0{\le}i{<}j{<}k{\le}n, \tag{2.3}$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{C}^{n+1}$ is a vector such that $\sum \alpha_j = 0$.

Solutions of system (2.2), (2.3) are called *hypergeometric functions on the group of unipotent matrices.*

In order to prove that system (2.2), (2.3) is a special case of the system (1.2), (1.3) we need the following simple lemma.

Lemma 2.2. *It follows from equations (2.3) that*

$$\prod_{(i,j)\colon l_{ij}>0} \left(\frac{\partial}{\partial z_{ij}}\right)^{l_{ij}} f = \prod_{(i,j)\colon l_{ij}<0} \left(\frac{\partial}{\partial z_{ij}}\right)^{-l_{ij}} f, \tag{2.4}$$

for all $l = (l_{ij})$, $0{\le}i{<}j{\le}n$, $l_{ij} \in \mathbb{Z}$ *such that* $\sum_i l_{ij} - \sum_k l_{jk} = 0$, $j = 0, 1, \ldots, n$.

Proof. It follows from (2.3) that

$$\frac{\partial f}{\partial z_{ij}} = \frac{\partial^{j-i} f}{\partial z_{ii+1}\, \partial z_{i+1i+2} \ldots \partial z_{j-1j}}$$

Now change in (2.4) all occurrences of $\frac{\partial}{\partial z_{ij}}$ to $\frac{\partial^{j-i}}{\partial z_{ii+1}\, \partial z_{i+1i+2} \ldots \partial z_{j-1j}}$. We get the same expressions in LHS and in RHS.

Let $P_n = P_{A_n^+}$ be the convex hull of the origin 0 and of e_{ij}, $0{\le}i{<}j{\le}n$. The first part of the following theorem is a special case of Theorem 1.4.

Theorem 2.3.

1. *The hypergeometric system (2.2), (2.3) gives a holonomic D-module. The number of linearly independent solutions of this system in a neighborhood of a generic point is equal to* $n!\,\mathrm{Vol}\,P_n$.
2. $n!\,\mathrm{Vol}\,P_n$ *is equal to the Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

3. Integral Expression for Hypergeometric Functions

In this section we present an integral expression for hypergeometric functions on unipotent matrices (see [GG1]).

Consider the following integral

$$f(z) = \int_C \exp\left(\sum z_{ij} t_i t_j^{-1}\right) t^{-\alpha} \frac{dt}{t}, \tag{3.1}$$

where the sum in exponent is over $0{\le}i{<}j{\le}n$; t is a point of torus $T = \{(t_0, \ldots, t_n) : t_0 \cdot \ldots \cdot t_n = 1\} \simeq (\mathbb{C}^*)^n$; $t^{-\alpha}\, dt/t = t_1^{-\alpha_1} \ldots t_n^{-\alpha_n}\, dt_1/t_1 \ldots dt_n/t_n$; and C is a real n-dimensional cycle in $2n$-dimensional space T.

Theorem 3.1. *The function $f(z)$ given by integral (3.1) is a solution of the hypergeometric system (2.2), (2.3).*

4. Γ-series and Admissible Bases

In this section we construct an explicit basis in the solution space of system (1.2), (1.3). In case of nonconfluent systems this construction was given in [GZK]. In this section we basically follow [GZK].

Recall that $A = \{a_1, a_2, \ldots, a_N\}$, where $a_j \in \mathbb{Z}^n$. Without loss of generality we can assume that vectors a_j generate the lattice $\mathbb{Z}^n$, i.e. $H_A = \mathbb{Z}^n$.

Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_N) \in \mathbb{C}^N$. Consider the following formal series

$$\Phi_\gamma(x) = \sum_{l \in L} \frac{x^{\gamma+l}}{\prod_{j=1}^N \Gamma(\gamma_j + l_j + 1)}, \tag{4.1}$$

where $x = (x_1, x_2, \ldots, x_N)$, L is the lattice such as in Definition 1.1, and $x^{\gamma+l} = \prod_{j=1}^N x_j^{\gamma_j + l_j}$.

Lemma 4.1. *The series $\Phi_\gamma(x)$ formally satisfies system (1.2), (1.3) with $\alpha = \sum_j \gamma_j a_j$.*

For a fixed vector of exponents $\alpha = (\alpha_1, \dots, \alpha_n)$ the vector $\gamma = (\gamma_1, \dots, \gamma_N)$ ranges over the affine $(N-n)$-dimensional plane $\Pi(\alpha) = \{(\gamma_1, \dots, \gamma_N) : \sum_j \gamma_j a_j = \alpha\}$. In this section we construct several vectors γ such that all series $\Phi_\gamma(x)$ converge in certain neighborhood and form a basis in the space of solutions of system (1.2), (1.3) in this neighborhood.

A subset $\mathcal{I} \in [N]$ is called a *base* if vectors a_j, $j \in \mathcal{I}$ form a basis of the linear space $H_A \otimes \mathbb{R}$. So we get a *matroid* on the set $[N]$. Let $\Delta_\mathcal{I}$ be the n-dimensional simplex with vertices 0 and a_j, $j \in \mathcal{I}$.

Let $\mathcal{I}$ be a base. By $\Pi(\alpha, \mathcal{I})$ denote the set of $\gamma \in \Pi(\alpha)$ such that $\gamma_j \in \mathbb{Z}$ for $j \notin \mathcal{I}$. It is clear that for every $l \in L$ (see Definition 1.1) $\Phi_\gamma(x) = \Phi_{\gamma+l}$.

