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Deformations of Coxeter Hyperplane Arrangements[1]

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We investigate several hyperplane arrangements that can be viewed as deformations of Coxeter arrangements. In particular, we prove a conjecture of Linial and Stanley that the number of regions of the arrangement $x_i - x_j = 1$, $1 \leqslant i < j \leqslant n$, is equal to the number of alternating trees on $n+1$ vertices. Remarkably, these numbers have several additional combinatorial interpretations in terms of binary trees, partially ordered sets, and tournaments. More generally, we give formulae for the number of regions and the Poincaré polynomial of certain finite subarrangements of the affine Coxeter arrangement of type A_{n-1}. These formulae enable us to prove a "Riemann hypothesis" on the location of zeros of the Poincaré polynomial. We give asymptotics of the Poincaré polynomials when n goes to infinity. We also consider some generic deformations of Coxeter arrangements of type A_{n-1}. © 2000 Academic Press

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1. INTRODUCTION

The *Coxeter arrangement* of type A_{n-1} is the arrangement of hyperplanes in $\mathbb{R}^n$ given by

$$x_i - x_j = 0, \qquad 1 \leqslant i < j \leqslant n. \tag{1.1}$$

This arrangement has $n!$ regions. They correspond to $n!$ different ways of ordering the sequence $x_1, ..., x_n$.

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In the paper we extend this simple, nevertheless important, result to the case of a general class of arrangements which can be viewed as deformations of the arrangement (1.1).

One special case of such deformations is the arrangement given by

$$x_i - x_j = 1, \qquad 1 \leqslant i < j \leqslant n. \tag{1.2}$$

We will call it the *Linial arrangement*. This arrangement was first considered by N. Linial and S. Ravid. They calculated its number of regions and the Poincaré polynomial for $n \leqslant 9$. On the basis of these numerical data the second author of the present paper made a conjecture that the number of regions of (1.2) is equal to the number of *alternating trees* on $n+1$ vertices (see [29]). A tree T on the vertices 1, 2, ..., $n+1$ is alternating if the vertices in any path in T alternate, i.e., form an up–down or down–up sequence. Equivalently, every vertex is either less than all its neighbors or greater than all its neighbors. These trees first appeared in [11], and in [20] a formula for the number of such trees on $n+1$ vertices was proved. In this paper we provide a proof of the conjecture on the number of regions of the Linial arrangement. Another proof was given by Athanasiadis [3, Thm. 4.1].

In fact, we prove a more general result for *truncated affine arrangements*, which are certain finite subarrangements of the affine hyperplane arrangement of type $\tilde{A}_{n-1}$ (see Section 9). As a byproduct we obtain an amazing theorem on the location of zeros of Poincaré polynomials of these arrangements. This theorem states that in one case all zeros are real, whereas in the other case all zeros have the same real part.

The paper is organized as follows. In Section 2 we give the basic notions of hyperplane arrangement, number of regions, Poincaré polynomial, and intersection poset. In Section 3 we describe the arrangements we will be concerned with in this paper—deformations of the arrangement (1.1). In Section 4 we review several general theorems on hyperplane arrangements. Then in Section 5 we apply these theorems to deformed Coxeter arrangements. In Section 6 we consider a “semigeneric” deformation of the braid arrangement (the Coxeter arrangement of type A_{n-1}) related to the theory of interval orders. In Section 7 we study the hyperplane arrangements which are related, in a special case, to interval orders (cf. [29]) and the Catalan numbers. We prove a theorem that establishes a relation between the numbers of regions of such arrangements. In Section 8 we formulate the main result on the Linial arrangement. We introduce several combinatorial objects whose numbers are equal to the number of regions of the Linial arrangement: alternating trees, local binary search trees, sleek posets, semiacyclic tournaments. We also prove a theorem on characterization of sleek posets in terms of forbidden subposets. In Section 9 we study truncated affine arrangements. We prove a functional equation for the generating function for the numbers of regions of such arrangements, deduce a formula for these numbers, and from it obtain a theorem on the location of zeros of the characteristic polynomial. In Sections 10 and 11 we study weighted trees and asymptotics of characteristic polynomials.

2. ARRANGEMENTS OF HYPERPLANES

First, we give several basic notions related to arrangements of hyperplanes. For more details, see [34, 16, 17].

A *hyperplane arrangement* is a discrete collection of affine hyperplanes in a vector space. We will be concerned here only with finite arrangements. Let $\mathcal{A}$ be a finite hyperplane arrangement in a real finite-dimensional vector space V. It will be convenient to assume that the vectors dual to hyperplanes in $\mathcal{A}$ span the vector space V^*; the arrangement $\mathcal{A}$ is then called *essential.* Denote by $r(\mathcal{A})$ the number of *regions* of $\mathcal{A}$, which are the connected components of the space $V - \bigcup_{H \in \mathcal{A}} H$. We will also consider the number $b(\mathcal{A})$ of "relatively bounded" regions of $\mathcal{A}$, which will just be the number of *bounded* regions when $\mathcal{A}$ is essential.

These numbers have a natural q-analogue. Let $\mathcal{A}_{\mathbb{C}}$ denote the complexified arrangement $\mathcal{A}$. In other words, $\mathcal{A}_{\mathbb{C}}$ is the collection of the hyperplanes $H \otimes \mathbb{C}$, $H \in \mathcal{A}$, in the complex vector space $V \otimes \mathbb{C}$. Let $C_{\mathcal{A}}$ be the complement to the union of the hyperplanes of $\mathcal{A}_{\mathbb{C}}$ in $V \otimes \mathbb{C}$, and let $H^k(\,\cdot\,; \mathbb{C})$ denote singular cohomology with coefficients in $\mathbb{C}$. Then one can define the *Poincaré polynomial* $\mathrm{Poin}_{\mathcal{A}}(q)$ of $\mathcal{A}$ as

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_{k \geqslant 0} \dim H^k(C_{\mathcal{A}}; \mathbb{C})\, q^k,$$

the generating function for the Betti numbers of $C_{\mathcal{A}}$.

The following theorem, proved in the paper of Orlik and Solomon [16], shows that the Poincaré polynomial generalizes the number of regions $r(\mathcal{A})$ and the number of bounded regions $b(\mathcal{A})$.

THEOREM 2.1. *We have* $r(\mathcal{A}) = \mathrm{Poin}_{\mathcal{A}}(1)$ *and* $b(\mathcal{A}) = \mathrm{Poin}_{\mathcal{A}}(-1)$.

Orlik and Solomon gave a combinatorial description of the cohomology ring $H^*(C_{\mathcal{A}}; \mathbb{C})$ (cf. Section 8.3) in terms of the *intersection poset* $L_{\mathcal{A}}$ of the arrangement $\mathcal{A}$.

The intersection poset is defined as follows: The elements of $L_{\mathcal{A}}$ are nonempty intersections of hyperplanes in $\mathcal{A}$ ordered by reverse inclusion. The poset $L_{\mathscr{A}}$ has a unique minimal element $\hat{0} = V$. This poset is always a meet-semilattice for which every interval is a geometric lattice. It will be a (geometric) lattice if and only if $L_{\mathscr{A}}$ contains a unique maximal element, i.e., the intersection of all hyperplanes in $\mathscr{A}$ is nonempty. (When $\mathscr{A}$ is essential, this intersection is $\{0\}$.) In fact, $L_{\mathscr{A}}$ is a geometric semilattice in the sense of Wachs and Walker [31], and thus for instance is a shellable and hence Cohen–Macaulay poset.

The *characteristic polynomial* of $\mathscr{A}$ is defined by

$$\chi_{\mathscr{A}}(q) = \sum_{z \in L_{\mathscr{A}}} \mu(\hat{0}, z)\, q^{\dim z}, \tag{2.1}$$

where μ denotes the Möbius function of $L_{\mathscr{A}}$ (see [27, Sect. 3.7]).

Let d be the dimension of the vector space V. Note that it follows from the properties of geometric lattices [27, Proposition 3.10.1] that the sign of $\mu(\hat{0}, z)$ is equal to $(-1)^{d-\dim z}$.

The following simple relation between the (topologically defined) Poincaré polynomial and the (combinatorially defined) characteristic polynomial was found in [16]:

$$\chi_{\mathscr{A}}(q) = q^d \operatorname{Poin}_{\mathscr{A}}(-q^{-1}). \tag{2.2}$$

Sometimes it will be more convenient for us to work with the characteristic polynomial $\chi_A(q)$ rather than the Poincaré polynomial.

A combinatorial proof of Theorem 2.1 in terms of the characteristic polynomial was earlier given by T. Zaslavsky in [34].

The number of regions, the number of (relatively) bounded regions, and, more generally, the Poincaré (or characteristic) polynomial are the most simple numerical invariants of a hyperplane arrangement. In this paper we will calculate these invariants for several hyperplane arrangements related to Coxeter arrangements

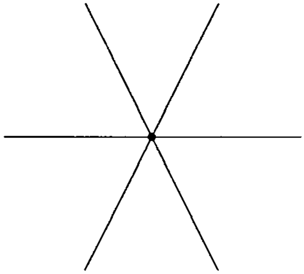

FIG. 1. The Coxeter hyperplane arrangement $\mathscr{A}_2$.

3. COXETER ARRANGEMENTS AND THEIR DEFORMATIONS

Let V_{n-1} denote the subspace (hyperplane) in $\mathbb{R}^n$ of all vectors $(x_1, ..., x_n)$ such that $x_1 + \cdots + x_n = 0$. All hyperplane arrangements that we consider below lie in V_{n-1}. The lower index $n-1$ will always denote dimension of an arrangement.

The *braid arrangement* or *Coxeter arrangement* (*of type* A_{n-1}) is the arrangement $\mathcal{A}_{n-1}$ of hyperplanes in $V_{n-1} \subset \mathbb{R}^n$ given by

$$x_i - x_j = 0, \qquad 1 \leqslant i < j \leqslant n \tag{3.1}$$

(see Fig. 1 for an example.)

It is clear that $\mathcal{A}_{n-1}$ has $r(\mathcal{A}_{n-1}) = n!$ regions (called Weyl chambers) and $b(\mathcal{A}_{n-1}) = 0$ bounded regions. Arnold [1] calculated the cohomology ring $H^*(C_{\mathcal{A}_{n-1}}; \mathbb{C})$. In particular, he proved that

$$\mathrm{Poin}_{\mathcal{A}_{n-1}}(q) = (1+q)(1+2q) \cdots (1+(n-1)\, q). \tag{3.2}$$

In this paper we will study *deformations* of the arrangement (3.1), which are hyperplane arrangements in $V_{n-1} \subset \mathbb{R}^n$ of the type

$$x_i - x_j = a_{ij}^{(1)}, ..., a_{ij}^{(m_{ij})}, \qquad 1 \leqslant i < j \leqslant n, \tag{3.3}$$

where m_{ij} are nonnegative integers and $a_{ij}^{(k)} \in \mathbb{R}$.

One special case is the arrangement given by

$$x_i - x_j = a_{ij}, \qquad 1 \leqslant i < j \leqslant n. \tag{3.4}$$

The following hyperplane arrangements of type (3.3) are worth mentioning:

- The *generic arrangement* (see the end of Section 5) given by

$$x_i - x_j = a_{ij}, \qquad 1 \leqslant i < j \leqslant n,$$

where the a_{ij}'s are generic real numbers.

- The *semigeneric arrangement* $\mathcal{G}_n$ (see Section 6) given by

$$x_i - x_j = a_i, \qquad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n,\ i \neq j,$$

where the a_i's are generic real numbers.

- The *Linial arrangement* $\mathcal{L}_{n-1}$ (see [29] and Section 8; also see Fig. 2) given by

$$x_i - x_j = 1, \qquad 1 \leqslant i < j \leqslant n. \tag{3.5}$$

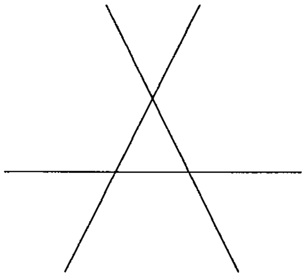

FIG. 2. Seven regions of the Linial arrangement $\mathscr{L}_2$.

- The *Shi arrangement* $\mathscr{S}_{n-1}$ (see [25, 26, 29] and Section 9.2) given by

$$x_i - x_j = 0, 1, \qquad 1 \leqslant i < j \leqslant n. \tag{3.6}$$

- The *extended Shi arrangement* $\mathscr{S}_{n-1,\,k}$ (see Section 9.2) given by

$$x_i - x_j = -k, -k+1, ..., k+1, \qquad 1 \leqslant i < j \leqslant n, \tag{3.7}$$

where $k \geqslant 0$ is fixed.

- The *Catalan arrangements* (see Section 7) $\mathscr{C}_{n-1}(1)$ given by

$$x_i - x_j = -1, 1, \qquad 1 \leqslant i < j \leqslant n, \tag{3.8}$$

and $\mathscr{C}^0_{n-1}(1)$ given by

$$x_i - x_j = -1, 0, 1, \qquad 1 \leqslant i < j \leqslant n. \tag{3.9}$$

- The *truncated affine arrangement* $\mathscr{A}^{ab}_{n-1}$ (see Section 9) given by

$$x_i - x_j = -a+1, -a+2, ..., b-1, \qquad 1 \leqslant i < j \leqslant n, \tag{3.10}$$

where a and b are fixed integers such that $a + b \geqslant 2$.

One can define analogous arrangements for any root system. Let V be a real d-dimensional vector space, and let R be a *root system* in V^* with a chosen set of *positive* roots $R_+ = \{\beta_1, \beta_2, ..., \beta_N\}$ (see, e.g., [6, Ch. VI]). The *Coxeter arrangement* $\mathscr{R}$ of type R is the arrangement of hyperplanes in V given by

$$\beta_i(x) = 0, \qquad 1 \leqslant i \leqslant N. \tag{3.11}$$

Brieskorn [7] generalized Arnold's formula (3.2). His formula for the Poincaré polynomial of (3.11) involves the exponents $e_1, ..., e_d$ of the corresponding Weyl group W:

$$\mathrm{Poin}_{\mathscr{R}}(q) = (1+e_1 q)(1+e_2 q) \cdots (1+e_d q).$$

Consider the hyperplane arrangement given by

$$\beta_i(x) = a_i^{(1)}, ..., a_i^{(m_i)}, \qquad 1 \leqslant i \leqslant N, \tag{3.12}$$

where $x \in V$, m_i are some nonnegative integers, and $a_i^{(k)} \in \mathbb{R}$. Many of the results of this paper have a natural counterpart in the case of an arbitrary root system. We will briefly outline several related results and conjectures in Section 9.4.

4. WHITNEY'S FORMULA AND THE NBC THEOREM

In this section we review several essentially well-known results on hyperplane arrangements that will be useful in what follows.

Consider the arrangement $\mathscr{A}$ of hyperplanes in $V \cong \mathbb{R}^d$ given by equations

$$h_i(x) = a_i, \qquad 1 \leqslant i \leqslant N, \tag{4.1}$$

where $x \in V$, the $h_i \in V^*$ are linear functionals on V, and the a_i are real numbers.

We call a subset I of $\{1, 2, ..., N\}$ *central* if the intersection of the hyperplanes $h_i(x) = a_i$, $i \in I$, is nonempty. For a subset $I = \{i_1, i_2, ..., i_l\}$, denote by $\mathrm{rk}(I)$ the dimension (rank) of the linear span of the vectors $h_{i_1}, ..., h_{i_l}$.

The following statement is a generalization of a classical formula of Whitney [32].

THEOREM 4.1. *The Poincaré and characteristic polynomials of the arrangement $\mathscr{A}$ are equal to*

$$\mathrm{Poin}_{\mathscr{A}}(q) = \sum_I (-1)^{|I| - \mathrm{rk}(I)}\, q^{\mathrm{rk}(I)}, \tag{4.2}$$

$$\chi_{\mathscr{A}}(q) = \sum_I (-1)^{|I|}\, q^{d - \mathrm{rk}(I)}, \tag{4.3}$$

where I ranges over all central subsets in $\{1, 2, ..., N\}$. *In particular,*

$$r(\mathcal{A}) = \sum_I (-1)^{|I| - \mathrm{rk}(I)} \tag{4.4}$$

$$b(\mathcal{A}) = \sum_I (-1)^{|I|}. \tag{4.5}$$

We also need the well-known cross-cut theorem (see [27, Corollary 3.9.4]).

THEOREM 4.2. *Let L be a finite lattice with minimal element $\hat{0}$ and maximal element $\hat{1}$, and let X be a subset of vertices in L such that* (a) $\hat{0} \notin X$, *and* (b) *if $y \in L$ and $y \neq \hat{0}$, then $x \leqslant y$ for some $x \in X$. Then*

$$\mu_L(\hat{0}, \hat{1}) = \sum_k (-1)^k n_k, \tag{4.6}$$

where n_k is the number of k-element subsets in X with join equal to $\hat{1}$.

