COMBINATORIAL GELFAND MODELS

RON M. ADIN, ALEXANDER POSTNIKOV, AND YUVAL ROICHMAN

Abstract. A combinatorial construction of a Gelafand model for the symmetric group and its Iwahori-Hecke algebra is presented.

1. Introduction

A complex representation of a group or an algebra A is called a *Gelfand model* for A, or simply a *model*, if it is equivalent to the multiplicity free direct sum of **all** A-irreducible representations.

Models (for compact Lie groups) were first constructed by Bernstein, Gelfand and Gelfand [6]. Constructions of models for the symmetric group, using induced representations from centralizers, were found by Klyachko [11, 12] and by Inglis, Richardson and Saxl [9]; see also [4, 16, 2, 1, 3]. Our goal is to determine an explicit and simple combinatorial action, which gives a model for the symmetric group and its Iwahori-Hecke algebra.

1.1. **Signed Conjugation.**

Let S_n be the symmetric group on n letters, S - the set of simple reflections in S_n, I_n - the set of involutions in S_n and $V_n :=$ $\text{span}_{\mathbb{Q}}\{C_w \mid w \in I_n\}$ be a vector space over $\mathbb{Q}$ spanned by the involutions.

Recall the standard length function on the symmetric group

$$\ell(\pi) := \min\{\ell |\ \pi = s_{i_1} s_{i_2} \cdots s_{i_\ell}\ ,\ (\forall j) s_{i_j} \in S\},$$

the descent set

$$\text{Des}(\pi) := \{s \in S |\ \ell(\pi s) < \ell(\pi)\},$$

and the descent number $\text{des}(\pi) := \#\text{Des}(\pi)$.

Define a map $\rho : S \mapsto GL(V_n)$ by

$$\rho(s) C_w = \text{sign}(s;\ w) \cdot C_{sws} \qquad (\forall s \in S, w \in I_n) \tag{1}$$

where

$$\text{sign}(s;\ w) := \begin{cases} -1 & \text{if } sws = w \text{ and } s \in \text{Des}(w) \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

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Theorem 1.1.1. *ρ determines an S_n-representation.*

Theorem 1.1.2. *ρ determines a Gelfand model for S_n.*

1.2. **Hecke Algebra Action.** Consider the Hecke algebra of the symmetric group S_n, $H_n(q)$, generated by $\{T_i|\ 1 \le i < n\}$ with the defining relations

$$(T_i + q)(T_i - 1) = 0 \qquad (\forall i)$$
$$T_iT_j = T_jT_i \qquad \text{if} \quad |i-j| > 1$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad (1 \le i < n-1).$$

In order to construct an extended signed conjugation, which gives a model for $H_n(q)$, we extend the standard notions of length and weak order. Recall that the (left) weak order on S_n is the reflexive and transitive closure of the relation $w \le_L ws$ if $s \in S$ and $\ell(w) + 1 = \ell(sw)$.

Definition 1.2.1. *Define the* involutive length *of an involution $w \in I_n$ with cycle type $(1^{n-2k}2^k)$ as*

$$\hat{\ell}(w) := \min\{\ell(v)|\ w = vs_1s_3s_{2k-1}v^{-1}\},$$

where $\ell(v)$ is the standard length of $v \in S_n$.

Define the involutive weak order *on I_n, $\prec_L$, as the reflexive and transitive closure of the relation $w \lessdot_L sws$ if $s \in S$ and $\hat{\ell}(w) + 1 = \hat{\ell}(sws)$.*

Define a map $\rho_q : S \mapsto GL(V_n)$ by

$$\rho_q(T_s)C_w = \begin{cases} -qC_w & \text{if } sws = w \text{ and } s \in \mathrm{Des}(w) \\ C_w & \text{if } sws = w \text{ and } s \notin \mathrm{Des}(w) \\ (1-q)C_w + qC_{sws} & w \prec_L sws \\ C_{sws} & \text{if } sws \prec_L w, \end{cases} \tag{3}$$

where $\mathrm{Des}(\cdot)$ is the standard descent set and $\prec_L$ is the left involutive weak order.

