Chern Forms on Flag Manifolds and Forests

(Extended Abstract)

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Summary

Let $\mathcal{A}_n$ be the ring generated by the Chern 2-forms of n standard hermitian line bundles over the flag manifold $\mathrm{SL}(n,\mathbb{C})/\mathrm{B}$. We prove a conjecture from [3] that the dimension of $\mathcal{A}_n$ is equal to the number of forests on n labelled vertices. We present an explicit construction for a monomial basis in $\mathcal{A}_n$. More generally, the results naturally extend to a wider class of rings, whose bases are labelled by generalized parking functions.

1 Main Results

Let $Fl_n = \mathrm{SL}(n, \mathbb{C})/\mathrm{B}$ be the manifold of complete flags in $\mathbb{C}^n$. The manifold Fl_n comes equipped with a flag of of tautological vector bundles $E_0 \subset E_1 \subset \cdots \subset E_n$ and associated sequence of line bundles $L_i = E_i/E_{i-1}, i = 1, \ldots, n$. The L_i possess natural hermitian structures induced from the standard hermitian metric $\sum z_i \bar{z}_i$ on $\mathbb{C}^n$. For $i = 1, \ldots, n$, we denote by w_i the 2-dimensional Chern form (or curvature form) on Fl_n of the hermitian line bundle L_i. The w_i are also called the curvature forms. They represent the Chern classes $c_1(L_i)$ in the 2-dimensional cohomology of Fl_n. The forms w_i are invariant under the action of the unitary group U_n on Fl_n.

The main purpose of the present paper is to investigate the ring $\mathcal{A}_n$ generated by the forms $w_1, \ldots, w_n$. As an additive group, $\mathcal{A}_n$ is a free abelian group. The ring $\mathcal{A}_n$ is graded: $\mathcal{A}_n = \mathcal{A}_n^0 \oplus \mathcal{A}_n^1 \oplus \mathcal{A}_n^2 \cdots$. The component $\mathcal{A}_n^k$ consists of $2k$-dimensional forms. The cohomology ring $\mathrm{H}^*(Fl_n, \mathbb{Z})$ of the flag manifold is a quotient of the ring $\mathcal{A}_n$, since the former is generated by the Chern classes $c_1(L_i)$.

Recall that a *forest* is a graph without cycles. For a forest F on vertices labelled $1, \ldots, n$, let us construct a tree T by adding a new vertex (root) connected with the maximal vertices in the connected components of F. An *inversion* in F is a pair $1 \le i < j \le n$ such that the vertex labelled j lies on the shortest path in T from the vertex labelled i to the root.

Our primary result is the following statement. Its first part was initially conjectured in [3] and the second part was then guessed by R. Stanley.

Theorem 1 *The dimension of the ring $\mathcal{A}_n$ is equal to the number of forests on n labelled vertices. Moreover, the dimension of a graded component $\mathcal{A}_n^k$ is equal to the number of forests on n labelled vertices with exactly $\binom{n}{2} - k$ inversions.*

To formulate our further results, we need some extra notation. We say that a sequence $a = (a_1, a_2, \ldots, a_n)$ of nonnegative integers is a *strictly parking function* if it satisfies the following two conditions:

1. For $r = 0, 1, \ldots, n$, we have $\#\{i \mid a_i \ge n - r\} \le r$.
2. For $r \in \{1, \ldots, n\}$ such that $\#\{i \mid a_i \ge n-r\} = r$, let $j = \min\{i \mid a_i \ge n-r\}$. Then $a_j = n - r$.

Let P_n be the set of all strictly parking functions. We remark that a sequence a that satisfies the first of these two condition is usually called a parking function.

For a sequence $a = (a_1, \ldots, a_n) \in \mathbb{Z}_+^n$, we denote $w^a = w_1^{a_1} w_2^{a_2} \cdots w_n^{a_n}$.

Theorem 2 *The monomial forms w^a, $a \in P_n$, form a linear basis in the ring $\mathcal{A}_n$.*

Theorem 2 together with the following combinatorial statement imply Theorem 1. Let $|a| = a_1 + a_2 + \ldots + a_n$.

Theorem 3 *The number of elements $a \in P_n$ such that $|a| = k$ is equal to the number of forests on n labelled vertices with exactly $\binom{n}{2} - k$ inversions.*

The proof of this theorem is based on an explicit bijection between forests and strictly parking functions.

A description of the ring $\mathcal{A}_n$ in terms of generators and relations was given in [3]. Let $\mathcal{I}_n$ be the ideal in the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ generated by $2^n - 1$ polynomials of the form

$$(x_{i_1} + \cdots + x_{i_r})^{r(n-r)+1}, \tag{1}$$

where $\{i_1, \ldots, i_r\}$ is any nonempty subset of $\{1, \ldots, n\}$.

Theorem 4 [3] *The ring $\mathcal{A}_n$ is canonically isomorphic, as a graded ring, to the quotient $\mathbb{Z}[x_1, \ldots, x_n]/\mathcal{I}_n$. The isomorphism is given by sending the generators w_i of $\mathcal{A}_n$ to the corresponding x_i.*

Let us also define the ideal $\mathcal{J}_n$ generated by $2^n - 1$ monomials of the form

$$(x_{i_1} \cdots x_{i_r})^{n-r} x_{i_1}, \tag{2}$$

where $1 \leq i_1 < \cdots < i_r \leq n$ is any nonempty subset of $\{1, \ldots, n\}$. Finally, let $\mathcal{B}_n = \mathbb{Z}[x_1, \ldots, x_n]/\mathcal{J}_n$.