Proof of Theorem 4.1. Let z be any element in the intersection poset $L_{\mathcal{A}}$, and let $L(z)$ be the subposet of all elements $x \in L_{\mathcal{A}}$ such that $x \leqslant z$, i.e., the subspace x contains z. In fact, $L(z)$ is a geometric lattice. Let X be the set of all hyperplanes from $\mathcal{A}$ which contain z. If we apply Theorem 4.2 to $L = L(z)$ and sum (4.6) over all $z \in L_{\mathcal{A}}$, we get the formula (4.3). Then by (2.2) we get (4.2). ∎

A *cycle* is a minimal subset I such that $\mathrm{rk}(I) = |I| - 1$. In other words, a subset $I = \{i_1, i_2, ..., i_l\}$ is a cycle if there exists a nonzero vector $(\lambda_1, \lambda_2, ..., \lambda_l)$, unique up to a nonzero factor, such that $\lambda_1 h_{i_1} + \lambda_2 h_{i_2} + \cdots + \lambda_l h_{i_l} = 0$. It is not difficult to see that a cycle I is central if, in addition, we have $\lambda_1 a_{i_1} + \lambda_2 a_{i_2} + \cdots + \lambda_l a_{i_l} = 0$. Thus, if $a_1 = \cdots = a_N = 0$ then all cycles are central, and if the a_i are generic then there are no central cycles.

A subset I is called *acyclic* if $|I| = \mathrm{rk}(I)$, i.e., I contains no cycles. It is clear that any acyclic subset is central.

COROLLARY 4.3. *In the case when the a_i are generic, the Poincaré polynomial is given by*

$$\mathrm{Poin}_{\mathcal{A}}(q) = \sum_I q^{\mathrm{rk}(I)},$$

where the sum is over all acyclic subsets I of $\{1, 2, ..., N\}$. *In particular, the number of regions $r(\mathcal{A})$ is equal to the number of acyclic subsets.*

Indeed, in this case a subset I is acyclic if and only if it is central.

Remark 4.4. The word "generic" in the corollary means that no k distinct hyperplanes in (4.1) intersect in an affine subspace of codimension less than k. For example, if $\mathscr{A}$ is defined over $\mathbb{Q}$ then it is sufficient to require that the a_i be linearly independent over $\mathbb{Q}$.

Let us fix a linear order ρ on the set $\{1, 2, ..., N\}$. We say that a subset I of $\{1, 2, ..., N\}$ is a *broken central circuit* if there exists $i \notin I$ such that $I \cup \{i\}$ is a central cycle and i is the minimal element of $I \cup \{i\}$ with respect to the order ρ.

The following, essentially well-known, theorem gives us the main tool for the calculation of Poincaré (or characteristic) polynomials. We will refer to it as the No Broken Circuit (NBC) Theorem.

THEOREM 4.5. *We have*

$$\mathrm{Poin}_{\mathscr{A}}(q) = \sum_{I} q^{|I|},$$

where the sum is over all acyclic subsets I of $\{1, 2, ..., N\}$ without broken central circuits.

Proof. We will deduce this theorem from Theorem 4.1 using the *involution principle*. In order to do this we construct an involution $\imath: I \to \imath(I)$ on the set of all central subsets I with a broken central circuit such that for any I we have $\mathrm{rk}(\imath(I)) = \mathrm{rk}(I)$ and $|\imath(I)| = |I| \pm 1$.

This involution is defined as follows: Let I be a central subset with a broken central circuit, and let $s(I)$ be the set of all $i \in 1, ..., N$ such that i is the minimal element of a broken central circuit $J \subset I$. Note that $s(I)$ is nonempty. If the minimal element s_* of $s(I)$ lies in I, then we define $\imath(I) = I \backslash \{s_*\}$. Otherwise, we define $\imath(I) = I \cup \{s_*\}$.

Note that $s(I) = s(\imath(I))$, thus $\imath$ is indeed an involution. It is clear now that all terms in (4.2) for I with a broken central circuit cancel each other and the remaining terms yield the formula in Theorem 4.5. ∎

Remark 4.6. Note that by Theorem 4.5 the number of subsets I without broken central circuits does not depend on the choice of the linear order ρ.

5. DEFORMATIONS OF GRAPHIC ARRANGEMENTS

In this section we show how to apply the results of the previous section to arrangements of type (3.3) and to give an interpretation of these results in terms of (colored) graphs.

With the hyperplane $x_i - x_j = a_{ij}^{(k)}$ of (3.3) one can associate the edge (i, j) that has the color k. We will denote this edge by $(i, j)^{(k)}$. Then a subset I of hyperplanes corresponds to a colored graph G on the set of vertices $\{1, 2, ..., n\}$. According to the definitions in Section 4., a circuit $(i_1, i_2)^{(k_1)}, (i_2, i_3)^{(k_2)}, ..., (i_l, i_1)^{(k_l)}$ in G is *central* if $a_{i_1, i_2}^{(k_1)} + a_{i_2, i_3}^{(k_2)} + \cdots + a_{i_l, i_1}^{(k_l)} = 0$. Clearly, a graph G is acyclic if and only if G is a *forest*.

Fix a linear order on the edges $(i, j)^{(k)}$, $1 \leqslant i < j \leqslant n$, $1 \leqslant k \leqslant m_{ij}$. We will call a subset of edges C a *broken A-circuit* if C is obtained from a central circuit by deleting the minimal element. (Here A stands for the collection $\{a_{ij}^{(k)}\}$). Note that it should not be confused with the classical notion of a broken circuit of a graph, which corresponds to the case when all $a_{ij}^{(k)}$ are zero.

We summarize below several special cases of the NBC Theorem (Theorem 4.5). Here $|F|$ denotes the number of edges in a forest F.

Corollary 5.1. *The Poincaré polynomial of the arrangement* (3.3) *is equal to*

$$\mathrm{Poin}_{\mathscr{A}}(q) = \sum_F q^{|F|},$$

where the sum is over all colored forests F on the vertices $1, 2, ..., n$ (*an edge* (i, j) *can have a color k, where* $1 \leqslant k \leqslant m_{ij}$) *without broken A-circuits. The number of regions of arrangement* (3.3) *is equal to the number of such forests.*

In the case of the arrangement (3.4) we have:

Corollary 5.2. *The Poincaré polynomial of the arrangement* (3.4) *is equal to*

$$\mathrm{Poin}_{\mathscr{A}}(q) = \sum_F q^{|F|},$$

where the sum is over all forests on the set of vertices $\{1, 2, ..., n\}$ *without broken A-circuits. The number of regions of the arrangement* (3.4) *is equal to the number of such forests.*

In the case when the $a_{ij}^{(k)}$ are generic these results become especially simple.

For a forest F on vertices $1, 2, ..., n$ we will write $m^F := \prod_{(i, j) \in F} m_{ij}$, where the product is over all edges (i, j), $i < j$, in F. Let $c(F)$ denote the number of connected components in F.

COROLLARY 5.3. *Fix nonnegative integers* m_{ij}, $1 \leqslant i < j \leqslant n$. *Let* $\mathscr{A}$ *be an arrangement of type* (3.3) *where the* $a_{ij}^{(k)}$ *are generic. Then*

1. $\mathrm{Poin}_{\mathscr{A}}(q) = \sum_F m^F q^{|F|}$,
2. $r(\mathscr{A}) = \sum_F m^F$,

where the sums are over all forests F on the vertices 1, 2, ..., *n*.

COROLLARY 5.4. *The number of regions of the arrangement* (3.4) *with generic* a_{ij} *is equal to the number of forests on n labelled vertices.*

This corollary is "dual" to the following known result (see, e.g., [27, Exercise 4.32(a)]). Let P_n be the permutohedron, i.e., the polyhedron with vertices $(\sigma_1, ..., \sigma_n) \in \mathbb{R}^n$, where $\sigma_1, ..., \sigma_n$ ranges over all permutations of $1, ..., n$.

PROPOSITION 5.5. *The number of integer lattice points in* P_n *is equal to the number of forests on n labelled vertices.*

The connected components of the $\binom{n}{2}$-dimensional space of all arrangements (3.4) correspond to (coherent) zonotopal tilings of the permutohedron P_n, i.e., certain subdivisions of P_n into parallelepipeds. The regions of a generic arrangement (3.4) correspond to the vertices of the corresponding tiling, which are all integer lattice points in P_n.

It is also well known that the volume of the permutohedron P_n is equal to the number of parallelepipeds in a tiling which, in turn, is equal to n^{n-2}—the number of trees on n labelled vertices. Dually, this implies the following result.

PROPOSITION 5.6. *The number of vertices* (*i.e., zero-dimensional intersections of hyperplanes*) *of the arrangement* (3.4) *with generic* a_{ij} *is equal to* n^{n-2}.

6. A SEMIGENERIC DEFORMATION OF THE BRAID ARRANGEMENT

Define the "semigeneric" deformation $\mathscr{G}_n$ of the braid arrangement (3.1) to be the arrangement

$$x_i - x_j = a_i, \qquad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n,\ i \neq j,$$

where the a_i's are generic real numbers (e.g., linearly independent over $\mathbb{Q}$). The significance of this arrangement to the theory of interval orders is discussed in [29, Sect. 3]. In [29, Thm. 3.1 and Cor. 3.3] a generating function for the number $r(\mathcal{G}_n)$ of regions and for the characteristic polynomial $\chi_{\mathcal{G}_n}(q)$ of $\mathcal{G}_n$ is stated without proof. In this section we provide the proofs.

THEOREM 6.1. *Let*

$$z = \sum_{n \geqslant 0} r(\mathcal{G}_n) \frac{x^n}{n!}$$

$$= 1 + x + 3\frac{x^2}{2!} + 19\frac{x^3}{3!} + 195\frac{x^4}{4!} + 2831\frac{x^5}{5!} + 53703\frac{x^6}{6!} + \cdots.$$

Define a power series

$$y = 1 + x + 5\frac{x^2}{2!} + 46\frac{x^3}{3!} + 631\frac{x^4}{4!} + 11586\frac{x^5}{5!} + 267369\frac{x^6}{6!} + \cdots$$

by the equation

$$1 = y(2 - e^{xy}).$$

Then z is the unique power series satisfying

$$\frac{z'}{z} = y^2, \qquad z(0) = 1.$$

Proof. We use the formula (4.4) to compute $R(\mathcal{G}_n)$. Given a central set I of hyperplanes $x_i - x_j = a_i$ in $\mathcal{G}_n$, define a directed graph G_I on the vertex set $1, 2, ..., n$ as follows: let $i \to j$ be a directed edge of G_I if and only if the hyperplane $x_i - x_j = a_i$ belongs to I. (By slight abuse of notation, we are using I to denote a set of hyperplanes, rather than the set of their indices.) Note that G_I cannot contain both the edges $i \to j$ and $j \to i$, since the intersection of the corresponding hyperplanes is empty. If $k_1, k_2, ..., k_r$ are distinct elements of $\{1, 2, ..., n\}$, then it is easy to see that if r is even then there are exactly two ways to direct the edges $k_1k_2, k_2k_3, ..., k_{r-1}k_r, k_rk_1$ so that the hyperplanes corresponding to these edges have nonempty intersection, while if r is odd then there are no ways. It follows that G_I, ignoring the direction of edges, is bipartite (i.e., all circuits have even length). Moreover, given an undirected bipartite graph on the vertices $1, 2, ..., n$ with blocks (maximal connected subgraphs that remain connected when any vertex is removed) $B_1, ..., B_s$, there are exactly two ways to direct the edges of each block so that the resulting directed graph G is the graph G_I of a *central* set I of hyperplanes. In addition, $\mathrm{rk}(I) = n - c(G)$, where $c(G)$

is the number of connected components of G. Letting $e(G)$ be the number of edges and $b(G)$ the number of blocks of G, it follows from Eq. (4.3) that

$$\chi_{\mathscr{G}_n}(q) = \sum_G (-1)^{e(G)}\, 2^{b(G)} q^{c(G)},$$

where G ranges over all bipartite graphs on the vertex set 1, 2, ..., n. This formula appears without proof in [29, Thm. 3.2]. In particular, putting $q = -1$ gives

$$r(\mathscr{G}_n) = (-1)^n \sum_G (-1)^{e(G)+c(G)}\, 2^{b(G)}. \tag{6.1}$$

To evaluate the generating function $z = \sum r(\mathscr{G}_n)(x^n/n!)$, we use the following strategy.

(a) Compute $A_n := \sum_G (-1)^{e(G)}$, where G ranges over *all* (undirected) bipartite graphs on 1, 2, ..., n.

(b) Use (a) and the exponential formula to compute $B_n := \sum_G (-1)^{e(G)}$, where now G ranges over all *connected* bipartite graphs on 1, 2, ..., n.

(c) Use (b) and the block-tree theorem to compute the sum $C_n := \sum_G (-1)^{e(G)}$, where G ranges over all bipartite *blocks* on 1, 2, ..., n.

(d) Use (c) and the block-tree theorem to compute the sum $D_n := \sum_G (-1)^{e(G)}\, 2^{b(G)}$, where G ranges over all *connected* bipartite graphs on 1, 2, ..., n.

(e) Use (d) and the exponential formula to compute the desired sum (6.1).

We now proceed to steps (a)–(e).

(a) Let $b_k(n)$ be the number of k-edge bipartite graphs on the vertex set 1, 2, ..., n. It is known (e.g., [28, Exercise 5.5]) that

$$\sum_{n \geqslant 0} \sum_{k \geqslant 0} b_k(n)\, q^k \frac{x^n}{n!} = \left[\sum_{n \geqslant 0} \left(\sum_{i=0}^{n} (1+q)^{i(n-i)} \binom{n}{i} \right) \frac{x^n}{n!} \right]^{1/2}.$$

Put $q = -1$ to get

$$\sum_{n \geqslant 0} A_n \frac{x^n}{n!} = \left(1 + \sum_{n \geqslant 1} 2 \frac{x^n}{n!} \right)^{1/2} = (2e^x - 1)^{1/2}.$$

(b) According to the exponential formula [12, p. 166], we have

$$\sum_{n\geqslant 1} B_n \frac{x^n}{n!} = \log \sum_{n\geqslant 0} A_n \frac{x^n}{n!}$$
$$= \frac{1}{2} \log(2e^x - 1).$$

(c) Let B'_n denote the number of *rooted* connected bipartite graphs on 1, 2, ..., n. Since $B'_n = nB_n$, we get

$$\sum_{n\geqslant 1} B'_n \frac{x^n}{n!} = x \frac{d}{dx} \sum_{n\geqslant 1} B_n \frac{x^n}{n!}$$
$$= \frac{x}{2 - e^{-x}}. \tag{6.2}$$

Suppose now that $\mathscr{B}$ is a set of nonisomorphic blocks B and w is a weight function on $\mathscr{B}$, so $w(B)$ denotes the weight of the block B. Let

$$T(x) = \sum_{B \in \mathscr{B}} w(B) \frac{x^{p(B)}}{p(B)!},$$

where $p(B)$ denotes the number of vertices of B. Let

$$u(x) = \sum_{G} \left(\prod_{B} w(B) \right) \frac{x^{p(G)}}{p(G)!},$$

where G ranges over all rooted connected graphs whose blocks are isomorphic (as unrooted graphs) to elements of $\mathscr{B}$, and where B ranges over all blocks of G. The *block-tree theorem* [13, (1.3.3); 28, Exercise 5.20(a)] asserts that

$$u = xe^{T'(u)}. \tag{6.3}$$

If we take $\mathscr{B}$ to be the set of all nonisomorphic bipartite blocks, $w(B) = (-1)^{e(B)}$, and $u = x/(2 - e^{-x})$, then it follows from (6.4) that

$$T(x) = \sum_{n\geqslant 1} C_n \frac{x^n}{n!}. \tag{6.4}$$

(d) Let D'_n be defined like D_n, except that G ranges over all *rooted* connected bipartite graphs on 1, 2, ..., n, so $D'_n = nD_n$. Let $v(x) = \sum_{n\geqslant 1} D'_n (x^n/n!)$. By the block-tree theorem we have

$$v = xe^{2T'(v)},$$

where $T(x)$ is given by (6.4). Write $f^{\langle -1\rangle}(x)$ for the compositional inverse of a power series $f(x)=x+a_2x^2+\cdots$, i.e., $f(f^{\langle -1\rangle}(x))=f^{\langle -1\rangle}(f(x))=x$. Substitute $v^{\langle -1\rangle}$ for x and use (6.3) to get

$$\begin{aligned} x &= v^{\langle -1\rangle}(x)\, e^{2T'(x)} \\ &= v^{\langle -1\rangle}(x) \left(\frac{x}{u^{\langle -1\rangle}(x)}\right)^2. \end{aligned}$$

Substitute $v(x)$ for x to obtain

$$x\, v(x) = u^{\langle -1\rangle}(v(x))^2.$$

Take the square roots of both sides and compose with $u(x)=x/(2-e^{-x})$ on the left to get

$$\frac{\sqrt{xv}}{2-e^{-\sqrt{xv}}} = v. \tag{6.5}$$

(e) Equation (6.1) and the exponential formula show that

$$\begin{aligned} z &= \exp\left(-\sum_{n\geq 1} (-1)^n D_n \frac{x^n}{n!}\right) \\ &= \exp\left(-\int \frac{v(-x)}{x}\right), \end{aligned} \tag{6.6}$$

where $\int$ denotes the formal integral, i.e.,

$$\int \sum a_n \frac{x^n}{n!} = \sum a_n \frac{x^{n+1}}{(n+1)!}.$$

(The first minus sign in (6.6) corresponds to the factor $(-1)^{c(G)}$ in (6.1).)

