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GENERALIZED PARKING FUNCTIONS, DESCENT NUMBERS, AND CHAIN POLYTOPES OF RIBBON POSETS

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Abstract. We consider the inversion enumerator $I_n(q)$, which counts labeled trees or, equivalently, parking functions. This polynomial has a natural extension to generalized parking functions. Substituting $q = -1$ into this generalized polynomial produces the number of permutations with a certain descent set. In the classical case, this result implies the formula $I_n(-1) = E_n$, the number of alternating permutations. We give a combinatorial proof of these formulas based on the involution principle. We also give a geometric interpretation of these identities in terms of volumes of generalized chain polytopes of ribbon posets. The volume of such a polytope is given by a sum over generalized parking functions, which is similar to an expression for the volume of the parking function polytope of Pitman and Stanley.

1. introduction

Let $\mathcal{T}_n$ be the set of all trees on vertices labeled 0, 1, 2, ..., n rooted at 0. For $T \in \mathcal{T}_n$, let $\mathrm{inv}(T)$ be the number of pairs $i > j$ such that j is a descendant of i in T. Define the n-th *inversion enumerator* to be the polynomial

$$I_n(q) := \sum_{T \in \mathcal{T}_n} q^{\mathrm{inv}(T)}.$$

Another way to define this polynomial is via parking functions. A sequence $(b_1, b_2, \ldots, b_n)$ of positive integers is a *parking function of length* n if for all $1 \le j \le n$, at least j of the b_i's do not exceed j. A classical bijection of Kreweras [3] establishes a correspondence between trees in $\mathcal{T}_n$ with k inversions and parking functions of length n whose components add up to $\binom{n+1}{2} - k$. Hence we can write

$$I_n(q) = \sum_{(b_1,\ldots,b_n) \in \mathcal{P}_n} q^{\binom{n+1}{2} - b_1 - b_2 - \cdots - b_n},$$

or

$$\sum_{(b_1,\ldots,b_n) \in \mathcal{P}_n} q^{b_1 + b_2 + \cdots + b_n - n} = q^{\binom{n}{2}} \cdot I_n(q^{-1}),$$

where $\mathcal{P}_n$ is the set of all parking functions of length n. Cayley's formula states that $|\mathcal{T}_n| = |\mathcal{P}_n| = (n+1)^{n-1}$, hence $I_n(1) = (n+1)^{n-1}$.

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Here we focus on the formula

$$I_n(-1) = E_n, \tag{1}$$

where E_n is the n-th *Euler number*, most commonly defined as the number of permutations $\sigma_1\sigma_2\dots\sigma_n$ of $[n] = \{1, 2, \dots, n\}$ such that $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \dots$, called *alternating* permutations. This formula can be obtained by deriving a closed form expression for the generating function $\sum_{n\geq 0} I_n(q)x^n/n!$ and showing that setting $q = -1$ yields $\tan x + \sec x = \sum_{n\geq 0} E_n x^n/n!$ (see the paper [1] by Gessel or Exercises 3.3.48–49 in [2]). A direct combinatorial proof was given by Pansiot [6]. In this paper we give two other ways to prove this fact, one of which, presented in Section 2, is an involution argument on the set of all but E_n members of $\mathcal{P}_n$. This involution is a special case of a more general argument valid for a broader version of parking functions, which we now describe.

Let $\vec{a} = (a_1, a_2, \dots, a_n)$ be a *non-decreasing* sequence of positive integers. Let us call a sequence $(b_1, b_2, \dots, b_n)$ of positive integers an *$\vec{a}$-parking function* if the increasing rearrangement $b'_1 \leq b'_2 \leq \dots \leq b'_n$ of this sequence satisfies $b'_i \leq a_i$ for all i. Note that $(1, 2, \dots, n)$-parking functions are the regular parking functions of length n. These $\vec{a}$-parking functions are $(a_1, a_2 - a_1, a_3 - a_2, \dots)$-parking functions in the original notation of Yan [10], but the present definition is consistent with later literature, such as the paper [4] of Kung and Yan. Let $\mathcal{P}_{\vec{a}}$ be the set of all $\vec{a}$-parking functions, and define

$$I_{\vec{a}}(q) := \sum_{(b_1,\dots,b_n)\in\mathcal{P}_{\vec{a}}} q^{b_1+b_2+\dots+b_n-n} \tag{2}$$

(this is the *sum enumerator* studied in [4]). For a subset $S \subseteq [n-1]$, let $\beta_n(S)$ be the number of permutations of size n with descent set S. In Section 2 (Theorem 2.4) we prove the following generalization of (1):

$$|I_{\vec{a}}(-1)| = \begin{cases} 0, & \text{if } a_1 \text{ is even;} \\ \beta_n(S), & \text{if } a_1 \text{ is odd,} \end{cases} \tag{3}$$

where

$$S = \Big\{ i \in [n-1] \;\Big|\; a_{i+1} \text{ is odd} \Big\}. \tag{4}$$

Indeed, for $\vec{a} = (1, 2, \dots, n)$ we have $S = \{2, 4, 6, \dots\} \cap [n-1]$, so that $\beta_n(S)$ counts alternating permutations of size n. The formula (3) arises in a more sophisticated algebraic context in the paper [5] of Pak and Postnikov.

