CHAINS IN THE BRUHAT ORDER

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Abstract. We study a family of polynomials whose values express degrees of Schubert varieties in the generalized complex flag manifold G/B. The polynomials are given by weighted sums over saturated chains in the Bruhat order. We derive several explicit formulas for these polynomials, and investigate their relations with Schubert polynomials, harmonic polynomials, Demazure characters, and generalized Littlewood-Richardson coefficients. In the second half of the paper, we concern with the case of to the classical flag manifold of Lie type A and discuss related combinatorial objects: flagged Schur polynomials, 312-avoiding permutations, generalized Gelfand-Tsetlin polytopes, the inverse Schubert-Kostka matrix, parking functions, and binary trees.

1. Introduction

The complex generalized flag manifold G/B embeds into projective space $\mathbb{P}(V_\lambda)$, for an irreducible representation V_λ of G. The degree of a Schubert variety $X_w \subset G/B$ in this embedding is a polynomial function of λ. The aim of this paper is to study the family of polynomials $\mathfrak{D}_w$ in $r = \mathrm{rank}(G)$ variables that express degrees of Schubert varieties. According to Chevalley's formula [Chev], also known as Monk's rule in type A, these polynomials are given by weighted sums over saturated chains from id to w in the Bruhat order on the Weyl group. These weighted sums over saturated chains appeared in Bernstein-Gelfand-Gelfand [BGG] and in Lascoux-Schützenberger [LS2]. Stembridge [Stem] recently investigated these sums in the case when $w = w_\circ$ is the longest element in the Weyl group. The value $\mathfrak{D}_w(\lambda)$ is also equal to the leading coefficient in the dimension of the Demazure modules $V_{k\lambda,w}$, as $k \to \infty$.

The polynomials $\mathfrak{D}_w$ are dual to the Schubert polynomials $\mathfrak{S}_w$ with respect a certain natural pairing on the polynomial ring. They form a basis in the space of W-harmonic polynomials. We show that Bernstein-Gelfand-Gelfand's results [BGG] easily imply two different formulas for the polynomials $\mathfrak{D}_w$. The first "top-to-bottom" formula starts with the top polynomial $\mathfrak{D}_{w_\circ}$, which is given by the Vandermonde product. The remaining polynomials $\mathfrak{D}_w$ are obtained from $\mathfrak{D}_{w_\circ}$ by applying differential operators associated with Schubert polynomials. The second "bottom-to-top" formula starts with $\mathfrak{D}_{id} = 1$. The remaining polynomials $\mathfrak{D}_w$ are obtained from $\mathfrak{D}_{id}$ by applying certain integration operators. Duan's recent

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result [Duan] about degrees of Schubert varieties can be deduced from the bottom-to-top formula.

Let $c_{u,v}^w$ be the generalized Littlewood-Richardson coefficient defined as the structure constant of the cohomology ring of G/B in the basis of Schubert classes. The coefficients $c_{u,v}^w$ are related to the polynomials $\mathfrak{D}_w$ in two different ways. Define a more general collection of polynomials $\mathfrak{D}_{u,w}$ as sums over saturated chains from u to w in the Bruhat order with similar weights. (In particular, $\mathfrak{D}_w = \mathfrak{D}_{id,w}$.) The polynomials $\mathfrak{D}_{u,w}$ extend the $\mathfrak{D}_w$ in the same way as the skew Schur polynomials extend the usual Schur polynomials. The expansion coefficients of $\mathfrak{D}_{u,w}$ in the basis of $\mathfrak{D}_v$'s are exactly the generalized Littlewood-Richardson coefficients: $\mathfrak{D}_{u,w} = \sum_v c_{u,v}^w\, \mathfrak{D}_v$. On the other hand, we have $\mathfrak{D}_w(y+z) = \sum_{u,v} c_{u,v}^w\, \mathfrak{D}_u(y)\, \mathfrak{D}_w(z)$, where $\mathfrak{D}_w(y+z)$ denote the polynomial in pairwise sums of two sets y and z of variables.

We pay closer attention to the Lie type A case. In this case, the Weyl group is the symmetric group $W = S_n$. Schubert polynomials for vexillary permutations, i.e., 2143-avoiding permutations, are known to be given by flagged Schur polynomials. From this we derive a more explicit formula for the polynomials $\mathfrak{D}_w$ for 3412-avoiding permutations w and, in particular, an especially nice determinant expression for $\mathfrak{D}_w$ in the case when w is 312-avoiding.

It is well-known that the number of 312-avoiding permutations in S_n is equal to the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Actually, these permutations are exactly the Kempf elements studied by Lakshmibai [Lak] (though her definition is quite different). We show that the characters $ch(V_{\lambda,w})$ of Demazure modules for 312-avoiding permutations are given by flagged Schur polynomials. (Here flagged Schur polynomials appear in a different way than in the previous paragraph.) This expression can be geometrically interpreted in terms of generalized Gelfand-Tsetlin polytopes $\mathcal{P}_{\lambda,w}$ studied by Kogan [Kog]. The Demazure character $ch(V_{\lambda,w})$ equals a certain sum over lattice points in $\mathcal{P}_{\lambda,w}$, and thus, the value $\mathfrak{D}_w(\lambda)$ equals the normalized volume of $\mathcal{P}_{\lambda,w}$. The generalized Gelfand-Tsetlin polytopes $\mathcal{P}_{\lambda,w}$ are related to the toric degeneration of Schubert varieties X_w constructed by Conciulea and Lakshmibai [GL].

One can expand Schubert polynomials as nonnegative sums of monomials using RC-graphs. We call the matrix K of coefficients in these expressions the *Schubert-Kostka matrix,* because it extends the usual Kostka matrix. It is an open problem to find a subtraction-free expression for entries of the inverse Schubert-Kostka matrix K^{-1}. The entries of K^{-1} are exactly the coefficients of monomials in the polynomials $\mathfrak{D}_w$ normalized by a product of factorials. On the other hand, the entries of K^{-1} are also the expansion coefficients of Schubert polynomials in terms of standard elementary monomials. We give a simple expression for entries of K^{-1} corresponding to 312-avoiding permutations and 231-avoiding permutations. Actually, these special entries are always equal to ± 1, or 0.

We illustrate our results by calculating the polynomial $\mathfrak{D}_w$ for the long cycle $w = (1, 2, \ldots, n) \in S_n$ in five different ways. First, we show that $\mathfrak{D}_w$ equals a sum over parking functions. This polynomial appeared in Pitman-Stanley [PS] as the volume of a certain polytope. Indeed, the generalized Gelfand-Tsetlin polytope $\mathcal{P}_{\lambda,w}$ for the long cycle w, which is a 312-avoiding permutation, is exactly the polytope studied in [PS]. Then the determinant formula leads to another simple expression for $\mathfrak{D}_w$ given by a sum of 2^n monomials. Finally, we calculate $\mathfrak{D}_w$ by counting saturated chains in the Bruhat order and obtain an expression for this polynomial as a sum over binary trees.

The general outline of the paper follows. In Section 2, we give basic notation related to root systems. In Section 3, we recall classical results about Schubert calculus for G/B. In Section 4, we define the polynomials $\mathfrak{D}_w$ and $\mathfrak{D}_{u,w}$ and discuss their geometric meaning. In Section 5, we discuss the pairing on the polynomial ring and harmonic polynomials. In Section 6, we prove the top-to-bottom and the bottom-to-top formulas for the polynomial $\mathfrak{D}_w$ and give several corollaries. In particular, we show how these polynomials are related to the generalized Littlewood-Richardson coefficients. In Section 7, we give several examples and deduce Duan's formula. In Section 8, we recall a few facts about the K-theory of G/B. In Section 9, we give a simple proof of the product formula for $\mathfrak{D}_{w_\circ}$. In Section 10, we mention a formula for the permanent of a certain matrix. The rest of the paper is concerned with the type A case. In Section 11, we recall Lascoux-Schützenberger's definition of Schubert polynomials. In Section 12, we specialize the results of the first half of the paper to type A. In Section 13, we discuss flagged Schur polynomials, vexillary and dominant permutations, and give a simple formula for the polynomials $\mathfrak{D}_w$, for 312-avoiding permutations. In Section 14, we give a simple proof of the fact that Demazure characters for 312-avoiding permutations are given by flagged Schur polynomials. In Section 15, we interpret this claim in terms of generalized Gelfand-Tsetlin polytopes. In Section 17, we discuss the inverse of the Schubert-Kostka matrix. In Section 18, we discuss the special case of the long cycle related to parking functions and binary trees.

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2. Notations

Let G be a complex semisimple simply-connected Lie group. Fix a Borel subgroup B and a maximal torus T such that $G \supset B \supset T$. Let $\mathfrak{h}$ be the corresponding Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G, and let r be its rank. Let $\Phi \subset \mathfrak{h}^*$ denote the corresponding *root system*. Let $\Phi^+ \subset \Phi$ be the set of positive roots corresponding to our choice of B. Then Φ is the disjoint union of Φ^+ and $\Phi^- = -\Phi^+$. Let $V \subset \mathfrak{h}^*$ be the linear space over $\mathbb{Q}$ spanned by Φ. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the associated set of *simple roots*. They form a basis of the space V. Let (x, y) denote the scalar product on V induced by the Killing form. For a root $\alpha \in \Phi$, the corresponding *coroot* is given by $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. The collection of coroots forms the dual root system $\Phi^\vee$.

The *Weyl group* $W \subset \mathrm{Aut}(V)$ of the Lie group G is generated by the reflections $s_\alpha : y \mapsto y - (y, \alpha^\vee)\,\alpha$, for $\alpha \in \Phi$ and $y \in V$. Actually, the Weyl group W is generated by *simple reflections* $s_1, \ldots, s_r$ corresponding to the simple roots, $s_i = s_{\alpha_i}$, subject to the *Coxeter relations:* $(s_i)^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$, where m_{ij} is half the order of the dihedral subgroup generated by s_i and s_j.

An expression of a Weyl group element w as a product of generators $w = s_{i_1} \cdots s_{i_l}$ of minimal possible length l is called a *reduced decomposition* for w. Its length l is called the *length* of w and denoted $\ell(w)$. The Weyl group W contains a unique *longest element* $w_\circ$ of maximal possible length $\ell(w_\circ) = |\Phi^+|$.

The *Bruhat order* on the Weyl group W is the partial order relation "$\leq$" which is the transitive closure of the following covering relation: $u \lessdot w$, for $u, w \in W$, whenever $w = u\, s_\alpha$, for some $\alpha \in \Phi^+$, and $\ell(u) = \ell(w) - 1$. The Bruhat order has the unique minimal element *id* and the unique maximal element $w_\circ$. This order can also be characterized, as follows. For a reduced decomposition $w = s_{i_1} \cdots s_{i_l} \in W$ and $u \in W$, $u \le w$ if and only if there exists a reduced decomposition $u = s_{j_1} \cdots s_{j_s}$ such that $j_1, \dots, j_s$ is a subword of $i_1, \dots, i_l$.

Let Λ denote the *weight lattice* $\Lambda = \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z}$ for any $\alpha \in \Phi\}$. It is generated by the *fundamental weights* $\omega_1, \dots, \omega_r$ that form the dual basis to the basis of simple coroots, i.e., $(\omega_i, \alpha_j^\vee) = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by $\Lambda^+ = \{\lambda \in \Lambda \mid (\lambda, \alpha^\vee) \ge 0$ for any $\alpha \in \Phi^+\}$. A dominant weight λ is called *regular* if $(\lambda, \alpha^\vee) > 0$ for any $\alpha \in \Phi^+$. Let $\rho = \omega_1 + \cdots + \omega_r = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ be the minimal regular dominant weight.

3. Schubert calculus

In this section, we recall some classical results of Borel [Bor], Chevalley [Chev], Demazure [Dem], and Bernstein-Gelfand-Gelfand [BGG].

The *generalized flag variety* G/B is a smooth complex projective variety. Let $H^*(G/B) = H^*(G/B, \mathbb{Q})$ be the *cohomology ring* of G/B with rational coefficients. Let $\mathbb{Q}[V^*] = \mathit{Sym}(V)$ be the algebra of polynomials on the space V^* with rational coefficients. The action of the Weyl group W on the space V induces a W-action on the polynomial ring $\mathbb{Q}[V^*]$. According to *Borel's theorem* [Bor], the cohomology of G/B is canonically isomorphic[1] to the quotient of the polynomial ring:

$$H^*(G/B) \simeq \mathbb{Q}[V^*]/\mathcal{I}_W, \tag{3.1}$$

where $\mathcal{I}_W = \left\langle f \in \mathbb{Q}[V^*]^W \mid f(0) = 0 \right\rangle$ is the ideal generated by W-invariant polynomials without constant term. Let us identify the cohomology ring $H^*(G/B)$ with this quotient ring. For a polynomial $f \in \mathbb{Q}[V^*]$, let $\bar f = f \pmod{\mathcal{I})_W}$ be its coset modulo $\mathcal{I}_W$, which we view as a class in the cohomology ring $H^*(G/B)$.

One can construct a linear basis of $H^*(G/B)$ using the following *divided difference operators* (also known as the Bernstein-Gelfand-Gelfand operators). For a root $\alpha \in \Phi$, let $A_\alpha : \mathbb{Q}[V^*] \to \mathbb{Q}[V^*]$ be the operator given by

$$A_\alpha : f \mapsto \frac{f - s_\alpha(f)}{\alpha}. \tag{3.2}$$

Notice that the polynomial $f - s_\alpha(f)$ is always divisible by α. The operators A_α commute with operators of multiplication by W-invariant polynomials. Thus the A_α preserve the ideal $\mathcal{I}_W$ and induce operators acting on $H^*(G/B)$, which we will denote by the same symbols A_α.

Let $A_i = A_{\alpha_i}$, for $i = 1, \dots, r$. The operators A_i satisfy the nilCoxeter relations

$$(A_i A_j)^{m_{ij}} = 1 \qquad \text{and} \qquad (A_i)^2 = 0.$$

For a reduced decomposition $w = s_{i_1} \cdots s_{i_l} \in W$, define $A_w = A_{i_1} \cdots A_{i_l}$. The operator A_w depends only on $w \in W$ and does not depend on a choice of reduced decomposition.

[1]The isomorphism is given by $c_1(\mathcal{L}_\lambda) \mapsto \lambda \pmod{\mathcal{I}_W}$, where $c_1(\mathcal{L}_\lambda)$ is the first Chern class of the line bundle $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$ over G/B, for $\lambda \in \Lambda^+$.

Let us define the *Schubert classes* $\sigma_w \in H^*(G/B)$, $w \in W$, by

$$\sigma_{w_\circ} = |W|^{-1} \prod_{\alpha \in \Phi^+} \alpha \pmod{\mathcal{I}_W}, \quad \text{for the longest element } w_\circ \in W;$$
$$\sigma_w = A_{w^{-1}w_\circ}(\sigma_{w_\circ}), \quad \text{for any } w \in W.$$

The classes σ_w have the following geometrical meaning. Let $X_w = \overline{BwB/B}$, $w \in W$, be the *Schubert varieties* in G/B. According to Bernstein-Gelfand-Gelfand [BGG] and Demazure [Dem], $\sigma_w = [X_{w_\circ w}] \in H^{2\ell(w)}(G/B)$ are the cohomology classes of the Schubert varieties. They form a linear basis of the cohomology ring $H^*(G/B)$. In the basis of Schubert classes, the divided difference operators can be expressed, as follows (see [BGG]):

$$A_i(\sigma_w) = \begin{cases} \sigma_{ws_i} & \text{if } \ell(ws_i) = \ell(w) - 1, \\ 0 & \text{if } \ell(ws_i) = \ell(w) + 1. \end{cases} \tag{3.3}$$

Remark 3.1. There are many possible choices for polynomial representatives of the Schubert classes. In type A_{n-1}, Lascoux and Schützenberger [LS1] introduced the polynomial representatives, called the *Schubert polynomials,* obtained from the monomial $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ by applying the divided difference operators. Here $x_1, \dots, x_n$ are the coordinates in the standard presentation for type A_{n-1} roots $\alpha_{ij} = x_i - x_j$ (see [Hum]). Schubert polynomials have many nice combinatorial properties; see Section 11 below.

For $\sigma \in H^*(G/B)$, let $\langle\sigma\rangle = \int_{G/B} \sigma$ be the coefficient of the top class $\sigma_{w_\circ}$ in the expansion of σ in the Schubert classes. Then $\langle\sigma \cdot \theta\rangle$ is the *Poincaré pairing* on $H^*(G/B)$. In the basis of Schubert classes the Poincaré pairing is given by

$$\langle\sigma_u \cdot \sigma_w\rangle = \delta_{u,\, w_\circ w}. \tag{3.4}$$

The *generalized Littlewood-Richardson coefficients* $c_{u,v}^w$, are given by

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{u,v}^w \, \sigma_w, \quad \text{for } u, v \in W.$$

Let $c_{u,v,w} = \langle\sigma_u \cdot \sigma_v \cdot \sigma_w\rangle$ be the triple intersection number of Schubert varieties. Then, according to (3.4), we have $c_{u,v}^w = c_{u,v,w_\circ w}$.

For a linear form $y \in V \subset \mathbb{Q}[V^*]$, let $\bar{y} \in H^*(G/B)$ be its coset[2] modulo $\mathcal{I}_W$. *Chevalley's formula* [Chev] gives the following rule for the product of a Schubert class σ_w, $w \in W$, with $\bar{y}$:

$$\bar{y} \cdot \sigma_w = \sum (y, \alpha^\vee) \, \sigma_{ws_\alpha}, \tag{3.5}$$

where the sum is over all roots $\alpha \in \Phi^+$ such that $\ell(w\,s_\alpha) = \ell(w) + 1$, i.e., the sum is over all elements in W that cover w in the Bruhat order. The coefficients $(y, \alpha^\vee)$, which are associated to edges in the Hasse diagram of the Bruhat order, are called the *Chevalley multiplicities.* Figure 1 shows the Bruhat order on the symmetric group $W = S_3$ with edges of the Hasse diagram marked by the Chevalley multiplicities, where $Y_1 = (y, \alpha_1^\vee)$ and $Y_2 = (y, \alpha_2^\vee)$.

We have, $\sigma_{id} = [G/B] = 1$. Chevalley's formula implies that $\sigma_{s_i} = \bar{\omega}_i$ (the coset of the fundamental weight ω_i).