Let $v(-x) = -xy^2$. Equation (6.5) becomes (one must take care to choose the right sign of the square root)

$$1 = y(2-e^{xy}),$$

while (6.6) shows that $z'/z = -v(-x)/x = y^2$. This completes the proof. ∎

Note. The semigeneric arrangement $\mathcal{G}_n$ satisfies the hypotheses of [29, Thm. 1.2]. It follows that

$$\sum_{n\geq 0} \chi_{\mathcal{G}_n}(q) \frac{x^n}{n!} = z(-x)^{-q},$$

as stated in [29, Cor. 3.3]. Here z is as defined in Theorem 6.1.

An arrangement closely related to $\mathscr{G}_n$ is given by

$$\mathscr{G}'_n: \qquad x_i - x_j = a_i, \qquad 1 \leqslant i < j \leqslant n,$$

where the a_i's are generic. The analogue of Eq. (6.1) is

$$r(\mathscr{G}'_n) = (-1)^n \sum_G (-1)^{e(G)+c(G)}\, 2^{b(G)},$$

where now G ranges over all bipartite graphs on the vertex set 1, 2, ..., n for which every block is *alternating*, i.e., every vertex is either less than all its neighbors or greater than all its neighbors. The first author of this paper has obtained a result analogous to Theorem 6.1.

7. CATALAN ARRANGEMENTS AND SEMIORDERS

Let us fix distinct real numbers $a_1, a_2, ..., a_m > 0$, and let $A = (a_1, ..., a_m)$. In this section we consider the arrangement $\mathscr{C}_{n-1} = \mathscr{C}_{n-1}(A)$ of hyperplanes in the space $V_{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$ given by

$$x_i - x_j = a_1, a_2, ..., a_m, \qquad i \neq j. \tag{7.1}$$

We consider also the arrangement $\mathscr{C}^0_{n-1} = \mathscr{C}^0_{n-1}(A)$ obtained from $\mathscr{C}_{n-1}$ by adjoining the hyperplanes $x_i = x_j$, i.e., $\mathscr{C}^0_n$ is given by

$$x_i - x_j = 0, a_1, a_2, ..., a_m, \qquad i \neq j. \tag{7.2}$$

Let

$$f_A(t) = \sum_{n \geqslant 0} r(\mathscr{C}_{n-1})\, \frac{t^n}{n!},$$

$$g_A(t) = \sum_{n \geqslant 0} r(\mathscr{C}^0_{n-1})\, \frac{t^n}{n!}$$

be the exponential generating functions for the numbers of regions of the arrangements $\mathscr{C}_{n-1}$ and $\mathscr{C}^0_{n-1}$.

The main result of this section is the following theorem, stated without proof in [29, Thm. 2.3].

THEOREM 7.1. *We have $f_A(t) = g_A(1 - e^{-t})$ or, equivalently,*

$$r(\mathscr{C}^0_{n-1}) = \sum_{k \geqslant 0} c(n, k)\, r(\mathscr{C}_{k-1}),$$

where $c(n, k)$ is the signless Stirling number of the first kind, i.e., the number of permutations of 1, 2, ..., n *with* k *cycles.*

Let us have a closer look at two special cases of arrangements (7.1) and (7.2). Consider the arrangement of hyperplanes in $V_{n-1} \subset \mathbb{R}^n$ given by the equations

$$x_i - x_j = \pm 1, \qquad 1 \leqslant i < j \leqslant n. \tag{7.3}$$

Consider also the arrangement given by

$$x_i - x_j = 0, \ \pm 1, \qquad 1 \leqslant i < j \leqslant n. \tag{7.4}$$

It is not difficult to check the following result directly from the definition.

PROPOSITION 7.2. *The number of regions of the arrangement* (7.4) *is equal to* $n!\, C_n$, *where* C_n *is the Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Theorem 7.1 then gives a formula for the number of regions of the arrangement (7.3).

Let R be a region of the arrangement (7.3), and let $(x_1, ..., x_n) \in R$ be any point in the region R. Consider the poset P on the vertices 1, ..., n such that $i >_P j$ if and only if $x_i - x_j > 1$. Clearly, distinct regions correspond to distinct posets. The posets that can be obtained in such a way are called *semiorders*. See [29] for more results on the relation between hyperplane arrangements and *interval orders* (which are a generalization of semiorders).

The symmetric group $\mathfrak{S}_n$ naturally acts on the space V_{n-1} by permuting the coordinates x_i. Thus it also permutes the regions of the arrangement (7.4). The region $x_1 < x_2 < \cdots < x_n$ is called the *dominant chamber*. Every $\mathfrak{S}_n$-orbit of regions of the arrangement (7.4) consists of $n!$ regions and has a unique representative in the dominant chamber. It is also clear that the regions of (7.4) in the dominant chamber correspond to *unlabelled* (i.e., nonisomorphic) semiorders on n vertices. Hence, Proposition 7.2 is equivalent to a well-known result of Wine and Freund [33] that the number of nonisomorphic semiorders on n vertices is equal to the Catalan number. In the special case of the arrangements (7.3) and (7.4), i.e., $A = (1)$, Theorem 7.1 gives a formula for the number of labelled semiorders on n vertices which was first proved by Chandon, Lemaire, and Pouget [8].

The following theorem, due to Scott and Suppes [24], presents a simple characterization of semiorders (cf. Theorem 8.4).

THEOREM 7.3. *A poset* P *is a semiorder if and only if it contains no induced subposet of either of the two types shown in Fig.* 3

FIG. 3. Forbidden subposets for semiorders.

Return now to the general case of the arrangements $\mathscr{C}_{n-1}$ and $\mathscr{C}^0_{n-1}$ given by (7.1) and (7.2). The symmetric group $\mathfrak{S}_n$ acts on the regions of $\mathscr{C}_{n-1}$ and $\mathscr{C}^0_{n-1}$ by permuting the coordinates of x_i. Let R_{n-1} denote the set of all regions of $\mathscr{C}_{n-1}$.

LEMMA 7.4. *The number of regions of $\mathscr{C}^0_{n-1}$ is equal to $n!$ times the number of $\mathfrak{S}_n$-orbits in R_{n-1}.*

Indeed, the number of regions of $\mathscr{C}^0_{n-1}$ is $n!$ times the number of those in the dominant chamber. They, in turn, correspond to $\mathfrak{S}_n$-orbits in R_{n-1}. As was shown in [29], the regions of $\mathscr{C}_{n-1}$ can be viewed as (labelled) generalized interval orders. On the other hand, the regions of $\mathscr{C}^0_{n-1}$ that lie in the dominant chamber correspond to unlabelled generalized interval orders. The statement now is tautological, that the number of unlabelled objects is the number of $\mathfrak{S}_n$-orbits.

Now we can apply the following well-known lemma of Burnside (actually first proved by Cauchy and Frobenius, as discussed, e.g., in [28, p. 404]).

LEMMA 7.5. *Let G be a finite group which acts on a finite set M. Then the number of G-orbits in M is equal to*

$$\frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g, M),$$

where $\mathrm{Fix}(g, M)$ *is the number of elements in M fixed by $g \in G$.*

By Lemmas 7.4 and 7.5 we have

$$r(\mathscr{C}^0_{n-1}) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{Fix}(\sigma, \mathscr{C}_{n-1}),$$

where $\mathrm{Fix}(\sigma, \mathscr{C}_{n-1})$ is the number of regions of $\mathscr{C}_{n-1}$ fixed by the permutation σ.

Theorem 7.1 now follows easily from the following lemma.

LEMMA 7.6. *Let $\sigma \in \mathfrak{S}_n$ be a permutation with k cycles. Then the number of regions of $\mathscr{C}_{n-1}$ fixed by σ is equal to the total number of regions of $\mathscr{C}_{k-1}$.*

Indeed, by Lemma 7.6, we have

$$r(\mathscr{C}^0_{n-1}) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{Fix}(\sigma, \mathscr{C}_{n-1}) = \sum_{k \geqslant 0} c(n, k)\, r(\mathscr{C}_{k-1}),$$

which is precisely the claim of Theorem 7.1.

Proof of Lemma 7.6. We will construct a bijection between the regions of $\mathscr{C}_{n-1}$ fixed by σ and the regions of $\mathscr{C}_{k-1}$.

Let R be any region of $\mathscr{C}_{n-1}$ fixed by a permutation $\sigma \in \mathfrak{S}_n$, and let $(x_1, ..., x_n)$ be any point in R. Then for any $i, j \in \{1, ..., n\}$ and any $s = 1, ..., m$ we have $x_i - x_j > a_s$ if and only if $x_{\sigma(i)} - x_{\sigma(j)} > a_s$.

Let $\sigma = (c_{11}\, c_{12} \cdots c_{1l_1})\,(c_{21}\, c_{22} \cdots c_{2l_2}) \cdots (c_{k1}\, c_{k2} \cdots c_{kl_k})$ be the cycle decomposition of the permutation σ. Write $X_i = (x_{c_{i1}}, x_{c_{i2}}, ...)$ for $i = 1, ..., k$. We will write $X_i - X_j > a$ if $x_{i'} - x_{j'} > a$ for any $x_{i'} \in X_i$ and $x_{j'} \in X_j$. The notation $X_i - X_j < a$ has an analogous meaning. We will show that for any two classes X_i and X_j and for any $s = 1, ..., m$ we have either $X_i - X_j > a_s$ or $X_i - X_j < a_s$.

Let x_{i*} be the maximal element in X_i and let x_{j*} be the maximal element in X_j. Suppose that $x_{i*} - x_{j*} > a_s$. Since R is σ-invariant, for any integer p we have the inequality $x_{\sigma^p(i*)} - x_{\sigma^p(j*)} > a_s$. Then, since x_{i*} is the maximal element of X_i, we have $x_{i*} - x_{\sigma^p(j*)} > a_s$. Again, for any integer q, we have $x_{\sigma^q(i*)} - x_{\sigma^{p+q}(j*)} > a_s$, which implies that $X_i - X_j > a_s$.

Analogously, suppose that $x_{i*} - x_{j*} < a_s$. Then for any integer p we have $x_{\sigma^p(i*)} - x_{\sigma^p(j*)} < a_s$. Since $x_{j*} \geqslant x_{\sigma^p(j*)}$, we have $x_{\sigma^p(i*)} - x_{j*} < a_s$. Finally, for any integer q we obtain $x_{\sigma^{p+q}(i*)} - x_{\sigma^q(j*)} < a_s$, which implies that $X_i - X_j < a_s$.

If we pick an element $x_{i'}$ in each class X_i we get a point $(x_{1'}, x_{2'}, ..., x_{k'})$ in $\mathbb{R}^k$. This point lies in some region R' of $\mathscr{C}_{k-1}$. The construction above shows that the region R' does not depend on the choice of $x_{i'}$ in X_i.

Thus we get a map $\phi\colon R \to R'$ from the regions of $\mathscr{C}_{n-1}$ invariant under σ to the regions of $\mathscr{C}_{k-1}$. It is clear that ϕ is injective. To show that ϕ is surjective, let $(x_{1'}, ..., x_{k'})$ be any point in a region R' of $\mathscr{C}_k$. Pick the point $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ such that $x_{c_{11}} = x_{c_{12}} = \cdots = x_{1'}$, $x_{c_{21}} = x_{c_{22}} = \cdots = x_{2'}$, ..., $x_{c_{k1}} = x_{c_{k2}} = \cdots = x_{k'}$. Then $(x_1, ..., x_n)$ is in some region R of $\mathscr{C}_{n-1}$ (here we use the condition $a_1, ..., a_m \neq 0$). According to our construction, we have $\phi(R) = R'$. Thus ϕ is a bijection.

This completes the proof of Lemma 7.6 and therefore also of Theorem 7.1. ∎

8. THE LINIAL ARRANGEMENT

As before, $V_{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$. Consider the arrangement $\mathcal{L}_{n-1}$ of hyperplanes in V_{n-1} given by the equations

$$x_i - x_j = 1, \qquad 1 \leqslant i < j \leqslant n. \tag{8.1}$$

Recall that $r(\mathcal{L}_{n-1})$ denotes the number of regions of the arrangement $\mathcal{L}_{n-1}$. This arrangement was first considered by Nati Linial and Shmulik Ravid. They calculated the numbers $r(\mathcal{L}_{n-1})$ and the Poincaré polynomials $\mathrm{Poin}_{\mathcal{L}_{n-1}}(q)$ for $n \leqslant 9$.

In this section we give an explicit formula and several different combinatorial interpretations for the numbers $r(\mathcal{L}_{n-1})$.

8.1. *Alternating Trees and Local Binary Search Trees*

We call a tree T on the vertices $0, 1, 2, ..., n$ *alternating* (see Fig. 4) if the vertices in any path $i_1, ..., i_k$ in T alternate, i.e., we have $i_1 < i_2 > i_3 < \cdots i_k$ or $i_1 > i_2 < i_3 > \cdots i_k$. In other words, there are no $i < j < k$ such that both (i, j) and (j, k) are edges in T. Equivalently, every vertex is either greater than all its neighbors or less than all its neighbors. Alternating trees first appear in [11] and were studied in [20], where they were called *intransitive trees* (see also [29]).

Let f_n be the number of alternating trees on the vertices $0, 1, 2, ..., n$, and let

$$f(x) = \sum_{n \geqslant 0} f_n \frac{x^n}{n!}$$

be the exponential generating function for the sequence f_n.

A plane binary tree B on the vertices $1, 2, ..., n$ is called a *local binary search tree* (see Fig. 5) if for any vertex i in T the left child of i is less than i and the right child of i is greater than i. These trees were first considered by Ira Gessel (private communication). Let g_n denote the number of local binary search trees on the vertices $1, 2, ..., n$. By convention, $g_0 = 1$.

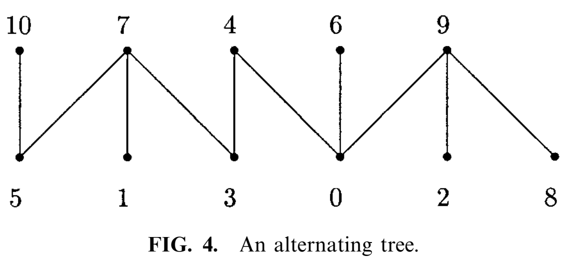

FIG. 4. An alternating tree.

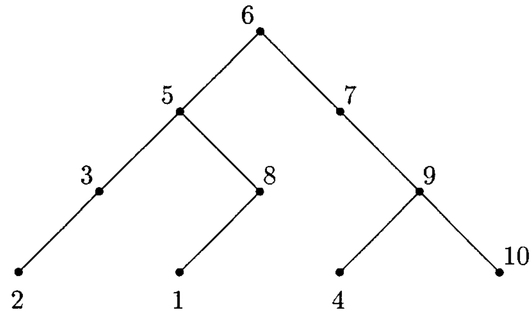

FIG. 5. A local binary search tree.

The following result was proved in [20] (see also [11, 29]).

THEOREM 8.1. *For $n \geqslant 1$ we have*

$$f_n = g_n = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1}$$

and $f = f(x)$ satisfies the functional equation

$$f = e^{x(1+f)/2}.$$

The first few numbers f_n are given in the table below.

n	0	1	2	3	4	5	6	7	8	9	10
f_n	1	1	2	7	36	246	2,104	21,652	260,720	3,598,120	56,010,096

The main result on the Linial arrangement is the following:

THEOREM 8.2. *The number $r(\mathscr{L}_{n-1})$ of regions of $\mathscr{L}_{n-1}$ is equal to the number f_n of alternating trees on the vertices* 0, 1, 2..., n, *and thus to the number g_n of local binary search trees on* 1, 2, ..., n.

This theorem was conjectured by the second author (thanks to the numerical data provided by Linial and Ravid) and was proved by the first author. A different proof was later given by C. Athanasiadis [3].

In Section 9 we will prove a more general result (see Theorems 9.1 and Corollary 9.9).

8.2. *Sleek Posets and Semiacyclic Tournaments*

Let R be a region of the arrangement $\mathscr{L}_{n-1}$, and let $(x_1, ..., x_n)$ be any point in R. Define $P = P(R)$ to be the poset on the vertices 1, 2, ..., n such that $i <_P j$ if and only if $x_i - x_j > 1$ and $i < j$ in the usual order on $\mathbb{Z}$.

We will call a poset P on the vertices $1, 2, ..., n$ *sleek* if P is the intersection of a semiorder (see Section 7.) with the chain $1<2<\cdots<n$.