In Section 3 we obtain a geometric interpretation of these results by considering generalized chain polytopes of ribbon posets. Given a subset $S \subseteq \{2, 3, \dots, n-1\}$, define $u_S = u_1u_2\dots u_{n-1}$ to be the monomial in non-commuting formal variables $\mathbf{a}$ and $\mathbf{b}$ with $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. Let $\mathrm{c}(S)$ be the composition $(1, \delta_1, \delta_2, \dots, \delta_{k-1})$ of n, where the δ_i's are defined by $u_S = \mathbf{a}^{\delta_1}\,\mathbf{b}^{\delta_2}\,\mathbf{a}^{\delta_3}\,\mathbf{b}^{\delta_4}\dots$. For example, for $n = 7$ and $S = \{2, 3, 4\}$ we have $u_S = \mathbf{a\,b\,b\,b\,a\,a} = \mathbf{a}\,\mathbf{b}^3\,\mathbf{a}^2$, so $\mathrm{c}(S) = (1, 1, 3, 2)$. Now define the polytope $\mathcal{Z}_S(d_1, d_2, \dots, d_k)$, where $0 < d_1 \leq d_2 \leq \dots \leq d_k$ are real numbers, to be the set of all points $(x_1, x_2, \dots, x_n)$ satisfying the inequalities $x_j \geq 0$ for $j \in [n]$, $x_1 \leq d_1$, and

$$x_{\delta_1+\delta_2+\dots+\delta_{i-1}+1} + x_{\delta_1+\delta_2+\dots+\delta_{i-1}+2} + \dots + x_{\delta_1+\delta_2+\dots+\delta_i+1} \leq d_{i+1}$$

for $1 \le i \le k-1$. Thus to the above example corresponds the polytope $\mathcal{Z}_S(d_1, d_2, d_3, d_4)$ in $\mathbb{R}^7_{\ge 0}$ defined by

$$x_1 \le d_1;$$
$$x_1 + x_2 \le d_2;$$
$$x_2 + x_3 + x_4 + x_5 \le d_3;$$
$$x_5 + x_6 + x_7 \le d_4.$$

We require $1 \notin S$ here to ensure that $\delta_1 \neq 0$, but there is no essential loss of generality because the chain polytope of the poset Z_S is defined by the same relations as $Z_{[n-1]-S}$.

For a poset P on n elements, the *chain polytope* $\mathcal{C}(P)$ is the set of points $(x_1, x_2, \dots, x_n)$ of the unit hypercube $[0,1]^n$ satisfying the inequalities $x_{p_1} + x_{p_2} + \cdots + x_{p_\ell} \le 1$ for every chain $p_1 < p_2 < \cdots < p_\ell$ in P; see [9]. Hence $\mathcal{Z}_S(1, 1, \dots)$ is the chain polytope of the *ribbon poset* Z_S, which is the poset on $\{z_1, z_2, \dots, z_n\}$ generated by the cover relations $z_i \cdot\!> z_{i+1}$ if $i \in S$ and $z_i <\!\cdot\, z_{i+1}$ if $i \notin S$. The volume of $\mathcal{C}(P)$ equals $1/n!$ times the number of linear extensions of P, which in the case $P = Z_S$ naturally correspond to permutations of size n with descent set S. Our main result concerning the polytope $\mathcal{Z}_S$ is a formula for its volume. For a composition γ of n, let K_γ denote the set of *weak* compositions $\alpha = (\alpha_1, \alpha_2, \dots)$ of n, meaning that α can have parts equal to 0, such that $\alpha_1 + \alpha_2 + \cdots + \alpha_i \ge \gamma_1 + \gamma_2 + \cdots + \gamma_i$ for all i. Define $\vec{a}(\gamma)$ to be the sequence consisting of γ_1 1's, followed by γ_2 2's, followed by γ_3 3's, and so on. Then α is in K_γ if and only if α is the content of an $\vec{a}(\gamma)$-parking function. (The *content* of a parking function is the composition whose i-th part is the number of components of the parking function equal to i.) In Section 3 (Theorem 3.1) we show that

$$n! \cdot \mathrm{Vol}\big(\mathcal{Z}_S(d_1, d_2, \dots, d_k)\big) = \left| \sum_{(b_1,\dots,b_n) \in \mathcal{P}_{\vec{a}(\mathrm{c}(S))}} \prod_{i=1}^{n} (-1)^{b_i} d_{b_i} \right| = \tag{5}$$