The following lemma was proven in [GZK].

Lemma 4.2. *Let $\mathcal{I}$ be a base. Then $|\Pi(\alpha,\mathcal{I})/L| = n!\operatorname{Vol}(\Delta_\mathcal{I})$.*

Definition 4.3. We call a base $\mathcal{I} \in [N]$ *admissible* if the $(n-1)$-dimensional simplex with vertices a_j, $j \in \mathcal{I}$ belongs to the boundary ∂P_A of the polyhedron P_A. In this case the simplex $\Delta_\mathcal{I}$ is also called *admissible.*

Remark 4.4. If vectors a_j satisfy condition (1.5) then all bases are admissible.

Let $B = \{b_1, b_2, \dots, b_{N-n}\}$ be a $\mathbb{Z}$-basis in the lattice L. We say that a base $\mathcal{I}$ is *compatible* with a basis B if whenever $l = (l_1, \dots, l_N) \in L$ such that $l_j \geq 0$ for $j \notin \mathcal{I}$ then l can be expressed as $l = \sum \lambda_k b_k$, where all $\lambda_k \geq 0$. Clearly, the set $\Pi_B(\alpha, \mathcal{I}) = \{\gamma \in \Pi(\alpha, \mathcal{I}) : \gamma = \sum \lambda_k b_k,\ \text{where } 0{\leq}\lambda_k{<}1\}$ is a set of representatives in $\Pi(\alpha, \mathcal{I})/L$.

Let $y_k = x^{b_k}$, $k = 1, 2 \dots, N-n$.

Proposition 4.5. *Let an admissible base $\mathcal{I}$ be compatible with a basis B. Then for all $\gamma \in \Pi_B(\alpha, \mathcal{I})$ the series $\Phi_\gamma(x)$ is of the form $\Phi_\gamma(x) = x^\gamma \sum_m c(m) y^m$, where the sum is over $m = (m_1, \dots, m_{N-n})$, $m_k \geq 0$. The series $\sum c(m) y^m$ converges for sufficiently small $|y_k|$.*

Proof. Let $b_k = (b_{k1}, \dots, b_{kN}) \in L$, $k = 1, \dots, N{-}n$. By definition, $\Phi_\gamma(x) = x^\gamma \sum_m c(m) y^m$, where $c(m) = \prod_j \Gamma(\gamma_j + \sum_k m_k b_{kj} + 1)^{-1}$, $m = (m_1, \dots, m_{N-n}) \in \mathbb{Z}^{N-n}$. Let $\gamma \in \Pi_B(\alpha, \mathcal{I})$. Then $\gamma_j + \sum_k m_k b_{kj} + 1 \in \mathbb{Z}$, for $j \notin \mathcal{I}$. Hence, if $c(m) \neq 0$ then $\gamma_j + \sum_k m_k b_{kj} + 1 \geq 0$, $j \notin \mathcal{I}$. Since $\mathcal{I}$ is compatible with B, we can deduce that $c(m) \neq 0$ only if $m_k \geq 0$, $k = 1, \dots, N{-}n$ (see details in [GZK]). Convergence of the series $\sum c(m) y^m$ follows from the next lemma.

Lemma 4.6. *Let $c(m) = \prod_j \Gamma(\mu_j(m) + \gamma_j + 1)^{-1}$, $m = (m_1, \dots, m_r)$, $m_k \geq 0$, where μ_j are linear functions of m such that $\sum \mu_j(m) = s_1 m_1 + \dots + s_r m_r$, $s_k \geq 0$. Then $|c(m)| \leq R\, c_1^{m_1} \dots c_r^{m_r}$ for some positive constants $R, c_1, \dots, c_r$.*

It is not difficult to prove this Lemma using Stiltjes formula.

Thus, by Proposition 4.5 for every admissible base I we have $n!\operatorname{Vol}(\Delta_\mathcal{I})$ series $\Phi_\gamma(x)$, $\gamma \in \Pi_B(\alpha, \mathcal{I})$ with nonempty common convergence domain.

Remark 4.7. It can be shown that if $\gamma \in \Pi(\alpha, \mathcal{I})$, where $\mathcal{I}$ is not admissible, then $\Phi_\gamma(x)$ diverges.

Recall that P_A is the convex hull of 0 and a_j, $j = 1, 2, \dots, N$.

Definition 4.8. The set of bases Θ is called a *local triangulation* of P_A if

1. $\cup_{\mathcal{I}\in\Theta}\Delta_{\mathcal{I}} = P_A$;
2. $\Delta_{\mathcal{I}_1}\cap\Delta_{\mathcal{I}_2}$ is the common face of $\Delta_{\mathcal{I}_1}$ and $\Delta_{\mathcal{I}_2}$ for all $\mathcal{I}_1,\mathcal{I}_2\in\Theta$.

We call such triangulation Θ local because all simplices $\Delta_{\mathcal{I}}$, $\mathcal{I}\in\Theta$ contain the origin 0.

Remark 4.9. Note that if Θ is a local triangulation then all bases $\mathcal{I}\in\Theta$ are admissible

Definition 4.10. A local triangulation Θ is called *coherent* if there exist a piecewise linear function ϕ on P_A such that ϕ is linear on simplices $\Delta_{\mathcal{I}}$, $\mathcal{I}\in\Theta$ and ϕ is strictly convex on P_A.

Lemma 4.11. *There exists a coherent local triangulation of* P_A.