The following proposition immediately follows from the definitions.

PROPOSITION 8.3. *The map $R \mapsto P(R)$ is a bijection between regions of $\mathscr{L}_{n-1}$ and sleek posets on $1, 2, ..., n$. Hence the number $r(\mathscr{L}_{n-1})$ is equal to the number of sleek posets on $1, 2, ..., n$.*

There is a simple characterization of sleek posets in terms of forbidden induced subposets (compare Theorem 7.3).

THEOREM 8.4. *A poset P on the vertices $1, 2, ..., n$ is sleek if and only if it contains no induced subposet of the four types shown in Fig. 6, where $a<b<c<d$.*

In the remaining part of this subsection we prove Theorem 8.4.

First, we give another description of regions in $\mathscr{L}_{n-1}$ (or, equivalently, sleek posets). A *tournament* on the vertices $1, 2, ..., n$ is a directed graph T without loops such that for every $i \neq j$ either $(i, j) \in T$ or $(j, i) \in T$. For a region R of $\mathscr{L}_{n-1}$ construct a tournament $T=T(R)$ on the vertices $1, 2, ..., n$ as follows: let $(x_1, ..., x_n) \in R$. If $x_i - x_j > 1$ and $i<j$, then $(i, j) \in T$; while if $x_i - x_j < 1$ and $i<j$, then $(j, i) \in T$.

Let C be a directed cycle in the complete graph K_n on the vertices $1, 2, ..., n$. We will write $C=(c_1, c_2, ..., c_m)$ if C has the edges (c_1, c_2), (c_2, c_3), ..., (c_m, c_1). By convention, $c_0 = c_m$. An *ascent* in C is a number $1 \leqslant i \leqslant m$ such that $c_{i-1} < c_i$. Analogously, a *descent* in C is a number $1 \leqslant i \leqslant m$ such that $c_{i-1} > c_i$. Let $\operatorname{asc}(C)$ denote the number of ascents and $\operatorname{des}(C)$ denote the number of descents in C. We say that a cycle C is *ascending* if $\operatorname{asc}(C) \geqslant \operatorname{des}(C)$. For example, the following cycles are ascending: $C_0=(a,b,c)$, $C_1=(a,c,b,d)$, $C_2=(a,d,b,c)$, $C_3=(a,b,d,c)$, $C_4=(a,c,d,b)$, where $a<b<c<d$. These cycles are shown in Fig. 7.

We call a tournament T on $1, 2, ..., n$ *semiacyclic* if it contains no ascending cycles. In other words, T is semiacyclic if for any directed cycle C in T we have $\operatorname{asc}(C) < \operatorname{des}(C)$.

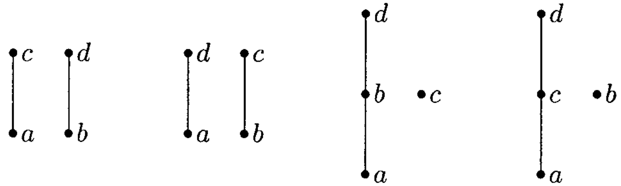

FIG. 6. Obstructions to sleekness.

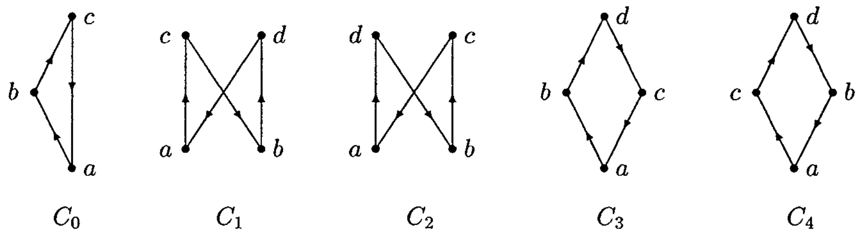

FIG. 7. Ascending cycles.

PROPOSITION 8.5. *A tournament T on $1, 2, ..., n$ corresponds to a region R in $\mathscr{L}_{n-1}$, i.e., $T = T(R)$, if and only if T is semiacyclic. Hence $r(\mathscr{L}_{n-1})$ is the number of semiacyclic tournaments on $1, 2, ..., n$.*

This fact was independently found by Shmulik Ravid.

For any tournament T on $1, 2, ..., n$ without cycles of type C_0 we can construct a poset $P = P(T)$ such that $i <_P j$ if and only if $i < j$ and $(i, j) \in T$. Now the four ascending cycles C_1, C_2, C_3, C_4 in Fig. 7 correspond to the four posets in Fig. 6. Therefore, Theorem 8.4 is equivalent to the following result.

THEOREM 8.6. *A tournament T on the vertices $1, 2, ..., n$ is semiacyclic if and only if it contains no ascending cycles of the types C_0, C_1, C_2, C_3, and C_4 shown in Fig. 7, where $a < b < c < d$.*

Remark 8.7. This theorem is an analogue of a well-known fact that a tournament T is acyclic if and only if it contains no cycles of length 3. For semiacyclicity we have obstructions of lengths 3 and 4.

Proof. Let T be a tournament on $1, 2, ..., n$. Suppose that T is not semiacyclic. We will show that T contains a cycle of type C_0, C_1, C_2, C_3, or C_4. Let $C = (c_1, c_2, ..., c_m)$ be an ascending cycle in T of minimal length. If $m = 3$ or 4, then C is of type C_0, C_1, C_2, C_3, or C_4. Suppose that $m > 4$.

LEMMA 8.8. *We have* $\operatorname{asc}(C) = \operatorname{des}(C)$.

Proof. Since C is ascending, we have $\operatorname{asc}(C) \geqslant \operatorname{des}(C)$. Suppose $\operatorname{asc}(C) > \operatorname{des}(c)$. If C has two adjacent ascents i and $i+1$ then $(c_{i-1}, c_{i+1}) \in T$ (otherwise we have an ascending cycle (c_{i-1}, c_i, c_{i+1}) of type C_0 in T). Then $C' = (c_1, c_2, ..., c_{i-1}, c_{i+1}, ..., c_m)$ is an ascending cycle in T of length $m-1$, which contradicts the fact that we chose C to be minimal. So for every ascent i in C the index $i+1$ is a descent. Hence $\operatorname{asc}(C) \leqslant \operatorname{des}(C)$, and we get a contradiction. ∎

We say that c_i and c_j are *on the same level* in C if the number of ascents between c_i and c_j is equal to the number of descents between c_i and c_j.

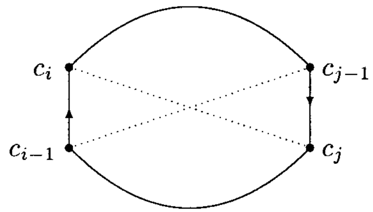

FIGURE 8

LEMMA 8.9. *We can find $i, j \in \{1, 2, ..., m\}$ such that* (a) *i is an ascent and j is a descent in C,* (b) *$i \not\equiv j \pm 1 \pmod m$, and* (c) *$c_i$ and c_{j-1} are on the same level* (*see Fig.* 8).

Proof. We may assume that for any $1 \leqslant s \leqslant m$ the number of ascents in $\{1, 2, ..., s\}$ is greater than or equal to the number of descents in $\{1, 2, ..., s\}$ (otherwise take some cyclic permutation of $(c_1, c_2, ..., c_m)$). Consider two cases.

1. There exists $1 \leqslant t \leqslant m-1$ such that c_t and c_m are on the same level. In this case, if the pair $(i, j) = (1, t)$ does not satisfy conditions (a)–(c) then $t = 2$. On the other hand, if the pair $(i, j) = (t+1, m)$ does not satisfy (a)–(c) then $t = m-2$. Hence, $m = 4$ and C is of type C_1 or C_2 shown in Figure 7.

2. There is no $1 \leqslant t \leqslant m-1$ such that c_t and c_m are on the same level. Then 2 is an ascent and $m-1$ is a descent. If the pair $(i, j) = (2, m-2)$ does not satisfy (a)–(c) then $m = 4$ and C is of type C_3 or C_4 shown in Fig. 7. ∎

Now we can complete the proof of Theorem 8.6. Let i, j be two numbers satisfying the conditions of Lemma 8.9. Then $c_{i-1}, c_i, c_{j-1}, c_j$ are four distinct vertices such that (a) $c_{i-1} < c_i$, (b) $c_{j-1} > c_j$, (c) c_i and c_{j-1} are on the same level, and (d) c_{i-1} and c_j are on the same level (see Fig. 8). We may assume that $i < j$.

If $(c_{j-1}, c_{i-1}) \in T$ then $(c_{i-1}, c_i, ..., c_{j-1})$ is an ascending cycle in T of length less than m, which contradicts the requirement that C is an ascending cycle on T of minimal length. So $(c_{i-1}, c_{j-1}) \in T$. If $c_{i-1} < c_{j-1}$ then $(c_{j-1}, c_j, ..., c_m, c_1, ..., c_{i-1})$ is an ascending cycle in T of length less than m. Hence, $c_{i-1} > c_{j-1}$.

Analogously, if $(c_i, c_j) \in T$ then $(c_j, c_{j+1}, ..., c_p, c_1, ..., c_i)$ is an ascending cycle in T of length less than m. So $(c_j, c_i) \in T$. If $c_i > c_j$ then $(c_i, c_{i+1}, ..., c_j)$ is an ascending cycle in T of length less than m. So $c_i < c_j$.

Now we have $c_{i-1} > c_{j-1} > c_j > c_i > c_{i-1}$, and we get an obvious contradiction.

We have shown that every minimal ascending cycle in T is of length 3 or 4 and thus have proved Theorem 8.6. ∎

8.3. *The Orlik–Solomon Algebra*

In [16] Orlik and Solomon gave the following combinatorial description of the cohomology ring of the complement of an arbitrary complex hyperplane arrangement. Consider a complex arrangement $\mathscr{A}$ of affine hyperplanes $H_1, H_2, ..., H_N$ in the complex space $V \cong \mathbb{C}^n$ given by

$$H_i : f_i(x) = 0, \qquad i = 1, ..., N,$$

where $f_i(x)$ are linear forms on V (with a constant term).

We say that hyperplanes $H_{i_1}, ..., H_{i_p}$ are *independent* if the codimension of the intersection $H_{i_1} \cap \cdots \cap H_{i_p}$ is equal to p. Otherwise, the hyperplanes are *dependent*.

Let $e_1, ..., e_N$ be formal variables associated with the hyperplanes $H_1, ..., H_N$. The *Orlik–Solomon algebra* $\mathrm{OS}(\mathscr{A})$ of the arrangement $\mathscr{A}$ is generated over the complex numbers by $e_1, ..., e_N$, subject to the relations

$$e_i e_j = -e_j e_i, \qquad 1 \leqslant i < j \leqslant N, \tag{8.2}$$

$$e_{i_1} \cdots e_{i_p} = 0, \qquad \text{if} \quad H_{i_1} \cap \cdots \cap H_{i_p} = \varnothing, \tag{8.3}$$

$$\sum_{j=1}^{p+1} (-1)^j e_{i_1} \cdots \widehat{e_{i_j}} \cdots e_{i_{p+1}} = 0, \tag{8.4}$$

whenever $H_{i_1}, ..., H_{i_{p+1}}$ are dependent. (Here $\widehat{e_{i_j}}$ denotes that e_{i_j} is missing.)

Let $C_{\mathscr{A}} = V - \bigcup_i H_i$ be the complement to the hyperplanes H_i of $\mathscr{A}$, and let $H^*_{DR}(C_{\mathscr{A}}, \mathbb{C})$ denote de Rham cohomology of $C_{\mathscr{A}}$.

THEOREM 8.10 (Orlik and Solomon [16]). *The map* $\phi\colon \mathrm{OS}(\mathscr{A}) \to H^*_{DR}(C_{\mathscr{A}}, \mathbb{C})$ *defined by*

$$\phi\colon e_i \mapsto [df_i/f_i]$$

is an isomorphism.

Here $[df_i/f_i]$ is the cohomology class in $H^*_{DR}(C_{\mathscr{A}}, \mathbb{C})$ of the differential form df_i/f_i.

We will apply Theorem 8.10 to the Linial arrangement. In this case hyperplanes $x_i - x_j = 1$, $i < j$, correspond to edges (i, j) of the complete graph K_n.

PROPOSITION 8.11. *The Orlik–Solomon algebra* $\mathrm{OS}(\mathscr{L}_{n-1})$ *of the Linial arrangement is generated by* $e_{vw} = e_{(v,w)}$, $1 \leqslant v < w \leqslant n$, *subject to relations* (8.2), (8.3), *and also to the relations*

$$\begin{aligned} &e_{ab}e_{bc}e_{ac} - e_{ab}e_{bc}e_{cd} + e_{ab}e_{ac}e_{cd} - e_{bc}e_{ac}e_{cd} = 0, \\ &e_{ac}e_{bc}e_{bd} - e_{ac}e_{bc}e_{ad} + e_{ac}e_{bd}e_{ad} - e_{bc}e_{bd}e_{ad} = 0. \end{aligned} \tag{8.5}$$

where $1 \leqslant a < b < c < d \leqslant n$ (*cf. Fig.* 7).

Proof. Let $C = (c_1, c_2, ..., c_p)$ be a cycle in K_n. We say that C is *balanced* if $\mathrm{asc}(C) = \mathrm{des}(C)$. We may assume that in Eq. (8.4) $i_1, i_2, ..., i_p$ are edges of a balanced cycle C. We will prove (8.4) by induction on p. If $p = 4$ then C is of type C_1, C_2, C_3, or C_4 (see Fig. 7). Thus C produces one of the relations (8.5). If $p > 4$, then we can find $r \neq s$ such that both $C' = (c_r, c_{r+1}, ..., c_s)$ and $C'' = (c_s, c_{s+1}, ..., c_r)$ are balanced. Equation (8.4) for C is the sum of the equations for C' and C''. Thus the statement follows by induction. ∎

Remark 8.12. This proposition is an analogue to the well-known description of the cohomology ring of the Coxeter arrangement (3.1), due to Arnold [1]. This cohomology ring is generated by $e_{vw} = e_{(v,w)}$, $1 \leqslant v < w \leqslant n$, subject to relations (8.2), (8.3) and also the "triangle" equation:

$$e_{ab}e_{bc} - e_{ab}e_{ac} + e_{bc}e_{ac} = 0,$$

where $1 \leqslant a < b < c \leqslant n$.

9. TRUNCATED AFFINE ARRANGEMENTS

In this section we study a general class of hyperplane arrangements which contains, in particular, the Linial and Shi arrangements.

Let a and b be two integers such that $a \geqslant 0$, $b \geqslant 0$, and $a + b \geqslant 2$. Consider the hyperplane arrangement $\mathscr{A}^{ab}_{n-1}$ in $V_{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$ given by

$$x_i - x_j = -a+1, -a+2, ..., b-1, \qquad 1 \leqslant i < j \leqslant n. \tag{9.1}$$

We call $\mathscr{A}^{ab}_{n-1}$ a *truncated affine arrangement* because it is a finite subarrangement of the affine arrangement of type $\tilde{A}_{n-1}$ given by $x_i - x_j = k$, $k \in \mathbb{Z}$.

As we will see the arrangement $\mathscr{A}^{ab}_{n-1}$ has different behavior in the *balanced case* ($a = b$) and the *unbalanced case* ($a \neq b$).

9.1. *Functional Equations*

Let $f_n = f_n^{ab}$ be the number of regions of the arrangement $\mathscr{A}_{n-1}^{ab}$, and let

$$f(x) = \sum_{n \geqslant 0} f_n \frac{x^n}{n!} \tag{9.2}$$

be the exponential generating function for f_n.

THEOREM 9.1. *Suppose $a, b \geqslant 0$.*

1. *The generating function $f = f(x)$ satisfies the functional equation*

$$f^{b-a} = e^{x \cdot \frac{f^a - f^b}{1-f}}. \tag{9.3}$$

2. *If $a = b \geqslant 1$, then $f = f(x)$ satisfies the equation:*

$$f = 1 + xf^a. \tag{9.4}$$

Note that Eq. (9.4) can be formally obtained from (9.3) by l'Hôpital's rule in the limit $a \to b$.

In the case $a = b$ the functional equation (9.4) allows us to calculate the numbers f_n^{aa} explicitly.

COROLLARY 9.2. *The number f_n^{aa} is equal to $an(an-1)\cdots(an-n+2)$.*

The functional equation (9.3) is especially simple in the case $a = b - 1$. We call the arrangement $\mathscr{A}_{n-1}^{a,a+1}$ the *extended Shi arrangement*. In this case we get:

COROLLARY 9.3. *Let $a \geqslant 1$. The number f_n of regions of the hyperplane arrangement in $\mathbb{R}^n$ given by*

$$x_i - x_j = -a+1, -a+2, ..., a, \qquad i < j,$$

is equal to $f_n = (a\,n+1)^{n-1}$, and the exponential generating function $f = \sum_{n \geqslant 0} f_n(x^n/n!)$ satisfies the functional equation $f = e^{x \cdot f^a}$.