$$= \left| \sum_{\alpha \in \mathrm{K}_{\mathrm{c}(S)}} \binom{n}{\alpha} \cdot (-1)^{\alpha_1 + \alpha_3 + \alpha_5 + \dots} \cdot d_1^{\alpha_1} d_2^{\alpha_2} \dots d_k^{\alpha_k} \right|,$$

where $\binom{n}{\alpha} = \frac{n!}{\alpha_1! \alpha_2! \dots \alpha_k!}$ and $k = \ell(\mathrm{c}(S))$ is the number of parts of $\mathrm{c}(S)$. For example, for $n = 5$ and $S = \{4\}$, we have

$$\mathrm{K}_{\mathrm{c}(S)} = \mathrm{K}_{(1,3,1)} = \{(1,3,1), (1,4,0), (2,2,1), (2,3,0), (3,1,1), (3,2,0), (4,0,1), (4,1,0), (5,0,0)\},$$

so we get from (5) that

$$5! \cdot \mathrm{Vol}\big(\mathcal{Z}_{\{1\}}(d_1, d_2, d_3)\big) = 20 d_1 d_2^3 d_3 - 5 d_1 d_2^4 - 30 d_1^2 d_2^2 d_3 + 10 d_1^2 d_2^3 + 20 d_1^3 d_2 d_3 - 10 d_1^3 d_2^2 - 5 d_1^4 d_3 + 5 d_1^4 d_2 - d_1^5.$$

Setting $d_i = q^{i-1}$ in (5), where we take $q \ge 1$ so that the sequence $d_1, d_2, \dots$ is non-decreasing, and recalling (2) gives

$$n! \cdot \mathrm{Vol}(1, q, q^2, \dots) = \left| \sum_{(b_1,\dots,b_n) \in \mathcal{P}_{\vec{a}(\mathrm{c}(S))}} (-q)^{b_1 + b_2 + \cdots + b_n - n} \right| = \left| I_{\vec{a}(\mathrm{c}(S))}(-q) \right|.$$

Specializing further by setting $q = 1$ yields the identity

$$\left|I_{\vec{a}(\mathrm{c}(S))}(-1)\right| = \beta_n(S).$$

Observe that this identity is consistent with (3). Indeed, the first part of $\mathrm{c}(S)$ is positive, and thus the first element of $\vec{a}(\mathrm{c}(S)) = (a_1, a_2, \ldots, a_n)$ is 1, i.e. an odd number. Comparing the sequence $(a_1, a_2, \ldots, a_n)$ with the letters of the word $\mathbf{b}\, u_S$ we see that $a_{i+1} = a_i + 1$ if the corresponding letters of u_S are different, and $a_{i+1} = a_i$ otherwise; in other words, changes of parity between consecutive elements of $(a_1, a_2, \ldots, a_n)$ correspond to letter changes in the word $\mathbf{b}\, u_S$. (The extra $\mathbf{b}$ in front corresponds to the first part 1 of $\mathrm{c}(S)$.) For example, for $n = 7$ and $S = \{2, 3, 4\}$, we have $\mathrm{c}(S) = (1, 1, 3, 2)$, $\mathbf{b}\, u_S = \mathbf{b}\,\mathbf{a}\,\mathbf{b}^3\,\mathbf{a}^2$, and $\vec{a}(\mathrm{c}(S)) = (1, 2, 3, 3, 3, 4, 4)$. It follows that the subset constructed from $\vec{a}(\mathrm{c}(S))$ according to the rule (4) of an earlier result is S, so the results agree.

Considering once more the case $S = \{2, 4, 6, \ldots\} \cap [n-1]$, let us point out the similarity between the formula (5) and the expression that Pitman and Stanley [7] derive for the volume of their *parking function polytope*. This polytope, which we denote by $\Pi_n(c_1, c_2, \ldots, c_n)$, is defined by the inequalities $x_i \geq 0$ and

$$x_1 + x_2 + \cdots + x_i \leq c_1 + c_2 + \cdots + c_i$$

for all $i \in [n]$. The volume-preserving change of coordinates $y_i = c_n + c_{n-1} + \cdots + c_{n+1-i} - (x_1 + x_2 + \cdots + x_i)$ transforms the defining relations above into $y_i \geq 0$ for $i \in [n]$, $y_1 \leq c_1$, and $y_i - y_{i+1} \geq c_i$ for $i \in [n-1]$, and these new relations look much like the ones defining $\mathcal{Z}_{\{2,4,6,\ldots\}}(c_1, c_2, \ldots, c_n)$: in essence we have here a difference instead of a sum. This similarity somewhat explains the close resemblance of the volume formulas for the two polytopes, as for $\Pi_n(c_1, c_2, \ldots, c_n)$ we have

$$n! \cdot \mathrm{Vol}(\Pi_n(c_1, c_2, \ldots, c_n)) = \sum_{(b_1, \ldots, b_n) \in \mathcal{P}_n} \prod_{i=1}^{n} c_{b_i} = \sum_{\alpha \in \mathrm{K}_{1^n}} \prod_{i=1}^{n} \binom{n}{\alpha} c_i^{\alpha_i}.$$

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2. An involution on $\vec{a}$-parking functions

The idea of the combinatorial argument presented in this section was first discovered by the second author and Igor Pak during their work on [5].