[2]Equivalently, $\bar{y} = c_1(\mathcal{L}_\lambda)$, if $y = \lambda$ is in the weight lattice Λ.

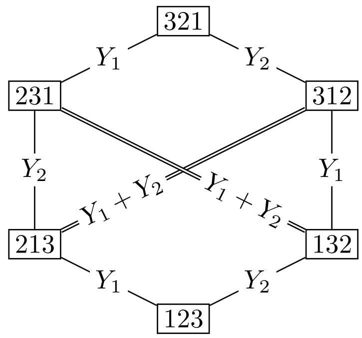

FIGURE 1. The Bruhat order on S_3 marked with the Chevalley multiplicities.

4. Degrees of Schubert varieties

For $y \in V$, let $m(u \lessdot us_\alpha) = (y, \alpha^\vee)$ denote the Chevalley multiplicity of a covering relation $u \lessdot u\, s_\alpha$ in the Bruhat order on the Weyl group W. Let us define the *weight* $m_C = m_C(y)$ of a saturated chain $C = (u_0 \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_l)$ in the Bruhat order as the product of Chevalley multiplicities:

$$m_C(y) = \prod_{i=1}^{l} m(u_{i-1} \lessdot u_i).$$

Then the weight $m_C \in \mathbb{Q}[V]$ is a polynomial function of $y \in V$.

For two Weyl group elements $u, w \in W$, $u \le w$, let us define the polynomial $\mathfrak{D}_{u,w}(y) \in \mathbb{Q}[V]$ as the sum

$$\mathfrak{D}_{u,w}(y) = \frac{1}{(\ell(w) - \ell(u))!} \sum_C m_C(y) \tag{4.1}$$

over all saturated chains $C = (u_0 \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_l)$ in the Bruhat order from $u_0 = u$ to $u_l = w$. In particular, $\mathfrak{D}_{w,w} = 1$. Let $\mathfrak{D}_w = \mathfrak{D}_{id,w}$. It is clear from the definition that $\mathfrak{D}_w$ is a homogeneous polynomial of degree $\ell(w)$ and $\mathfrak{D}_{u,w}$ is homogeneous of degree $\ell(w) - \ell(u)$.

Example 4.1. For $W = S_3$, we have $\mathfrak{D}_{id,231} = \frac{1}{2}(Y_1Y_2 + Y_2(Y_1 + Y_2))$ and $\mathfrak{D}_{132,321} = \frac{1}{2}((Y_1 + Y_2)Y_1 + Y_1Y_2)$, where $Y_1 = (y, \alpha_1^\vee)$ and $Y_2 = (y, \alpha_2^\vee)$ (see Figure 1).

According to Chevalley's formula (3.5), the values of the polynomials $\mathfrak{D}_{u,w}(y)$ are the expansion coefficients in the following product in the cohomology ring $H^*(G/B)$:

$$[e^y] \cdot \sigma_u = \sum_{w \in W} \mathfrak{D}_{u,w}(y) \cdot \sigma_w, \text{ for any } y \in V, \tag{4.2}$$

where $[e^y] := 1 + \bar{y} + \bar{y}^2/2! + \bar{y}^3/3! + \cdots \in H^*(G/B)$. Note that $[e^y]$ involves only finitely many nonzero summands, because $H^k(G/B) = 0$, for sufficiently large k. Equation (4.2) is actually equivalent to definition (4.1) of the polynomials $\mathfrak{D}_{u,w}$.

The values of the polynomials $\mathfrak{D}_w(\lambda)$ at dominant weights $\lambda \in \Lambda^+$ have the following natural geometric interpretation. For $\lambda \in \Lambda^+$, let V_λ be the *irreducible representation* of the Lie group G with the highest weight λ, and let $v_\lambda \in V_\lambda$ be a highest weight vector. Let $e : G/B \to \mathbb{P}(V_\lambda)$ be the map given by $gB \mapsto g(v_\lambda)$, for

$g \in G$. If the weight λ is regular, then e is a projective embedding $G/B \hookrightarrow \mathbb{P}(V_\lambda)$. Let $w \in W$ be an element of length $l = \ell(w)$. Let us define the λ-*degree* $\deg_\lambda(X_w)$ of the Schubert variety $X_w \subset G/B$ as the number of points in the intersection of $e(X_w)$ with a generic linear subspace in $\mathbb{P}(V_\lambda)$ of complex codimension l. The pull-back of the class of a hyperplane in $H^2(\mathbb{P}(V_\lambda))$ is $\bar{\lambda} = c_1(\mathcal{L}_\lambda) \in H^2(G/B)$. Then the λ-degree of X_w is equal to the Poincaré pairing $\deg_\lambda(X_w) = \langle [X_w] \cdot \bar{\lambda}^l \rangle$. In other words, $\deg_\lambda(X_w)$ equals the coefficient of the Schubert class σ_w, which is Poincaré dual to $[X_w] = \sigma_{w_\circ w}$, in the expansion of $\bar{\lambda}^l$ in the basis of Schubert classes. Chevalley's formula (3.5) implies the following well-known statement; see, e.g., [BL].

Proposition 4.2. *For $w \in W$ and $\lambda \in \Lambda^+$, the λ-degree $\deg_\lambda(X_w)$ of the Schubert variety X_w is equal to the sum $\sum m_C(\lambda)$ over saturated chains C in the Bruhat order from id to w. Equivalently,*

$$\deg_\lambda(X_w) = \ell(w)! \cdot \mathfrak{D}_w(\lambda).$$

If $\lambda = \rho$, we will call $\deg(X_w) = \deg_\rho(X_w)$ simply the *degree* of X_w.

5. Harmonic polynomials

We discuss harmonic polynomials and the natural pairing on polynomials defined in terms of partial derivatives. Constructions in this section are essentially well-known; cf. Bergeron-Garsia [BG].

The space of polynomials $\mathbb{Q}[V]$ is the graded dual to $\mathbb{Q}[V^*]$, i.e., the corresponding finite-dimensional graded components are dual to each other.

Let us pick a basis $v_1, \ldots, v_r$ in V, and let $v_1^*, \ldots, v_r^*$ be the dual basis in V^*. For $f \in \mathbb{Q}[V^*]$ and $g \in \mathbb{Q}[V]$, let $f(x_1, \ldots, x_r) = f(x_1 v_1^* + \cdots + x_r v_r^*)$ and $g(y_1, \ldots, y_r) = g(y_1 v_1 + \cdots + y_r v_r)$ be polynomials in the variables $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$, correspondingly. For each $f \in \mathbb{Q}[V^*]$, let us define the differential operator $f(\partial/\partial y)$ that acts on the polynomial ring $\mathbb{Q}[V]$ by

$$f(\partial/\partial y) : g(y_1, \ldots, y_r) \longmapsto f(\partial/\partial y_1, \ldots, \partial/\partial y_r) \cdot g(y_1, \ldots, y_r),$$

where $\partial/\partial y_i$ denotes the partial derivative with respect to y_i. The operator $f(\partial/\partial y)$ can also be described without coordinates as follows. Let $d_v : \mathbb{Q}[V] \to \mathbb{Q}[V]$ be the differentiation operator in the direction of a vector $v \in V$ given by

$$d_v : g(y) \mapsto \left. \frac{d}{d\,t}\, g(y + t\, v) \right|_{t=0}. \tag{5.1}$$

The linear map $v \mapsto d_v$ extends to the homomorphism $f \mapsto d_f$ from the polynomial ring $\mathbb{Q}[V] = \mathit{Sym}(V^*)$ to the ring of operators on $\mathbb{Q}[V^*]$. Then $d_f = f(\partial/\partial y)$

One can extend the usual pairing between V and V^* to the following pairing between the spaces $\mathbb{Q}[V^*]$ and $\mathbb{Q}[V]$. For $f \in \mathbb{Q}[V^*]$ and $g \in \mathbb{Q}[V]$, let us define the *D-pairing* $(f, g)_D$ by

$$(f, g)_D = \mathrm{CT}\left(f(\partial/\partial y) \cdot g(y)\right) = \mathrm{CT}\left(g(\partial/\partial x) \cdot f(x)\right),$$

where the notation CT means taking the constant term of a polynomial.

A graded basis of a polynomial ring is a basis that consists of homogeneous polynomials. Let us say that a graded $\mathbb{Q}$-basis $\{f_u\}_{u \in U}$ in $\mathbb{Q}[V^*]$ is *D-dual* to a graded $\mathbb{Q}$-basis $\{g_u\}_{u \in U}$ in $\mathbb{Q}[V]$ if $(f_u, g_v)_D = \delta_{u,v}$, for any $u, v \in U$.

Example 5.1. Let $x^a = x_1^{a_1} \cdots x_r^{a_r}$ and $y^{(a)} = \frac{y_1^{a_1}}{a_1!} \cdots \frac{y_r^{a_r}}{a_r!}$, for $a = (a_1, \ldots, a_r)$. Then the monomial basis $\{x^a\}$ of $\mathbb{Q}[V^*]$ is D-dual to the basis $\{y^{(a)}\}$ of $\mathbb{Q}[V]$.

This example shows that the D-pairing gives a non-degenerate pairing of corresponding graded components of $\mathbb{Q}[V^*]$ and $\mathbb{Q}[V]$ and vanishes on different graded components. Thus, for a graded basis in $\mathbb{Q}[V^*]$, there exists a unique D-dual graded basis in $\mathbb{Q}[V]$ and vice versa.

For a graded space $A = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$, let A_∞ be the space of formal series $a_0 + a_1 + a_2 + \cdots$, where $a_i \in A^i$. For example, $\mathbb{Q}[V]_\infty = \mathbb{Q}[[V]]$ is the ring of formal power series. The exponential $e^{(x,y)} = e^{x_1 y_1 + \cdots + x_r y_r}$ given by its Taylor series can be regarded as an element of $\mathbb{Q}[[V^*, V]] := \mathbb{Q}[[V^*]] \otimes \mathbb{Q}[[V]]$, where (x, y) is the standard pairing between $x \in V^*$ and $y \in V$.

Proposition 5.2. *Let $\{f_u\}_{u\in U}$ be a graded basis for $\mathbb{Q}[V^*]$, and let $\{g_u\}_{u\in U}$ be a collection of formal power series in $\mathbb{Q}[[V]]$ labeled by the same set U. Then the following two conditions are equivalent:*

1. *The g_u are the homogeneous polynomials in $\mathbb{Q}[V]$ that form the D-dual basis to $\{f_u\}$.*
2. *The equality $e^{(x,y)} = \sum_{u\in U} f_u(x) \cdot g_u(y)$ holds identically in the ring of formal power series $\mathbb{Q}[[V^*, V]]$.*

Proof. For $f \in \mathbb{Q}[V^*]$, the action of the differential operator $f(\partial/\partial y)$ on polynomials extends to the action on the ring of formal power series $\mathbb{Q}[[V]]$ and on $\mathbb{Q}[[V^*, V]]$. The D-pairing $(f, g)_D$ makes sense for any $f \in \mathbb{Q}[V^*]$ and $g \in \mathbb{Q}[[V]]$. Let $C = \sum_{u\in U} f_u(x) \cdot g_u(y) \in \mathbb{Q}[[V^*, V]]$. Then $\mathrm{CT}\,(f_u(\partial/\partial y) \cdot C) = \sum_{v\in U} (f_u, g_v)_D\, f_v(x)$, for any $u \in U$.

Condition (1) is equivalent to the condition $\mathrm{CT}\,(f(\partial/\partial y) \cdot C) = f(x)$, for any basis element $f = f_u$ of $\mathbb{Q}[V^*]$. The latter condition is equivalent to condition (2), which says that $C = e^{(x,y)}$. Indeed, the only element $E \in \mathbb{Q}[[V^*, V]]$ that satisfies $\mathrm{CT}\,(f(\partial/\partial y) \cdot E) = f(x)$, for any $f \in \mathbb{Q}[V^*]$, is the exponent $E = e^{(x,y)}$. □

Let $I \subseteq \mathbb{Q}[V^*]$ be a graded ideal. Define the space of *I-harmonic polynomials* as

$$\mathcal{H}_I = \{g \in \mathbb{Q}[V] \mid f(\partial/\partial y) \cdot g(y) = 0, \text{ for any } f \in I\}.$$

Lemma 5.3. *The space $\mathcal{H}_I \subseteq \mathbb{Q}[V]$ is the orthogonal subspace to $I \subseteq \mathbb{Q}[V^*]$ with respect to the D-pairing. Thus $\mathcal{H}_I$ is the graded dual to the quotient space $\mathbb{Q}[V^*]/I$.*

Proof. The ideal I is orthogonal to $I^\perp := \{g \mid (f, g)_D = 0, \text{ for any } f \in I\}$. Clearly, $\mathcal{H}_I \subseteq I^\perp$. On the other hand, if $(f, g)_D = \mathrm{CT}\,(f(\partial/\partial y) \cdot g(y)) = 0$, for any $f \in I$, then $f(\partial/\partial y) \cdot g(y) = 0$, for any $f \in I$, because I in an ideal. Thus $\mathcal{H}_I = I^\perp$. □

Let $\bar{f} := f \pmod I$ denote the coset of a polynomial $f \in \mathbb{Q}[V^*]$ modulo the ideal I. For $g \in \mathcal{H}_I$, the differentiation $\bar{f}(\partial/\partial y) \cdot g := f(\partial/\partial y) \cdot g$ does not depend on the choice of a polynomial representative f of the coset $\bar{f}$. Thus we have correctly defined a D-pairing $(\bar{f}, g)_D := (f, g)_D$ between the spaces $\mathbb{Q}[V^*]/I$ and $\mathcal{H}_I$. Let us say that a graded basis $\{\bar{f}_u\}_{u\in U}$ of $\mathbb{Q}[V^*]/I$ and a graded basis $\{g_u\}_{u\in U}$ of $\mathcal{H}_I$ are *D-dual* if $(\bar{f}_u, g_v)_D = \delta_{u,v}$, for any $u, v \in U$.

Proposition 5.4. *Let $\{\bar{f}_u\}_{u\in U}$ be a graded basis of $\mathbb{Q}[V^*]/I$, and let $\{g_u\}_{u\in U}$ be a collection of of formal power series in $\mathbb{Q}[[V]]$ labeled by the same set U. Then the following two conditions are equivalent:*

(1) *The* g_u *are the polynomials that form the graded basis of* $\mathcal{H}_I$ *such that the bases* $\{\bar{f}_u\}_{u\in U}$ *and* $\{g_u\}_{u\in U}$ *are D-dual.*
(2) *The equality* $e^{(x,y)} = \sum_{u\in U} f_u(x)\cdot g_u(y)$ *modulo* $I_\infty \otimes \mathbb{Q}[[V]]$ *holds identically.*

Proof. Let us augment the set $\{f_u\}_{u\in U}$ by a graded $\mathbb{Q}$-basis $\{f_u\}_{u\in U'}$ of the ideal I. Then $\{f_u\}_{u\in U\cup U'}$ is a graded basis of $\mathbb{Q}[V^*]$. A collection $\{g_u\}_{u\in U}$ is the basis of $\mathcal{H}_I$ that is D-dual to $\{\bar{f}_u\}_{u\in U}$ if and only if there are elements $g_u \in \mathbb{Q}[V]$, for $u\in U'$, such that $\{f_u\}_{u\in U\cup U'}$ and $\{g_u\}_{u\in U\cup U'}$ are D-dual bases of $\mathbb{Q}[V^*]$ and $\mathbb{Q}[V]$, correspondingly. The claim now follows from Proposition 5.2. □

The *product map* $M : \mathbb{Q}[V^*]/I\otimes\mathbb{Q}[V^*]/I \to \mathbb{Q}[V^*]/I$ is given by $M : \bar{f}\otimes\bar{g} \mapsto \bar{f}\cdot\bar{g}$. Let us define *coproduct map* $\Delta : \mathcal{H}_I \to \mathcal{H}_I \otimes \mathcal{H}_I$ as the D-dual map to M. For $h \in \mathbb{Q}[V]$, the polynomial $h(y+z)$ of the sum of two vector variables $y, z \in V$ can be regarded as an element of $\mathbb{Q}[V]\otimes\mathbb{Q}[V]$.

Proposition 5.5. *The coproduct map* $\Delta : \mathcal{H}_I \to \mathcal{H}_I \otimes \mathcal{H}_I$ *is given by*

$$\Delta : g(y) \mapsto g(y+z),$$

for any $g \in \mathcal{H}_I$.

Proof. Let $\{\bar{f}_u\}_{u\in U}$ be a graded basis in $\mathbb{Q}[V^*]/I$ and let $\{g_u\}_{u\in U}$ be its D-dual basis in $\mathcal{H}_I$. We need to show that the two expressions

$$\bar{f}_u(x)\cdot\bar{f}_v(x) = \sum_{w\in U} a^w_{u,v}\,\bar{f}_w(x) \quad\text{and}\quad \bar{g}_w(y+z) = \sum_{u,v\in U} b^w_{u,v}\, g_u(y)\cdot g_v(z)$$

have the same coefficients $a^w_{u,v} = b^w_{u,v}$. Here $x\in V^*$ and $y,z\in V$. Indeed, according to Proposition 5.4, we have

$$\sum_{u,v,w} a^w_{u,v}\,\bar{f}_w(x)\cdot g_u(y)\cdot g_v(z) = \left(\sum_u \bar{f}_u(x)\cdot g_u(y)\right)\cdot\left(\sum_v \bar{f}_v(x)\cdot g_v(z)\right) =$$
$$= e^{(x,y)}\,e^{(x,z)} = e^{(x,y+z)} = \sum_w \bar{f}_w(x)\cdot g_w(y+z) = \sum_{u,v,w} b^w_{u,v}\,\bar{f}_w(x)\cdot g_u(y)\cdot g_v(z)$$

in the space $(\mathbb{Q}[V^*]/I\otimes\mathbb{Q}[V]\otimes\mathbb{Q}[V])_\infty$. This implies that $a^w_{u,v} = b^w_{u,v}$, for any $u,v,w\in U$. □

In what follows, we will assume and $I = \mathcal{I}_W \subset \mathbb{Q}[V^*]$ is the ideal generated by W-invariant polynomials without constant term, and $\mathbb{Q}[V^*]/I = H^*(G/B)$ is the cohomology ring of G/B. Let $\mathcal{H}_W = \mathcal{H}_{\mathcal{I}_W} \subset \mathbb{Q}[V]$ be its dual space with respect to the D-pairing. We will call $\mathcal{H}_W$ the *space of W-harmonic polynomials* and call its elements *W-harmonic polynomials* in $\mathbb{Q}[V]$.