In order to prove Theorem 9.1 we need several new definitions. A *graded graph* is a graph G on a set V of vertices labelled by natural numbers together with a function $h: V \to \{0, 1, 2, ...\}$, which is called a *grading*. For $r \geqslant 0$ the vertices v of G such that $h(v) = r$ form the rth *level* of G. Let $e = (u, v)$ be an edge in G, $u < v$. We say that the *type* of the edge e is the integer $t = h(v) - h(u)$ and that a graded graph G is *of type* (a, b) if the types of all edges in G are in the interval $[-a+1, b-1] = \{-a+1, -a+2, ..., b-1\}$.

Choose a linear order on the set of all triples (u, t, v), $u, v \in V$, $t \in [-a+1, b-1]$. Let C be a graded cycle of type (a, b). Every edge (u, v) of C corresponds to a triple (u, t, v), where t is the type of the edge (u, v). Choose the edge e of C with the minimal triple (u, t, v). We say that $C \backslash \{e\}$ is a *broken circuit of type* (a, b).

Let (F, h) be a graded forest. We say that (F, h) is *grounded* or that h is a *grounded* grading on the forest F if each connected component in F contains a vertex on the 0th level.

PROPOSITION 9.4. *The number f_n of regions of the arrangement* (9.1) *is equal to the number of grounded graded forests of type* (a, b) *on the vertices* $1, 2, ..., n$ *without broken circuits of type* (a, b).

Proof. By Corollary 5.1, the number f_n is equal to the number of colored forests F on the vertices $1, 2, ..., n$ without broken A-circuits. Every edge (u, v), $u < v$, in F has a color which is an integer from the interval $[-a+1, b-1]$. Consider the grounded grading h on F such that for every edge (u, v), $u < v$, in F of color t we have that $t = h(v) - h(u)$ is the type of (u, v). It is clear that such a grading is uniquely defined. Then (F, h) is a grounded graded forest of type (a, b). Clearly, this gives a correspondence between colored and graded forests. Then broken A-circuits correspond to broken graded circuits. The proposition easily follows. ∎

From now on we fix the lexicographic order on triples (u, t, v), i.e., $(u, t, v) < (u', t', v')$ if and only if $u < u'$, or ($u = u'$ and $t < t'$), or ($u = u'$ and $t = t'$ and $v < v'$). Note the order of u, t, and v. We will call a graded tree T *solid* if T is of type (a, b) and T contains no broken circuits of type (a, b).

Let T be a solid tree on $1, 2, ..., n$ such that vertex 1 is on the rth level. If we delete the minimal vertex 1, then the tree T decomposes into connected components $T_1, T_2, ..., T_m$. Suppose that each component T_i is connected with 1 by an edge $(1, v_i)$ where v_i is on the r_ith level.

LEMMA 9.5. *Let $T, T_1, ..., T_m, v_1, ..., v_m$, and $r_1, ..., r_m$ be as above. The tree T is solid if and only if* (a) *all $T_1, T_2, ..., T_m$ are solid,* (b) *for all i the r_ith level is the minimal nonempty level in T_i such that $-a+1 \leqslant r_i - r \leqslant b-1$, and* (c) *the vertex v_i is the minimal vertex on its level in T_i*

Proof. First, we prove that if T is solid then the conditions (a)–(c) hold. Condition (a) is trivial, because if some T_i contains a broken circuit of type (a, b) then T also contains this broken circuit. Assume that for some i there is a vertex v_i' on the r_i'th level in T_i such that $r_i' < r_i$ and $r_i' - r \geqslant -a+1$. Then the minimal chain in T that connects vertex 1 with vertex v_i' is a broken circuit of type (a, b). Thus condition (b) holds. Now suppose that for some i vertex v_i is not the minimal vertex v_i'' on its level. Then the minimal chain in T that connects vertex 1 with v_i'' is a broken circuit of type (a, b). Therefore, condition (c) holds too.

Now assume that conditions (a)–(c) are true. We prove that T is solid. For suppose not. Then T contains a broken circuit $B = C \setminus \{e\}$ of type (a, b), where C is a graded circuit and e is its minimal edge. If B does not pass through vertex 1 then B lies in T_i for some i, which contradicts condition (a). We can assume that B passes through vertex 1. Since e is the minimal edge in C, $e = (1, v)$ for some vertex v' on level r' in T. Suppose $v \in T_i$ If v' and v_i are on different levels in T_i then by (b), $r_i < r$. Thus the minimal edge in C is $(1, v_i)$ and not $(1, v')$. If v' and v_i are on the same level in T_i then by (c) we have $v_i < v'$. Again, the minimal edge in C is $(1, v_i)$ and not $(1, v')$. Therefore, the tree T contains no broken circuit of type (a, b), i.e., T is solid. ∎

Let s_i be the minimal nonempty level in T_i and let l_i be the maximal nonempty level in T_i By Lemma 9.5, the vertex 1 can be on the rth level, $r \in \{s_i - b + 1, s_i - b + 1, ..., l_i + a - 1\}$, and for each such r there is exactly one way to connect 1 with T_i

Let p_{nkr} denote the number of solid trees (not necessarily grounded) on the vertices 1, 2, ..., n which are located on levels 0, 1, ..., k such that vertex 1 is on the rth level, $0 \leqslant r \leqslant k$.

Let

$$p_{kr}(x) = \sum_{n \geqslant 1} p_{nkr} \frac{x^n}{n!}, \qquad p_k(x) = \sum_{r=0}^{k} p_{kr}(x).$$

By the exponential formula (see [12, p. 166]) and Lemma 9.5, we have

$$p'_{kr}(x) = \exp b_{kr}(x), \tag{9.5}$$

where $b_{kr}(x) = \sum_{n \geqslant 1} b_{nkr}(x^n/n!)$ and b_{nkr} is the number of solid trees T on n vertices located on the levels 0, 1, ..., k such that at least one of the levels $r - a + 1, r - a + 2, ..., r + b - 1$ is nonempty, $0 \leqslant r \leqslant k$. The polynomial $b_{kr}(x)$ enumerates the solid trees on levels 1, 2, ..., k minus trees on levels 1, ..., $r - a$ and trees on levels $r + b, ..., k$. Thus we obtain

$$b_{kr}(x) = p_k(x) - p_{r-a}(x) - p_{k-r-b}(x).$$

By (9.5), we get

$$p'_{kr}(x) = \exp(p_k(x) - p_{r-a}(x) - p_{k-r-b}(x)),$$

where $p_{-1}(x)=p_{-2}(x)=\cdots=0$, $p_0(x)=x$, $p_k(0)=0$ for $k\in\mathbb{Z}$. Hence

$$p'_k(x)=\sum_{r=0}^{k}\exp(p_k(x)-p_{r-a}(x)-p_{k-r-b}(x)).$$

Equivalently,

$$p'_k(x)\exp(-p_k(x))=\sum_{r=0}^{k}\exp(-p_{r-a}(x))\exp(-p_{k-r-b}(x)).$$

Let $q_k(x)=\exp(-p_k(x))$. We have

$$q'_k(x)=-\sum_{r=0}^{k}q_{r-a}(x)\,q_{k-r-b}(x), \tag{9.6}$$

$q_{-1}=q_{-2}=\cdots=1$, $q_0=e^{-x}$, $q_k(0)=1$ for $k\in\mathbb{Z}$.

The following lemma describes the relation between the polynomials $q_k(x)$ and the number of regions of the arrangement $\mathscr{A}^{ab}_{n-1}$.

LEMMA 9.6. *The quotient* $q_{k-1}(x)/q_k(x)$ *tends to* $\sum_{n\geqslant 0}f_n(x^n/n!)$ *as* $k\to\infty$.

Proof. Clearly, $p_k(x)-p_{k-1}(x)$ is the exponential generating function for the numbers of grounded solid trees of height less than or equal to k. By the exponential formula (see [12, p. 166]) $q_{k-1}(x)/q_k(x)=\exp(p_k(x)-p_{k-1}(x))$ is the exponential generating function for the numbers of grounded solid forests of height less than or equal to k. The lemma obviously follows from Proposition 9.4. ∎

All previous formulae and constructions are valid for arbitrary a and b. Now we will take advantage of the condition $a,b\geqslant 0$. Let

$$q(x,y)=\sum_{k\geqslant 0}q_k(x)\,y^k.$$

By (9.6), we obtain the following differential equation for $q(x,y)$,

$$\frac{\partial}{\partial x}q(x,y)=-(a_y+y^a q(x,y))\cdot(b_y+y^b q(x,y)),$$

$$q(0,y)=(1-y)^{-1},$$

where $a_y:=(1-y^a)/(1-y)$.

This differential equation has the solution

$$q(x, y) = \frac{b_y \exp(-x \cdot b_y) - a_y \exp(-x \cdot a_y)}{y^a \exp(-x \cdot a_y) - y^b \exp(-x \cdot b_y)}. \tag{9.7}$$

Let us fix some small x. Since $Q(y) := q(x, y)$ is an analytic function of y, then $\gamma = \gamma(x) = \lim_{k \to \infty} q_{k-1}/q_k$ is the pole of $Q(y)$ closest to 0 (γ is the radius of convergence of $Q(y)$ if x is a small positive number). By (9.7), $\gamma^a \exp(-x \cdot a_\gamma) - \gamma^b \exp(-x \cdot b_\gamma) = 0$. Thus, by Lemma 9.6, $f(x) = \sum_{n \geqslant 0} f_n(x^n/n!) = \gamma(x)$ is the solution of the functional equation

$$f^a \, e^{-x \cdot \frac{1-f^a}{1-f}} = f^b \, e^{-x \cdot \frac{1-f^b}{1-f}},$$

which is equivalent to (9.3).

This completes the proof of Theorem 9.1. ∎

9.2. *Formulae for the Characteristic Polynomial*

Let $\mathscr{A} = \mathscr{A}^{ab}_{n-1}$ be the truncated affine arrangement given by (9.1). Consider the characteristic polynomial $\chi^{ab}_n(q)$ of the arrangement $\mathscr{A}^{ab}_{n-1}$. Recall that $\chi^{ab}_n(q) = q^{n-1} \operatorname{Poin}_{\mathscr{A}^{ab}_{n-1}}(-q^{-1})$.

Let $\chi^{ab}(x, q)$ be the exponential generating function

$$\chi^{ab}(x, q) = 1 + \sum_{n>0} \chi^{ab}_{n-1}(q) \frac{x^n}{n!}.$$

According to [29,Theorem 1.2], we have

$$\chi^{ab}(x, q) = f(-x)^{-q}, \tag{9.8}$$

where $f(x) = \chi^{ab}(-x, -1)$ is the exponential generating function (9.2) for numbers of regions of $\mathscr{A}^{ab}_{n-1}$.

Let S be the *shift operator* S: $f(q) \mapsto f(q-1)$.

THEOREM 9.7. *Assume that* $0 \leqslant a < b$. *Then*

$$\chi^{ab}_n(q) = (b-a)^{-n} \, (S^a + S^{a+1} + \cdots + S^{b-1})^n \cdot q^{n-1}.$$

Proof. The theorem can be easily deduced from Theorem 9.1 and (9.8) (using, e.g., the Lagrange inversion formula). ∎

In the limit $b \to a$, using l'Hôpital's rule, we obtain

$$\chi^{aa}_n(q) = \left(S^a \, \frac{\log S}{1-S} \right)^n \cdot q^{n-1}.$$

In fact, there is an explicit formula for $\chi^{aa}(q)$. The following statement easily follows from Corollary 9.2 and appears in [10, proof of Prop. 3.1].

THEOREM 9.8. *We have*

$$\chi_n^{aa}(q) = (q+1-an)(q+2-an)\cdots(q+n-1-an).$$

There are several equivalent ways to reformulate Theorem 9.7, as follows:

COROLLARY 9.9. *Let* $r = b-a$.

1. *We have*

$$\chi_n^{ab}(q) = r^{-n} \sum (q - \phi(1) - \cdots - \phi(n))^{n-1},$$

where the sum is over all functions ϕ: $\{1, ..., n\} \to \{a, ..., b-1\}$.

2. *We have*

$$\chi_n^{ab}(q) = r^{-n} \sum_{s,\, l \geqslant 0} (-1)^l \, (q - s - an)^{n-1} \binom{n}{l} \binom{s+n-rl-1}{n-1}.$$

3. *We have*

$$\chi_n^{ab}(q) = r^{-n} \sum \binom{n}{n_1, ..., n_r} (q - an_1 - \cdots - (b-1)\, n_r)^{n-1},$$

where the sum is over all nonnegative integers $n_1, n_2, ..., n_r$ *such that* $n_1 + n_2 + \cdots + n_r = n$.

EXAMPLES. 9.10. 1. ($a=1$ and $b=2$) The Shi arrangement $\mathcal{S}_{n-1}$ given by (3.6) is the arrangement $\mathcal{A}_{n-1}^{12}$. By Corollary 9.9.1, we get the following formula of Headley [14, Thm. 2.4] (generalizing the formula $r(\mathcal{S}_{n-1}) = (n+1)^{n-1}$ due to Shi [25, Cor. 7.3.10; 26]):

$$\chi_n^{12}(q) = (q-n)^{n-1}. \tag{9.9}$$

2. ($a \geqslant 1$ and $b = a+1$) More generally, for the extended Shi arrangement $\mathcal{S}_{n-1,\, k}$ given by (3.7), we have (cf. Corollary 9.3)

$$\chi_n^{a,\, a+1}(q) = (q-an)^{n-1}.$$

3. ($a=0$ and $b=2$) In this case we get the Linial arrangement $\mathcal{L}_{n-1}=\mathcal{A}_{n-1}^{02}$ (see Section 8.). By Corollary 9.9.3, we have (cf. Theorem 8.2)

$$\chi_n^{02}(q)=2^{-n}\sum_{k=0}^{n}\binom{n}{k}(q-k)^{n-1}, \tag{9.10}$$

4. ($a\geqslant 0$ and $b=a+2$) More generally, for the arrangement $\mathcal{A}_{n-1}^{a,\ a+2}$, we have

$$\chi_n^{a,\ a+2}(q)=2^{-n}\sum_{k=0}^{n}\binom{n}{k}(q-an-k)^{n-1}. \tag{9.11}$$

We will call this arrangement the *extended Linial arrangement.*

Formula (9.10) for the characteristic polynomial $\chi_n^{02}(q)$ was earlier obtained by C. Athanasiadis [3, Theorem 5.2] (see also [4, Sect. 3]). He used a different approach based on a combinatorial interpretation of the value of $\chi_n(q)$ for sufficiently large primes q.

9.3. *Roots of the Characteristic Polynomial*

Theorem 9.7 has one surprising application concerning the location of roots of the characteristic polynomial $\chi_n^{ab}(q)$.

We start with the balanced case ($a=b$). One can reformulate Theorem 9.8 in the following way:

Corollary 9.11. *Let $a\geqslant 1$. The roots of the polynomial $\chi_n^{aa}(q)$ are the numbers $an-1, an-2, ..., an-n+1$ (each with multiplicity 1). In particular, the roots are symmetric to each other with respect to the point $(2a-1)\,n/2$.*

Now assume that $a\neq b$, with $a\geqslant 0$ and $b\geqslant 0$ as before (unbalanced case). The characteristic polynomial $\chi_n^{ab}(q)$ satisfies the following "Riemann hypothesis":

Theorem 9.12. *Let $a+b\geqslant 2$. All the roots of the characteristic polynomial $\chi_n^{ab}(q)$ of the truncated affine arrangement $\mathcal{A}_{n-1}^{ab}$, $a\neq b$, have real part equal to $(a+b-1)\,n/2$. They are symmetric to each other with respect to the point $(a+b-1)\,n/2$.*

Thus in both cases the roots of the polynomial $\chi_n^{ab}(n)$ are symmetric to each other with respect to the point $(a+b-1)\,n/2$, but in the case $a=b$ all roots are real, whereas in the case $a\neq b$ the roots are on the same vertical line in the complex plane $\mathbb{C}$. Note that in the case $a=b-1$ the polynomial $\chi_n^{ab}(q)$ has only one root $an=(a+b-1)\,n/2$ of multiplicity $n-1$.

The following lemma is implicit in a paper of Auric [5] and also follows from a problem posed by Pólya [18] and solved by Obreschkoff [15] (repeated in [19, Problem V. 196.1, pp. 70 and 251]). For the sake of completeness we give a simple proof.