Let $\vec{a} = (a_1, a_2, \ldots, a_n)$ be a non-decreasing sequence of positive integers. As a first step in the construction of our involution on $\vec{a}$-parking functions, let $Y_{\vec{a}}$ be the Young diagram whose column lengths from left to right are a_n, a_{n-1}, $\ldots$, a_1. Define a *horizontal strip* H inside $Y_{\vec{a}}$ to be a set of cells of $Y_{\vec{a}}$ satisfying the following conditions:

(i) for every $i \in [n]$, the set H contains exactly one cell σ_i from column i (we number the columns 1, 2, $\ldots$, n from left to right);

(ii) for $i < j$, the cell σ_i is in the same or in a lower row than the cell σ_j.

For a horizontal strip H, let us call a filling of the cells of H with numbers 1, 2, ..., n *proper* if the numbers in row i are in increasing order if i is odd, or in decreasing order if i is even (we number the rows 1, 2, ..., from top to bottom). Let $\mathcal{H}_{\vec{a}}$ denote the set of all properly filled horizontal strips inside $Y_{\vec{a}}$.

For an $\vec{a}$-parking function $\vec{b} = (b_1, b_2, \ldots, b_n)$, define $H(\vec{b}) \in \mathcal{H}_{\vec{a}}$ in the following way. Let $I_j \subseteq [n]$ be the set of indices i such that $b_i = j$. Construct the filled horizontal strip $H(\vec{b})$ by first writing the elements of I_1 in *increasing* order in the $|I_1|$ *rightmost* columns in row 1 of $Y_{\vec{a}}$, then writing the elements of I_2 in *decreasing* order in the next $|I_2|$ columns from the right in row 2, then writing the elements of I_3 in *increasing* order in the next $|I_3|$ columns from the right in row 3, and so on, alternating between increasing and decreasing order.

Lemma 2.1. *A sequence $\vec{b}$ is an $\vec{a}$-parking function if and only if the horizontal strip $H(\vec{b})$ produced in the above construction fits into $Y_{\vec{a}}$.*

Proof. Let $(b'_1, b'_2, \ldots, b'_n)$ be the increasing rearrangement of $\vec{b}$. Then the cell of $H(\vec{b})$ in the i-th column from the right belongs to row b'_i. Thus the condition of the lemma is equivalent to $b'_i \leq a_i$ for all i. □

Clearly, the filling of $H(\vec{b})$ from the above procedure is proper, and hence $\vec{b} \leftrightarrow H(\vec{b})$ is a bijection between $\mathcal{P}_{\vec{a}}$ and $\mathcal{H}_{\vec{a}}$. We will describe our involution on $\vec{a}$-parking functions in terms of the corresponding filled horizontal strips.

Let H be a properly filled horizontal strip in $\mathcal{H}_{\vec{a}}$. In what follows we use σ_i to refer to both the cell of H in column i and to the number written in it. Let $r(\sigma_i)$ be the number of the row containing σ_i. We begin by defining the *assigned direction* for each of the σ_i's. For the purpose of this definition it is convenient to imagine that H contains a cell labeled $\sigma_{n+1} = n+1$ in row 1 and column $n+1$, that is, just outside the first row $Y_{\vec{a}}$ on the right. Let

$$\epsilon_i = \operatorname{sgn}(\sigma_i - \sigma_{i+1})(-1)^{r(\sigma_i)},$$

where $\operatorname{sgn}(x)$ equals 1 if $x > 0$, or -1 if $x < 0$. Define the assigned direction for σ_i to be *up* if $\epsilon_i = -1$ and *down* if $\epsilon_i = 1$.

Let us say that σ_i is *moveable down* if the assigned direction for σ_i is down, σ_i is not the bottom cell of column i, and moving σ_i to the cell immediately below it would not violate the rules of a properly filled horizontal strip. The latter condition prohibits moving σ_i down if there is another cell of H immediately to the left of it, or if moving σ_i down by one row would violate the rule for the relative order of the numbers in row $r(\sigma_i) + 1$. Let us say that σ_i is *moveable up* if the assigned direction for σ_i is up. Note that we do not need any complicated conditions in this case: if σ_i has another cell of H immediately to its right, or if σ_i is in the top row, then the assigned direction for σ_i is down.