6. Expressions for polynomials $\mathfrak{D}_{u,w}$

In this section, we give two different expressions for the polynomials $\mathfrak{D}_{u,w}$ and derive several corollaries.

Formula (4.2), for $u = id$, and Proposition 5.4 imply the following statement.

Corollary 6.1. (cf. Bernstein-Gelfand-Gelfand [BGG, Theorem 3.13]) *The collection of polynomials* $\mathfrak{D}_w$, $w\in W$, *forms a linear basis of the space* $\mathcal{H}_W \subset \mathbb{Q}[V]$ *of W-harmonic polynomials. This basis is D-dual to the basis* $\{\sigma_w\}_{w\in W}$ *of Schubert classes in* $H^*(G/B)$.

This basis of W-harmonic polynomials appeared in Bernstein-Gelfand-Gelfand [BGG, Theorem 3.13] (in somewhat disguised form) and more recently in Kriloff-Ram [KR, Sect. 2.2]; see Remark 6.6 below.

By the definition, the polynomial $\mathfrak{D}_{u,w}$ is given by a sum over saturated chains in the Bruhat order. However, this expression involves many summands and is difficult to handle. The following theorem given a more explicit formula for $\mathfrak{D}_{u,w}$.

Let $\sigma_w(\partial/\partial y)$ be the differential operator on the space of W-harmonic polynomials $\mathcal{H}_W$ given by $\sigma_w(\partial/\partial y) : g(y) \mapsto \mathfrak{S}_w(\partial/\partial y) \cdot g(y)$, where $\mathfrak{S}_w \in \mathbb{Q}[V^*]$ is any polynomial representative of the Schubert class σ_w. According to Section 5, $\sigma_w(\partial/\partial y)$ does not depend on the choice of a polynomial representative $\mathfrak{S}_w$.

Theorem 6.2. *For any $w \in W$, we have*

$$\mathfrak{D}_{u,w}(y) = \sigma_u(\partial/\partial y)\, \sigma_{w_\circ w}(\partial/\partial y) \cdot \mathfrak{D}_{w_\circ}(y).$$

In particular, all polynomials $\mathfrak{D}_{u,w}$ are W-harmonic.

Proof. According to (4.2), we have $\mathfrak{D}_{u,w}(\lambda) = \left\langle [e^\lambda] \cdot \sigma_u \cdot \sigma_{w_\circ w} \right\rangle$, for any weight $\lambda \in \Lambda$. Thus the W-harmonic polynomial $\mathfrak{D}_{u,w}$ is uniquely determined by the identity $(\sigma, \mathfrak{D}_{u,w})_D = \langle \sigma \cdot \sigma_u \cdot \sigma_{w_\circ w}\rangle$, for any $\sigma \in H^*(G/B)$. Let us show that the same identity holds for the W-harmonic polynomial $\tilde{\mathfrak{D}}_{u,w}(y) = \sigma_u(\partial/\partial y)\, \sigma_{w_\circ w}(\partial/\partial y) \cdot \mathfrak{D}_{w_\circ}(y)$. Indeed, $(\sigma, \tilde{\mathfrak{D}}_{u,w})_D$ equals

$$\mathrm{CT}\left(\sigma(\partial/\partial y) \cdot \sigma_u(\partial/\partial y) \cdot \sigma_{w_\circ w}(\partial/\partial y) \cdot \mathfrak{D}_{w_\circ}(y)\right) = (\sigma \cdot \sigma_u \cdot \sigma_{w_\circ w}, \mathfrak{D}_{w_\circ})_D.$$

Since $\{\mathfrak{D}_w\}_{w \in W}$ is the D-dual basis to $\{\sigma_w\}_{w\in W}$, the last expression is equal to triple intersection number $\langle \sigma \cdot \sigma_u \cdot \sigma_{w_\circ w}\rangle$, as needed. □

Corollary 9.2 below gives a simple multiplicative Vandermonde-like expression for $\mathfrak{D}_{w_\circ}$. Theorem 6.2, together with this expression, gives an explicit "top-to-bottom" differential formula for the W-harmonic polynomials $\mathfrak{D}_w$. Let us give an alternative "bottom-to-top" integral formula for these polynomials.

For $\alpha \in \Phi$, let I_α be the operator that acts on polynomials $g \in \mathbb{Q}[V]$ by

$$I_i : g(y) \mapsto \int_0^{(y,\alpha^\vee)} g(y - \alpha t)\, dt. \tag{6.1}$$

In other words, the operator I_α integrates a polynomial g on the line interval $[y, s_\alpha(y)] \subset V$. Clearly, this operator increases the degree of polynomials by 1.

Recall that $A_\alpha : \mathbb{Q}[V^*] \to \mathbb{Q}[V^*]$ is the BGG operator given by (3.2).

Lemma 6.3. *For $\alpha \in \Phi$, the operator I_α is adjoint to the operator A_α with respect to the D-pairing. In other words,*

$$(f, I_\alpha(g))_D = (A_\alpha(f), g)_D, \tag{6.2}$$

for any polynomials $f \in \mathbb{Q}[V^*]$ and $g \in \mathbb{Q}[V]$.

Proof. Let us pick a basis $v_1, \ldots, v_r$ in V and its dual basis $v_1^*, \ldots, v_r^*$ in V^* such that $v_1 = \alpha$ and $(v_i, \alpha) = 0$, for $i = 2, \ldots, r$. Let $f(x_1, \ldots, x_r) = f(x_1 v_1^* + \cdots + x_r v_r^*)$ and $g(y_1, \ldots, y_r) = g(y_1 v_1 + \cdots + y_r v_r)$, for $f \in \mathbb{Q}[V^*]$ and $g \in \mathbb{Q}[V]$. In these coordinates, the operators A_α and I_α can be written as

$$A_\alpha : f(x_1,\dots,x_r) \mapsto \frac{f(x_1,x_2,\dots,x_r) - f(-x_1,x_2,\dots,x_r)}{x_1}$$

$$I_\alpha : g(y_1,\dots,y_r) \mapsto \int_{-y_1}^{y_1} g(t,y_2,\dots,y_r)\,dt.$$

These operators are linear over $\mathbb{Q}[x_2,\dots,x_r]$ and $\mathbb{Q}[y_2,\dots,y_r]$, correspondingly. It is enough to verify identity (6.2) for $f = x_1^{m+1}$ and $g = y_1^m$. For these polynomials, we have $A_\alpha(f) = 2x_1^m$, $I_\alpha(g) = \frac{2}{m+1} y_1^{m+1}$, if m is even; and $A_\alpha(f) = 0$, $I_\alpha(g) = 0$, if m is odd. Thus $(f, I_\alpha(g))_D = (A_\alpha(f), g)_D$ in both cases. □

Let $I_i = I_{\alpha_i}$, for $i = 1,\dots,r$.

Corollary 6.4. *The operators I_i satisfy the nilCoxeter relations $(I_iI_j)^{m_{ij}} = 1$ and $(I_i)^2 = 0$. Also, if $I_\alpha(g) = 0$, then g is an anti-symmetric polynomial with respect to the reflection s_α, and thus, g is divisible by the linear form $(y, \alpha^\vee) \in \mathbb{Q}[V]$.*

Proof. The first claim follows from the fact that the BGG operators A_i satisfy the nilCoxeter relations. The second claim is clear from the formula for I_α given in the proof of Lemma 6.3. □

For a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, let us define $I_w = I_{i_1} \cdots I_{s_l}$. The operator I_w depends only on w and does not depend on the choice of reduced decomposition. Lemma 6.3 implies that the operator $A_w : \mathbb{Q}[V^*] \to \mathbb{Q}[V^*]$ is adjoint to the operator $I_{w^{-1}} : \mathbb{Q}[V] \to \mathbb{Q}[V]$ with respect to the D-pairing.

Theorem 6.5. (cf. Bernstein-Gelfand-Gelfand [BGG, Theorem 3.12]) *For any $w \in W$ and $i = 1,\dots,r$, we have*

$$I_i \cdot \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{ws_i} & \text{if } \ell(ws_i) = \ell(w) + 1, \\ 0 & \text{if } \ell(ws_i) = \ell(w) - 1. \end{cases}$$

Thus the polynomials $\mathfrak{D}_w$ are given by

$$\mathfrak{D}_w = I_{w^{-1}}(1).$$

Proof. Follows from Bernstein-Gelfand-Gelfand formula (3.3), Corollary 6.1, and Lemma 6.3. □

Remark 6.6. Theorem 6.5 is essentially contained in [BGG]. However, Bernstein-Gelfand-Gelfand treated the $\mathfrak{D}_w$ not as (harmonic) polynomials but as linear functionals on $\mathbb{Q}[V^*]/\mathcal{I}_W$ obtained from Id by appying the operators adjoint to the divided difference operators operators (with respect to the natural pairing between a linear space and its dual). It is immediate that these functionals form a basis in $(\mathbb{Q}[V^*]/\mathcal{I}_W)^* \simeq \mathcal{H}_W$; see [BGG, Theorem 3.13] and [KR, Sect. 2.2]. Note that there are several other constructions of bases of $\mathcal{H}_W$; see, e.g., Hulsurkar [Hul].

In the next section we show that Duan's recent result [Duan] about degrees of Schubert varieties easily follows from Theorem 6.5. Note that this integral expression for the polynomials $\mathfrak{D}_w$ can be formulated in the general Kac-Moody setup. Indeed, unlike the previous expression given by Theorem 6.2, it does not use the longest Weyl group element $w_\circ$, which exists in finite types only.

For $I \subseteq \{1, \dots, r\}$, let W_I be the parabolic subgroup in W generated by s_i, $i \in I$. Let $\Phi_I^+ = \{\alpha \in \Phi^+ \mid s_\alpha \in W_I\}$.

Proposition 6.7. *Let $w \in W$. Let $I = \{i \mid \ell(ws_i) < \ell(w)\}$ be the descent set of w. Then the polynomial $\mathfrak{D}_w(y)$ is divisible by the product $\prod_{\alpha\in\Phi_I^+}(y, \alpha^\vee) \in \mathbb{Q}[V]$.*

Proof. According to Corollary 6.4, it is enough to check that $I_\alpha(\mathfrak{D}_w) = 0$, for any $\alpha \in \Phi_I^+$. We have $I_\alpha(\mathfrak{D}_w) = I_\alpha I_{w^{-1}}(1)$. The operator $I_\alpha I_{w^{-1}}$ is adjoint to $A_w A_\alpha$ with respect to the D-pairing. Let us show that $A_w A_\alpha = 0$, identically. Notice that $s_i A_\alpha = A_{s_i(\alpha)} s_i$, where s_i is regarded as an operator on the polynomial ring $\mathbb{Q}[V^*]$. Also $A_i = s_i A_i = -A_i s_i$. Thus, for any i in the descent set I, we can write

$$A_w A_\alpha = A_{w'} A_i A_\alpha = -A_{w'} A_i s_i A_\alpha = -A_{w'} A_i A_{s_i(\alpha)} s_i = -A_w A_{s_i(\alpha)} s_i,$$

where $w' = ws_i$. Since $s_\alpha \in W_I$, there is a sequence $i_1, \dots, i_l \in I$ and $j \in I$ such that $s_{i_1} \cdots s_{i_l}(\alpha) = \alpha_j$. Thus

$$A_w A_\alpha = \pm A_w A_j s_{i_1} \cdots s_{i_l} = \pm A_{w''} A_j A_j s_{i_1} \cdots s_{i_l} = 0,$$

as needed. □

Corollary 6.8. *Fix $I \subseteq \{1, \dots, r\}$. Let w_I be the longest element in the parabolic subgroup W_I. Then*

$$\mathfrak{D}_{w_I}(y) = \mathrm{Const} \cdot \prod_{\alpha\in\Phi_I^+} (y, \alpha^\vee),$$

where Const $\in \mathbb{Q}$.

Proof. Proposition 6.7 says that the polynomial $\mathfrak{D}_{w_I}(y)$ is divisible by the product $\prod_{\alpha\in\Phi_I^+}(y, \alpha^\vee)$. Since the degree of this polynomial equals

$$\deg \mathfrak{D}_{w_I} = \ell(w_I) = |\Phi_I^+| = \deg \prod_{\alpha\in\Phi_I^+} (y, \alpha^\vee),$$

we deduce the claim. □

In Section 9 below, we will give an alternative derivation for this multiplicative expression for $\mathfrak{D}_{w_I}$; see Corollary 9.2. We will show that the constant Const in Corollary 6.8 is given by the condition $\mathfrak{D}_{w_I}(\rho) = 1$.

We can express the generalized Littlewood-Richardson coefficients $c_{u,v}^w$ using the polynomials $\mathfrak{D}_{u,w}$ in two different ways.

Corollary 6.9. *For any $u \leq w$ in W, we have*

$$\mathfrak{D}_{u,w} = \sum_{v\in W} c_{u,v}^w \, \mathfrak{D}_v.$$

The polynomials $\mathfrak{D}_{u,w}$ extend the polynomials $\mathfrak{D}_v$ in the same way as the skew Schur polynomials extend the usual Schur polynomials. Compare Corollary 6.9 with a similar formula for the skew Schubert polynomials of Lenart and Sottile [LeS].

Proof. Let us expand the W-harmonic polynomial $\mathfrak{D}_{u,w}$ in the basis $\{\mathfrak{D}_v \mid v \in W\}$. According to Theorem 6.2, the coefficient of $\mathfrak{D}_v$ in this expansion is equal to the coefficient of $\sigma_{w_\circ v}$ in the expansion of the product $\sigma_u \cdot \sigma_{w_\circ w}$ in the Schubert classes. This coefficient equals $c_{u,w_\circ w}^{w_\circ v} = c_{u,v,w_\circ w} = c_{u,v}^w$. □

Proposition 5.5 implies the following statement.

Corollary 6.10. *For $w \in W$, we have the equality*[3]

$$\mathfrak{D}_w(y+z) = \sum_{u,v\in W} c^w_{u,v}\, \mathfrak{D}_u(y)\cdot \mathfrak{D}_v(z)$$

of polynomials in $y, z \in V$.

Compare Corollary 6.10 with the coproduct formula [EC2, Eq. (7.66)] for Schur polynomials.

7. Examples and Duan's formula

Let us calculate several polynomials $\mathfrak{D}_w$ using Theorem 6.5. Let $Y_1, \dots, Y_r$ be the generators of $\mathbb{Q}[V]$ given by $Y_i = (y, \alpha_i^\vee)$, and let $a_{ij} = (\alpha_i^\vee, \alpha_j)$ be the Cartan integers, for $1 \le i, j \le r$. For a simple reflection $w = s_i$, we obtain

$$\mathfrak{D}_{s_i} = I_i(1) = \int_0^{(y,\alpha_i^\vee)} 1\cdot dt = (y, \alpha_i^\vee) = Y_i.$$

For $w = s_i s_j$, we obtain

$$\mathfrak{D}_{s_i s_j} = I_j I_i(1) = I_j(Y_i) = I_j((y,\alpha_i^\vee)) = \int_0^{(y,\alpha_j^\vee)} (y - t\,\alpha_j,\, \alpha_i^\vee)\, dt =$$

$$= (y,\alpha_i^\vee)\int_0^{(y,\alpha_j^\vee)} dt - (\alpha_j, \alpha_i^\vee)\int_0^{(y,\alpha_j^\vee)} t\, dt = Y_i Y_j - a_{ij}\,\frac{Y_j^2}{2}.$$

We can further iterate this procedure. The following lemma is obtained immediately from the definition of I_j's, as shown above.

Lemma 7.1. *For any $1 \le i_1, \dots, i_n, j \le r$ and $c_1, \dots, c_n \in \mathbb{Z}_{\ge 0}$, the operator I_j maps the monomial $Y_{i_1}^{c_1}\cdots Y_{i_n}^{c_n}$ to $I_j(Y_{i_1}^{c_1}\cdots Y_{i_n}^{c_n}) =$*

$$\sum_{k_1+\cdots+k_n=k} (-1)^k \binom{c_1}{k_1}\cdots\binom{c_n}{k_n} a_{i_1 j}^{k_1}\cdots a_{i_n j}^{k_n}\, Y_{i_1}^{c_1-k_1}\cdots Y_{i_n}^{c_n-k_n}\,\frac{Y_j^{k+1}}{k+1},$$

where the sum is over $k_1, \dots, k_n$ such that $\sum k_i = k$, $0 \le k_i \le c_i$, for $i = 1, \dots, n$.

For example, for $w = s_i s_j s_k$, we obtain

$$\mathfrak{D}_{s_i s_j s_k} = I_k I_j I_i(1) = I_k(Y_i Y_j - a_{ij}\,\tfrac{Y_j^2}{2}) = Y_i Y_j Y_k - a_{ik}\, Y_j\,\tfrac{Y_k^2}{2} -$$

$$- a_{jk}\, Y_i\,\tfrac{Y_k^2}{2} + a_{ik}a_{jk}\,\tfrac{Y_k^3}{3} - a_{ij}\,\tfrac{Y_j^2}{2} Y_k + a_{ij}a_{jk} Y_j\,\tfrac{Y_k^2}{2} - a_{ij}a_{jk}^2\tfrac{1}{2}\,\tfrac{Y_k^3}{3}.$$

Let us fix $w \in W$ together with its reduced decomposition $w = s_{i_1}\cdots s_{i_l}$. Applying Lemma 7.1 repeatedly for the calculation of $\mathfrak{D}_w = I_{i_l}\cdots I_{i_1}(1)$, and transferring the sequences of integers $(k_1, \dots, k_n)$, $n = 1, 2, \dots, l-1$, to the columns of a triangular array (k_{pq}), we deduce the following result.

Corollary 7.2. [Duan] *For a reduced decomposition $w = s_{i_1}\cdots s_{i_l} \in W$, we have*

$$\mathfrak{D}_w(y) = \sum_{(k_{pq})} \prod_{1\le p<q\le l} \frac{(-a_{i_p i_q})^{k_{pq}}}{k_{pq}!} \prod_{s=1}^{l} \frac{K_{*s}!\, Y_p^{K_{*s}+1-K_{s*}}}{(K_{*s}+1-K_{s*})!},$$

[3]Here $y+z$ denotes the usual sum of two vectors. This notation should not be confused with the λ-ring notation for symmetric functions, where $y+z$ means the union of two sets of variables.

where the sum is over collections of nonnegative integers $(k_{pq})_{1\leq p<q\leq l}$ *such that* $K_{*s}+1\geq K_{s*}$, *for* $s=1,\dots,l$; *and* $K_{*s}=\sum_{p<s}k_{ps}$ *and* $K_{s*}=\sum_{q>s}k_{sq}$.