LEMMA 9.13. *Let $P(q) \in \mathbb{C}[q]$ have the property that every root has real part a. Let z be a complex number satisfying $|z| = 1$. Then every root of the polynomial $R(q) = (S+z)\, P(q) = P(q-1) + zP(q)$ has real part $a + \frac{1}{2}$.*

Proof. We may assume that $P(q)$ is monic. Let

$$P(q) = \prod_j (q - a - b_j i), \qquad b_j \in \mathbb{R},$$

where $i^2 = -1$. If $R(w) = 0$, then $|P(w)| = |P(w-1)|$. Suppose that $w = a + \frac{1}{2} + c + di$, where $c, d \in \mathbb{R}$. Thus

$$\left| \prod_j \left(\tfrac{1}{2} + c + (d - b_j)\, i\right) \right| = \left| \prod_j \left(-\tfrac{1}{2} + c + (d - b_j)\, i\right) \right|.$$

If $c > 0$ then $|\frac{1}{2} + c + (d - b_j)\, i| > |-\frac{1}{2} + c + (d - b_j)\, i|$. If $c < 0$ then we have strict inequality in the opposite direction. Hence $c = 0$, so w has real part $a + \frac{1}{2}$. ∎

Proof of Theorem 9.12. All the roots of the polynomial q^{n-1} have real part 0. The operator $T = (S^a + S^{a+1} + \cdots + S^{b-1})^n$ can be written as

$$T = S^{an} \prod_{j=1}^{b-1-a} (S - z_j)^n,$$

where each z_j is a complex number of absolute value one (in fact, a root of unity). The proof now follows from Theorem 9.7 and Lemma 9.13. ∎

Note. We have been considering the truncated affine arrangement $\mathscr{A}_{n-1}^{ab}$ only in the case $a \geqslant 0$ and $b \geqslant 0$. We do not have any interesting results otherwise. For instance, the arrangement $\mathscr{A}_3^{-1,4}$ (with hyperplanes $x_i - x_j = 2, 3$ for $1 \leqslant i < j \leqslant 4$) has characteristic polynomial $q^4 - 12q^3 + 60q^2 - 116$. The roots of this polynomial are given approximately by 0, 4.33, and $3.83 \pm 3.48i$, so the Riemann hypothesis fails.

9.4. *Other Root Systems*

The results of Sections 9.1–9.3 extend, partly conjecturally, to all the other root systems, as well as to the nonreduced root system BC_n (the union of B_n and C_n, which satisfies all the root system axioms except the axiom stating that if α and β are roots satisfying $\alpha = c\beta$, then $c = \pm 1$). Henceforth in this section when we use the term "root system," we also include the case BC_n.

Given a root system R in $\mathbb{R}^n$ and integers $a \geqslant 0$ and $b \geqslant 0$ satisfying $a + b \geqslant 2$, we define the *truncated R-affine arrangement* $\mathscr{A}^{ab}(R)$ to be the collection of hyperplanes

$$\langle \alpha, x \rangle = -a+1,\ -a+2, ..., b-1,$$

where α ranges over all positive roots of R (with respect to some fixed choice of simple roots). Here $\langle\ ,\ \rangle$ denotes the usual scalar product on $\mathbb{R}^n$, and $x = (x_1, ..., x_n)$. As in the case $R = A_{n-1}$ we refer to the *balanced case* $(a = b)$ and *unbalanced case* $(a \neq b)$.

The characteristic polynomial for the balanced case was found by Edelman and Reiner [10, proof of Prop. 3.1] for the root system A_{n-1} (see Theorem 9.9), and conjectured (Conjecture 3.3) by them for other root systems. This conjecture was proved by Athanasiadis [2, Cor. 7.2.3 and Thm. 7.7.6; 4, Prop. 5.3] for types A, B, C, BC, and D. For types A, B, C, and D the result is also stated in [3, Thm. 5.5]. We will not say anything more about the balanced case here.

For the unbalanced case, we have considerable evidence (discussed below) to support the following conjecture.

Conjecture 9.14. Let R be an irreducible root system in $\mathbb{R}^n$. Suppose that the unbalanced truncated affine arrangement $\mathscr{A} = \mathscr{A}^{ab}(R)$ has $h(\mathscr{A})$ hyperplanes. Then all the roots of the characteristic polynomial $\chi_{\mathscr{A}}(q)$ have real part equal to $h(\mathscr{A})/n$.

Note. (a) If all the roots of $\chi_{\mathscr{A}}(q)$ have the same real part, then this real part must equal $h(\mathscr{A})/n$, since for any arrangement $\mathscr{A}$ in $\mathbb{R}^n$ the sum of the roots of $\chi_{\mathscr{A}}(q)$ is equal to $h(\mathscr{A})$.

(b) Conjecture 9.14 implies the "functional equation"

$$\chi_{\mathscr{A}}(q) = (-1)^n\, \chi_{\mathscr{A}}(-q + 2h(\mathscr{A})/n). \tag{9.12}$$

Thus $\chi_{\mathscr{A}}(q)$ is determined by around half of its coefficients (or values).

(c) Let $a + b \geqslant 2$ and $R = A_n$, B_n, C_n, BC_n, or D_n. Athanasiadis [4, Sects. 3–5] has shown that

$$\chi_R^{ab}(q) = \chi_R^{0,\, b-a}(q - ak), \tag{9.13}$$

where k denotes the Coxeter number of R (suitably defined for $R = BC_n$). These results and conjectures reduce Conjecture 9.14 to the case $a = 0$ when R is a classical root system. A similar reduction is likely to hold for the exceptional root systems.

(d) Conjecture 9.14 is true for all the classical root systems (A_n, B_n, C_n, BC_n, D_n). This follows from explicit formulas found for $\chi_R^{ab}(q)$ by Athanasiadis [4] together with Lemma 9.13. The result of Athanasiadis is the following.

THEOREM 9.15. *Up to a constant factor, we have the following characteristic polynomials of the indicated arrangements. (If the formula has the form $F(S)\,q^n$ or $F(S)(q-1)^n$, then the factor is $1/F(1)$.)*

$$\mathcal{A}^{0,2k+2}(B_n):\qquad (1+S^2+\cdots+S^{2k})^2\,(1+S^2+\cdots+S^{4k+2})^{n-1}\,(q-1)^n$$

$$\mathcal{A}^{0,2k+2}(C_n):\qquad \text{same as for } \mathcal{A}^{0,2k+2}(B_n)$$

$$\mathcal{A}^{0,2k+1}(B_n):\qquad (1+S+\cdots+S^{2k})^2\,(1+S^2+\cdots+S^{4k})^{n-1}\,q^n$$

$$\mathcal{A}^{0,2k+1}(C_n):\qquad \text{same as for } \mathcal{A}^{0,2k+1}(B_n)$$

$$\mathcal{A}^{0,2k+2}(D_n):\qquad (1+S^2)(1+S^2+\cdots+S^{2k})^4$$
$$(1+S^2+\cdots+S^{4k+2})^{n-3}\,(q-1)^n$$

$$\mathcal{A}^{0,2k+1}(D_n):\qquad (1+S+\cdots+S^{2k})^4\,(1+S^2+\cdots+S^{4k})^{n-3}\,q^n$$

$$\mathcal{A}^{0,2k+2}(BC_n):\qquad (1+S^2+\cdots+S^{2k})(1+S^2+\cdots+S^{4k+2})^n\,(q-1)^n$$

$$\mathcal{A}^{0,2k+1}(BC_n):\qquad (1+S+\cdots+S^{2k})(1+S^2+\cdots+S^{4k})^n\,q^n.$$

We also checked Conjecture 9.14 for the arrangements $\mathcal{A}^{02}(F_4)$ and $\mathcal{A}^{02}(E_6)$ (as well as the almost trivial case $\mathcal{A}^{ab}(G_2)$, $a\neq b$). The characteristic polynomials are

$$\mathcal{A}^{02}(F_4):\qquad q^4-24q^3+258q^2-1368q+2917$$

$$\mathcal{A}^{02}(E_6):\qquad q^6-36q^5+630q^4-6480q^3+40185q^2-140076q+212002.$$

The formula for $\chi_{F_4}^{02}(q)$ has the remarkable alternative form:

$$\mathcal{A}^{02}(F_4):\qquad \tfrac{1}{8}((q-1)^4+3(q-5)^4+3(q-7)^4+(q-11)^4)-48.$$

Note that the numbers 1, 5, 7, 11 are the exponents of the root system F_4. For E_6 the analogous formula is given by

$$\mathcal{A}^{02}(E_6):\qquad \frac{1}{1008}\,P(q)-210,$$

where

$$P(q)=61(q-1)^6+352(q-4)^6+91(q-5)^6+91(q-7)^6$$
$$+352(q-8)^6+61(q-11)^6,$$

which is not as intriguing as the F_4 case. It is not hard to see that the symmetry of the coefficient sequences (1, 3, 3, 1) and (61, 352, 91, 91, 352, 61) is a consequence of Eq. (9.12) and the fact that if $e_1 < e_2 < \cdots < e_n$ are the exponents of an irreducible root system R, then $e_i + e_{n+1-i}$ is independent of i.

10. CHARACTERISTIC POLYNOMIALS AND WEIGHTED TREES

In this section we present an interpretation of the characteristic polynomial $\chi_n^{ab}(q)$ of a truncated affine arrangement as a weight enumerator of trees.

10.1. *Weighted Trees*

The differentiation operator $D\colon f(q) \mapsto df/dq$ is related to the shift operator $S\colon f(q) \mapsto f(q-1)$ via Taylor's formula $\exp(-D) = S$. By Theorem 9.7 we can express the characteristic polynomial $\chi_n^{ab}(q)$, for $0 \leqslant a < b$, as

$$(-1)^{n-1}(b-a)^n \chi_n^{ab}(-q) = (e^{aD} + e^{(a+1)D} + \cdots + e^{(b-1)D})^n \cdot q^{n-1}.$$

We can generalize this expression as follows.

Let $s(t)$ be a formal exponential power series

$$s(t) = s_0 + s_1 t + s_2 t^2/2! + \cdots + s_k t^k/k! + \cdots,$$

where the s_i are arbitrary numbers and s_0 is nonzero.

We define the polynomials $f_n(q)$, $n > 0$, by the formula

$$f_n(q) = (s(D))^n q^{n-1}, \tag{10.1}$$

where $D = d/dq$. The polynomials $f_n(q)$ are correctly defined even if the series $s(t)$ does not converge, since the expression for $f_n(q)$ involves only a finite sum of nonzero terms.

Let $\mathcal{T}_n$ be the set of all trees on the vertices 0, 1, 2, ..., n. We will regard the vertex 0 as the root of a tree and orient the edges away from the root. By $d_i = d_i(T)$ we denote the outdegree of the vertex i in a tree $T \in \mathcal{T}_n$. For $i \neq 0$, d_i is the degree of the vertex i minus 1. Define the weight $w_q(T)$ of a tree T by

$$w_q(T) = q^{d_0 - 1} s_{d_1} s_{d_2} \cdots s_{d_n}.$$

Let us also define the weighting $\tilde{w}$ on trees $T \in \mathcal{T}_n$ by $\tilde{w}(T) = s_{d_0} s_{d_1} \cdots s_{d_n}$. And let $g_n = \sum_{T \in \mathcal{T}_n} \tilde{w}(T)$ be the weighted sum of all trees in $\mathcal{T}_n$.

THEOREM 10.1. 1. *The polynomial* $f_n(q)$ *is the* w_q*-weight enumerator for trees on* $n+1$ *vertices, i.e.,*

$$f_n(q) = \sum_{T \in \mathscr{T}_n} w_q(T).$$

In particular, $g_n = f_{n+1}(0)/(n+1)$.

2. *The coefficient of* q^k *in* $f_n(q)$ *is equal to*

$$\sum s_{k_1} \cdots s_{k_n} \binom{n-1}{k, k_1, ..., k_n},$$

where the sum is over all $k_1, ..., k_n \geqslant 0$ *such that* $k + k_1 + \cdots + k_n = n-1$.

3. *Let* $f(x, q)$ *and* $g(x)$ *be the exponential generating functions*:

$$f(x, q) = 1 + q \sum_{n \geqslant 1} f_n(q) \frac{x^n}{n!} \qquad \text{and} \qquad g(x) = \sum_{n \geqslant 0} g_n \frac{x^{n+1}}{n!}.$$

Then $f(x, q) = \exp(q\, g(x))$ *and the series* $g = g(x)$ *satisfies the functional equation*

$$g = xs(g). \tag{10.2}$$

Proof. By (10.1), we have

$$\begin{aligned} f_n(q) &= s(D)^n q^{n-1} = s(D)^{n-1} \sum_{k_1 \geqslant 0} s_{k_1} \frac{D^{k_1}}{k_1!} q^{n-1} \\ &= s(D)^{n-1} \sum_{k_1 \geqslant 0} s_{k_1} \binom{n-1}{k_1} q^{n-1-k_1} = \cdots \\ &= \sum_{k_1, ..., k_n \geqslant 0} s_{k_1} \cdots s_{k_n} \binom{n-1}{k, k_1, k_2, ..., k_n} q^k, \end{aligned}$$

where $k = n - 1 - k_1 - \cdots - k_n$. This proves 2. Using Prüfer's coding of trees [22; 28, Thm. 5.3.4], we obtain the statement 1. A standard exponential formula argument yields the statement 3. ▮

Now we give several examples for Theorem 10.1.

EXAMPLE 10.2 (cf. Example 9.10.1). For the Shi arrangement ($a=1$ and $b=2$), we have $s(t) = e^t$ and $w_q(T) = q^{d_0 - 1}$. Theorem 10.1 claims that $(-1)^{n-1} \chi_n^{12}(-q) = (q+n)^{n-1}$ is the q-enumerator for all trees in $\mathscr{T}_n$ according to the degree of the root. Of course, this is a well-known statement.

EXAMPLE 10.3 (cf. Example 9.10.3). For the Linial arrangement ($a=0$ and $b=2$) we have $s(t)=1+e^t$, i.e., $s_0=2$ and $s_i=1$ for $i\geqslant 1$. Thus $w_q(T)=2^{\mathrm{ep}(T)}\,q^{d_0-1}$, where $\mathrm{ep}(T)$ is the number of endpoints i, $i\neq 0$, of T. In this case we obtain the following statement.

COROLLARY 10.4. *For the Linial arrangement $\mathscr{L}_{n-1}$, we have*

$$(-1)^{n-1}\,\chi_n^{02}(-q)=\sum_{T\in\mathscr{T}_n} 2^{\mathrm{ep}(T)-n}\,q^{d_0-1}.$$

In particular, the number of regions of the Linial arrangement $\mathscr{L}_{n-1}$ is equal to $\sum_{T\in\mathscr{T}_n} 2^{\mathrm{ep}(T)-n}$.

10.2. *Odd Degree Trees*

Let us introduce the following shift of the characteristic polynomial of the Linial arrangement:

$$b_n(q)=2^{n-1}\chi_n^{02}((q+n)/2). \tag{10.3}$$

The Riemann hypothesis (Theorem 9.12) implies that all roots of $b_n(q)$ are purely imaginary. By Theorem 9.7, we have

$$b_n(q)=\left(\frac{S+S^{-1}}{2}\right)^n q^{n-1}=2^{-n}\sum_{k=0}^{n}\binom{n}{k}(q+n-2k)^{n-1}. \tag{10.4}$$

The first ten polynomials $b_n(q)$ are given below:

$$b_1(q)=1$$

$$b_2(q)=q$$

$$b_3(q)=q^2+3$$

$$b_4(q)=q^3+12q$$

$$b_5(q)=q^4+30q^2+65$$

$$b_6(q)=q^5+60q^3+480q$$

$$b_7(q)=q^6+105q^4+1995q^2+3787$$

$$b_8(q)=q^7+168q^5+6160q^3+41216q$$

$$b_9(q)=q^8+252q^6+15750q^4+242172q^2+427905$$

$$b_{10}(q)=q^9+360q^7+35280q^5+1021440q^3+6174720q$$

We can express $b_n(q)$ via the differentiation operator $D=d/dq$ as

$$b_n(q)=\cosh(D)^n\, q^{n-1}. \tag{10.5}$$

Thus the sequence of polynomials $b_n(q)$ is a special case of (10.1) for $s(t)=\cosh(t)$. Equivalently, $s_i=1$ for even i's and $s_i=0$ for odd i's.

We say that a tree T on the vertices 0, 1, ..., n is an *odd degree tree* if the degrees of the vertices 1, ..., n in T are odd. Let $d_0(T)$ denote the degree of the root 0 in a tree T. Note that, for an odd degree tree, $d_0(T)$ has the same parity as n.

Theorem 10.1 implies the following statement.

COROLLARY 10.5. 1. *For $n\geqslant 1$, we have*

$$b_n(q)=\sum_T q^{d_0(T)-1},$$

where the sum is over all odd degree trees on the vertices 0, 1, ..., n.

2. *The coefficient of q^k in $b_n(q)$ is equal to the sum of multinomial coefficients*

$$\binom{n-1}{k, k_1, ..., k_n}$$

over all nonnegative even $k_1, ..., k_n$ such that $k+k_1+\cdots+k_n=n-1$.