It is a good time to consider an example. Figure 1 shows the diagram $Y_{\vec{a}}$ and a properly filled horizontal strip $H \in \mathcal{H}_{\vec{a}}$ for $\vec{a} = (3, 3, 6, 7, 7, 7, 8)$. The horizontal strip shown is $H(\vec{b})$, where $\vec{b} = (5, 7, 2, 5, 1, 5, 2) \in \mathcal{P}_{\vec{a}}$. Moveable cells are equipped with arrows pointing in their assigned directions. Note that the assigned direction for $\sigma_7 = 5$ is down because of the "imaginary" $\sigma_8 = 8$ next to it, but it is not moveable down because the numbers 7, 3, 5 in row 2 would not be ordered properly. The assigned direction for $\sigma_3 = 4$ and $\sigma_4 = 6$ is also down, but these cells are not moveable down because moving them down would not produce a horizontal strip.

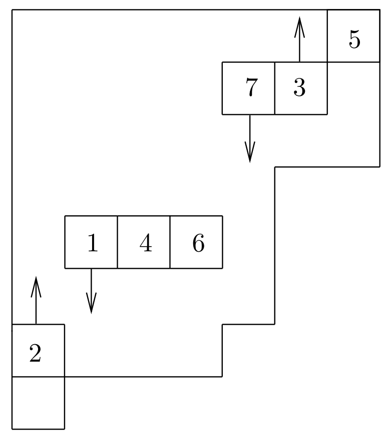

FIGURE 1. A properly filled horizontal strip with assigned directions for its cells

The validity of the involution we are about to define depends on the following simple but crucial fact.

Lemma 2.2. *Let σ_i be a moveable cell of $H \in \mathcal{H}_{\vec{a}}$, and let $H' \in \mathcal{H}_{\vec{a}}$ be the horizontal strip obtained from H by moving σ_i by one row in its assigned direction. Then*

(a) σ_i is moveable in the opposite direction in H';

(b) if $j \neq i$ and σ_j is moveable in H, then σ_j is moveable in the same direction in H';

(c) if σ_j is not moveable in H, then it is not moveable in H'.

Proof. Follows by inspection of the moving rules. □

Let $\tilde{\mathcal{H}}_{\vec{a}}$ be the set of all $H \in \mathcal{H}_{\vec{a}}$ such that H has at least one moveable cell (up or down). Define the map $\psi : \tilde{\mathcal{H}}_{\vec{a}} \to \tilde{\mathcal{H}}_{\vec{a}}$ as follows: given $H \in \tilde{\mathcal{H}}_{\vec{a}}$, let $\psi(H)$ be the horizontal strip obtained from H by choosing the *rightmost* moveable cell of H and moving it by one row in its assigned direction. In view of Lemma 2.2, ψ is an involution.

For $\vec{b} = (b_1, b_2, \ldots, b_n) \in \mathcal{P}_{\vec{a}}$ and $H = H(\vec{b})$, define $s(\vec{b}) = s(H) := b_1 + b_2 + \cdots + b_n - n$. Observe that $s(\vec{b})$ is the number of cells of $Y_{\vec{a}}$ that lie above one of the cells of $H(\vec{b})$. In the example in Figure 1, we have $s(\vec{b}) = s(H) = 6 + 4 + 4 + 4 + 1 + 1 + 0 = 20$. Clearly,

$s(H) = s(\psi(H)) \pm 1$ for $H \in \tilde{\mathcal{H}}_{\vec{a}}$. Since ψ is fixed-point free, it follows that

$$\sum_{H \in \tilde{\mathcal{H}}_{\vec{a}}} (-1)^{s(H)} = 0,$$

and that

$$I_{\vec{a}}(-1) = \sum_{H \in \tilde{\mathcal{H}}_{\vec{a}} - \mathcal{H}_{\vec{a}}} (-1)^{s(H)} \tag{6}$$

(cf. (2)). It remains to examine the members of $\mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}}$ in order to evaluate the right hand side of (6).

Lemma 2.3. *If a_1 is even, then $\mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}} = \varnothing$. If a_1 is odd, then $\mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}}$ consists of all filled horizontal strips H in $Y_{\vec{a}}$ such that the cell σ_i of H is at the bottom of column i for all $i \in [n]$, and the permutation $\sigma_n \dots \sigma_2\sigma_1$ has descent set*

$$S = \left\{ i \in [n-1] \;\middle|\; a_{i+1} \textit{ is odd} \right\}$$

(cf. (4)).