Lemma 5 *The monomials x^a, $a \in P_n$, form a linear basis of the ring $\mathcal{B}_n$.*

Our proof of Theorem 2 is based on a construction of a sequence of rings that interpolates between $\mathcal{A}_n$ and $\mathcal{B}_n$. We then prove by induction an analogous statement for all rings in this sequence. Lemma 5 provides the base of the induction. In particular, we deduce the following statement.

Corollary 6 *The rings $\mathcal{A}_n$ and $\mathcal{B}_n$ have the same Hilbert series.*

Let us also consider a more general ring $\mathcal{A}_{nk}$, $1 \leq k \leq n$, generated by the first k Chern forms $w_1, w_2, \ldots, w_k$ (see [3]). It is not hard to show that $\mathcal{A}_{nk}$ is isomorphic to the ring generated by any k-tuple of the Chern forms $w_{j_1}, \ldots, w_{j_k}$. It is clear that $\mathcal{A}_{nn} = \mathcal{A}_n$ and $\mathcal{A}_{n\,n-1} = \mathcal{A}_n$. (The latter is due to the identity $w_1 + \cdots + w_n = 0$.)

Theorem 7 *The dimension of the ring A_{nk} is equal to the number of forests on $2n - k$ labelled vertices such that the first $n - k$ vertices belong to $n - k$ different connected components.*

Let $f_n(q)$ be the generating function $f_n(q) = \sum_F q^{d(F)}$, where the sum is over all forests F on $n + 1$ labelled vertices and $d(F)$ is the degree of the first vertex, i.e. the number of edges that emanate from it. The previous theorem is equivalent to the following statement.

Corollary 8 *The dimension of $\mathcal{A}_{nk}$ is equal to $f_n(n-k)$.*

The ring $\mathcal{A}_{nk}$ is canonically isomorphic to the quotient of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ modulo the ideal generated by $2^k - 1$ polynomials of the form (1), where $i_1, \ldots, i_r \leq k$.

Analogously, let $\mathcal{B}_{nk}$ be the quotient of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ modulo the ideal generated by $2^k - 1$ polynomials of the form (2), where $i_1, \ldots, i_r \leq k$. Clearly, both $\mathcal{A}_{nk}$ and $\mathcal{B}_{nk}$ are graded rings. Corollary 6 can be generalized as follows.

Theorem 9 *The rings $\mathcal{A}_{nk}$ and $\mathcal{B}_{nk}$ have the same Hilbert series.*

2 Remarks and Open Problems

A natural open problem is to extend the results to a partial flag manifold SL_n/P, where P is a parabolic subgroup.

Let $E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_r = \mathbb{C}^n$, $\dim E_k = i_1 + i_2 + \ldots + i_k$, be the tautological bundles over SL_n/P and denote by $L_k(P)$ the quotients E_k/E_{k-1}, $k = 1, \ldots, r$. The standard hermitian form in $\mathbb{C}^n$ induces a hermitian structure on each $L_k(P)$ and gives rise to the Chern forms $w_k^1, w_k^2, \ldots, w_k^{i_k}$ of dimensions $2, 4, \ldots, 2i_k$. Explicit formulas for these forms can be found in [1].

Let $\mathcal{A}_n(P)$ be the ring generated by the forms w_k^j. It is a natural extension of the ring $\mathcal{A}_n$. The cohomology ring $\mathrm{H}^*(\mathrm{SL}_n/P, \mathbb{Z})$ is a quotient of the ring $\mathcal{A}_n(P)$.

Problem 10 *Investigate the ring $\mathcal{A}_n(P)$. Find a description for this ring in terms of generators and relations. Find the dimension of the ring $\mathcal{A}_n(P)$ and its Hilbert series.*

Remark 11 In the case of the Grassmannian $G_{n,k}$ (i.e. when P is a maximal parabolic subgroup) the ring $\mathcal{A}_n(P)$ coincides with the cohomology ring $\mathrm{H}^*(G_{n,k})$, cf. [1]. In particular, its dimension is equal to $\binom{n}{k}$.

The ring $\mathcal{A}_n$ is related to the ring of all U_n-invariant forms, which recently appeared in [4, 5]. The latter ring has an additive basis that consists of Eulerian digraphs on n labelled vertices.

Problem 12 *Find an explicit description in terms of generators and relations for the ring of all U_n-invariant forms.*

There is an analogy between the cohomology ring $\mathrm{H}^*(Fl_n, \mathbb{Z})$ of the flag manifold and the Orlik-Solomon algebra OS_n of the braid hyperplane arrangement, which consists of all hyperplanes $\{x_i = x_j\}$, $1 \le i < j \le n$. For example, the dimensions of these two algebras are equal to each other.

Let $\widetilde{\mathrm{OS}}_n$ be the Orlik-Solomon algebra of a generic affine deformation of the braid arrangement, which consists of the hyperplanes $\{x_i = x_j + \epsilon_{ij}\}$, where ϵ_{ij} are generic real numbers. The analogy between $\mathrm{H}^*(Fl_n, \mathbb{Z})$ and OS_n seems to extend to the ring $\mathcal{A}_n$ on one side and $\widetilde{\mathrm{OS}}_n$ on the other side. For example, the dimension of $\widetilde{\mathrm{OS}}_n$ is equal to the number of forests on n labelled vertices, see [2]. It would be interesting to clarify and study this relationship.

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