Lemma 4.12. *Let* Θ *be a coherent local triangulation of* P_A. *Then there exist a basis* B *of* H_A *such that* B *is compatible with every base* $\mathcal{I}$ *in* Θ.

Theorem 4.13. *Let* Θ *be a coherent local triangulation of* P_A*; and* $B=\{b_1,b_2,\dots\dots,b_{N-n}\}$ *a basis such as in Lemma 4.12. Let* $y_k = x^{b_k}$. *Then for every* $\gamma\in\Pi_B(\alpha,\mathcal{I})$, $\mathcal{I}\in\Theta$ *the series* $\Phi_\gamma(x)$ *is equal* x^γ *times a series of variables* y_k*, which converges for sufficiently small* $|y_k|$. *If exponents* $\alpha_1,\alpha_2,\dots,\alpha_n$ *are generic then all these series* $\Phi_\gamma(x)$ *are linearly independent.*

Hence, for generic $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ we constructed $n!\operatorname{Vol}(P_A)$ independent solutions of system (1.2), (1.3), which converge in common domain. Therefore, by Theorem 1.4, these series form a basis in the space of solutions of system (1.2), (1.3).

5. ADMISSIBLE TREES

In this section we describe admissible bases in the case of the hypergeometric system (2.2), (2.3).

It is well known that a subset $\mathcal{I}\subset\{(i,j):0\le i<j\le n\}$ is a base in the set of positive roots $A=A_n^+$ if and only if $\mathcal{I}$ is the set of edges of a tree $T_{\mathcal{I}}$ on $[0,n]$.

Definition 5.1. A tree T on the set $[0,n]$ is called *admissible* if there are no $0\le i<j<k\le n$ such that both (i,j) and (j,k) are edges of T.

Proposition 5.2. *A subset* $\mathcal{I}\subset\{(i,j):0\le i<j\le n\}$ *is an admissible base in* $A=A_n^+$ *if and only if* $T_{\mathcal{I}}$ *is an admissible tree.*

Lemma 5.3. $n!\operatorname{Vol}\Delta_{\mathcal{I}}=1$ *for any base* $\mathcal{I}$.

Therefore, by Lemma 4.2 $|\Pi(\alpha,\mathcal{I})/L|=1$ and by Proposition 4.5 for every admissible tree T we have a series $\Phi_T(z)=\Phi_\gamma(z)$, where $\gamma\in\Pi(\alpha,\mathcal{I})$, $T=T_{\mathcal{I}}$. The series $\Phi_T(z)$ converges in some domain and presents a solution of the system (2.2), (2.3).

There exists a formula for the number of all admissible trees on the set $[0,n]$.

Theorem 5.4. *The number* F_n *of admissible trees on the set of vertices* $[0,n]$ *is equal to*

$$F_n=\frac{1}{2^n(n+1)}\sum_{k=1}^{n+1}\binom{n+1}{k}k^n.$$

The proof of this formula is given in [Po].

First few numbers F_n are given below.

n	0	1	2	3	4	5	6	7
F_n	1	1	2	7	36	246	2104	21652

6. Standard Triangulation of P_n

Recall that P_n is the convex hull of 0 and e_{ij}, $0{\le}i{<}j{\le}n$.

In this section we construct a coherent triangulation of the polyhedron P_n. This will give us an explicit basis in the solution space of system (2.2), (2.3).

Let T be a tree on the set $[0, n]$. We say that two edges (i, j) and (k, l) in T form an *intersection* if $i{<}k{<}j{<}l$.

Definition 6.1. A tree T on the set $[0, n]$ is called *standard* if T is admissible and does not have intersections. The corresponding base $\mathcal{I} \subset \{(i, j) : 0{\le}i{<}j{\le}n\}$ is also called *standard*.

Example 6.2. All standard trees for $n = 0, 1, 2, 3$ are shown on Figure 6.1.

Figure 6.1. Standard trees.

Theorem 6.3. *The set Θ_n of standard bases forms a coherent local triangulation of the polyhedron P_n.*

Theorem 6.4. *The number of standard trees on the set $[0, n]$ is equal to the Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

As a consequence of these two theorems we get Theorem 2.3.(2).

Proof of Theorem 6.4. Construct by induction an explicit 1–1 correspondence ψ_n between the set ST_n of standard trees on $[0, n]$ and the set BT_n of binary trees with n unmarked vertices $\psi_n : \mathrm{ST}_n \to \mathrm{BT}_n$.

If $n = 1$ then ψ_1 maps a unique element of ST_1 to a unique element of BT_1.

Let $n > 1$. Every standard tree $T \in \mathrm{ST}_n$ has the edge $(0, n)$. Delete this edge. Then T splits into two standard trees $T_1 \in \mathrm{BT}_k$ and $T_2 \in \mathrm{BT}_l$, $k+l+1 = n$ on the sets $[0, k]$ and $[k+1, n]$. Let as define $\psi_n(T)$ as the binary tree whose left and right branches are equal to $\psi_k(T_1)$ and $\psi_l(T_2)$ correspondingly. See example on Figure 6.2.

It is well known that the number of binary trees is equal to the Catalan number (e.g. see [SW]).

FIGURE 6.2. Bijection between standard and binary trees.

Now prove Theorem 6.3.

Proof of Theorem 6.3. Recall that $\epsilon_0, \epsilon_1, \dots, \epsilon_n$ is the standard basis in $\mathbb{Z}^{n+1}$; and $e_{ij} = \epsilon_i - \epsilon_j$.