This result is equivalent to Duan's recent result [Duan] about degrees $\deg(X_w)=\ell(w)!\,\mathfrak{D}_w(\rho)$ of Schubert varieties. Note that the approach and notations of [Duan] are quite different from ours.

8. K-theory and Demazure modules

In this section, we recall a few facts about the K-theory for G/B.

Denote by $K(G/B)=K(G/B,\mathbb{Q})$ the *Grothendieck ring* of coherent sheaves on G/B with rational coefficients. Let $\mathbb{Q}[\Lambda]$ be the group algebra of the weight lattice Λ. It has a linear basis of formal exponents $\{e^\lambda \mid \lambda\in\Lambda\}$ with multiplication $e^\lambda\cdot e^\mu=e^{\lambda+\mu}$, i.e., $\mathbb{Q}[\Lambda]$ is the algebra of Laurent polynomials in the variables $e^{\omega_1},\cdots,e^{\omega_r}$. The action of the Weyl group on Λ extends to a W-action on $\mathbb{Q}[\Lambda]$. Let $\epsilon:\mathbb{Q}[\Lambda]\to\mathbb{Q}$ be the linear map such that $\epsilon(e^\lambda)=1$, for any $\lambda\in\Lambda$, i.e., $\epsilon(f)$ is the sum of coefficients of exponents in f. Then the Grothendieck ring $K(G/B)$ is canonically isomorphic[4] to the quotient ring:

$$K(G/B)\simeq\mathbb{Q}[\Lambda]/\mathcal{J}_W,$$

where $\mathcal{J}_W=\left\langle f\in\mathbb{Q}[\Lambda]^W \mid \epsilon(f)=0\right\rangle$ is the ideal generated by W-invariant elements $f\in\mathbb{Q}[\Lambda]$ with $\epsilon(f)=0$. Let us identify the Grothendieck ring $K(G/B)$ with the quotient $\mathbb{Q}[\Lambda]/\mathcal{J}_W$ via this isomorphism. Since ϵ annihilates the ideal $\mathcal{J}_W$, it induces the map $\epsilon:K(G/B)\to\mathbb{Q}$, which we denote by the same letter.

The *Demazure operators* $T_i:\mathbb{Q}[\Lambda]\to\mathbb{Q}[\Lambda]$, $i=1,\dots,r$, are given by

$$T_i: f\mapsto\frac{f-e^{-\alpha_i}\,s_i(f)}{1-e^{-\alpha_i}}. \tag{8.1}$$

The Demazure operators satisfy the Coxeter relations $(T_i\,T_j)^{m_{ij}}=1$ and $(T_i)^2=T_i$. For a reduced decomposition $w=s_{i_1}\cdots s_{i_l}\in W$, define $T_w=T_{i_1}\cdots T_{i_l}$. The operator T_w depends only on $w\in W$ and does not depend on a choice of reduced decomposition. The operators T_i commute with operators of multiplication by W-invariant elements. Thus the T_i preserve the ideal $\mathcal{J}_W$ and induce operators acting on the Grothendieck ring $K(G/B)$, which we will denote by same symbols T_i.

The *Grothendieck classes* $\gamma_w\in K(G/B)$, $w\in W$, can be constructed, as follows.

$$\gamma_{w_\circ}=|W|^{-1}\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})\pmod{\mathcal{J}_W};$$
$$\gamma_w=T_{w^{-1}w_\circ}(\gamma_{w_\circ}),\quad\text{for any } w\in W.$$

According to Demazure [Dem], the classes γ_w are the K-theoretic classes $[\mathcal{O}_X]_K$ of the structure sheaves of Schubert varieties $X=X_{w_\circ w}$. In particular, $\gamma_{id}=[\mathcal{O}_{G/B}]_K=1$. The classes γ_w, $w\in W$, form a linear basis of $K(G/B)$.

Moreover, we have (see [Dem])

$$T_i(\gamma_w)=\begin{cases}\gamma_{ws_i} & \text{if } \ell(ws_i)=\ell(w)-1,\\ \gamma_w & \text{if } \ell(ws_i)=\ell(w)+1.\end{cases} \tag{8.2}$$

The *Chern character* is the ring isomorphism $\chi:K(G/B)\to H^*(G/B)$ induced by the map $\chi:e^\lambda\mapsto[e^\lambda]$, for $\lambda\in\Lambda$, where $[e^\lambda]:=1+\bar\lambda+\bar\lambda^2/2!+\cdots\in H^*(G/B)$

[4]The isomorphism is given by sending the K-theoretic class $[\mathcal{L}_\lambda]_K\in K(G/B)$ of the line bundle $\mathcal{L}_\lambda$ to the coset $e^\lambda\pmod{\mathcal{J}_W}$, for any $\lambda\in\Lambda$.

and $\bar{\lambda} = c_1(\mathcal{L}_\lambda)$, as before. The isomorphism χ relates the Grothendieck classes γ_w with the Schubert classes σ_w by a triangular transformation:

$$\chi : \gamma_w \mapsto \sigma_w + \text{higher degree terms.} \tag{8.3}$$

For a dominant weight $\lambda \in \Lambda^+$, let V_λ denote the finite dimensional irreducible representation of the Lie group G with highest weight λ. For $\lambda \in \Lambda^+$ and $w \in W$, the *Demazure module* $V_{\lambda,w}$ is the B-module that is dual to the space of global sections of the line bundle $\mathcal{L}_\lambda$ on the Schubert variety X_w:

$$V_{\lambda,w} = H^0(X_w, \mathcal{L}_\lambda)^*.$$

For the longest Weyl group element $w = w_\circ$, the space $V_{\lambda,w_\circ} = H^0(G/B, \mathcal{L}_\lambda)^*$ has the structure of a G-module. The classical *Borel-Weil theorem* says that $V_{\lambda,w_\circ}$ is isomorphic to the irreducible G-module V_λ. Formal characters of Demazure modules are given by $ch(V_{\lambda,w}) = \sum_{\mu\in\Lambda} m_{\lambda,w}(\mu)\, e^\mu \in \mathbb{Z}[\Lambda]$, where $m_{\lambda,w}(\mu)$ is the multiplicity of weight μ in $V_{\lambda,w}$. They generalize characters of irreducible representations $ch(V_\lambda) = ch(V_{\lambda,w_\circ})$. *Demazure's character formula* [Dem] says that the character $ch(V_{\lambda,w})$ is given by

$$ch(V_{\lambda,w}) = T_w(e^\lambda). \tag{8.4}$$

9. Asymptotic expression for degree

Proposition 9.1. *For any $w \in W$, the dimension of the Demazure module $V_{\lambda,w}$ is a polynomial in λ of degree $\ell(w)$. The polynomial $\mathfrak{D}_w$ is the leading homogeneous component of the polynomial* $\dim V_{\lambda,w} \in \mathbb{Q}[V]$. *In other words, the value $\mathfrak{D}_w(\lambda)$ equals*

$$\mathfrak{D}_w(\lambda) = \lim_{k\to\infty} \frac{\dim V_{k\lambda,w}}{k^{\ell(w)}},$$

for any $\lambda \in \Lambda^+$.

Proposition 9.1 together with Weyl's dimension formula implies the following statement, which was derived by Stembridge using Standard Monomial Theory.

Corollary 9.2. [Stem, Theorem 1.1] *For the longest Weyl group element $w = w_\circ$, we have*

$$\mathfrak{D}_{w_\circ}(y) = \prod_{\alpha\in\Phi^+} \frac{(y, \alpha^\vee)}{(\rho, \alpha^\vee)}.$$

Proof. Weyl's formula says that the dimension of $V_{\lambda,w_\circ} = V_\lambda$ is

$$\dim V_\lambda = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\rho, \alpha^\vee)}{(\rho, \alpha^\vee)}.$$

Taking the leading homogeneous component of this polynomial in λ, we prove the claim for $y = \lambda \in \Lambda^+$, and thus, for any $y \in V$. □

In order to prove Proposition 9.1 we need the following lemma.

Lemma 9.3. *The map $\epsilon : K(G/B) \to \mathbb{Q}$ is given by $\epsilon(\gamma_w) = \delta_{w,id}$, for any $w \in W$.*

Proof. It follows directly from the definitions that the Chern character χ translates ϵ to the map $\epsilon \cdot \chi^{-1} : H^*(G/B) \to \mathbb{Q}$ given by $\epsilon \cdot \chi^{-1} : \bar{f} \mapsto f(0)$, for a polynomial representative $f \in \mathbb{Q}[\mathfrak{h}]$ of $\bar{f}$. Thus $\epsilon \cdot \chi^{-1}(\sigma_w) = \delta_{w,id}$. Indeed, $\sigma_{id} = 1$ and all other Schubert classes σ_w have zero constant term, for $w \neq id$. Triangularity (8.3) of the Chern character implies the needed statement. □

Proof of Proposition 9.1. The preimage of identity (4.2), for $u = id$, under the Chern character χ is the following expression in $K(G/B)$:

$$e^\lambda = \sum_{w \in W} \mathfrak{D}_w(\lambda)\, \chi^{-1}(\sigma_w) = \sum_{w \in W} \hat{\mathfrak{D}}_w(\lambda)\, \gamma_w,$$

Triangularity (8.3) implies that $\chi^{-1}(\sigma_w) = \gamma_w + \sum_{\ell(u)>\ell(w)} c_{w,u}\, \gamma_u$ and $\hat{\mathfrak{D}}_w = \mathfrak{D}_w + \sum_{\ell(u)<\ell(w)} c_{u,w}\, \mathfrak{D}_u$, for some coefficients $c_{w,u} \in \mathbb{Q}$. Thus the homogeneous polynomial $\mathfrak{D}_w$ is the leading homogeneous component of the polynomial $\hat{\mathfrak{D}}_w$. Applying the map $\epsilon \cdot T_w$ to both sides of the previous expression and using Lemma 9.3, we obtain

$$\epsilon(T_w(e^\lambda)) = \sum_{u \leq w} \hat{\mathfrak{D}}_u(\lambda).$$

Indeed, according to (8.2), the coefficient of γ_{id} in $T_w(\gamma_u)$ is equal to 1 if $u \leq w$, and 0 otherwise. Thus $\epsilon(T_w(e^\lambda))$ is a polynomial in λ of degree $\ell(w)$ and its leading homogeneous component is again $\mathfrak{D}_w$. But, Demazure's character formula says that $T_w(e^\lambda)$ is the character of $V_{\lambda,w}$ and $\epsilon(T_w(e^\lambda)) = \dim V_{\lambda,w}$. □

Lakshmibai reported the following simple geometric proof of Proposition 9.1. Assume that λ is a dominant regular weight. We have $V^*_{w,k\lambda} = H^0(X_w, \mathcal{L}_{k\lambda}) = R_k$, where R_k is the k-th graded component of the coordinate ring R of the image of X_w in $\mathbb{P}(V_\lambda)$. The Hilbert polynomial of the coordinate ring has the form $\mathrm{Hilb}_R(k) = \dim R_k = A\, k^l/l! + (\text{lower degree terms})$, where $l = \dim_{\mathbb{C}} X_w = \ell(w)$, and $A = \deg_\lambda(X_w)$ is the degree of X_w in $\mathbb{P}(V_\lambda)$. Thus $\lim_{k\to\infty} \dim V_{k\lambda,w}/k^{\ell(w)} = A/l! = \deg_\lambda(X_w)/\ell(w)! = \mathfrak{D}_w(\lambda)$.

10. Permanent of the matrix of Cartan integers

Let us give a curious consequence of Theorem 6.2.

Corollary 10.1. *Let $A = (a_{\alpha,\beta})$ be the $N \times N$-matrix, $N = |\Phi^+|$, formed by the Cartan integers $a_{\alpha,\beta} = (\alpha, \beta^\vee)$, for $\alpha, \beta \in \Phi^+$. Then the permanent of the matrix A equals*

$$\mathrm{per}(A) = |W| \cdot \prod_{\alpha \in \Phi^+} (\rho, \alpha^\vee).$$

The matrix A should not be confused with the Cartan matrix. The latter is a certain $r \times r$-submatrix of A.

Proof. According to Theorem 6.2 and Corollary 9.2, we have

$$1 = \mathfrak{D}_{id} = \sigma_{w_\circ}(\partial/\partial y) \cdot \mathfrak{D}_{w_\circ}(y) = \left(\frac{1}{|W|} \prod_{\alpha \in \Phi^+} d_\alpha\right) \cdot \left(\prod_{\beta \in \Phi^+} \frac{(y, \beta^\vee)}{(\rho, \beta^\vee)}\right),$$

where d_α is the operator of differentiation with respect to a root α given by (5.1). Using the product rule for differentiation and the fact that $d_\alpha \cdot (y, \beta^\vee) = (\alpha, \beta^\vee)$, we derive the claim. □

For type A_{n-1}, we obtain the following result.

Corollary 10.2. *Let $B = (b_{ij,k})$ be the $\binom{n}{2} \times n$-matrix with rows labeled by pairs $1 \le i < j \le n$ and columns labeled by $k = 1, \ldots, n$ such that $b_{ij,k} = \delta_{i,j} - \delta_{j,k}$. Then*

$$\mathrm{per}(B \cdot B^T) = 1!\,2! \cdots n!.$$

Proof. For type A_{n-1}, the matrix A in Corollary 10.1 equals $B \cdot B^T$. □

This claim can be also derived from the Cauchy-Binet formula for permanents. For example, for type A_3, we have

$$\mathrm{per}\left(\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}\right) = 1!\,2!\,3!\,4!.$$

Note that the rank of the $\binom{n}{2} \times \binom{n}{2}$-matrix $B \cdot B^T$ is at most $n-1$. Thus the determinant of this matrix is zero, for $n \ge 3$. It would be interesting to find a combinatorial proof of Corollary 10.2.

11. Schubert polynomials

In the rest of the paper we will be mainly concerned with the case $G = SL_n$.

The root system Φ associated to SL_n is of the type A_{n-1}. In this case, the spaces V can be presented as $V = \mathbb{Q}^n/(1, \ldots, 1)\mathbb{Q}$. Then $\Phi = \{\varepsilon_i - \varepsilon_j \in V \mid 1 \le i \ne j \le n\}$, where the ε_i are images of the coordinate vectors in $\mathbb{Q}^n$. The Weyl group is the symmetric group $W = S_n$ of order n that acts on V by permuting the coordinates in $\mathbb{Q}^n$. The Coxeter generators are the adjacent transpositions $s_i = (i, i+1)$. The length $\ell(w)$ of a permutation $w \in S_n$ is the number of inversions in w. The longest permutation in S_n is $w_\circ = n, n-1, \cdots, 2, 1$.

The quotient SL_n/B is the classical complex flag variety. Its cohomology ring $H^*(SL_n/B)$ over $\mathbb{Q}$ is canonically identified with the quotient

$$H^*(SL_n/B) = \mathbb{Q}[x_1, \ldots, x_n]/\mathcal{I}_n,$$

where $\mathcal{I}_n = \langle e_1, \ldots, e_n \rangle$ is the ideal generated by the elementary symmetric polynomials e_i in the variables $x_1, \ldots, x_n$. The divided difference operators A_i act on the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ by

$$A_i : f(x_1, \ldots, x_n) \mapsto \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+1}, \ldots, x_n)}{x_i - x_{i+1}}.$$

For a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, let $A_w = A_{i_1} \cdots A_{i_l}$.

Lascoux and Schützenberger [LS1] defined the *Schubert polynomials* $\mathfrak{S}_w$, for $w \in S_n$, by

$$\mathfrak{S}_{w_\circ} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \quad \text{and} \quad \mathfrak{S}_w = A_{w^{-1}w_\circ}(\mathfrak{S}_{w_\circ}).$$

Then the cosets of Schubert polynomials $\mathfrak{S}_w$ modulo the ideal $\mathcal{I}_n$ are the Schubert classes $\sigma_w = \bar{\mathfrak{S}}_w$ in $H^*(SL_n/B)$.

This particular choice of polynomial representatives for the Schubert classes has the following *stability property.* The symmetric group S_n is naturally embedded into S_{n+1} as the set of order $n+1$ permutations that fix the element $n+1$. Then the Schubert polynomials remain the same under this embedding.

Let S_∞ be the injective limit of symmetric groups $S_1 \hookrightarrow S_2 \hookrightarrow S_3 \hookrightarrow \cdots$. In other words, S_∞ is the group of infinite permutations $w : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that $w(i) = i$ for almost all i's. We think of S_n as the subgroup of infinite permutations $w \in S_\infty$ that fix all $i > n$. Let $\mathbb{Q}[x_1, x_2, \dots]$ be the polynomial ring in infinitely many variables $x_1, x_2, \dots$. The stability of the Schubert polynomials under the embedding $S_n \hookrightarrow S_{n+1}$ implies that the Schubert polynomials $\mathfrak{S}_w \in \mathbb{Q}[x_1, x_2, \dots]$ are consistently defined for any $w \in S_\infty$. Moreover, $\{\mathfrak{S}_w\}_{w \in S_\infty}$ is a basis of the polynomial ring $\mathbb{Q}[x_1, x_2, \dots]$.

12. Degree polynomials for type A

Let us summarize properties of the polynomials $\mathfrak{D}_{u,w}$ for type A_{n-1}.

Let $y_1, \dots, y_n$ be independent variables. Let us assign to each edge $w \lessdot ws_{ij}$ in the Hasse diagram of the Bruhat order on S_n the weight $m(w, ws_{ij}) = y_i - y_j$. For a saturated chain $C = (u_0 \lessdot u_1 \lessdot u_2 \lessdot \cdots \lessdot u_l)$ in the Bruhat order, we define its weight as $m_C(y) = m(u_0, u_1)\, m(u_1, u_2) \cdots m(u_{l-1}, u_l)$.

For $u, w \in S_n$ such that $u \le w$, the polynomial $\mathfrak{D}_{u,w} \in \mathbb{Q}[y_1, \dots, y_n]$ is defined as the sum

$$\mathfrak{D}_{u,w} = \frac{1}{\ell(w)!} \sum_C m_C(y)$$

over all saturated chains $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_l)$ from $u_0 = u$ to $u_l = w$ in the Bruhat order. Also $\mathfrak{D}_w := \mathfrak{D}_{id,w}$.