Let odd_n be the number of all odd degree trees on the vertices 0, 1, ..., n. By Corollary 10.8, $\text{odd}_n=b_n(1)$. We have

n	0	1	2	3	4	5	6	7	8	9	10
odd_n	1	1	1	4	13	96	541	5,888	47,545	686,080	7,231,801

If n is odd then the degrees of all vertices (including the root) of an odd degree tree are odd. The first ten numbers $\text{odd}_1, \text{odd}_3, \text{odd}_5$, ... appear in [23] without further references.

Note that $\text{odd}_{2m}=b_{2m+1}(0)/(2m+1)$ and $\text{odd}_{2m-1}=b'_{2m}(0)/(2m-1)$ for $m\geqslant 1$. Indeed, by Corollary 10.5, $b_{2m+1}(0)$ is the number of odd degree trees on the vertices 0, 1, ..., $2m+1$ such that the degree of the root 0 is one. Removing the only edge incident to 0, we obtain an odd degree tree on the vertices 1, ..., $2m+1$ with the root at any of its $2m+1$ vertices. The number of such trees is $(2m+1)\,\text{odd}_{2m}$.

Also $b'_{2m}(0)$ is the number of odd degree trees on the vertices 0, 1, ..., $2m$ such that the degree of the root 0 is two. Let e be the edge of such a tree that connects the root 0 with the component which does not contain the vertex 1. Contracting the edge e we obtain an odd degree tree on the vertices 1, ..., $2m$ with the root at any vertex except 1. The number of such trees is $(2m-1)\,\mathrm{odd}_{2m-1}$.

Theorem 10.1.3 gives a functional equation for the generating functions.

Corollary 10.6. *Let $f(x, q)$ and $g(x)$ be the exponential generating functions:*

$$f(x, q) = 1 + q \sum_{n \geqslant 1} b_n(q) \frac{x^n}{n!} \qquad \text{and} \qquad g(x) = \sum_{m \geqslant 0} \mathrm{odd}_{2m} \frac{x^{2m+1}}{(2m)!}.$$

Then $f(x, q) = \exp(q\, g(x))$ and $g = g(x)$ satisfies the functional equation

$$g = x \cosh(g).$$

11. ASYMPTOTICS

11.1. *Asymptotics of the Characteristic Polynomial*

In this section we find the asymptotics of the characteristic polynomial $\chi_n^{a,\, a+2}(q)$ of the extended Linial arrangement. By (9.11), we have

$$(-1)^{n-1}\, \chi_n^{a,\, a+2}(q) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} (an + k - q)^{n-1}. \qquad (11.1)$$

We will use this formula to define the polynomial $\chi_n^{a,\, a+2}(q)$ for an arbitrary real a.

Recall that two sequences a_n and b_n are said to be *asymptotically equal* (in symbols, $a_n \sim b_n$) if $\lim_{n \to \infty} a_n / b_n = 1$.

Theorem 11.1. *For any $a \in \mathbb{R}$, $a \geqslant 0$, and $q \in \mathbb{C}$, the value of the polynomial $(-1)^{n-1}\, \chi_n^{a,\, a+2}(q)$ is asymptotically equal to*

$$(-1)^{n-1}\, \chi_n^{a,\, a+2}(q) \sim A \cdot B^{q+a+\beta} \cdot C^n \cdot (n+1)^{n-1}, \qquad (11.2)$$

where β is the unique solution to the equation

$$\beta/(1-\beta) = e^{1/(\beta + a)}, \quad 0 < \beta < 1. \qquad (11.3)$$

and

$$A = (\beta + 2a\beta + a^2)^{-1/2},$$
$$B = \beta^{-1}(1-\beta),$$
$$C = 2^{-1} \beta^{-\beta} (1-\beta)^{\beta - 1} (\beta + a).$$

Moreover, the asymptotical equality remains valid for the m*th derivatives of both sides with respect to* q.

COROLLARY 11.2. *For any* $a \in \mathbb{R}$, $a \geqslant 0$, *and* $q \in \mathbb{C}$, *we have*

$$\lim_{n \to \infty} \frac{\chi_n^{a,\ a+2}(q)}{\chi_n^{a,\ a+2}(0)} = \left(\frac{1-\beta}{\beta}\right)^q,$$

where β *is given by* (11.3). *Moreover, for any* $q_0 \in \mathbb{C}$ *the Taylor expansion of* $\chi_n^{a,\ a+2}(q)/\chi_n^{a,\ a+2}(0)$ *at* $q = q_0$ *converges termwise to the Taylor expansion of the right-hand side at* $q = q_0$.

EXAMPLE 11.3. For the characteristic polynomial of the Linial arrangement (case $a = 0$) we have

$$\begin{aligned} \beta &\approx 0.7821882, \\ A = \beta^{-1/2} &\approx 1.1306920, \\ B = \beta^{-1}(1-\beta) &\approx 0.2784645, \\ C = 2^{-1}\beta^{-\beta+1}(1-\beta)^{\beta-1} &\approx 0.6605498, \\ D = A \cdot B^{\beta-1} &\approx 1.4937570. \end{aligned}$$

The number f_n of regions of the Linial arrangement $\mathscr{L}_{n-1}$ is asymptotically equal to

$$f_n = (-1)^{n-1}\, \chi_n^{02}(-1) \sim D \cdot C^n\, (n+1)^{n-1}.$$

Recall that f_n is the number of alternating trees on $n+1$ vertices (see Section 8.1). The total number of trees on $n+1$ labelled vertices is $(n+1)^{n-1}$.

COROLLARY 11.4. *The probability that a uniformly chosen tree on* $n+1$ *labelled vertices is an alternating tree is asymptotically equal to*

$$D \cdot C^n \approx 1.4937570 \cdot 0.6605498^n.$$

Compare the result that the probability that a uniformly chosen permutation $w_1, w_2, ..., w_n$ of $1, 2, ..., n$ is alternating (i.e., $a_1 > a_2 < a_3 > a_4 < \cdots$) is asymptotically equal to

$$\left(\frac{2}{\pi}\right)^{n+1} \approx 0.6366198^{n+1}.$$

By Theorem 2.1, the number of bounded regions of the arrangement $\mathcal{A}_{n-1}^{a,a+2}$ is equal to $(-1)^{n-1}\chi_n^{a,a+2}(1)$. By (11.2) this number is asymptotically equal to $B^2 \cdot (-1)^{n-1}\chi_n^{a,a+2}(-1)$.

COROLLARY 11.5. *The probability that a uniformly chosen region in the extended Linial arrangement* $\mathcal{A}_{n-1}^{a\,a+2}$ *is bounded tends to* B^2 *as* $n \to \infty$. *For the Linial arrangement,* $B^2 \approx 0.0775425$. *Thus, for large n, approximately* 7.75425% *of the regions of the Linial arrangement* $\mathcal{L}_{n-1}$ *are bounded.*

Note that by (9.9) the portion of the bounded regions in the Shi arrangement $\mathcal{S}_{n-1}$ is equal to $(n-1)^{n-1}/(n+1)^{n-1}$ and tends to $e^{-2} \approx 0.1353353$.

In the proof of Theorem 11.1 we use methods described in [9]. The general outline of the proof is the following: (a) Use the Stirling formula for the Γ-function to approximate the summands in (11.1); (b) approximate the summation by integration; (c) use the Laplace method to approximate the integral. The Laplace method amounts to the following statement; see [9, Sect. 4.2].

PROPOSITION 11.6. *Suppose that* $g(x)$ *and* $h(x)$ *are real smooth functions on the interval* $[a, b]$. *Suppose that* β, $a < \beta < b$, *is the absolute maximum of* $h(x)$. *We also require that* $h(x) < h(\beta)$ *for* $x \neq \beta$. *Moreover, there exist positive numbers c and d such that* $h(x) \leqslant h(\beta) - c$ *for* $|x-\beta| \geqslant d$. *Also suppose that* $h''(\beta)$ *exists and* $h''(\beta) < 0$ *and that* $b(\beta) \neq 0$. *Then*

$$\int_a^b g(x)\, e^{n\,h(x)}\, dx \sim (2\pi)^{1/2}\, g(\beta)\, (-n\, h''(\beta))^{-1/2}\, e^{n\,h(\beta)} \qquad (\text{as } n \to \infty).$$

Now we give more details.

Proof of Theorem 11.1. Let us express the kth summand $a_n(k)$ in (11.1) via the Γ-function as

$$a_n(k) = \frac{\Gamma(n+1)\,(k+an-q)^{n-1}}{2^n\,\Gamma(k+1)\,\Gamma(n-k+1)}$$

and view it as a continuous function of k on the interval $[0, n]$. Elementary calculations show that $|a_n(k)|$ has a unique absolute maximum $k = m_n$ on the interval $[0, n]$. And, for sufficiently large n, we have $1/2 < m_n/n < (1+e^{-2/(1+2a)})^{-1}$. Actually, m_n/n approaches β as given by (11.3).

Let us fix ε such that $0 < \varepsilon < 1 - (1+e^{-2/(1+2a)})^{-1}$. Then we can write

$$\sum_{k=0}^{n} a_n(k) = (1 + r_n(\varepsilon)) \cdot \sum_{k=\lceil \varepsilon n \rceil}^{\lfloor (1-\varepsilon)\,n \rfloor} a_n(k)\,, \tag{11.4}$$

where $|r_n(\varepsilon)| \leqslant 2\varepsilon$ for sufficiently large n. The Stirling formula claims that

$$\Gamma(z) = z^{z-1/2} e^{-z} (2\pi)^{1/2} (1 + O(1/z)).$$

Therefore, the $a_n(k)$ can be written as

$$a_n(k) = \frac{\Gamma(n+1)\,(k+an-q)^{n-1}}{2^n\,\Gamma(k+1)\,\Gamma(n-k+1)}$$

$$= \frac{e\,(n+1)^{n+1/2}}{2^n\,(2\pi)^{1/2}} \cdot \frac{(an+k-q)^{n-1}}{(k+1)^{k+1/2}\,(n-k+1)^{n-k+1/2}}\,(1+O_{nk}),$$

where O_{nk} is an abbreviation for $O((k+1)^{-1} + (n-k+1)^{-1})$. For $\varepsilon n \leqslant k \leqslant (1-\varepsilon)\,n$, we have $O_{nk} = O(1/n)$. Let $x = \frac{k+1/2}{n+1}$. Making transformations, we can write, for $\varepsilon \leqslant x \leqslant 1-\varepsilon$,

$$\frac{(an+k-q)^{n-1}}{(k+1)^{k+1/2}\,(n-k+1)^{n-k+1/2}}$$

$$= \frac{(x+a)^{n-1}}{(x^x\,(1-x)^{1-x})^{n+1}}$$

$$\cdot \frac{1}{(n+1)^2} \cdot \frac{\left(1 - \dfrac{q+a+1/2}{x+a}\,\dfrac{1}{n+1}\right)^{n-1}}{\left(1+\dfrac{1/2}{k+1/2}\right)^{k+1/2}\left(1+\dfrac{1/2}{n-k+1/2}\right)^{n-k+1/2}}$$

$$= \frac{(x+a)^{n-1}}{(x^x\,(1-x)^{1-x})^{n+1}} \cdot \frac{1}{(n+1)^2} \cdot \frac{e^{-(q+a+1/2)/(x+a)}}{e^{1/2}\,e^{1/2}}\,(1+O(1/n)).$$

Let us introduce two functions

$$g(x) = e^{-(q+a+1/2)/(x+a)}\,(x+a)^{-1}\,x^{-x}\,(1-x)^{x-1},$$

$$h(x) = \log(x+a) - x\log(x) - (1-x)\log(1-x)$$

on the interval $[\varepsilon, 1-\varepsilon]$. The function $h(x)$ has a unique maximum $\beta \in\ ]\,\varepsilon, 1-\varepsilon[$ given by $h'(\beta) = 1/(\beta+a) - \log(\beta) + \log(1-\beta) = 0$. This equation is equivalent to (11.3). We have $g(\beta) \neq 0$. Thus the functions $g(x)$ and $h(x)$ satisfy the conditions of Proposition 11.6.

Then, for $k \in [\varepsilon n, (1-\varepsilon)\,n]$, the function $a_n(k)$ can be written as

$$a_n(k) = A_n(x) = \frac{(n+1)^{n-3/2}}{2^n\,(2\pi)^{1/2}} \cdot g(x)\,e^{n\,h(x)}\,(1+O(1/n))\,. \qquad (11.5)$$

Since the function $|a_n(k)|$ has a unique maximum, we have

$$\left| \sum_{k=\lceil \varepsilon n \rceil}^{\lfloor (1-\varepsilon)\, n \rfloor} a_n(k) - \int_{\varepsilon n}^{(1-\varepsilon)\, n} a_n(k)\, dk \right| \leqslant \max_{k \in [0,\, n]} |a_n(k)|\,. \tag{11.6}$$

We have

$$\int_{\varepsilon n}^{(1-\varepsilon)\, n} a_n(k)\, dk \sim (n+1) \int_{\varepsilon}^{1-\varepsilon} A_n(x)\, dx \sim \frac{(n+1)^{n-1/2}}{2^n\, (2\pi)^{1/2}} \cdot \int_{\varepsilon}^{1-\varepsilon} g(x)\, e^{n\, h(x)}\, dx.$$

By Proposition 11.6, this expression is asymptotically equal to

$$\frac{(n+1)^{n-1/2}}{2^n\, (2\pi)^{1/2}} \cdot (2\pi)^{1/2}\, g(\beta)\, (-n\, h''(\beta))^{-1/2}\, e^{n\, h(\beta)}. \tag{11.7}$$

This expression shows that

$$\max_{k \in [0,\, n]} |a_n(k)| \sim A_n(\beta) \sim \text{Constant} \cdot n^{-1/2} \int_{\varepsilon n}^{(1-\varepsilon)\, n} a_n(k)\, dk. \tag{11.8}$$

Using (11.6) and simplifying (11.7), we obtain

$$\sum_{k=\lceil \varepsilon n \rceil}^{\lfloor (1-\varepsilon)\, n \rfloor} a_n(k) \sim \int_{\varepsilon n}^{(1-\varepsilon)\, n} a_n(k)\, dk \sim \frac{(n+1)^{n-1}}{2^n}\, g(\beta)(-h''(\beta))^{-1/2}\, e^{n\, h(\beta)}. \tag{11.9}$$

Since ε can be arbitrarily small, from (11.4) we conclude that $\sum_{k=0}^{n} a_n(k)$ is asymptotically equal to the right-hand side of (11.9). Finally, the explicit calculation of $g(\beta)$, $h(\beta)$, and $h''(\beta)$, left as an exercise for the reader, produces the formula (11.2).

To prove the statement about derivatives of the characteristic polynomial, we remark that the mth derivative of $a_n(k)$ with respect to q is obtained by multiplying the expression (11.5) by $(-1/(x+a))^m$. Exactly the same argument as that above shows that the asymptotic behavior of the sum of the mth derivatives of $a_n(k)$ is given by the expression (11.9) times $(-1/(\beta+a))^m$, which is equal to the mth derivative of the right-hand side of (11.9). ∎

11.2. *Asymptotics of Odd Degree Trees*

In this section we find the asymptotics of the shifted characteristic polynomial $b_n(q) = 2^{n-1} \chi_n^{02}(\frac{q+n}{2})$ introduced in Section 10.2. Recall that $b_n(q)$ is given by the sum (10.4), and it is also the enumerator for the odd degree trees according to the degree of the root. The behavior of the polynomials

$b_n(q)$ depends on the parity of n. For example, $b_n(q)$ is an even function for odd n and is an odd function for even n.

THEOREM 11.7. *Let $\alpha \approx 1.1996786$ be the unique positive solution of the equation*

$$\cosh(\alpha) = \alpha \sinh(\alpha) \qquad \text{or, equivalently,} \qquad (\alpha - 1)\, e^{2\alpha} = (\alpha + 1), \tag{11.10}$$

and let $C = \sinh(\alpha)/e \approx 0.5550857$. Then we have two asymptotic equalities

$$\begin{aligned} b_n(q) &\sim 2e^{-1} \cdot \cosh(\alpha q) \cdot C^n \cdot (n+1)^{n-1}, \qquad n \text{ is odd, } n \to \infty, \\ b_n(q) &\sim 2e^{-1} \cdot \sinh(\alpha q) \cdot C^n \cdot (n+1)^{n-1}, \qquad n \text{ is even, } n \to \infty, \end{aligned} \tag{11.11}$$

for any $q \in \mathbb{C}$ such that the right-hand side is nonzero. Moreover, the asymptotic equalities remain valid for the mth derivatives of both sides with respect to q provided that the mth derivative of the right-hand side is nonzero.

Note that we can simplify the right-hand sides in (11.11) and replace them by asymptotically equal expressions $2\cosh(\alpha q)\, C^n\, n^{n-1}$ and $2\sinh(\alpha q)\, C^n\, n^{n-1}$, respectively. Numerical calculations, however show that these expressions are worse approximations for $b_n(q)$ than (11.11).