Proof. Let $H \in \mathcal{H}_{\vec{a}}$, and consider the cell σ_n in H. Since $\sigma_n < \sigma_{n+1} = n+1$, it follows that σ_n is moveable up if $r(\sigma_n)$ is even, or moveable down if $r(\sigma_n)$ is odd, unless in the latter case σ_n is at the bottom of column n. Thus if a_1 is even, that is, if the rightmost column of $Y_{\vec{a}}$ has even height, σ_n is always moveable and $\mathcal{H}_{\vec{a}} = \tilde{\mathcal{H}}_{\vec{a}}$.

Suppose that a_1 is odd, and let $H \in \mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}}$. Then no cells of H are moveable, and hence the assigned direction for every cell is down.

First, let us show that all cells of H are at the bottom of their respective columns. Suppose it is not so, and choose the *leftmost* cell σ_i of H such that the cell immediately below it is in $Y_{\vec{a}}$. Our choice guarantees that σ_i does not have another cell of H immediately to its left, so the only way σ_i can be not moveable down is if σ_{i-1} is in column $i-1$ one row below σ_i and $\operatorname{sgn}(\sigma_{i-1} - \sigma_i) = -(-1)^{r(\sigma_{i-1})}$. But in this case the assigned direction for σ_{i-1} is up — a contradiction.

Now let us compute the descent set of $\sigma_n\sigma_{n-1}\dots\sigma_1$. We just proved that $r(\sigma_{n+1-i}) = a_i$ for all $i \in [n]$. For $i \in [n-1]$, we have

$$1 = \epsilon_{n-i} = \operatorname{sgn}(\sigma_{n-i} - \sigma_{n+1-i})(-1)^{r(\sigma_{n-i})},$$

and hence $\sigma_{n+1-i} > \sigma_{n-i}$, that is, $\sigma_n\sigma_{n-1}\cdots\sigma_1$ has a descent in position i, if and only if $r(\sigma_{n-i}) = a_{i+1}$ is odd. □

Note that from Lemma 2.3 it follows that for all $\mathcal{H}_{\vec{a}} - \tilde{\mathcal{H}}_{\vec{a}}$, the value of $s(H)$ is the same, namely, $a_1 + a_2 + \cdots + a_n - n$. Combining with (6), we obtain the following theorem (cf. (3)).

Theorem 2.4 (cf. [5]). *For a non-decreasing sequence $\vec{a} = (a_1, a_2, \ldots, a_n)$ of positive integers, we have*

$$I_{\vec{a}}(-1) = \begin{cases} 0, & \text{if } a_1 \text{ is even;} \\ (-1)^{a_1+\cdots+a_n-n} \cdot \beta_n(S), & \text{if } a_1 \text{ is odd,} \end{cases}$$

where $S = \Big\{ i \in [n-1] \;\Big|\; a_{i+1} \text{ is odd} \Big\}$, and $\beta_n(S)$ is the number of permutations of $[n]$ with descent set S.

3. Generalized chain polytopes of ribbon posets

In this section we prove the formula (5) of Section 1:

Theorem 3.1. *For a positive integer n, a subset $S \subseteq [n-1]$ such that*

$$\mathrm{c}(S) = (1, \delta_1, \delta_2, \ldots, \delta_{k-1}),$$

and a sequence $0 < d_1 \le d_2 \le \cdots \le d_k$ of real numbers, we have

$$\begin{aligned} n! \cdot \mathrm{Vol}\big(\mathcal{Z}_S(d_1, d_2, \ldots, d_k)\big) &= (-1)^{1+\delta_2+\delta_4+\cdots} \sum_{(b_1,\ldots,b_n)\in\mathcal{P}_{\vec{a}(\mathrm{c}(S))}} \prod_{i=1}^{n} (-1)^{b_i} d_{b_i} \\ &= (-1)^{1+\delta_2+\delta_4+\cdots} \sum_{\alpha \in \mathrm{K}_{\mathrm{c}(S)}} \binom{n}{\alpha} \cdot (-1)^{\alpha_1+\alpha_3+\alpha_5+\cdots} \cdot d_1^{\alpha_1} \cdots d_k^{\alpha_k}. \end{aligned} \tag{7}$$

Proof. First, note that the expression in the right hand side of (7) is obtained from the middle one by grouping together the terms corresponding to all $\binom{n}{\alpha}$ $\vec{a}$-parking functions of content α; each of these terms equals

$$(-1)^{\alpha_1+2\alpha_2+3\alpha_3+\cdots} \cdot d_1^{\alpha_1} \cdots d_k^{\alpha_k} = (-1)^{\alpha_1+\alpha_3+\alpha_5+\cdots} \cdot d_1^{\alpha_1} \cdots d_k^{\alpha_k}.$$