The subspace $\mathcal{H}_n$ of S_n-harmonic polynomials in $\mathbb{Q}[y_1, \dots, y_n]$ is given by

$$\mathcal{H}_n = \{g \in \mathbb{Q}[y_1, \dots, y_n] \mid f(\partial/\partial y_1, \dots, \partial/\partial y_n) \cdot g(y_1, \dots, y_n) = 0 \text{ for any } f \in \mathcal{I}_n\}.$$

Corollary 12.1. (1) *The polynomials $\mathfrak{D}_w$, $w \in S_n$, form a basis of $\mathcal{H}_n$.*

(2) *The polynomials $\mathfrak{D}_{u,w}$, $u, w \in S_n$, can be expressed as*

$$\mathfrak{D}_{w_\circ} = \frac{1}{1!\,2! \cdots (n-1)!} \prod_{1 \le i < j \le n} (y_i - y_j) = \det\left(\left(y_i^{(n-j)}\right)_{i,j=1}^{n}\right),$$

$$\mathfrak{D}_{u,w} = \mathfrak{S}_u(\partial/\partial y_1, \dots, \partial/\partial y_n)\, \mathfrak{S}_{w_\circ w}(\partial/\partial y_1, \dots, \partial/\partial y_n) \cdot \mathfrak{D}_{w_\circ},$$

where $a^{(b)} = \frac{a^b}{b!}$.

(3) *The polynomials $\mathfrak{D}_w$, $w \in S_n$, can be also expressed as*

$$\mathfrak{D}_w = I_{w^{-1}}(1),$$

where $I_w = I_{i_1} \cdots I_{i_l}$, for a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, and the operators $I_1, \dots, I_{n-1}$ on $\mathbb{Q}[y_1, \dots, y_n]$ are given by

$$I_i : g(y_1, \dots, y_n) \mapsto \int_0^{y_i - y_{i+1}} g(y_1, \dots, y_{i-1}, y_i - t, y_{i+1} + t, y_{i+2}, \dots, y_n)\, dt.$$

The following symmetries are immediate from the definition of the polynomials $\mathfrak{D}_w$.

Lemma 12.2. (1) *For any $w \in S_n$, we have*

$$\mathfrak{D}_w(y_1, \ldots, y_n) = \mathfrak{D}_{w_\circ w w_\circ}(-y_n, \ldots, -y_1).$$

(2) *Also* $\mathfrak{D}_w(y_1 + c, \ldots, y_n + c) = \mathfrak{D}_w(y_1, \ldots, y_n)$, *for any constant* c.

The spaces $\mathcal{H}_n$ of S_n-harmonic polynomials are embedded in the polynomial ring $\mathbb{Q}[y_1, y_2, \ldots]$ in infinitely many variables: $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 \subset \cdots \subset \mathbb{Q}[y_1, y_2, \ldots]$. Moreover, the union of all $\mathcal{H}_n$'s is exactly this polynomial ring. It is clear from the definition that the polynomials $\mathfrak{D}_w$ are stable under the embedding $S_n \hookrightarrow S_{n+1}$. Thus the polynomials $\mathfrak{D}_w \in \mathbb{Q}[y_1, y_2, \ldots]$ are consistently defined for any $w \in S_\infty$.

Corollary 12.3. (1) *The set of polynomials $\mathfrak{D}_w$, $w \in S_\infty$, forms a linear basis of the polynomial ring $\mathbb{Q}[y_1, y_2, \ldots]$.*

(2) *The basis $\{\mathfrak{S}_w\}_{w \in S_\infty}$ of Schubert polynomials in $\mathbb{Q}[x_1, x_2, \ldots]$ is D-dual*[5] *to the basis $\{\mathfrak{D}_w\}_{w \in S_\infty}$ in $\mathbb{Q}[y_1, y_2, \ldots]$, i.e., $(\mathfrak{S}_u, \mathfrak{D}_w)_D = \delta_{u,w}$, for any $u, w \in S_\infty$.*

Proof. Let $u, v \in S_\infty$. Then, for sufficiently large n, we have $u, v \in S_n$. Now the identity $(\mathfrak{S}_u, \mathfrak{D}_w)_D = \delta_{u,w}$ follows from Corollary 6.1. □

13. Flagged Schur polynomials

Let $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_1 \geq \cdots \geq \mu_n \geq 0$, be a partition, $\beta = (\beta_1, \ldots, \beta_m)$ be a nonnegative integer sequence, and $a = (a_1 \leq \cdots \leq a_n)$ and $b = (b_1 \leq \cdots \leq b_n)$ be two weakly increasing positive integer sequences. A *flagged semistandard Young tableau* of shape μ, weight β, with flags a and b is an array of positive integers $T = (t_{ij})$, $i = 1, \ldots, n$, $j = 1, \ldots, \mu_i$, such that

(1) entries strictly increase in the columns: $t_{1j} < t_{2j} < t_{3j} < \cdots$;
(2) entries weakly increase in the rows: $t_{i1} \leq t_{i2} \leq t_{i3} < \cdots$;
(3) $\beta_k = \#\{(i,j) \mid t_{ij} = k\}$ is the number of entries k in T, for $k = 1, \ldots, m$;
(4) for all entries in the i-th row, we have $a_i \leq t_{ij} \leq b_i$.

The *flagged Schur polynomial* $s_\mu^{a,b} = s_\mu^{a,b}(x) \in \mathbb{Q}[x_1, x_2, \ldots]$ is defined as the sum

$$s_\mu^{a,b}(x) = \sum x^T$$

over all flagged semistandard Young tableaux T of shape μ with flags a and b and arbitrary weight, where $x^T := x_1^{\beta_1} \cdots x_m^{\beta_m}$ and $\beta = (\beta_1, \ldots, \beta_m)$ is the weight of T.

Note that $s_\mu^{(1,\ldots,1),(n,\ldots,n)}$ is the usual Schur polynomial $s_\mu(x_1, \ldots, x_n)$. Flagged Schur polynomials were originally introduced by Lascoux and Schützenberger [LS1].

The polynomial $s_\mu^{a,b}(x)$ does not depend on the flag a provided that $a_i \leq i$, for $i = 1, \ldots, n$. Indeed, entries in the i-th row of any semistandard Young tableaux (of a standard shape) are greater than or equal to i. Thus the condition $a_i \leq t_{ij}$ is redundant. Let

$$s_\mu^b(x) := s_\mu^{(1,\ldots,1),b}(x) = s_\mu^{(1,\ldots,n),b}(x).$$

Flagged semistandard Young tableaux can be presented as collections of n non-crossing lattice paths on $\mathbb{Z} \times \mathbb{Z}$ that connect points $A_1, \ldots, A_n$ with $B_1, \ldots, B_n$, where $A_i = (-i, a_i)$ and $B_i = (\mu_i - i, b_i)$. Let us assign the weight x_i to each edge $(i,j) \to (i, j{+}1)$ in a lattice path and weight 1 to an edge $(i,j) \to (i{+}1, j)$. Then the product of weights over all edges in the collection of lattice paths corresponding to a

[5] Note that D-pairing between polynomials in n variables is stable under the embedding $\mathbb{Q}[x_1, \ldots, x_n] \subset \mathbb{Q}[x_1, \ldots, x_{n+1}]$. Thus D-pairing is consistently defined for polynomials in infinitely many variables.

flagged tableau T equals x^T. According to the method of Gessel and Viennot [GV] for counting non-crossing lattice paths, the flagged Schur polynomial $s_\mu^{a,b}(x)$ equals the determinant

$$s_\mu^{a,b}(x) = \det\left(h_{\mu_i - i + j}^{[a_j, b_i]}\right)_{i,j=1}^{n}, \tag{13.1}$$

where, for $k \le l$,

$$h_m^{[k,l]} = h_m(x_k, x_{k+1}, \ldots, x_l) = \sum_{k \le i_1 \le \cdots \le i_m \le l} x_{i_1} \cdots x_{i_m}$$

is the complete homogeneous symmetric polynomial of degree m in the variables $x_k, \ldots, x_l$; and $h_m^{[k,l]} = 0$, for $k > l$. Another proof of this result was given by Wachs [Wac].

For permutations $w = w_1 \cdots w_n$ in S_n and $\sigma = \sigma_1 \cdots \sigma_r$ in S_r, let us say that w is *σ-avoiding* if there is no subset $I = \{i_1 < \cdots < i_r\} \subseteq \{1, \ldots, n\}$ such that the numbers $w_{i_1}, \ldots, w_{i_r}$ have the same relative order as the numbers $\sigma_1, \ldots, \sigma_r$. Let $S_n^\sigma \subseteq S_n$ be the set of σ-avoiding permutations in S_n. For example, a permutation $w = w_1 \cdots w_n$ is *312-avoiding* if there are no $i < j < k$ such that $w_i > w_k > w_j$. It is well-known that, for any permutation $\sigma \in S_3$ of size 3, the number of σ-avoiding permutations in S_n equals the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. A permutation w is called *vexillary* if it is 2143-avoiding.

Lascoux and Schützenberger [LS1] stated that Schubert polynomials for vexillary permutations are certain flagged Schur polynomials. This claim was clarified and proved by Wachs [Wac].

For a permutation $w = w_1 \cdots w_n$ is S_n, the inversion sets $\mathrm{Inv}_i(w)$, $i = 1, \ldots, n$, are defined as

$$\mathrm{Inv}_i(w) = \{j \mid i < j \le n \text{ and } w_i > w_j\}.$$

The *code* of permutation w is the sequence $\mathrm{code}(w) = (c_1, \ldots, c_n)$ given by

$$c_i = c_i(w) = |\mathrm{Inv}_i(w)| = \#\{j \mid j > i,\ w_j < w_i\} \text{ for } i = 1, \ldots, n.$$

The map $w \mapsto \mathrm{code}(w)$ is a bijection between the set of permutations S_n and the set of vectors $\{(c_1, \ldots, c_n) \in \mathbb{Z}^n \mid 0 \le c_i \le n - i, \text{ for } i = 1, \ldots, n\}$.

The *shape* of permutation $w \in S_n$ is the partition $\mu = (\mu_1 \ge \cdots \ge \mu_m)$ given by nonzero components c_i of its code arranged in decreasing order. The *flag* of permutation $w \in S_n$ is the sequence $b = (b_1 \le \cdots \le b_m)$ given by the numbers $\min \mathrm{Inv}_i(w) - 1$, for non-empty $\mathrm{Inv}_i(w)$, arranged in increasing order.

Proposition 13.1. [Wac], cf. [LS1] *Assume that $w \in S_n^{2143}$ is a vexillary permutation. Let μ be its shape and b be its flag. Then the Schubert polynomial $\mathfrak{S}_w(x)$ is the following flagged Schur polynomial: $\mathfrak{S}_w(x) = s_\mu^b(x)$.*

We remark that not every flagged Schur polynomial is a Schubert polynomial.

Let $\mathcal{C}_n$ be the set of partitions $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_1 \ge \cdots \ge \mu_n \ge 0$, such that $\mu_i \le n - i$, for $i = 1, \ldots, n$, i.e., $\mathcal{C}_n$ is the set of partitions whose Young diagrams fit inside the staircase shape $(n-1, n-2, \ldots, 0)$. These partitions are in an obvious correspondence with Catalan paths. Thus $|\mathcal{C}_n| = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number.

A permutation w is called *dominant* if $\mathrm{code}(w) = (c_1, \ldots, c_n)$ is a partition, i.e., $c_1 \ge \cdots \ge c_n$. The next claim is essentially well known; see, e.g., [Man].

Proposition 13.2. *A permutation $w = w_1 \cdots w_n \in S_n$ is dominant if and only if it is 132-avoiding.*

The map $w \mapsto \mathrm{code}(w)$ is a bijection between the set S_n^{132} of dominant permutations and the set $\mathcal{C}_n$. We have $w_i > w_{i+1}$ if and only if $c_i > c_{i+1}$, and $w_i < w_{i+1}$ if and only if $c_i = c_{i+1}$.

For $w \in S_n^{132}$, we have $\mathrm{Inv}_i(w) = \{k \mid w_k < \min\{w_1, \dots, w_i\}\}$ and $c_i(w) = \min\{w_1, \dots, w_i\} - 1$.

The inverse map $c \mapsto w(c)$ from $\mathcal{C}_n$ to S_n^{132} is given recursively by $w_1 = c_1 + 1$ and $w_i = \min\{j > c_i \mid j \neq w_1, \dots, w_{i-1}\}$, for $i = 2, \dots, n$. In particular, if $c_i < c_{i-1}$ then $w_i = c_i + 1$.

Proof. Let us assume that w is 132-avoiding and show that $\mathrm{code}(w)$ is weakly decreasing. Indeed, if $w_i > w_{i+1}$ then $c_i > c_{i+1}$. If $w_i < w_{i+1}$ then there is no $j > i+1$ such that $w_i < w_j < w_{i+1}$, because w is 132-avoiding. Thus $c_i = c_{i+1}$ in this case.

On the other hand, assume that $w \in S_n$ is not a 132-avoiding permutation. Say that (i, j, k) is a 132-triple of indices if $i < j < k$ and $w_i < w_k < w_j$. Let us find a 132-triple (i, j, k) such that the difference $j - i$ is as small as possible. We argue that $j = i + 1$. Otherwise, pick any l such that $i < l < j$. If $w_l < w_k$ then (l, j, k) is a 132-triple, and if $w_l > w_k$ then (i, l, k) is a 132-triple. Both these triples have a smaller difference. This shows that we can always find a 132-triple of the form $(i, i+1, k)$. Then $c_i(w) < c_{i+1}(w)$. Thus $\mathrm{code}(w)$ is not weakly decreasing. This proves that $w \mapsto \mathrm{code}(w)$ is a bijection between S_n^{132} and $\mathcal{C}_n$.

Let $w \in S_n^{132}$. Fix an index i and find $1 \leq j \leq i$ such that $w_j = \min\{w_1, \dots, w_i\}$. Since w is 132-avoiding, there is no $k > i$ such that $w_i > w_k > w_j$. Thus the conditions $k > i$, $w_k < w_i$ imply that $w_k < w_j$. On the other hand, if $w_k < w_j$ for some $k \in \{1, \dots, n\}$ then $k > i$ because of our choice of j. This shows that the i-th inversion set of the permutation w is $\mathrm{Inv}_i(w) = \{k \mid w_k < \min\{w_1, \dots, w_i\}\}$. Thus $c_i(w) = |\mathrm{Inv}_i(w)| = \min\{w_1, \dots, w_i\} - 1$.

Let $w \in S_n^{132}$ and $\mathrm{code}(w) = (c_1, \dots, c_n)$. We have $w_1 = c_1 + 1$. Let us derive the identity $w_i = \min\{j > c_i \mid j \neq w_1, \dots, w_{i-1}\}$, for $i = 2, \dots, n$. Indeed, if $c_i < c_{i-1}$ then $w_i = c_i + 1$, as needed. Otherwise, if $c_i = c_{i-1}$, then $w_i > w_{i-1}$. Let k be the index such that $w_k = \min\{j > c_i \mid j \neq w_1, \dots, w_{i-1}\}$. If $k \neq i$ then $k > i$ and $w_k < w_i$. Thus $w_{i-1} < w_k < w_i$. This is impossible because we assumed that w is 132-avoiding. $\square$

The following claim is also well known; see, e.g., [Man].

Proposition 13.3. *For a dominant permutation $w \in S_n^{132}$, the Schubert polynomial is given by the monomial $\mathfrak{S}_w(x) = x_1^{c_1(w)} \cdots x_n^{c_n(w)}$.*

This claim follows from Proposition 13.1, because the set of dominant permutations is a subset of vexillary permutations.

Proof. Let $\mu = \mathrm{code}(w) = (k_1^{m_1}, k_2^{m_2}, \dots)$, $k_1 > k_2 > \cdots$, be the shape of w. According to Proposition 13.2, the flag of w is $b = (m_1^{m_1}, (m_1 + m_2)^{m_2}, \dots)$. For this shape and flag, there exists only one flagged semistandard Young tableau $T = (t_{ij})$, which is given by $t_{ij} = i$. Thus $\mathfrak{S}_w(x) = s_\mu^b(x) = x_1^{\mu_1} \cdots x_n^{\mu_m}$. $\square$

A permutation w is 3412-avoiding if and only if $w_\circ w$ is vexillary. Also a permutation w is 312-avoiding if and only if $w_\circ w$ is 132-avoiding. The next claim follows from Theorem 6.2, Proposition 13.1, and Corollary 13.3.

Theorem 13.4. *Let $w \in S_n^{3412}$ be a 3412-avoiding permutation. Let μ and b be the shape and flag of the vexillary permutation $w_\circ w$. Then*

$$\mathfrak{D}_w(y_1,\dots,y_n) = \frac{1}{1!\,2!\,\cdots(n-1)!}\, s_\mu^b(\partial/\partial y_1,\dots,\partial/\partial y_n)\cdot\prod_{i<j}(y_i-y_j).$$

In particular, for a 312-avoiding permutation $w \in S_n^{312}$ and $(c_1,\dots,c_n) = \mathrm{code}(w_\circ w)$, we have

$$\mathfrak{D}_w(y_1,\dots,y_n) = \frac{1}{1!\,2!\,\cdots(n-1)!}\left(\prod_{k=1}^n(\partial/\partial y_k)^{c_k}\right)\cdot\prod_{i<j}(y_i-y_j)$$
$$= \det\left(\left(y_i^{(n-c_i-j)}\right)_{i,j=1}^n\right),$$

where $a^{(b)} = a^b/b!$, for $b\ge 0$, and $a^{(b)}=0$, for $b<0$.

Applying Lemma 12.2(1), we obtain the determinant expression for $\mathfrak{D}_w$, for 231-avoiding permutations w, as well.

Corollary 13.5. *For a 231-avoiding permutation $w \in S_n^{231}$ and $(c_1,\dots,c_n) = \mathrm{code}(ww_\circ)$, we have*

$$\mathfrak{D}_w(y_1,\dots,y_n) = \det\left(\left((-y_{n-i+1})^{(n-c_i-j)}\right)_{i,j=1}^n\right).$$

14. Demazure characters for 312-avoiding permutations

In the previous section we gave a simple determinant formula for the polynomial $\mathfrak{D}_w$, for a 312-avoiding permutation $w\in S_n^{312}$. We remark that 312-avoiding permutations are exactly the *Kempf elements* that were studied by Lakshmibai in [Lak]. In this and the following sections, we give some additional nice properties of 312-avoiding permutations. In this section, we show how Weyl's character formula can be easily deduced from Demazure's character formula by induction on some sequence of 312-avoiding permutations that interpolates between 1 and $w_\circ$.