COROLLARY 11.8. *For any $q \in \mathbb{C}$, we have*

$$\lim_{n \text{ is odd},\, n \to \infty} b_n(q)/b_n(0) = \cosh(\alpha\, q),$$

$$\lim_{n \text{ is even},\, n \to \infty} b_n(q)/b'_n(0) = \alpha^{-1} \sinh(\alpha\, q),$$

where α is given by (11.10). *Moreover, for any $q_0 \in \mathbb{C}$ the Taylor expansions at $q = q_0$ of the terms in the left-hand side converge termwise to the Taylor expansion of the right-hand side at $q = q_0$.*

Recall that the roots of the polynomials $b_n(q)$ are located on the purely imaginary axis in $\mathbb{C}$. Theorem 11.7 gives an approximation for the roots of $b_n(q)$.

COROLLARY 11.9. *Let us fix a positive number R. Then the roots of the polynomials $b_n(q)$ located in the interval $\mathcal{I} =]-i\,R, i\,R[$*

(a) *approach the points $\{\alpha\,\pi\,(1/2 + m)\, i \mid m \in \mathbb{Z}\} \cap \mathcal{I}$ as $n \to \infty$ (n is odd),*

(b) *approach the points $\{\alpha\,\pi\, m\, i \mid m \in \mathbb{Z}\} \cap \mathcal{I}$ as $n \to \infty$ (n is even),*

where α is given by (11.10) *and $i = \sqrt{-1}$.*

Remark 11.10. Clearly, we also obtain an approximation for the roots of the characteristic polynomials $\chi_n^{02}(q)$ of Linial arrangements by the numbers $2^{-1}(n+\alpha\,\pi\,(1/2+m)\,i)$ for odd n, and by the numbers $2^{-1}(n+\alpha\,\pi\,m\,i)$ for even n, where $m\in\mathbb{Z}$.

Proof of Theorem 11.7. We will follow the proof of Theorem 11.1. If n is odd then by (10.4) we can write $b_n(q)$ as

$$b_n(q)=\sum_{k=0}^{(n-1)/2} 2^{-n}\binom{n}{k}\left((n-2k+q)^{n-1}+(n-2k-q)^{n-1}\right).$$

Let us express the kth summand $a_n(k)$ in the above sum via the Γ-function as

$$a_n(k)=\frac{\Gamma(n+1)\,((n-2k+q)^{n-1}+(n-2k-q)^{n-1})}{2^n\,\Gamma(k+1)\,\Gamma(n-k+1)}$$

and view it as a continuous function of k on the interval $[0,(n-1)/2]$. Again, $|a_n(k)|$ has a unique absolute maximum m_n on $[0,(n-1)/2]$. Calculations shows that, for sufficiently large n, we have $0.08<m_n/n<0.09$.

Let us fix ε such that $0<\varepsilon<0.08$. Then

$$\sum_{k=0}^{(n-1)/2} a_n(k)=(1+r_n(\varepsilon))\cdot\sum_{k=\lceil\varepsilon n\rceil}^{\lfloor(1/2-\varepsilon)\,n\rfloor} a_n(k), \tag{11.12}$$

where $|r_n(\varepsilon)|\leqslant 4\varepsilon$ for sufficiently large n. We can approximate $a_n(k)$, for $k\in[\varepsilon n,(1/2-\varepsilon)\,n]$, via the Stirling formula as

$$a_n(k)=\frac{e\,(n+1)^{n-3/2}}{2^n\,(2\pi)^{1/2}}\cdot\frac{(1-2x)^{n-1}}{(x^x\,(1-x)^{1-x})^{n+1}}$$
$$\cdot\frac{\left(1+\frac{q}{1-2x}\,\frac{1}{n+1}\right)^{n-1}+\left(1-\frac{q}{1-2x}\,\frac{1}{n+1}\right)^{n-1}}{\left(1+\frac{1/2}{k+1/2}\right)^{k+1/2}\left(1+\frac{1/2}{n-k+1/2}\right)^{n-k+1/2}}\,(1+O(n^{-1}))$$
$$=\frac{e\,(n+1)^{n-3/2}}{2^n\,(2\pi)^{1/2}}\cdot\frac{(1-2x)^{n-1}}{(x^x\,(1-x)^{1-x})^{n+1}}$$
$$\cdot\frac{e^{q/(1-2x)}+e^{-q/(1-2x)}}{e^{1/2}\,e^{1/2}}\,(1+O(n^{-1})),$$

where, as before, $x=(k+1/2)/(n+1)$. Let us define two functions

$$g(x)=(e^{q/(1-2x)}+e^{-q/(1-2x)})\,(1-2x)^{-1}\,x^{-x}\,(1-x)^{x-1},$$
$$h(x)=\log(1-2x)-x\log(x)-(1-x)\log(1-x)$$

on the interval $[\varepsilon, 1/2-\varepsilon]$. Then we can write $a_n(k)$ as

$$a_n(k)=A_n(x)=\frac{(n+1)^{n-3/2}}{2^n\,(2\pi)^{1/2}}\cdot g(x)\,e^{n\,h(x)}\,(1+O(1/n))\,.$$

Let $\beta\approx 0.0832217$ be the unique maximum of $h(x)$ on the interval $[\varepsilon, 1/2-\varepsilon]$ given by the equation $h'(\beta)=-2/(1-2\beta)-\log(\beta)+\log(1-\beta)=0$. And let $\alpha=1/(1-2\beta)$. The equation for β transforms into the defining equation (11.10) for α.

If $g(\beta)\neq 0$ or, equivalently, $\cosh(\alpha\, q)\neq 0$, then the functions $g(x)$ and $f(x)$ satisfy the conditions of Proposition 11.6. Using exactly the same argument as that in the proof of Theorem 11.1, we can write

$$\sum_{k=\lceil \varepsilon n\rceil}^{\lfloor (1/2-\varepsilon)\,n\rfloor} a_n(k)\sim\int_{\varepsilon n}^{(1/2-\varepsilon)\,n} a_n(k)\,dk=(n+1)\int_{\varepsilon}^{1/2-\varepsilon} A_n(x)\,dx$$
$$\sim\frac{(n+1)^{n-1}}{2^n}\,g(\beta)\,(-h''(\beta))^{-1/2}\,e^{n\,h(\beta)}$$
$$=2e^{-1}\cosh(\alpha\,q)\,C^n\,(n+1)^{n-1}.$$

Since ε can be chosen arbitrarily small, from (11.12) we conclude that $b_n(q)$ is asymptotically equal to $2e^{-1}\cosh(\alpha\,q)\,C^n\,(n+1)^{n-1}$.

For asymptotics of the mth derivative of the polynomials $b_n(q)$ we need to replace the function $g(x)=\cosh(\frac{q}{1-2x})\times\langle$terms that do not depend on $q\rangle$ by its mth derivative with respect to q. If the value of this derivative for $x=\beta$ and certain $q\in\mathbb{C}$ is nonzero, then we can apply Proposition 11.6 and obtain the required statement.

If n is even then by (10.4) we can write $b_n(q)$ as

$$b_n(q)=\sum_{k=0}^{n/2-1}\binom{n}{k}\,((n-2k+q)^{n-1}-(n-2k-q)^{n-1})+\binom{n}{n/2}\,q^{n-1}\,.$$

The proof in this case goes exactly along the same lines. The additional term $\binom{n}{n/2}\,q^{n-1}$ is infinitesimally small with respect to $b_n(q)$; cf. (11.8). In this case we obtain an analogous expression for the asymptotics of $b_n(q)$ with $g(x)=(e^{q/(1-2x)}-e^{-q/(1-2x)})\,(1-2x)^{-1}\,x^{-x}\,(1-x)^{x-1}$ and exactly the same $h(x)$. This means that in the resulting expression we just replace $\cosh(\alpha\,q)$ by $\sinh(\alpha\,q)$. The argument about q-derivatives is the same. ∎

11.3. *Distribution of Degrees of Random Trees*

In this section we study a probability distribution on labelled trees inspired by Section 10.1.

Recall that in Section 10.1, for an arbitrary power series $s(t) = s_0 + s_1 t + s_2 t^2/2! + s_3 t^3/3! + \cdots$, $s_0 \neq 0$, we introduced the weighting $\tilde{w}(T) = s_{d_0} s_{d_1} \cdots s_{d_n}$ on the set $\mathcal{T}_n$ of trees on the vertices 0, 1, ..., n, where $d_0, d_1, ..., d_n$ are the outdegrees of the vertices of a tree $T \in \mathcal{T}_n$. We also defined the numbers $g_n = \sum_{T \in \mathcal{T}_n} \tilde{w}(T)$.

Let us assume that the s_i are nonnegative. Let I be the set of indices n for which $g_n > 0$. For $n \in I$, consider the probability distribution on the set $\mathcal{T}_n$ given by $P_T = \tilde{w}(T)/g_n$ for $T \in \mathcal{T}_n$. Let $P_n(k)$ be the probability that a uniformly chosen random vertex of a random tree in $\mathcal{T}_n$ has outdegree k, i.e.,

$$P_n(k) = \sum_{T \in \mathcal{T}_n} \frac{\tilde{w}(T)}{g_n} \frac{m_k(T)}{n+1},$$

where $m_k(T)$ is the number of vertices in T with outdegree k.

THEOREM 11.11. *Assume that the series $s(t)$ converges to a holomorphic nonlinear function on $\mathbb{C}$. Let us fix $k \geqslant 0$ and assume that there exists the limit $P(k) = \lim_{n \to \infty} P_n(k)$ over $n \in I$. Then*

$$P(k) = \frac{s_k\, \alpha^k}{s(\alpha)\, k!},$$

where α is the unique positive solution of the equation

$$s(\alpha) = \alpha\, s'(\alpha)\,. \tag{11.13}$$

We can interpret $P(k)$ as the probability that a "random vertex" of an "infinite random tree" has outdegree k.

Remark 11.12. It is interesting to find conditions on the function $s(t)$ that would guarantee that the sequence $P_n(k)$, $n \in I$, converges to a limit.

EXAMPLE 11.13. Suppose that $s_0 = s_1 = s_2 = \cdots = 1$. In this case we have the uniform distribution on trees in $\mathcal{T}_n$. We have $s(t) = e^t$ and $\alpha = 1$. Theorem 11.11 predicts the Poisson distribution for outdegrees of an infinite random tree:

$$P(k) = e^{-1}/k!\,.$$

In this case it is not hard to calculate $P_n(k)$ explicitly. For example, $P_n(0) = n\, n^{n-2}/(n+1)^{n-1}$ tends to $1/e$ as $n \to \infty$.

EXAMPLE 11.14. Suppose that $s_0 = s_2 = 1$ and $s_i = 0$ for $i = 1, 3, 4, 5, \ldots$. In this case we have the uniform distribution on trees such that each vertex has outdegree 0 (endpoint) or 2. We have $s(t) = 1 + t^2/2$ and $\alpha = \sqrt{2}$. Theorem 11.11 predicts the following distribution of outdegrees:

$$P(0) = P(2) = 1/2.$$

Actually, any tree in $\mathcal{T}_{2m}$ with outdegrees 0 or 2 has $m+1$ endpoints. Thus the probability that a random vertex is an endpoint tends to 1/2 as $m \to \infty$.

EXAMPLE 11.15. Assume that $s_{2m} = 1$ and $s_{2m+1} = 0$, $m \geqslant 0$. Then $s(t) = \cosh(t)$. In this case I is the set of nonnegative even numbers. We have the uniform distribution on the trees in $\mathcal{T}_n$ with even outdegrees. These are exactly odd degree trees if n is even. Thus $g_n = \mathrm{odd}_n$ for even n and $g_n = 0$ for odd n. Theorem 11.11 predicts the following distribution of outdegrees of an infinite random odd degree tree,

$$P(2m) = \frac{\alpha^{2m}}{\cosh(\alpha)\,(2m)!},$$

where $\alpha \approx 1.1996786$ is the unique positive solution of the equation

$$\sinh(\alpha)\,\alpha = \cosh(\alpha)\,.$$

Note that we have exactly the same α as the α in Theorem 11.7.

Theorem 11.11 does not guarantee that the limit $P(2m)$ exists. We can prove that the sequence $P_n(2m)$, $n = 0, 2, 4, \ldots$ converges to a limit using the results of Section 11.2. For example, the argument with removing an edge incident to an endpoint shows that, for even n,

$$P_n(0) \sim \frac{(n+1)\,\mathrm{odd}_n}{\mathrm{odd}_{n+1}} = \frac{(n+1)\,b_n(1)}{b_{n+1}(1)}\,.$$

By Theorem 11.7, we have, for even n,

$$\frac{(n+1)\,b_n(1)}{b_{n+1}(1)} \sim \frac{\sinh(\alpha)}{\cosh(\alpha)\,C} \cdot \frac{(n+1)^n}{(n+2)^n} \sim \frac{\sinh(\alpha)}{\cosh(\alpha)\,C\,e} = \frac{1}{\cosh(\alpha)}.$$

Thus the sequence $P_n(0)$ converges to $1/\cosh(\alpha) \approx 0.5524341$. In other words, for large n, around 55.24341% of the vertices of a uniformly chosen random odd degree tree are endpoints.

In order to prove Theorem 11.11, we need the following trivial statement.

LEMMA 11.16. *Let I be an infinite subset of nonnegative integers. Also let $a(x)=\sum_{n\in I} a_n x^n$ and $b(x)=\sum_{n\in I} b_n x^n$ be two power series and $x_c>0$ such that*

(a) *both series $a(x)$ and $b(x)$ converge for $0<x<x_c$ and diverge at $x=x_c$;*

(b) *we have $a_n, b_n>0$, $n\in I$, and there exists the limit $\lambda=\lim_{n\to\infty, n\in I} a_n/b_n$.*

Then there exists the limit $\lim_{x\to x_c-0} a(x)/b(x)$ and it is equal to λ.

Proof of Theorem 11.11. Note that $I=\{n\geqslant 0 \mid g_n>0\}$ is an infinite set unless $s_i=0$ for all $i\geqslant 1$. Let

$$a(x)=\sum_{n\in I} (n+1)\, P_n(k)\, g_n x^n/n!,$$

$$b(x)=\sum_{n\in I} (n+1)\, g_n x^n/n!.$$

Then $P_n(k)$ is the ratio of the coefficients of x^n in $a(x)$ and $b(x)$. By our assumption $P_n(k)$ converges to the limit $P(k)$. Thus the series $a(x)$ and $b(x)$ satisfy condition (b) of Lemma 11.16.

We have $b(x)=g'(x)$. Recall that $g=g(x)$ satisfies $g=x\,s(g)$; see (10.2). Thus

$$b(x)=s(g)+xs'(g)\,d(x),$$

$$b(x)=\frac{s(g)}{1-xs'(g)}. \tag{11.14}$$

Let $g_{(k)}(x,y)$ be the following exponential generating function

$$g_{(k)}(x,y)=\sum_{n\geqslant 0}\sum_{T\in\mathcal{T}_n} \tilde{w}(T)\, y^{m_k(T)} x^{n+1}/n!.$$

Clearly,

$$a(x)=x^{-1}\left|\frac{\partial g_{(k)}}{\partial y}\right|_{y=1}(x).$$

The function $g_{(k)}=g_{(k)}(x,y)$ satisfies the equation

$$g_{(k)}=x\,(s(g_{(k)})+(y-1)\,s_k g_{(k)}^k/k!).$$

Then

$$a(x) = x\, s'(g)\, a(x) + s_k g^k/k!,$$

$$a(x) = \frac{s_k g^k}{k!\,(1 - xs'(g))}. \tag{11.15}$$

Let $0 < R \leqslant \infty$ be the radius of convergence of $g(x)$. All coefficients of the expansion of $s'(g(x))$ are nonnegative and at least one of them is nonzero. Thus $r(x) = 1 - x\, s'(g(x))$ is decreasing for positive x, $r(0) = 1$, and $r(x) < 0$ for sufficiently large x. This implies that there exists a unique $x_c \in \,]0, R[$ such that

$$1 - x_c\, s'(g(x_c)) = 0. \tag{11.16}$$

Then (11.14) and (11.15) imply that $a(x)$ and $b(x)$ converge for $0 < x < x_c$ and diverge for $x = x_c$. This shows that the series $a(x)$ and $b(x)$ satisfy the condition (a) of Lemma 11.16.

Now we show that Eq. (11.13) correctly defines α. All coefficients of the expansion of $p(t) = s(t) - ts'(t)$ are nonpositive except the constant term $s_0 > 0$. Then, as before, $p(t)$ is decreasing for positive t, $p(0) > 0$, and $p(t) < 0$ for sufficiently large t. Thus $p(t) = 0$ has a unique positive solution $t = \alpha$. Moreover, $\alpha = g(x_c)$. Indeed, by (10.2), $x = g/s(g)$. Thus (11.16) is equivalent to (11.13).

Therefore, by Lemma 11.16, we have

$$P(k) = \lim_{x \to x_c - 0} \frac{a(x)}{b(x)} = \frac{s_k\, g(x_c)^k}{s(g(x_c))\, k!} = \frac{s_k\, \alpha^k}{s(\alpha)\, k!}. \quad \blacksquare$$

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