Let $\widetilde{P}_n \subset \mathbb{Z}^{n+1} \otimes \mathbb{R}$ denote the cone with vertex at 0 generated by all positive roots e_{ij}, $i < j$. Let $\widetilde{\Delta}_{\mathcal{I}}$ denote the simplicial cone generated by e_{ij}, $(i, j) \in \mathcal{I}$, where $\mathcal{I}$ is a base (the cone over the simplex $\Delta_{\mathcal{I}}$).

First, prove that the collection of cones $\widetilde{\Delta}_{\mathcal{I}}$, where $\mathcal{I}$ range over all standard bases, is a conic triangulation of $\widetilde{P}_n$. Then it follows that Θ_n is a local triangulation.

It is not difficult to show that the cone $\widetilde{P}_n$ is the set of $v = (v_0, v_1, \dots, v_n) \in \mathbb{R}^{n+1}$ such that

$$v_0 + v_1 + \dots + v_i \geq 0, \qquad i = 1, 2, \dots, n-1; \tag{6.1}$$

$$v_0 + v_1 + \dots + v_n = 0. \tag{6.2}$$

We must show that every generic point v subject to (6.1), (6.2) can be uniquely presented in the form

$$v = \sum_{(ij) \in \mathcal{I}} \rho_{ij} e_{ij}, \quad \rho_{ij} \geq 0, \tag{6.3}$$

for some standard base $\mathcal{I}$.

Prove it by induction on n.

Let $v' = (v'_0, v'_1, \dots, v'_{n-1}) \in \mathbb{R}^n$ be a vector such that $v'_i = v_i$, $i = 0, 1, \dots, n-2$, and $v'_{n-1} = v_{n-1} + v_n$. Then $v' \in \widetilde{P}_{n-1}$. By induction we may assume that v' is expressed in the form

$$v' = \sum_{(ij) \in \mathcal{I}'} \rho'_{ij} e_{ij}, \quad \rho'_{ij} \geq 0,$$

for a standard base $\mathcal{I}' \subset \{(i, j) : 0 \leq i < j \leq n-1\}$.

Let $i_1 < i_2 < \dots < i_s$ be all vertices of $T' = T_{\mathcal{I}'}$ connected with the vertex $n-1$ in T'.

Consider two cases.

1. $v_{n-1} \geq 0$. Define $\mathcal{I} = \mathcal{I}' \cup \{(n{-}1, n)\} \cup \{(i_k, n) : k \in [s]\} \setminus \{(i_k, n{-}1) : k \in [s]\}$. And $\rho_{ij} = \rho'_{ij}$ for $0{\leq}i{<}j{\leq}n{-}2$; $\rho_{i_k n} = \rho'_{i_k n-1}$ for $k \in [s]$; $\rho_{n-1n} = v_{n-1}$. Then we get expression (6.3) for v.

2. $v_{n-1} < 0$. Then $-v_n \leq \sum_{k=1}^{s} \rho'_{i_k n-1}$. Let t be the minimal integer $0{\leq}t{\leq}s$ such that $\sum_{k=1}^{t} \rho'_{i_k n-1} \geq -v_n$. Then define $\mathcal{I} = \mathcal{I}' \cup \{(i_k, n) : k \in [t]\} \setminus \{(i_k, n{-}1) : k \in [t{-}1]\}$. And $\rho_{ij} = \rho_{ij}$ for $0{\leq}i{<}j{\leq}n{-}2$; $\rho_{i_k n} = \rho'_{i_k n-1}$ for $k \in [t{-}1]$; $\rho_{i_t n} = -\sum_{k=1}^{t-1} \rho'_{i_k n-1} - v_n$; $\rho_{i_k n-1} = \rho'_{i_k n-1}$ for $k \in [t{+}1, s]$; $\rho_{i_t n-1} = -\sum_{k=t+1}^{s} \rho'_{i_k n-1} - v_{n-1}$. Then we get expression (6.3) for v.

Therefore, Θ_n is a local triangulation.

Prove that Θ_n is coherent triangulation (see Definition 4.10). We must present a piecewise linear function ϕ on P_n such that ϕ is linear on all simplices in Θ_n and ϕ is strictly convex on P_n.

It is sufficient to define ϕ on vertices of P_n. Let $\phi(0) = 0$ and $\phi(\epsilon_{ij}) = (i - j)^2$. It is not difficult to show that such ϕ satisfy the condition of Definition 4.10.

Now we can complete the proof of Theorem 2.3.

Proof of Theorem 2.3.

The first part of Theorem 2.3 is a special case of Theorem 1.4.

The second part follows from Theorems 6.3, 6.4 and Lemma 5.3.

In conclusion of this section we present a construction of another coherent triangulation of P_n.

Let T be a tree on the set $[0, n]$. We say that two edges (i, j) and (k, l) in T are *enclosed* if $i{<}k{<}l{<}j$.

Definition 6.5. A tree T on the set $[0, n]$ is called *anti-standard* if T is admissible and does not have enclosed edges. The corresponding base $\mathcal{I} \subset \{(i, j) : 0{\leq}i{<}j{\leq}n\}$ is also called *anti-standard*.

Theorem 6.6. *The set of anti-standard bases forms a coherent local triangulation of the polyhedron P_n.*

The proof of this theorem is analogous to the proof of Theorem 6.3.