Let $z_1,\dots,z_n$ be independent variables, and let T_i, $i=1,\dots,n-1$, be the operator that acts on the polynomial ring $\mathbb{Q}[z_1,\dots,z_n]$ by

$$T_i : f(z_1,\dots,z_n)\mapsto \frac{z_i\, f(z_1,\dots,z_n) - z_{i+1}\, f(z_1,\dots,z_{i-1},z_{i+1},z_i,z_{i+2},\dots,z_n)}{z_i-z_{i+1}}.$$

For $\lambda=(\lambda_1\ge\cdots\ge\lambda_n)$ and a reduced decomposition $w = s_{i_1}\cdots s_{i_l}\in S_n$, let

$$ch_{\lambda,w}(z_1,\dots,z_n) = T_{i_1}\cdots T_{i_l}(z_1^{\lambda_1}\cdots z_n^{\lambda_n}).$$

The polynomials $ch_{\lambda,w}$ do not depend on choice of reduced decomposition for w because the T_i satisfy the Coxeter relations. Let us map the ring $\mathbb{Q}[z_1,\dots,z_n]$ to the group algebra $\mathbb{Q}[\Lambda]$ of the type A_{n-1} weight lattice Λ by $z_i\mapsto e^{\omega_i-\omega_{i-1}}$, for $i=1,\dots,n$, where we assume that $\omega_0=\omega_n=0$. Then the operators T_i specialize to the Demazure operators (8.1) and the polynomials $ch_{\lambda,w}$ map to the characters of Demazure modules $ch(V_{\lambda,w})$; cf. the Demazure character formula (8.4). The polynomials $ch_{\lambda,w}$ were studied by Lascoux and Schützenberger [LS1], who called them *essential polynomials*, and by Reiner and Shimozono [RS], who called them *key polynomials*. To avoid confusion, we will call the polynomials $ch_{\lambda,w}$ simply *Demazure characters*.

For a given partition $\lambda = (\lambda_1, \dots, \lambda_n)$, the number of nonzero flagged Schur polynomials $s^b_\lambda(z_1, \dots, z_n)$ in n variables equals the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Indeed, such a polynomial is nonzero if and only if the flag $b = (b_1, \dots, b_n)$ satisfies $b_1 \leq \cdots \leq b_n \leq n$ and $b_i \geq i$, for $i = 1, \dots, n$. Let us denote by $\tilde{\mathcal{C}}_n$ the set of such flags b. The map $(b_1, \dots, b_n) \mapsto (c_1, \dots, c_n)$ given by $c_i = n - b_i$, for $i = 1, \dots, n$, is a bijection between the sets $\tilde{\mathcal{C}}_n$ and $\mathcal{C}_n$. The next theorem says that the flagged Schur polynomials $s^b_\lambda(z_1, \dots, z_n)$ are exactly the Demazure characters $ch_{\lambda,w}$, for 312-avoiding permutations $w \in S_n$.

Recall that the map $w \mapsto \mathrm{code}(w)$ is a bijection between the sets S_n^{132} and $\mathcal{C}_n$ (see Proposition 13.2). Then the map $w \mapsto b(w) = (b_1, \dots, b_n)$ given by $b_i = n - c_i(w_\circ w)$, for $i = 1, \dots, n$, is a bijection between the sets S_n^{312} and $\tilde{\mathcal{C}}_n$. Note that $\ell(w) = b_1 + \cdots + b_n - \binom{n+1}{2}$. The inverse map $\tilde{\mathcal{C}}_n \to S_n^{312}$ can be described recursively, as follows: $w_1 = b_1$ and $w_i = \max\{j \mid j \leq b_i,\ j \neq w_1, \dots, w_{i-1}\}$, for $i = 2, \dots, n$; cf. Proposition 13.2.

Theorem 14.1. *Let $w \in S_n^{312}$ be a 312-avoiding permutation. Let $b = b(w)$ be the corresponding element of $\tilde{\mathcal{C}}_n$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition. Then the Demazure character $ch_{\lambda,w}$ equals the flagged Schur polynomial:*

$$ch_{\lambda,w}(z_1, \dots, z_n) = s^b_\lambda(z_1, \dots, z_n).$$

This theorem follows from a general result by Reiner and Shimozono [RS], who expressed any flagged skew Schur polynomial as a combination of Demazure characters (key polynomials). Theorem 14.1 implies that every Schubert polynomial $\mathfrak{S}_w$, for a vexillary permutation $w \in S_n^{2143}$, is equal to some Demazure character $ch_{\lambda,u}$, for a certain 312-avoiding permutation $u \in S_m^{312}$, $m < n$, associated with w. Let us give a simple proof of Theorem 14.1.

Let $b = (b_1, \dots, b_n) \in \tilde{\mathcal{C}}_n$. Let us say that $k \in \{1, \dots, n-1\}$ is an *isolated entry* in b if k appears in the sequence b exactly once. Let us write $b \xrightarrow{k} b'$ if k is an isolated entry in b and $b' \in \tilde{\mathcal{C}}_n$ is obtained from b by adding 1 to this entry. In other words, we have $b_{i-1} < b_i = k < b_{i+1}$, for some $i \in \{1, \dots, n-1\}$ (assuming that $b_0 = 0$), and $b' = (b_1, \dots, b_{i-1}, b_i + 1, b_{i+1}, \dots, b_n)$.

Lemma 14.2. *If $b \xrightarrow{k} b'$, then $T_k \cdot s^b_\lambda(z_1, \dots, z_n) = s^{b'}_\lambda(z_1, \dots, z_n)$.*

Proof. The claim follows from the formula $s^b_\lambda = \det\left(h_{\lambda_i - i + j}(z_1, \dots, z_{b_i})\right)_{i,j=1}^n$, the fact that the operator T_k commutes with multiplication by $h_m(x_1, \dots, x_l)$ for $k \neq l$; and $T_k \cdot h_m(x_1, \dots, x_k) = h_m(x_1, \dots, x_{k+1})$. □

Let us also write $w \xrightarrow{k} w'$, for $w, w' \in S_n$, if $w' = s_k\, w$ and $\ell(w') = \ell(w) + 1$.

Lemma 14.3. *For $w, w' \in S_n^{312}$, if $b(w) \xrightarrow{k} b(w')$ then $w \xrightarrow{k} w'$.*

Proof. Assume $b(w) = b$, $b(w') = b'$, and $b \xrightarrow{k} b'$. Let $b_i = k$ be the isolated entry in b that we increase. The construction of the map $b \mapsto w$ implies that $w_i = k$ and $w_j = k + 1$ for some $j > i$. It also implies that $b' \mapsto s_k w$. The permutation $s_k w$ is obtained from w by switching w_i and w_j, and its length is $\ell(w) + 1$. Thus $w \xrightarrow{k} w'$. □

Exercise 14.4. *Check that $b(w) \xrightarrow{k} b(w')$ if and only if $w \xrightarrow{k} w'$.*

Proof of Theorem 14.1. Let $b = b(w) \in \tilde{\mathcal{C}}_n$. We claim that there is a directed path $b^{(0)} \xrightarrow{k_1} b^{(1)} \xrightarrow{k_2} \cdots \xrightarrow{k_l} b^{(l)}$ from $b^{(0)} = (1, \dots, n)$ to $b^{(l)} = b$. In other words, we can obtain the sequence b from the sequence $(1, \dots, n)$ by repeatedly adding 1's to some isolated entries. One possible choice of such a path is given by the following rule. We have $b_n = n$. Let us first increase the $(n-1)$-st entry until we obtain b_{n-1}; then increase the $(n-2)$-nd entry until we obtain b_{n-2}, etc.

For example, for the sequence $b = (3, 3, 3, 5, 5)$ that corresponds to $w = 32154$, we obtain the path

$$(1,2,3,4,5) \xrightarrow{4} (1,2,3,5,5) \xrightarrow{2} (1,3,3,5,5) \xrightarrow{1} (2,3,3,5,5) \xrightarrow{2} (3,3,3,5,5).$$

This path gives the reduced decomposition $s_2 s_1 s_2 s_4$ for $w = 32154$.

If $w = id$ then $b(w) = (1, \dots, n)$ and $ch_{id,\lambda} = s_\lambda^{(1,\dots,n)} = z_1^{\lambda_1} \dots z_n^{\lambda_n}$. In general, according to Lemmas 14.2 and 14.3, we have $w = s_{k_l} \cdots s_{k_1}$, and thus, $s_\lambda^b = T_w(s_\lambda^{(1,\dots,n)}) = T_w(x^\lambda) = ch_{w,\lambda}$. □

Remark 14.5. Lemma 14.3, together with the exercise, gives a bijective correspondence between paths $(1, \dots, n) \xrightarrow{k_1} \cdots \xrightarrow{k_l} b(w)$ and the special class of reduced decompositions $w = s_{k_l} \cdots s_{k_1}$ such that all truncated decompositions $s_{k_i} \cdots s_{k_1}$ give 312-avoiding permutations, for $i = 1, \dots, l$.

Corollary 14.6. *Let us use the notation of Theorem 14.1. The dimension of the Demazure module is given by the following matrix of binomial coefficients:*

$$\dim V_{\lambda,w} = \det\left(\binom{\lambda_i + b_i - i}{b_i - j} \right)_{i,j=1}^{n}.$$

Proof. We have $\dim V_{\lambda,w} = ch_{\lambda,w}(1, \dots, 1)$. The claim follows from the determinant expression (13.1) for the flagged Schur polynomial $ch_{\lambda,w} = s_\lambda^{(1,\dots,n),b}$ and the fact that $h_m^{[k,l]}(1, \dots, 1) = \binom{l-k+m}{l-k}$. □

Corollary 14.6 presents $\dim V_{\lambda,w}$ as a polynomial of degree $\sum (b_i - i) = \ell(w)$. According to Proposition 9.1, the leading homogeneous component of this polynomial equals $\mathfrak{D}_w(\lambda)$. Thus Corollary 14.6 produces the same determinant expression $\mathfrak{D}_w(\lambda) = \det\left(\lambda_i^{(b_i - j)}\right)$ for a 312-avoiding permutation w as Theorem 13.4.

Let us give another expression for the Demazure characters $ch_{\lambda,w}$ that generalizes the Weyl character formula. It is not hard to prove it by induction similar to the above argument.

Proposition 14.7. *Let $w \in S_n^{312}$ be a 312-avoiding permutation and let $b(w) = (b_1, \dots, b_n)$. Let $W_b = \{u \in S_n \mid u_i \le b_i,$ for any $i = 1, \dots, n\}$, and let $\Phi^+_{u,b} = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le b_{u^{-1}(i)}\} \subseteq \Phi^+$. Then*

$$ch_{\lambda,w}(z_1, \dots, z_n) = \sum_{u \in W_b} (-1)^{\ell(u)}\, z^{u(\lambda+\rho)-\rho} \prod_{\alpha \in \Phi^+_{u,b}} (1 - z^{-\alpha})^{-1}.$$

The set W_b is in one-to-one correspondence with rook placements in the Young diagram of shape $(b_n, b_{n-1}, \dots, b_1)$. We have $|W_b| = b_1\,(b_2-1)\,(b_3-2) \cdots (b_n-n+1)$. For any $u \in W_b$, we have $|\Phi^+_{u,b}| = \ell(w)$.

15. Generalized Gelfand-Tsetlin polytope

In this section we show how flagged Schur functions and Demazure characters are related to generalized Gelfand-Tsetlin polytopes studied by Kogan [Kog].

A *Gelfand-Tsetlin pattern* P of size n is a triangular array of real numbers $P = (p_{ij})_{n\geq i\geq j\geq 1}$ that satisfy the inequalities $p_{i-1\,j-1} \geq p_{ij} \geq p_{i-1\,j}$. These patterns are usually arranged on the plane as follows:

$$\begin{array}{ccccccccccccccccc}
p_{n1} & & & & p_{n2} & & & & p_{n3} & & \cdots & & \cdots & & \cdots & & p_{nn} \\
 & \geq & & \geq & & \geq & & \geq & & \geq & & & & & & \geq & \\
 & & \ddots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \iddots & & \\
 & & & \geq & & \geq & & \geq & & \geq & & \geq & & \geq & & & \\
 & & & & p_{31} & & & & p_{32} & & & & p_{33} & & & & \\
 & & & & & \geq & & \geq & & \geq & & \geq & & & & & \\
 & & & & & & p_{21} & & & & p_{22} & & & & & & \\
 & & & & & & & \geq & & \geq & & & & & & & \\
 & & & & & & & & p_{11} & & & & & & & &
\end{array}$$

The *shape* $\lambda = (\lambda_1, \ldots, \lambda_n)$ of a Gelfand-Tsetlin pattern P is given by $\lambda_i = p_{ni}$, for $i = 1, \ldots, n$, i.e., the shape is the top row of a pattern. The *weight* $\beta = (\beta_1, \ldots, \beta_n)$ of a Gelfand-Tsetlin pattern P is given by $\beta_1 = p_{11}$ and $\beta_i = p_{i1} + \ldots p_{ii} - p_{i-1\,1} - \cdots - p_{i-1\,i-1}$, for $i = 2, \ldots, n$, i.e., the i-th row sum $p_{i1} + \cdots + p_{ii}$ equals $\beta_1 + \cdots + \beta_i$.

The *Gelfand-Tsetlin polytope* $\mathcal{P}_\lambda \in \mathbb{R}^{\binom{n}{2}}$ is the set of all Gelfand-Tsetlin patterns of shape λ. This is a a convex polytope. A Gelfand-Tsetlin pattern $P = (p_{ij})$ is called *integer* if all p_{ij} are integers. The integer Gelfand-Tsetlin patterns are the lattice points of the polytope $\mathcal{P}_\lambda$.

The integer Gelfand-Tsetlin patterns $P = (p_{ij})$ of shape λ and weight β are in one-to-one correspondence with semistandard Young tableaux $T = (t_{ij})$ of shape λ and weight β. This correspondence is given by setting $p_{ij} = \#\{k \mid t_{kj} \leq i\}$, i.e., p_{ij} is the number of entries less than or equal to i in the j-th row of T. The proof of the following claim is immediate from the definitions.

Lemma 15.1. *A semistandard Young tableau T is a flagged tableau with flags $(1, \ldots, 1)$ and $(b_1, \ldots, b_n)$ if and only if the corresponding Gelfand-Tsetlin pattern $P = (p_{ij})$ satisfies the conditions $p_{ni} = p_{n-1\,i} = \cdots = p_{b_i\,i}$, for $i = 1, \ldots, n$.*

Let $w \in S_n^{312}$ be a 312-avoiding permutation, let $b = (b_1, \ldots, b_n) = b(w) \in \tilde{\mathcal{C}}_n$, and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. Let us define the *generalized Gelfand-Tsetlin polytope* $\mathcal{P}_{\lambda,w}$ as the set of all Gelfand-Tsetlin patterns $P = (p_{ij})$ of size n such that $\lambda_i = p_{ni} = p_{n-1\,i} = \cdots = p_{b_i\,i}$, for $i = 1, \ldots, n$. Note that $b_1 + \cdots + b_n - \binom{n+1}{2} = \ell(w)$ is the number of unspecified entries in a pattern. Thus $\mathcal{P}_{\lambda,w}$ is a convex polytope naturally embedded into $\mathbb{R}^{\ell(w)}$. These polytopes were studied by Kogan [Kog].

According to Theorem 14.1, the Demazure character $ch_{\lambda,w}$, for a 312-avoiding permutation w, is given by counting lattice points of the generalized Gelfand-Tsetlin polytope $\mathcal{P}_{\lambda,w}$.

Corollary 15.2. *For $w \in S_n^{312}$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we have*

$$ch_{\lambda,w}(z_1, \ldots, z_n) = s_\lambda^b(z_1, \ldots, z_n) = \sum_{P \in \mathcal{P}_{\lambda,w} \cap \mathbb{Z}^{\ell(w)}} z^P,$$

where the sum is over lattice points in the polytope $\mathcal{P}_{\lambda,w}$, $z^P = z_1^{\beta_1}\cdots z_n^{\beta_n}$, *and* $\beta = (\beta_1,\dots,\beta_n)$ *is the weight of* P. *In particular, the dimension of the Demazure module* $V_{\lambda,w}$ *is equal to the number of lattice points in the polytope* $\mathcal{P}_{\lambda,w}$*:*

$$\dim V_{\lambda,w} = \#(\mathcal{P}_{\lambda,w}\cap \mathbb{Z}^{\ell(w)}).$$

Finally, the λ*-degree of the Schubert variety* X_w *divided by* $\ell(w)!$ *equals the volume of the generalized Gelfand-Tsetlin polytope* $\mathcal{P}_{\lambda,w}$*:*

$$\frac{1}{\ell(w)!}\,\deg_\lambda(X_w) = \mathfrak{D}_w(\lambda) = \mathrm{Vol}(\mathcal{P}_{\lambda,w}),$$

where Vol *denotes the usual volume form on* $\mathbb{R}^{\ell(w)}$ *such that the volume of the unit* $\ell(w)$*-hypercube equals* 1.

The following claim is also straightforward from the definition of the polytopes $\mathcal{P}_{\lambda,w}$.

Proposition 15.3. *The polytope* $\mathcal{P}_{\lambda,w}$ *is the Minkowski sum of the polytopes* $\mathcal{P}_{\omega_i,w}$ *for the fundamental weights:*

$$\mathcal{P}_{\lambda,w} = a_1\mathcal{P}_{\omega_1,w} + \cdots + a_{n-1}\mathcal{P}_{\omega_{n-1},w},$$

where $\lambda = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$.

The last claim implies that $\dim V_{\lambda,w}$ is the mixed lattice point enumerator of the polytopes $\mathcal{P}_{\omega_i,w}$, $i = 1,\dots,n-1$.

Remark 15.4. Toric degenerations of Schubert varieties X_w for Kempf elements (312-avoiding permutations in our terminology), were constructed by Gonciulea and Lakshmibai [GL], and were studied by Kogan [Kog] and Kogan-Miller [KM]. According to [Kog, KM], these toric degenerations are associated with generalized Gelfand-Tsetlin polytopes $\mathcal{P}_{\lambda,w}$. It is a standard fact that the degree of a toric variety is equal to the normalized volume of the corresponding polytope.