In what follows we prove the equality between the left and the right hand sides of (7). For $i \in [k]$, let $\rho_i = 1 + \delta_1 + \delta_2 + \cdots + \delta_{i-1}$. The volume of $\mathcal{Z}_S(d_1, d_2, \ldots, d_k)$ can be expressed as the following iterated integral:

$$\begin{aligned} &\int_0^{d_1} \int_0^{d_2-x_1} \int_0^{d_2-x_1-x_2} \cdots \int_0^{d_2-x_1-x_2-\cdots-x_{\rho_2-1}} \\ &\int_0^{d_3-x_{\rho_2}} \int_0^{d_3-x_{\rho_2}-x_{\rho_2+1}} \cdots \int_0^{d_3-x_{\rho_2}-x_{\rho_2+1}-\cdots-x_{\rho_3-1}} \\ &\cdots \\ &\int_0^{d_k-x_{\rho_{k-1}}} \int_0^{d_k-x_{\rho_{k-1}}-x_{\rho_{k-1}+1}} \cdots \int_0^{d_k-x_{\rho_{k-1}}-x_{\rho_{k-1}+1}-\cdots-x_{\rho_k-1}} dx_n\, dx_{n-1} \cdots dx_1 \end{aligned} \tag{8}$$

(Similar integral formulas appear in [4] and in [8, Sec. 18].) Note that the assumption $d_1 \le d_2 \le \cdots \le d_k$ validates the upper limits of those integrals taken with respect to variables $x_2, x_{\rho_2+1}, x_{\rho_3+1}, \ldots, x_{\rho_{k-1}+1}$: for $2 \le i \le k-1$, the condition

$$x_{\rho_i+1} \le d_i - x_{\rho_{i-1}+1} - x_{\rho_{i-1}+2} - \cdots - x_{\rho_i}$$

implies that

$$d_{i+1} - x_{\rho_i+1} \geq 0,$$

and $x_1 \leq d_1$ implies $d_2 - x_1 \geq 0$.

For $\ell \in [n]$, let J_ℓ denote the evaluation of the $n+1-\ell$ inside integrals of (8), that is, the integrals with respect to the variables $x_n, x_{n-1}, \ldots, x_\ell$.

Lemma 3.2. *For $i \in [k]$, we have*

$$\begin{aligned} J_{\rho_i+1} &= (-1)^{\delta_{i+1}+\delta_{i+3}+\delta_{i+5}+\cdots} \\ &\quad \cdot \sum_{\alpha \in \mathrm{K}_{(0,\delta_i,\delta_{i+1},\ldots,\delta_{k-1})}} (-1)^{\alpha_1+\alpha_3+\alpha_5+\cdots} \cdot \frac{1}{\alpha_1!\ \alpha_2!\ \cdots} \cdot x_{\rho_i}^{\alpha_1} d_{i+1}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots . \end{aligned} \tag{9}$$

Proof. We prove the lemma by induction on i, starting with the trivial base case of $i = k$, in which we have $J_{\rho_k+1} = J_{n+1} = 1$. Now suppose the claim is true for some i. By straightforward iterated integration one can show that for non-negative integers r and s,

$$\int_0^a \int_0^{a-y_r} \int_0^{a-y_r-y_{r-1}} \cdots \int_0^{a-y_r-y_{r-1}-\cdots-y_2} y_1^s \ dy_1 \ \cdots \ d_{y_{r-1}} \ d_{y_r} = \frac{s!\ a^{r+s}}{(r+s)!}. \tag{10}$$

Using (10) to integrate the term of (9) corresponding to a particular $\alpha \in \mathrm{K}_{(0,\delta_i,\delta_{i+1},\ldots,\delta_{k-1})}$, we get

$$\begin{aligned} &\int_0^{d_i - x_{\rho_{i-1}}} \int_0^{d_i - x_{\rho_{i-1}} - x_{\rho_{i-1}+1}} \cdots \int_0^{d_i - x_{\rho_{i-1}} - x_{\rho_{i-1}+1} - \cdots - x_{\rho_i - 1}} \\ &\qquad (-1)^{\alpha_1+\alpha_3+\cdots} \cdot \frac{1}{\alpha_1!\ \alpha_2!\ \cdots} \cdot x_{\rho_i}^{\alpha_1} d_{i+1}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots \ dx_{\rho_i} \ \cdots \ dx_{\rho_{i-1}+1} \\ &= (-1)^{\alpha_1+\alpha_3+\cdots} \cdot \frac{1}{\alpha_1!\ \alpha_2!\ \cdots} \cdot \frac{\alpha_1!\ (d_i - x_{\rho_{i-1}})^{\delta_{i-1}+\alpha_1}}{(\delta_{i-1}+\alpha_1)!} \cdot d_{i+1}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots \\ &= (-1)^{\alpha_1+\alpha_3+\cdots} \cdot \sum_{\substack{j,m \geq 0\ :\\ j+m=\delta_{i-1}+\alpha_1}} \frac{1}{\alpha_2!\ \alpha_3! \cdots} \cdot (-1)^j \cdot \frac{x_{\rho_{i-1}}^j d_i^m}{j!\ m!} \cdot d_{i+1}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots \end{aligned}$$

$$= \sum_{\substack{j,m\ :\ j+m=\delta_{i-1}+\alpha_1, \\ (j,m,\alpha_2,\alpha_3,\ldots) \in \\ \mathrm{K}_{(0,\delta_{i-1},\delta_i,\ldots)}}} (-1)^{j+\alpha_1+\alpha_3+\cdots} \cdot \frac{1}{j!\ m!\ \alpha_2!\ \alpha_3!\ \cdots} \cdot x_{\rho_{i-1}}^j d_i^m d_{i+1}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots \tag{11}$$