Corollary 6.7. *The number of anti-standard trees on the set $[0, n]$ is equal to the Catalan number C_n.*

7. Coordinate Strata

Let Z_n be the group of unipotent matrices z_{ij}, $0{\leq}i{\leq}j{\leq}n$, $z_{ii} = 1$ (see Section 2).

Consider a subset $S \subset \{(i, j) : 0{\leq}i{<}j{\leq}n\}$. By Z_S denote the set of all $z = \{z_{ij}\} \in Z_n$ such that $z_{ij} \neq 0$ if and only if $(i, j) \in S$. We call Z_S *coordinate strata* in the space Z_n. Let $\overline{Z}_S \simeq \mathbb{C}^{|S|}$ be the closure of the stratum Z_S.

We can construct two sheaves of hypergeometric functions on the manifold $\overline{Z}_S$, where $S \subset \{(i, j) : 0{\leq}i{<}j{\leq}n\}$.

First, the sheaf Res_S of restrictions of hypergeometric functions on Z_n to the manifold $\overline{Z}_S$.

Second, the sheaf Sol_S of solutions of the hypergeometric system (1.2), (1.3) associated with $A = A_S = \{e_{ij} : (i, j) \in S\}$ (equivalently, associated with action (2.1) of torus on $\overline{Z}_S$).

The question is: when these two sheaves coincide?

Definition 7.3. Let $\mathcal{P} = \{b_0, b_1, \dots, b_n\}$ be a partially ordered set (poset) such that if $b_i <_{\mathcal{P}} b_j$ then $i < j$. Consider the set $S_{\mathcal{P}} = \{(i,j) : b_i <_{\mathcal{P}} b_j\}$. We call this set *associated* with poset $\mathcal{P}$

Theorem 7.4. *Let $S = S_{\mathcal{P}}$ be the set associated with a poset. Then sheaf* Res_S *coincides with sheaf* Sol_S *for generic exponents $\alpha_0, \dots, \alpha_n$, $\sum \alpha_i = 0$.*

Remark 7.5. By Theorem 1.4 the dimension of Sol_S in a neighborhood of a generic point is equal to $m!\,\mathrm{Vol}_{H(S)}\,P(S)$, where $H(S)$ is the lattice generated by e_{ij}, $(i,j) \in S$, $m = \dim H_S$, and $P(S)$ is the convex hull of the origin and e_{ij}, $(i,j) \in S$.

Proposition 7.6. *A set $S \subset \{(i,j) : 0{\leq}i{<}j{\leq}n\}$ is associated with a poset $\mathcal{P}$ if and only if there exists a cone C with vertex at 0 such that $S = \{(i<j) : e_{ij} \in C\}$.*

Proof. A set S is associated with a poset if and only if S satisfies the following transitivity: if $(i,j), (j,k) \in S$ then $(i,k) \in S$. The set $S = \{(i<j) : e_{ij} \in C\}$ satisfies transitivity because if $e_{ij}, e_{jk} \in C$ then $e_{ik} = e_{ij} + e_{jk} \in C$. Inversely, let C be the cone generated by all e_{ij}, $(i,j) \in S$. If S satisfy transitivity then $S = \{(i<j) : e_{ij} \in C\}$.

Now we can prove Theorem 7.4

Proof of Theorem 7.4. Clearly, Res_S is a subsheaf of Sol_S. Suppose for simplicity that e_{ij}, $(i,j) \in S$ generate $\mathbb{Z}^n$. The dimension of the sheaf Sol_S at a generic point is equal to $n!\,\mathrm{Vol}(P(S))$ (see Remark 7.5). Hence, it is sufficient to prove that the dimension of Res_S at a generic point is greater than or equal to $n!\,\mathrm{Vol}(P(S))$.

Let Θ be a coherent local triangulation of $P(A)$. It follows from Proposition 7.6 that Θ extends to a coherent local triangulation Θ' of P_n. Consider $n!\,\mathrm{Vol}(P(S))$ Γ-series $\Phi_\gamma(z)$ on Z_n, where $\gamma \in \Pi(\alpha, \mathcal{I})$, $\mathcal{I} \in \Theta \subset \Theta'$. By Theorem 4.13 these series linearly independent and have common convergence domain. Then restrictions of these series to $\overline{Z}_S$ give $n!\,\mathrm{Vol}(P(S))$ independent sections of the sheaf Res_S in some neighborhood. Therefore, $\mathrm{Res}_S = \mathrm{Sol}_S$.

8. FACE STRATA

Describe faces of the polyhedron P_n.

Let $I, J \subset [0,n]$, $I \cap J = \emptyset$. Let S_{IJ} be the set of all (i,j), $0{\leq}i{<}j{\leq}n$ such that $i \in I$ and $j \in J$.

Proposition 8.1. *Faces f of the polyhedron P_n such that $0 \notin f$ are in 1–1 correspondence with sets S_{IJ}. And $(i,j) \in S_{IJ}$ whenever e_{ij} is a vertex of the corresponding face f.*

Clearly, we may assume that $\min(I{\cup}J) \in I$ and $\max(I{\cup}J) \in J$ (if S_{IJ} is nonempty).

Construct a coordinate stratum associated with a face f of P_n $0 \notin f$.

Let $S = S_{IJ}$. By Z_{IJ} denote the stratum Z_S (see Section 3). We will call such strata *face strata.*

Note that condition (1.5) holds for vectors e_{ij}, $(i,j) \in S_{IJ}$, because all such e_{ij} belong to a supporting hyperplane of the corresponding face f.