Remark 15.5. We can extend the definition of generalized Gelfand-Tsetlin polytopes $\mathcal{P}_{w,\lambda}$ to a larger class of permutations, as follows. For a 231-avoiding permutation w, define $\mathcal{P}_{w,\lambda} = \mathcal{P}_{w_\circ w w_\circ,\,(-\lambda_n,\dots,-\lambda_1)}$, cf. Lemma 12.2(1). Let $w = w^1\times\cdots\times w^k \in S_{n_1}\times\cdots\times S_{n_k}\subset S_n$ be a permutation such that all blocks $w^i\in S_{n_i}$ are either 312-avoiding or 231-avoiding, and let λ be the concatenation of partitions $\lambda^1,\dots,\lambda^k$ of lengths $n_1,\dots,n_k$. We have $ch(V_{\lambda,w}) = \prod ch(V_{\lambda^i,\,w^i})$ and $\mathfrak{D}_w(\lambda) = \prod \mathfrak{D}_{w^i}(\lambda^i)$. Let us define $\mathcal{P}_{w,\lambda} = \mathcal{P}_{w^1,\,\lambda^1}\times\cdots\times\mathcal{P}_{w^k,\,\lambda^k}$. Then Corollary 15.2 and Proposition 15.3 remain valid for this more general class of permutations with 312- or 231-avoiding blocks. These claims extend results of Dehy and Yu [DY].

16. A conjectured value of $\mathfrak{D}_w$

In this section we give a conjectured value of $\mathfrak{D}_w$ for a special class of permutations w.

Let w be a permutation whose code has the form

$$\mathrm{code}(w) = (n,*,n-1,*,n-2,\cdots,*,2,*,1,0,0,\dots),$$

where each $*$ is either 0 or empty. We call such a permutation *special*. For instance, $w = 761829543$ is special, with $\mathrm{code}(w) = (6,5,0,4,0,3,2,1,0,\dots)$. Note also that $w_\circ$ is special. Suppose that w is special with $\mathrm{code}(w) = (c_1,c_2,\dots)$. Let $c_1 = n$, and let k be the number of 0's in $\mathrm{code}(w)$ that are preceded by a nonzero number, i.e, c_i=0, $c_{i-1} > 0$. Let $a_1 < \cdots < a_k = n+k$ be the positions of these 0's, so $c_{a_1} = \cdots = c_{a_k} = 0$. Define

$$\begin{aligned} a_\delta(y_1, \ldots, y_n) &= \prod_{1 \le i < j \le n} (y_i - y_j) \\ &= \sum_{w \in S_n} (-1)^{\ell(w)}\, y_1^{w(1)-1} \cdots y_n^{w(n)-1}. \end{aligned}$$

An n-element subset $J = \{j_1, \ldots, j_n\}$ of $\{1, 2, \ldots, n+k\}$ is said to be *valid* (with respect to w) if

$$\#(J \cap \{a_{i-1}+1, a_{i-1}+2, \ldots, a_i\}) = a_i - a_{i-1} - 1$$

for $1 \le i \le k$ (where we set $a_0 = 0$). For instance if $\mathrm{code}(w) = (3,0,2,1,0)$, then the valid sets are 134, 135, 145, 234, 235, 245. Clearly the number of valid sets in general is equal to $(a_1-1)(a_2-a_1-1)\cdots(a_k-a_{k-1}-1)$. If J is a valid set, then define the *sign* ε_J of J by $\varepsilon_J = (-1)^{d_J}$, where

$$d_J = \binom{n+k+1}{2} - 1 - (a_1+1) - \cdots - (a_{k-1}+1) - \sum_{i \in J} i.$$

Note that the quantity $\binom{n+k+1}{2} - 1 - (a_1+1) - \cdots - (a_{k-1}+1)$ appearing above is just $\sum_{i \in L} i$ for the valid subset L with largest element sum, viz.,

$$L = \{1, 2, \ldots, n+k\} - \{1, a_1+1, a_2+1, \ldots, a_{k-1}+1\}.$$

In particular, $d_L = 0$ and $\varepsilon_L = 1$.

Conjecture 16.1. Let w be special as above. Then

$$\mathfrak{D}_w = C_{nk} \sum_{J = \{j_1, \ldots, j_k\}} \varepsilon_J\, a_\delta(y_{n+k-j_1+1}, y_{n+k-j_2+1}, \ldots, y_{n+k-j_k+1}),$$

where

$$C_{nk} = \frac{(n+1)!\,(n+2)! \cdots (n+k-1)!}{\binom{n+1}{2}!}$$

and J ranges over all valid subsets of $\{1, 2, \ldots, n+k\}$.

As an example of Conjecture 16.1, let $w = 41532$, so $\mathrm{code}(w) = (3,0,2,1,0)$. Write $y_1 = a$, $y_2 = b$, etc. Then

$$\mathfrak{D}_w = \frac{1}{30}(a_\delta(a,b,d) - a_\delta(a,b,e) - a_\delta(a,c,d) + a_\delta(a,c,e) + a_\delta(b,c,d) - a_\delta(b,c,e)).$$

We have verified Conjecture 16.1 for $n \le 5$.

17. Schubert-Kostka matrix and its inverse

In this section we discuss the following three equivalent problems:

(1) Express the polynomials $\mathfrak{D}_w$ as linear combinations of monomials.
(2) Express monomials as linear combinations of Schubert polynomials $\mathfrak{S}_w$.
(3) Express Schubert polynomials as linear combination of standard elementary monomials $e_{a_1}(x_1)e_{a_2}(x_1,x_2)e_{a_3}(x_1,x_2,x_3)\cdots$.

Let $\mathbb{N}^\infty$ be the set of "infinite compositions" $a = (a_1, a_2, \dots)$ such that all $a_i \in \mathbb{N} = \mathbb{Z}_{\geq 0}$ and $a_i = 0$, for almost all i's. For $a \in \mathbb{N}^\infty$, let $x^a = x_1^{a_1} x_2^{a_2} \cdots$ and $y^{(a)} = \frac{y_1^{a_1}}{a_1!} \frac{y_2^{a_2}}{a_2!} \cdots$. The polynomial ring $\mathbb{Q}[x_1, x_2, \dots]$ in infinitely many variables has the linear bases $\{x^a\}_{a\in\mathbb{N}^\infty}$ and $\{\mathfrak{S}_w\}_{w\in S_\infty}$; also the polynomial ring $\mathbb{Q}[y_1, y_2, \dots]$ has the linear basis $\{y^{(a)}\}_{a\in\mathbb{N}^\infty}$ and $\{\mathfrak{D}_w\}_{w\in S_\infty}$, where $\mathfrak{S}_w = \mathfrak{S}_w(x_1, x_2, \dots)$ and $\mathfrak{D}_w = \mathfrak{D}_w(y_1, y_2, \dots)$.

Let us define the *Schubert-Kostka matrix* $K = (K_{w,a})$, $w \in S_\infty$ and $a \in \mathbb{N}^\infty$, by

$$\mathfrak{S}_w = \sum_{a\in\mathbb{N}^\infty} K_{w,a}\, x^a.$$

The numbers $K_{w,a}$ are nonnegative integers. They can be combinatorially interpreted in terms of *RC-graphs;* see [FK] and [BJS]. For grassmannian permutations w, the numbers $K_{w,a}$ are equal to the usual Kostka numbers, which are the coefficients of monomials in Schur polynomials.

The matrix K is invertible, because every monomial x^a can be expressed as a finite linear combination of Schubert polynomials. Let $K^{-1} = (K^{-1}_{a,w})$ be the inverse of the Schubert-Kostka matrix. We have

$$x^a = \sum_{w\in S_\infty} K^{-1}_{a,w}\, \mathfrak{S}_w.$$

The basis $\{x^a\}_{a\in\mathbb{N}^\infty}$ is D-dual to $\{y^{(a)}\}_{a\in\mathbb{N}^\infty}$, and the basis $\{\mathfrak{S}_w\}_{w\in S_\infty}$ is D-dual to $\{\mathfrak{D}_w\}_{w\in S_\infty}$; see Corollary 6.1. Thus the previous two formulas are equivalent to the following statement.

Proposition 17.1. *We have*

$$y^{(a)} = \sum_{w\in S_\infty} K_{w,a}\, \mathfrak{D}_w \quad \textit{and, equivalently,} \quad \mathfrak{D}_w = \sum_{a\in\mathbb{N}^\infty} K^{-1}_{a,w}\, y^{(a)}.$$

This claim shows that an explicit expression for the polynomials $\mathfrak{D}_w$ in terms of monomials is equivalent to a formula for entries of the inverse Schubert-Kostka matrix K^{-1}. We remark that a combinatorial interpretation of the inverse of the usual Kostka matrix was given by Egecioglu and Remmel [ER]. It would be interesting to give a subtraction-free combinatorial interpretation for entries of the inverse of the Schubert-Kostka matrix. Notice that the matrix K^{-1} has both positive and negative entries. Although we do not know such a formula in general, it is not hard to give an alternating formula for the entries of K^{-1}, as follows.

Let us fix a positive integer n. Let $w_\circ$ be the longest permutation in S_n, let $\mathbb{N}^n$ be the set of compositions $a = (a_1, \dots, a_n)$, $a_i \in \mathbb{N}$, naturally embedded into $\mathbb{N}^\infty$, and let $\rho = (n-1, n-2, \dots, 0) \in \mathbb{N}^n$.

Lemma 17.2. *If $w \in S_n$, then $K^{-1}_{a,w} = 0$, unless $a \in \mathbb{N}^n$.*

Proof. Follows from Proposition 17.1 and the fact that $\mathfrak{D}_w$ involves only $y_1, \dots, y_n$, for $w \in S_n$. □

Assume by convention that $K_{w,a} = 0$ if some entries a_i are negative.

Proposition 17.3. *Assume that $w \in S_n$. Then, for any $a \in \mathbb{N}^n$, we have*

$$K^{-1}_{a,w} = \sum_{u\in S_n} (-1)^{\ell(u)}\, K_{w_\circ w, u(\rho)-a}.$$

Proof. Follows from Corollary 12.1(2), for $u = id$, and Proposition 17.1. □

For a 312-avoiding permutation w, Proposition 17.3 implies a more explicit expression for $K^{-1}_{a,w}$. Indeed, in this case, $\mathfrak{S}_{w_\circ w} = x^c$, where $c = \mathrm{code}(w_\circ w)$. In other words, $K_{w_\circ w, u(\rho)-a}$ equals 1, if $u(\rho) - a = c$, and 0, otherwise. We obtain the following result.

Corollary 17.4. *For a* 312*-avoiding permutation* $w \in S_n^{312}$ *with* $c = \mathrm{code}(w_\circ w)$, *and an arbitrary* $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, *we have*

$$K^{-1}_{a,w} = \begin{cases} (-1)^{\ell(u)} & \text{if } a + c = u(\rho), \text{ for some permutation } u \in S_n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this expression for $K^{-1}_{a,w}$ is stable under the embedding $S_n \hookrightarrow S_{n+1}$. More generally, we can give an expression for $K^{-1}_{a,w}$, for any 3412-avoiding permutation w, as a sum over flagged semistandard tableaux; cf. Theorem 13.4. Also Conjecture 16.1 implies a conjecture for values $K^{-1}_{a,w}$, for special permutations w, as defined in Section 16.

Recall that the involution $y_i \mapsto -y_{n+1-i}$ sends $\mathfrak{D}_w$ to $\mathfrak{D}_{w_\circ w w_\circ}$ (see Lemma 12.2). If $w \in S_n$, then the second identity in Proposition 17.1 involves only terms with $a \in \mathbb{N}^n$. Applying the above involution to this identity, we deduce that the inverse Schubert-Kostka matrix has the following symmetry.

Lemma 17.5. *For any* $w \in S_n$ *and* $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, *we have*

$$K^{-1}_{a,w} = (-1)^{|a|} K^{-1}_{\bar{a}, w_\circ w\, w_\circ},$$

where $|a| = a_1 + \cdots + a_n$ *and* $\bar{a} = (a_n, \dots, a_1)$.

Remark 17.6. The matrix K does not have this kind of symmetry. For example, $\mathfrak{S}_{s_1} = x_1$ and $\mathfrak{S}_{s_{n-1}} = x_1 + \cdots + x_{n-1} \neq -x_1$. Thus $K_{s_1,(1\,0^{n-1})} = 1$ and $K_{w_\circ s_1 w_\circ, (0^{n-1} 1)} = 0$. An argument similar to the above does not work for matrix K, because the first identity in Proposition 17.1 may involve terms with $w \in S_\infty \backslash S_n$ even if $a \in \mathbb{N}^n$.

Applying this symmetry to Corollary 17.4, we obtain an explicit expression for $K^{-1}_{a,w}$, for 231-avoiding permutations w, as well.

Corollary 17.7. *For a* 231*-avoiding permutation* $w \in S_n^{231}$ *with* $\mathrm{code}(w\, w_\circ) = (c_1, \dots, c_n)$ *and an arbitrary* $a = (a_1, \dots, a_n) \in \mathbb{N}^n$, *we have*

$$K^{-1}_{a,w} = \begin{cases} (-1)^{\ell(u)+|a|} & \text{if } (c_1 + a_n, \dots, c_n + a_1) = u(\rho), \text{ for some } u \in S_n, \\ 0 & \text{otherwise.} \end{cases}$$

Say that a permutation w is *strictly dominant* if its code $\mathrm{code}(w) = (c_1, \dots, c_n)$ is a strict partition, i.e., $c_1 > c_2 > \cdots > c_k = c_{k+1} = \cdots = c_n = 0$, for some $k = 1, \dots, n$.

Exercise 17.8. (A) *Show that the following conditions are equivalent:*

(1) w *is strictly dominant;*
(2) $ww_\circ$ *is strictly dominant;*
(3) w *is of the form* $w_1 > w_2 > \cdots > w_k < w_{k+1} < \cdots < w_n$*;*

(4) *w is both 132-avoiding and 231-avoiding.*

(B) *There are exactly 2^{n-1} strictly dominant permutations in S_n.*
(C) *If w is strictly dominant with* $\mathrm{code}(w) = (c_1 > \cdots > c_{k-1} > 0 = \cdots = 0)$, *then* $\mathrm{code}(ww_\circ) = (c'_1 > \cdots > c'_{n-k} > 0 = \cdots = 0)$, *where the set $\{c'_1, \dots, c'_{n-k}\}$ is the complement to the set $\{c_1, \dots, c_{k-1}\}$ in $\{1, \dots, n-1\}$.*

Let us specialize Corollary 17.7 to strictly dominant permutations.

Corollary 17.9. *Let w be a strictly dominant permutation with* $\mathrm{code}(w) = (c_1 > \cdots > c_{k-1} > c_k = \cdots = 0)$. *Assume that $a = (a_1, \dots, a_k, 0, \dots, 0)$. Then*

$$K^{-1}_{a,w} = \begin{cases} (-1)^{\ell(\sigma)} & \text{if } (a_1, \dots, a_k) = (c_{\sigma_1}, \dots, c_{\sigma_k}), \text{ for some } \sigma \in S_k, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, we have $\mathfrak{D}_w(y_1, \dots, y_k, 0, \dots, 0) = \sum_{\sigma \in S_k} (-1)^{\ell(\sigma)}\, y_{\sigma_1}^{(c_1)} \cdots y_{\sigma_k}^{(c_k)}$.

Proof. We have $\mathrm{code}(ww_\circ) = (c'_1 > \cdots > c'_{n-k} > 0 = \cdots = 0)$, where $\{c'_1, \dots, c'_{n-k}\}$ is the set complement $\{0, \dots, n-1\} \setminus \{c_1, \dots, c_k\}$. According to Corollary 17.7, $K^{-1}_{a,w} = 0$, unless $c'_1, \dots, c'_{n-k}, a_k, \dots, a_1$ is a permutation of $0, \dots, n-1$; or, equivalently, $a_1, \dots, a_k$ is a permutation of $c_1, \dots, c_k$. We leave it as an exercise for the reader to check that the signs agree. □

According to Lemma 17.2, for the strictly dominant permutation $w = (k, k-1, \dots, 1, k+1, k+2, \dots, n) \in S_k \subset S_n$, the assertion of Corollary 17.9 is true for an arbitrary a, without the assumption that $a = (a_1, \dots, a_k, 0, \dots, 0)$. However, if we skip this assumption, for other permutations, we will have more cases. For example, for $w = (k+1, k-1, \dots, 1, k, k+2, k+3, \dots, n)$ with $\mathrm{code}(w) = (k, k-2, \dots, 1, 0, \dots, 0)$, Corollary 17.7 implies that

$$K^{-1}_{a,w} = \begin{cases} (-1)^{\ell(\sigma)} & \text{if } a = (c_{\sigma_1}, \dots, c_{\sigma_k}, 0, \dots, 0), \text{ for some } \sigma \in S_k, \\ (-1)^{\ell(\tau)+1} & \text{if } a = (k - \tau_1, \dots, k - \tau_k, 1, \dots, 0), \text{ for some } \tau \in S_k, \\ 0 & \text{otherwise.} \end{cases}$$

The polynomial ring $\mathbb{Q}[x_1, x_2, \dots]$ has the following basis of *standard elementary monomials:* $e_a := e_{a_2}(x_1)\, e_{a_3}(x_1, x_2)\, e_{a_4}(x_1, x_2, x_3) \cdots$, where $a = (a_1, a_2, \dots) \in \mathbb{N}^\infty$ such that $0 \le a_i \le i - 1$, for $i = 1, 2, \dots$. This basis was originally introduced by Lascoux and Schützenberger [LS1]; see also [FGP, Proposition 3.3].

Remark 17.10. Expressions for Schubert polynomials in the basis of standard elementary monomials play an important role in calculation of Gromov-Witten invariants for the small quantum cohomology ring of the flag manifold; see [FGP].