Observe that $(j, m, \alpha_2, \alpha_3, \ldots) \in \mathrm{K}_{(0,\delta_{i-1},\delta_i,\ldots)}$ if and only if $(\alpha_1, \alpha_2, \ldots) \in \mathrm{K}_{(0,\delta_i,\delta_{i+1},\ldots)}$, where $\alpha_1 = j + m - \delta_{i-1}$. Hence summing the above equation over all $\alpha \in \mathrm{K}_{(0,\delta_i,\delta_{i+1},\ldots)}$, we get

$$
\begin{aligned}
&J_{\rho_{i-1}+1}\\
&= \int_0^{d_i - x_{\rho_{i-1}}} \int_0^{d_i - x_{\rho_{i-1}} - x_{\rho_{i-1}+1}} \cdots \int_0^{d_i - x_{\rho_{i-1}} - x_{\rho_{i-1}+1} - \cdots - x_{\rho_i - 1}} J_{\rho_i+1}\ dx_{\rho_i}\ \cdots\ dx_{\rho_{i-1}+1}\\
&= (-1)^{\delta_i+\delta_{i+2}+\delta_{i+4}+\cdots}\\
&\quad \cdot \sum_{\substack{(j,m,\alpha_2,\alpha_3,\dots)\\ \in \mathrm{K}_{(0,\delta_{i-1},\delta_i,\dots)}}} (-1)^{j+\alpha_2+\alpha_4+\cdots} \cdot \frac{1}{j!\ m!\ \alpha_2!\ \alpha_3!\ \cdots} \cdot x_{\rho_{i-1}}^j d_i^m d_{i+1}^{\alpha_2} d_{i+2}^{\alpha_3} \cdots .
\end{aligned}
$$

Note that the signs are consistent: taking into account the factor $(-1)^{\delta_{i+1}+\delta_{i+3}+\delta_{i+5}+\cdots}$ omitted from (11), the total sign of a term of (11) is

$$(-1)^{\delta_{i+1}+\delta_{i+3}+\delta_{i+5}+\cdots} \cdot (-1)^{j+\alpha_1+\alpha_3+\cdots} = (-1)^{\delta_i+\delta_{i+2}+\delta_{i+4}+\cdots} \cdot (-1)^{j+\alpha_2+\alpha_4+\cdots},$$

which is true because

$$\alpha_1 + \alpha_2 + \cdots = \delta_i + \delta_{i+1} + \cdots = n - \rho_i,$$

and hence all the exponents on both sides add up to $2(n-\rho_i)+2j$, i.e. an even number. □

To finish the proof of Theorem 3.1, set $i = 1$ in Lemma 3.2 and integrate with respect to x_n:

$$
\begin{aligned}
&n! \cdot J_n\\
&= n! \int_0^{d_1} J_1\ dx_1\\
&= (-1)^{\delta_1+\delta_3+\delta_5+\cdots}\ n!\\
&\quad \cdot \sum_{\alpha \in \mathrm{K}_{(0,\delta_1,\delta_2,\dots)}} (-1)^{\alpha_2+\alpha_4+\alpha_6+\cdots} \cdot \frac{1}{(\alpha_1+1)!\ \alpha_2!\ \alpha_3!\ \cdots} \cdot d_1^{\alpha_1+1} d_2^{\alpha_2} d_3^{\alpha_3} \cdots\\
&= (-1)^{\delta_1+\delta_3+\cdots} \sum_{\substack{(\alpha_1+1,\alpha_2,\alpha_3,\dots)\\ \in \mathrm{K}_{(1,\delta_1,\delta_2,\dots)}}} (-1)^{\alpha_2+\alpha_4+\cdots} \binom{n}{\alpha_1+1,\ \alpha_2,\ \alpha_3,\dots} \cdot d_1^{\alpha_1+1} d_2^{\alpha_2} d_3^{\alpha_3} \cdots ,
\end{aligned}
$$

and it is clear that $(\alpha_1+1, \alpha_2, \alpha_3, \dots) \in \mathrm{K}_{(1,\delta_1,\delta_2,\dots)}$ if and only if $\alpha \in \mathrm{K}_{(0,\delta_1,\delta_2,\dots)}$. □

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