Definition 8.2. The *Hypergeometric System on* $\overline{Z}_{IJ}$ is the hypergeometric system (1.2), (1.3) associated with the set of vectors $A = \{e_{ij} : (i,j) \in S_{IJ}\}$. Solutions of this system are called *Hypergeometric Functions on* $\overline{Z}_{IJ}$.

Remark 8.3. Let $0 \le p < n$, $I = \{0, 1, \dots, p\}$, and $J = \{p+1, p+2, \dots, n\}$. Then $\overline{Z}_{IJ}$ is the space of rectangular matrices $z = \{z_{ij}\}$, $i \in [0,p]$, $j \in [p+1,n]$. The hypergeometric system on $\overline{Z}_{IJ}$ is also called the *Hypergeometric System on the Grassmannian* $G_{n+1\,p+1}$. This system was studied in the works [GGR1, GGR2, GGR3].

It is clear that the set $S = S_{IJ}$ is associated with a poset (see Definition 7.3). Therefore, by Theorem 8.4, the sheaf Res_S coincides with the sheaf Sol_S of hypergeometric functions on $\overline{Z}_{IJ}$ (for generic α).

We will find the dimension of this sheaf in a neighborhood of a generic point. Denote this dimension by D_{IJ}. In other words, D_{IJ} is the number of independent solutions of the hypergeometric system on $\overline{Z}_{IJ}$ in a neighborhood of a generic point.

Let P_{IJ} be the convex hull of 0 and e_{ij}, $(i,j) \in S_{IJ}$. Let H_{IJ} be the sublattice generated by e_{ij}, $(i,j) \in S_{IJ}$, and $m = \dim H_{IJ}$. By Theorem 1.4 the number D_{IJ} is equal to $m!\,\mathrm{Vol}_{H_{IJ}}(P_{IJ})$.

We present an explicit combinatorial interpretation of this number D_{IJ}.

Definition 8.4.

(1) A *word* w of *type* (p,q) is the sequence $w = (w_1, w_2, \dots, w_{p+q})$, $w_r \in \{1, 0\}$ such that $|\{r : w_r = 0\}| = p$ and $|\{r : w_r = 1\}| = q$.

(2) Let $w = (w_1, w_2, \dots, w_{p+q})$ and $w' = (w'_1, w'_2, \dots, w'_{p+q})$ be two words of type (p,q). We say that w' is *exceeds* w if $w'_1 + \dots + w'_r \ge w_1 + \dots + w_r$ for all $r = 1, 2, \dots, p+q$.

$$w = (0,0,1,1,0,0,1,0,0,0,1,1,0,0,1,0,1)$$
$$w' = (0,0,1,1,0,0,1,0,0,0,1,1,0,0,1,0,1)$$

FIGURE 8.1. The word w' exceeds the word w.

We can present a word w of type (p,q) as the path $\pi = (\pi_0, \pi_1, \dots, \pi_{p+q})$ in $\mathbb{Z}^2$ such that $\pi_s = (i_s, j_s)$ for all $s = 0, 1, \dots, p+q$, where i_s (correspondingly, j_s) is the number of 0's (correspondingly, 1's) in $w_1, w_2, \dots, w_s$. See example for $(p,q) = (10,7)$ on Fig. 8.1.

Clearly, a word w' exceeds a word w if and only if the path π' corresponding to w' is above the path π corresponding to w. (See Fig. 8.1.)

Let $a = \min I$ and $b = \max J$. Then $D_{IJ} \neq 0$ if ond only if $a < b$.

Suppose that $a < b$, $I = \{a\} \cup I'$ and $J = \{b\} \cup J'$, where $I', J' \subset [a{+}1, b{-}1]$, $I' \cap J' = \emptyset$. Let $|I'| = p$, $|J'| = q$ and $I' \cup J' = \{t_1{<}t_2{<}\cdots < t_{p+q}\}$. Associate with the pair (I, J) the word $w_{IJ} = (w_1, \ldots, w_{p+q})$ of type (p, q) such that $w_r = 0$ if $t_r \in I$ and $w_r = 1$ if $t_r \in J$ for all $r = 1, 2, \ldots, p{+}q$.

Theorem 8.5. *The number D_{IJ} is equal to the number of words w' of type (p, q) which exceed the word $w = w_{IJ}$. In other words, D_{IJ} is the number of paths π' from $(0, 0)$ to (p, q) such that π' is above the path $\pi = \pi_{IJ}$ corresponding to w_{IJ}.*

Corollary 8.6. *Let $I = \{0, 2, 4, \ldots, 2k\}$ and $J = \{1, 3, 5, \ldots, 2k{+}1\}$ then D_{IJ} is equal to the Catalan number C_k.*

Proof. Words $w' = (w'_1, w'_2, \ldots, w'_{2k})$ of type (k, k) which exceed the word $w = (1, 0, 1, 0, \ldots, 1, 0)$ are called *Dyck words.* It is well know (see e.g. [SW]) that the Catalan number C_k is equal to the number of Dyck words.

9. STANDARD TRIANGULATION OF P_{IJ}

Let $I, J \subset [0, n]$, $I \cap J = \emptyset$ be two subsets such that $\min(I{\cup}J) \in I$ and $\max(I{\cup}J) \in J$ (see Section 8).

Recall that $P_{IJ} = \mathrm{Conv}(0, e_{ij} : (i, j) \in S_{IJ})$.