The Cauchy formula (Lascoux [La1], see also, e.g., [Man])

$$\sum_{w \in S_n} \mathfrak{S}_w(x) \cdot \mathfrak{S}_{ww_\circ}(y) = \prod_{i+j \le n} (x_i + y_j) = \prod_{k=1}^{n-1} \sum_{i=0}^{k} y_{n-k}^{k-i} e_i(x_1, \dots, x_k)$$

implies that

$$e_{w_\circ(\rho - a)} = \sum_{w \in S_n} K_{w,a}\, \mathfrak{S}_{ww_\circ},$$

for $a \in \mathbb{N}^n$. Equivalently,

$$\mathfrak{S}_{ww_\circ} = \sum_a K^{-1}_{a,w}\, e_{w_\circ(\rho-a)}.$$

This shows that the problem of inverting the Schubert-Kostka matrix is equivalent to the problem of expressing a Schubert polynomial in the basis of standard elementary monomials.

Let us assume, by convention, that $e_a = 0$, unless $0 \le a_i \le i-1$, for $i \ge 1$. Proposition 17.3 implies the following claim.

Corollary 17.11. *For $w \in S_n$, the Schubert polynomial $\mathfrak{S}_w$ can be expressed as*

$$\mathfrak{S}_w = \sum_{u \in S_n,\, a \in \mathbb{N}^n} (-1)^{\ell(u)} K_{w_\circ w w_\circ,\, w_\circ(a)+u(\rho)-\rho}\, e_a.$$

In particular, for 213-avoiding permutations, we obtain the following result.

Corollary 17.12. *For a 213-avoiding permutation $w \in S_n$ and $c = \mathrm{code}(w_\circ w w_\circ)$, the Schubert $\mathfrak{S}_w$ polynomial can be expressed as*

$$\mathfrak{S}_w = \sum_{u \in S_{n-1}} (-1)^{\ell(u)}\, e_{w_\circ(c+\rho-u(\rho))}.$$

Let us also give a (not very difficult) alternating expression for the generalized Littlewood-Richardson coefficients.

Corollary 17.13. *Let $u, v, w \in S_n$. Then the generalized Littlewood-Richardson coefficient $c_{u,v,w}$ is equal to*

$$c_{u,v,w} = \sum_{a,b} K_{u,a}\, K_{v,b}\, K^{-1}_{a+b,w_\circ w} = \sum_{z,a,b,c} (-1)^{\ell(z)} K_{u,a}\, K_{v,b}\, K_{w,c},$$

where the second sum is over permutations $z \in S_n$ and compositions $a, b, c \in \mathbb{N}^n$ such that $a + b + c = z(\rho)$.

Proof. We have $\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{a,b} K_{u,a}\, K_{v,b}\, x^{a+b} = \sum_{a,b,w} K_{u,a}\, K_{v,b}\, K^{-1}_{a+b,w_\circ w}\, \mathfrak{S}_{w_\circ w}$, which implies the first claim. Now apply Proposition 17.3. □

Let us identify the polynomial rings $\mathbb{Q}[x_1, x_2, \dots] = \mathbb{Q}[y_1, y_2, \dots]$. The transition matrix between the bases $\{\mathfrak{S}_w\}$ and $\{x^a\}$ is K; the transition matrix between the bases $\{x^a\}$ and $\{x^{(a)}\}$ is the diagonal matrix D with products of factorials; and the transition matrix between the bases $\{x^{(a)}\}$ and $\{\mathfrak{D}_w\}$ is K^T. Thus the transition matrix between the bases $\{\mathfrak{S}_w\}$ and $\{\mathfrak{D}_u\}$ is KDK^T. In other words, we obtain the following result.

Corollary 17.14. *We have $\mathfrak{S}_u = \sum_{w \in S_\infty} L_{u,w}\, \mathfrak{D}_w$, where*

$$L_{u,w} = \sum_{a \in \mathbb{N}^\infty} K_{u,a}\ K_{w,a}\, a_1!\, a_2! \cdots = (\mathfrak{S}_u, \mathfrak{S}_u)_D.$$

Notice that the matrix L is symmetric, i.e., the coefficient of $\mathfrak{D}_w$ in $\mathfrak{S}_u$ equals the coefficient of $\mathfrak{D}_u$ in $\mathfrak{S}_w$.

18. Parking functions

Let $n = r + 1$. Assume that $w = (1, 2, \ldots, r+1) = s_1 s_2 \cdots s_r \in S_{r+1}$ is the long cycle. In this section we calculate the corresponding polynomial $\mathfrak{D}_r = \mathfrak{D}_{s_1 \ldots s_r}$ in five different ways.

Let us use the coordinates $Y_i = (y, \alpha_i^\vee)$, $i = 1, \ldots, r$, from Section 7. These coordinates are related to the coordinates $y_1, \ldots, y_{r+1}$ from Section 12 by $Y_i = y_i - y_{i+1}$, for $i = 1, \ldots, r$. In the notation of Corollary 7.2, for $w = s_1 \cdots s_r$, we have $(i_1, \ldots, i_l) = (1, \ldots, r)$, and the Cartan integer $a_{i_p i_q}$ is -1, if $q = p + 1$, and 0, if $q > p+1$. Thus the sum in Corollary 7.2 involves only terms corresponding to arrays (k_{pq}) with $k_{pq} = 0$, unless $q = p+1$. In this case, the product $\prod k_{pq}!$ cancels with the product $\prod K_{*s}!$. More explicitly, Corollary 7.2 gives

$$\mathfrak{D}_r = \sum_{c_1, \ldots, c_r} \frac{Y_1^{c_1}}{c_1!} \cdots \frac{Y_r^{c_r}}{c_r!},$$

where the sum is over nonnegative integer sequence $(c_1, \ldots, c_r)$ such that $c_1 \leq 1, c_1 + c_2 \leq 2, c_1 + c_2 + c_3 \leq 3, \ldots,\ c_1 + \cdots + c_{r-1} \leq r - 1,\ c_1 + \cdots + c_r = r$. There are exactly the Catalan number $\frac{1}{r+1}\binom{2r}{r}$ of such sequences.

A *parking function* of length r is a sequence of positive integers $(b_1, \ldots, b_r)$, $1 \leq b_i \leq r$, such that $\#\{i \mid b_i \leq k\} \geq k$, for $k = 1, \ldots, r$. The number of parking functions of length r equals $(r+1)^{r-1}$. Recall that the number $(r+1)^{r-1}$ also equals the number of spanning trees in the complete graph K_{r+1}. Let us define the r-th *parking polynomial* by

$$P_r(Y_1, \ldots, Y_r) = \sum_{(b_1, \ldots, b_r)} Y_{b_1} \cdots Y_{b_r},$$

where the sum is over parking functions $(b_1, \ldots, b_r)$ of length r. For example,

$$P_3 = 6\, Y_1 Y_2 Y_3 + 3\, Y_1^2 Y_2 + 3\, Y_1 Y_2^2 + 3\, Y_1^2 Y_3 + Y_1^3.$$

The polynomial $\frac{1}{r!} P_r(Y_1, \ldots, Y_r)$ appeared in [PS] as the volume of a certain polytope; see Corollary 18.7 below. According to [PS], for a partition $\lambda = (\lambda_1, \ldots, \lambda_{r+1})$, the value $P_r(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_r - \lambda_{r+1})$ equals the number of λ-parking functions, which generalize the usual parking functions.

We can write the above expression for $\mathfrak{D}_r$ in terms of the parking polynomial.

Proposition 18.1. *We have* $\mathfrak{D}_r = \frac{1}{r!} P_r(Y_r, \ldots, Y_2, Y_1)$. *In particular, the degree of the Schubert variety* $X_{s_1 \ldots s_r}$ *equals the number of trees*

$$\deg(X_{s_1 \cdots s_r}) = P_r(1, \ldots, 1) = (r+1)^{r-1}.$$

Remark 18.2. Proposition 18.1 is true for an arbitrary Weyl group W and $w = s_{i_1} \cdots s_{i_r} \in W$ such that $(\alpha_{i_p}^\vee, \alpha_{i_{p+1}}) = 1$ and $(\alpha_{i_p}^\vee, \alpha_{i_q}) = 0$, for $q > p + 1$; see Corollary 7.2.

Remark 18.3. Let us weight the covering relation $u \lessdot u s_{ij}$, $i < j$, in the Bruhat order on S_{r+1} by $j - i$. According to Proposition 18.1, the weighted sum over saturated chains from id to $s_1 \cdots s_r$ equals the number $(r+1)^{r-1}$ of trees. Compare this with the fact that the total number of decompositions of the cycle $s_1 \cdots s_r$ into a product of r transpositions also equals $(r+1)^{r-1}$.

Let us write the polynomial $\mathfrak{D}_r = \mathfrak{D}_r(y_1, \dots, y_{r+1})$ in terms of the variables $y_1, \dots, y_{r+1}$. According to Corollary 12.1(3), the polynomial $\mathfrak{D}_r$ is recursively given by the integration $\mathfrak{D}_r = I_r(\mathfrak{D}_{r-1})$. In other words,

$$\mathfrak{D}_r(y_1, \dots, y_{r+1}) = \int_{y_{r+1}}^{y_r} \mathfrak{D}_{r-1}(y_1, \dots, y_{r-1}, t)\, dt. \tag{18.1}$$

$$\mathfrak{D}_r(y_1, \dots, y_{r+1}) = \int_{y_{r+1}}^{y_r} dt_r \int_{t_r}^{y_{r-1}} dt_{r-1} \cdots \int_{t_3}^{y_2} dt_2 \int_{t_2}^{y_1} dt_1. \tag{18.2}$$

Equivalent integral formulas for the parking polynomials were given by Kung and Yan [KY]. The right-hand side of the second formula is easily seen to be equal to the volume of the polytope from [PS], see below.

The long cycle $w = s_1 \cdots s_r$ is a 312-avoiding permutation in S_{r+1}. The code of the permutation $w_\circ w$ equals $\mathrm{code}(w_\circ w) = (r-1, r-2, \dots, 1, 0, 0)$. According to Theorem 13.4, the polynomial $\mathfrak{D}_r$ is given by the determinant of the following almost lower-triangular $(r+1) \times (r+1)$-matrix:

$$\mathfrak{D}_r(y_1, \dots, y_{r+1}) = \det \begin{pmatrix} y_1 & 1 & 0 & \cdots & 0 & 0 \\ y_2^{(2)} & y_2 & 1 & \cdots & 0 & 0 \\ y_3^{(3)} & y_3^{(2)} & y_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y_r^{(r)} & y_r^{(r-1)} & y_r^{(r-2)} & \cdots & y_r & 1 \\ y_{r+1}^{(r)} & y_{r+1}^{(r-1)} & y_{r+1}^{(r-2)} & \cdots & y_{r+1} & 1 \end{pmatrix}, \tag{18.3}$$

where, as before, $y_i^{(a)} = \frac{y_i^a}{a!}$.

Remark 18.4. Determinant (18.3) is closely related to the formula found by Steck [Steck] and Gessel [Ges] that can be written in our notation as

$$\mathfrak{D}_r(y_1, \dots, y_r, 0) = \det \left(y_i^{(j-i+1)} \right)_{i,j=1}^{r}. \tag{18.4}$$

Since $\mathfrak{D}_r(y_1 + c, \dots, y_{r+1} + c) = \mathfrak{D}_r(y_1, \dots, y_{r+1})$, expression (18.4) defines the polynomial $\mathfrak{D}_r$. Expression (18.4) is obtained from (18.3) by setting $y_{r+1} = 0$. On the other hand, we can obtain expression (18.3) for $\mathfrak{D}_{r-1}$ by differentiating (18.4) with respect to y_r. This implies that

$$\mathfrak{D}_{r-1}(y_1, \dots, y_r) = \frac{\partial}{\partial y_r} \mathfrak{D}_r(y_1, \dots, y_r, 0),$$

which is equivalent to (18.1). Kung and Yan [KY, Sect. 3] derived this expression in terms of Gončarov polynomials.

Expanding the determinant (18.3), we obtain the following result.

Proposition 18.5. *We have*

$$\mathfrak{D}_r = \sum (-1)^{r+1-k}\, y_{i_1}^{(i_1)}\, y_{i_1+i_2}^{(i_2)} \cdots y_{i_1+\cdots+i_{k-1}}^{(i_{k-1})}\, y_{i_1+\cdots+i_k}^{(i_k-1)},$$

where the sum is over 2^{r+1} sequences $(i_1, \dots, i_k)$ such that $i_1, \dots, i_k \geq 1$ and $i_1 + \cdots + i_k = r+1$. (Notice that the power of the last term is decreased by 1.)

Corollary 18.6. *For $a \in \mathbb{N}^{r+1}$, the element $K^{-1}_{a,\, s_1\cdots s_r}$ of the inverse Schubert-Kostka matrix equals $(-1)^{r+1-k}$, if the sequence $(a_1, \ldots, a_r, a_{r+1}+1)$ is the concatenation of k sequences of the form $(0, \ldots, 0, l)$ with $l-1$ zeros, for $l \geq 1$; otherwise $K^{-1}_{a,\, s_1\cdots s_r} = 0$.*

For example, we have $K^{-1}_{(1,0,0,3,0,2,1,0,0,2),\, s_1\cdots s_9} = (-1)^{10-5}$.

The generalized Gelfand-Tsetlin polytope $\mathcal{P}_{\lambda,w}$ from Section 15, for the 312-avoiding permutation $w = s_1 \cdots s_r$, is given by the inequalities:

$$\mathcal{P}_{\lambda,s_1\cdots s_r} = \{(t_1, \ldots, t_r) \in \mathbb{R}^r \mid \lambda_i \geq t_i, \text{ for } i = 1, \ldots, r;\ t_1 \geq t_2 \geq \cdots \geq t_r \geq \lambda_{r+1}\}.$$

This polytope is exactly the polytope studied in [PS]. According to Corollary 15.2, $\mathfrak{D}_r(\lambda)$ equals the volume of the polytope $\mathcal{P}_{\lambda,s_1\cdots s_r}$. Also, as we already mentioned, this volume equals the right-hand side of (18.2), for $(y_1, \ldots, y_{r+1}) = (\lambda_1, \ldots, \lambda_{r+1})$. We recover the following result from [PS] about the relation of this polytope with the parking polynomial P_r.

Corollary 18.7. *We have $\mathrm{Vol}(\mathcal{P}_{\lambda,s_1\cdots s_r}) = \frac{1}{r!} P_r(Y_r, \ldots, Y_1)$, where $Y_i = \lambda_i - \lambda_{i+1}$, for $i = 1, \ldots, r$.*

Let us also calculate the polynomial $\mathfrak{D}_r$ using just its definition in terms of saturated chains in the Bruhat order.

For an arbitrary Weyl group W and $w = s_{i_1} \cdots s_{i_l} \in W$ with distinct $i_1, \ldots, i_l$, the interval $[id, w] \subset W$ in the Bruhat order consists of the elements $u = s_{j_1} \cdots s_{j_s}$ such that $j_1, \ldots, j_s$ is a subword of $i_1, \ldots, i_l$; see Section 2. Thus the interval $[id, w]$ is isomorphic to the Boolean lattice of order l.

In particular, this is true for the long cycle $w = s_1 \cdots s_r = (1, \ldots, r+1)$ in S_{r+1}. The elements u covered by w are of the form $u = s_1 \cdots \widehat{s_k} \cdots s_r = w\, s_{k,r+1} = (1, 2, \ldots, k)(k+1, k+2, \ldots, r+1)$, for some $k \in \{1, \ldots, r\}$. Moreover, for such u, the Chevalley multiplicity equals $m(u \lessdot w) = y_k - y_{r+1} = Y_k + Y_{k+1} + \cdots + Y_r$. The interval $[id, (1, \ldots, k)(k+1, \ldots, r+1)]$ in the Bruhat order is isomorphic to the product of two intervals $[id, (1, \ldots, k)] \times [id, (k+1, \ldots, r+1)]$. Thus we obtain the following recurrence relation for the parking polynomial P_r (related to $\mathfrak{D}_r$ by Proposition 18.1):

$$P_r(Y_1, \ldots, Y_r) = \sum_{k=1}^{r} (Y_1 + \cdots + Y_k) \cdot P_{k-1}(Y_1, \ldots, Y_{k-1}) \cdot P_{r-k}(Y_{k+1}, \ldots, Y_r).$$

Also $P_0 = 1$ and $P_1(Y_1) = Y_1$. This relation follows from results of Kreweras [Kre]. It implies the following combinatorial interpretation of the parking polynomial $P_r(Y_1, \ldots, Y_r)$.

An *increasing binary tree* is a directed rooted tree with an increasing labeling of vertices by the integers $1, \ldots, r$ such that each vertex has at most one left successor and at most one right successor. Let $\mathcal{T}_r$ be the set of such trees with r vertices. It is well known that $|\mathcal{T}_r| = r!$; see [EC1]. Let us define the weight of a tree in $\mathcal{T}_r$ as follows. For $T \in \mathcal{T}_r$, let $\tilde{T}$ be the binary tree obtained from T by adding two leaves (left and right) to each vertex of T without successors and one left (resp., right) leaf to each vertex of T with only a right (resp., left) successor. Then $\tilde{T}$ has $r+1$ leaves. Let us label these leaves by the variables $Y_1, \ldots, Y_{r+1}$ from left to right. For each vertex v in T, define the weight $\mathrm{wt}(v)$ as the sum of Y_i's corresponding to all leaves of $\tilde{T}$ in the left branch of v. Let us define the weight of $T \in \mathcal{T}_r$ as the product $\mathrm{wt}(T) = \prod \mathrm{wt}(v)$ over all vertices v of T.

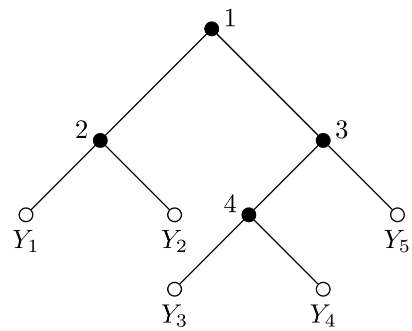

FIGURE 2. A tree in $\mathcal{T}_4$ of weight $(Y_1+Y_2)\,Y_1\,(Y_3+Y_4)\,Y_3$.

Figure 2 shows an example of a tree $T \in \mathcal{T}_4$ of weight $\mathrm{wt}(T) = (Y_1+Y_2)\,Y_1\,(Y_3+Y_4)\,Y_3$. The vertices of T are shown by black circles, and the added leaves of $\tilde{T}$ are shown by white circles. The above recurrence relation for P_r implies the following result.

Proposition 18.8. *The parking polynomial P_r equals the sum*

$$P_r(Y_1,\ldots,Y_r) = \sum_{T\in\mathcal{T}_r} \mathrm{wt}(T).$$

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