In this section we present a coherent local triangulation of the polyhedron P_{IJ} and prove Theorem 8.5.

Definition 9.1. Let T be a tree on the set $I \cup J$. We say that T is *of type* (I, J) if for every edge (i, j) in T $i \in I$ and $j \in J$. The base $\mathcal{I} \subset \{(i, j) : 0{\le}i{<}j{\le}n\}$ corresponding to T is also called *of type* (I, J). (Do not confuse $\mathcal{I}$ with I.)

Clearly, all trees of type (I, J) are admissible (see Definition 5.1).

Theorem 9.2. *The set Θ_{IJ} of all standard (see Definition 6.1) bases of type (I, J) forms a coherent local triangulation of the polyhedron P_{IJ}.*

The proof of this theorem is essentially the same as the proof of Theorem 6.3.

It is clear that $D_{IJ} = m!\,\mathrm{Vol}(P_{IJ})$ is equal to the number of all standard bases (trees) of type (I, J). Prove that this number coincides with the number given by Theorem 8.5.

Theorem 9.3. *Let $|I| = p{+}1$ and $|J| = q{+}1$. Then the number of all standard trees T of type (I, J) is equal to the number of words w' of type (p, q) which exceed the word $w = w_{IJ}$.*

Proof. Let D_{IJ} be the number of all standard trees of type (I, J) and $\widetilde{D}_{IJ}$ be the number of words w' of type (p, q) which exceed the word $w = w_{IJ}$ (we use the same notation as in Theorem 8.5).

We prove that $D_{IJ} = \widetilde{D}_{IJ}$ by induction on $p + q$. Obviously, this is true for $p = q = 0$.

Let d be the minimal element of J and c be the maximal element of I such that $c \le d$. Let $\widetilde{I} = I \setminus \{c\}$ and $\widetilde{J} = J \setminus \{d\}$.

Prove that if $p + q > 0$ then

$$D_{IJ} = D_{\widetilde{I}J} + D_{I\widetilde{J}}. \tag{9.1}$$

Every standard tree of type (I, J) has the edge (c, d). In every such tree either c or d is an end-point. The first choice corresponds to the term $D_{\tilde{I}J}$ and the second choice corresponds to the term $D_{I\tilde{J}}$ in (9.1).

The numbers $\widetilde{D}_{IJ}$ also satisfy the relation (9.1). The first term corresponds to the case when the word w' starts with 0 and the second term to the case when w' starts with 1.

Therefore, we get by induction $D_{IJ} = \widetilde{D}_{IJ}$.

Theorem 8.5 is a corollary of Theorem 9.3.

10. Examples

In this and the next sections we present several examples which illustrate the notions introduced in the paper and show the direction for following study.

10.1. Case $n = 2$.

In this case the solutions f of the system (2.2), (2.3) are functions of variables z_{01}, z_{02}, z_{12}.

Let $\beta_1 = \frac{1}{3}(\alpha_2 - 2\alpha_0)$ and $\beta_2 = \frac{1}{3}(2\alpha_2 - \alpha_0)$. Because of homogeneous conditions (1.4) we can write $f(z_{01}, z_{02}, z_{12}) = z_{01}^{\beta_1} z_{12}^{\beta_2}\, F(y)$, where $y = \frac{z_{02}}{z_{01}z_{12}}$. Now system (2.2), (2.3) is equivalent to the following equation on $F(y)$.

$$\frac{dF}{dy} = \left(y\frac{d}{dy} - \beta_1\right)\left(y\frac{d}{dy} - \beta_2\right) F. \tag{10.1}$$

This is the degenerate hypergeometric equation and its solutions can be written in terms of the degenerate hypergeometric function ${}_1F_1$ (see [BE]).

This system has two dimensional space of solutions, which is compatible with the fact that $C_2 = 2$.

10.1. Upper triangular matrices.

Let $I = \{0, 2, \dots, 2n\}$ and $J = \{1, 3, \dots, 2n{+}1\}$. It is natural to identify the space $\overline{Z}_{IJ}$ with the space of all upper triangular matrices with arbitrary elements on the diagonal. Consider the hypergeometric system on $\overline{Z}_{IJ}$. The call this system *the hypergeometric system on upper triangular matrices.*

This system has the same dimension C_n of solution space as system (2.2), (2.3) (see Corollary 8.6). But it is nonconfluent unlikely system (2.2), (2.3).

If fact, system (2.2), (2.3) can be obtained as a limit of the hypergeometric system on upper triangular matrices.

For example, if $I = \{0, 2, 4\}$ and $J = \{1, 3, 5\}$ then the corresponding hypergeometric system on $\overline{Z}_{IJ}$ can be reduced to the Gauss hypergeometric equation. And equation (10.1) is a limit of the Gauss hypergeometric equation.

11. Concluding Remarks and Open Problems

11.1. Characteristic manifold.

We do not prove here Theorem 1.4. There exist a proof of this theorem generalizing the proof from [GZK] for nonconfluent case.

This proof is based on consideration of *characteristic manifold* Ch for system (1.2), (1.3). The characteristic manifold for system (1.2), (1.3) is the submanifold in the space $\mathbb{C}^N \times \mathbb{C}^N$ with coordinates (x, ξ), $x = (x_1, \dots, x_N)$, $\xi = (\xi_1, \dots, \xi_n)$ given by the following algebraic equations.

$$\sum_{j=1}^{N} a_{ij} x_j \xi_j = 0, \qquad i = 1, 2, \dots, n;$$

$$\prod_{j:\, l_j > 0} {\xi_j}^{l_j} = \prod_{j:\, l_j < 0} {\xi_j}^{-l_j} \qquad \text{if } \sum_j l_j = 0;$$

$$\prod_{j:\, l_j > 0} {\xi_j}^{l_j} = 0 \qquad \text{if } \sum_{j:\, l_j > 0} l_j > \sum_{j:\, l_j < 0} l_j,$$

where $l = (l_1, l_2, \dots, l_N)$ ranges over the lattice L of integer vectors such that $l_1 a_1 + l_2 a_2 + \cdots + l_N a_N = 0$.

Then system (1.2), (1.3) is holonomic if $\dim Ch = N$. The number of independent solutions at a generic point is equal to degree of Ch along the zero section $\{(0, \xi) : \xi \in \mathbb{C}^N\}$ (see [Ka]).

11.2. Other root systems.

We can define (see Section 2) the hypergeometric system for arbitrary root system R.

It is interesting to find analogues of all results in this paper for other root systems.

Let P_{R^+} be the convex hull of 0 and all positive roots $r \in R^+$. Then by Theorem 1.4 the dimension of the system at a generic point is equal to $D(R) = n! \operatorname{Vol}(P_{R^+})$, where n is the dimension of R.

These numbers $D(R)$ can be viewed as a generalization of the Catalan numbers for arbitrary root system.

11.3. Discriminant and Triangulations of P_n.

We can associate with system (2.2), (2.3) the discriminant $\mathcal{D}_n(z)$. The discriminant $\mathcal{D}_n(z)$ is a polynomial of $z = (z_{ij})$, $0 \le i < j \le n$ such that $\mathcal{D}_n(z) = 0$ if and only if there exists $(z, \xi) \in Ch$ such that $\xi \neq 0$, where Ch is the characteristic manifold for system (2.2), (2.3).

It is an interesting problem to find an explicit expression for $\mathcal{D}_n(x)$ and describe all monomials in $\mathcal{D}_n(x)$.

The Newton polytope S_n for $\mathcal{D}_n(x)$ is called *Secondary polytope.* Vertices of S_n correspond to coherent local triangulations of P_n (cf. [GKZ]).

In Section 6 we constructed two coherent local triangulations of P_n. The important problem is to find all such triangulations.

Analogously, one can define discriminant $\mathcal{D}_{IJ}(z)$ associated with face strata Z_{IJ} (see Section 8). Vertices of the Newton polyhedron for $\mathcal{D}_{IJ}(z)$ correspond to coherent triangulations of P_{IJ}. (Note that all triangulations of P_{IJ} are local.) How to describe triangulations of P_{IJ}?

The special case of this problem for the pair (I, J) such as in Remark 8.3 (the hypergeometric system on the grassmannian) is connected with triangulations of the product of two simplices $\Delta^p \times \Delta^q$, $p+q = n+1$. In this case $\mathcal{D}_{IJ}$ is the product of all minors of $(p+1) \times (q+1)$-matrix z (see [GKZ], cf. [SZ, BZ]).

References

[Ad] Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. Jour. **73** (1994), no. 2, 269–290.

[BE] H. Bateman, A Erdélyi, *Higher Transcendent Functions, Vol. 1*, Mc Graw-Hill Book Company, Inc., New York, 1953.

[BZ] D. Bernstein, A. Zelevinsky, *Combinatorics of maximal minors*, Jour. of Algebraic Combinatorics **2** (1993), 111–121.

[Bo] N. Bourbaki, *Éléments de Mathématique, Groupes et Algèbres de Lie, Ch.* 6, Hermann, Paris, 1968.

[GG1] I. M. Gelfand, M. I. Graev, *Hypergeometric functions on flag manifolds*, Doklady Acad. Nauk SSSR **338** (1994), no. 3, 298–301.

[GGR1] I. M. Gelfand, M. I. Graev, V. S. Retakh, Γ*-series and general hypergeometric functions on the manifold of* $k \times m$*-matrices*, Preprint IPM no. 64 (1990).

[GGR2] I. M. Gelfand, M. I. Graev, V. S. Retakh, *General hypergeometric systems of equations and series of hypergeometric type*, Uspekhi Mat. Nauk SSSR **47** (1992), no. 4(286), 3–82.

[GGR3] I. M. Gelfand, M. I. Graev, V. S. Retakh, *Reduction formulas for hypergeometric functions associated with the grassmannian* G_{nk} *and description of these functions on strata of small codimension in* G_{nk}, Russian Jour. of Math. Physics **1** (1993), no. 1, 19–56.

[GGZ] I. M. Gelfand, M. I. Graev, A. V. Zelevinsky, *Holonomic systems of equations and series of hypergeometric type*, Doklady Acad. Nauk SSSR **295** (1987), no. 1, 14–19.

[GKZ] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[GZK] I. M. Gelfand, A. V. Zelevinsky, M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funct. Anal. and its Appl. **23** (1989), no. 2, 12–26.

[Ka] M. Kashiwara, *System of Microdifferential Equations*, Birkhäuser, Boston, 1983.

[Po] A. Postnikov, *Intransitive trees*, preprint (1994).

[SW] D. Stanton, D. White, *Constructive Combinatorics*, Springer-Verlag, 1986.

[SZ] B. Sturmfels, A. Zelevinsky, *Maximal minors and their leading terms*, Advances in Math. **98** (1993), no. 1, 65–112.