Theorem 1.2.2. *For generic q, ρ_q is a Gelfand model for $H_n(q)$; namely,*

(1). *ρ_q is an $H_n(q)$-representation.*
(2). *ρ_q is equivalent to the multiplicity free sum of all $H_n(q)$-irreducible representations.*

The proof involves Lusztig's version of Tits' Deformation Theorem [13]. For other versions of this theorem see [7, §4], [8, §68.A] and [5].

Let $\mu = (\mu_1, \mu_2, \dots, \mu_t)$ be a partition of n and let $a_j := \sum_{i=0}^{j} \mu_i$. A permutation $\pi \in S_n$ is *μ-unimodal* if for every $0 \le j < t$ there exists $1 \le d \le a_{j+1} - a_j$ such that

$$\pi_{a_j+1} < \pi_{a_j+2} < \cdots < \pi_{a_j+d} > \pi_{a_j+d+1} > \pi_{a_j+d+2} > \cdots > \pi_{a_{j+1}}.$$

The character of ρ_q may be expressed as a generating function of the descent number over μ-unimodal involutions.

Proposition 1.2.3.

$$\mathrm{Tr}\ (\rho_q(T_\mu)) = \sum_{\{w\in I_n \mid\ w\ is\ \mu\text{-}unimodal\}} (-q)^{\mathrm{des}(w)}$$

where $T_\mu := T_1T_2\cdots T_{\mu_1-1}T_{\mu_1+1}\cdots\cdots T_{\mu_1+\ldots+\mu_t-1}$ *is the subproduct of* $T_1T_2\cdots T_{n-1}$ *omitting* $T_{\mu_1+\cdots+\mu_i}$ *for all* $1\le i<t$.

2. Proof of Theorem 1.1.1

2.1. **First Proof.** This proof relies on a variant of the inversion number, which is introduced in this section. Recall the definition of the inversion set of a permutation $\pi\in S_n$

$$\mathrm{Inv}(\pi) := \{\ (i,j):\ i<j \text{ and } \pi(i)>\pi(j)\}.$$

Definition 2.1.1. *For an involution* $w\in I_n$ *let* $\mathrm{Pair}(w)$ *be the set of 2-cycles of* w. *For a permutation* $\pi\in S_n$ *and an involution* $w\in I_n$ *let*

$$\mathrm{Inv}_w(\pi) := \mathrm{Inv}(\pi)\cap \mathrm{Pair}(w),$$

and

$$\mathrm{inv}_w(\pi) := \#\,\mathrm{Inv}_w(\pi).$$

Define a map $\psi: S_n \mapsto GL(V_n)$ by

$$\psi(\pi)C_w = (-1)^{\mathrm{inv}_w(\pi)}\cdot C_{\pi w\pi^{-1}} \qquad (\forall \pi\in S_n, w\in I_n). \tag{4}$$

Note that for every Coxeter generator $s\in S$ and every involution $w\in I_n$,

$$\mathrm{inv}_w(s) = \begin{cases} 1 & \text{if } sws=w \text{ and } s\in \mathrm{Des}(w) \\ 0 & \text{otherwise.}\end{cases}$$

Thus, ψ and ρ coincide on the Coxeter generators. In order to prove that ρ is an S_n-representation it suffices to prove that ψ is a group homomorphism.

Indeed, for every pair of permutations $\sigma,\pi\in S_n$ and every pair $1\le i<j\le n$

$$\chi\{\sigma\pi(i)>\sigma\pi(j)\} = \chi\{\pi(i)>\pi(j)\}\cdot\chi\{\sigma(\pi(i))>\sigma(\pi(j)),$$

where $\chi\{\text{ event }\}=1$ if the event holds and -1 otherwise. Hence, for every pair of permutations $\sigma,\pi\in S_n$ and every involution $w\in I_n$

$$(-1)^{\mathrm{inv}_w(\sigma\pi)} = (-1)^{\mathrm{inv}_w(\pi)}\cdot(-1)^{\mathrm{inv}_{\pi w\pi^{-1}}(\sigma)},$$

and thus

$$\psi(\sigma\pi)C_w = (-1)^{\mathrm{inv}_w(\sigma\pi)}\cdot C_{\sigma\pi w(\sigma\pi)^{-1}} = (-1)^{\mathrm{inv}_w(\pi)}\cdot(-1)^{\mathrm{inv}_{\pi w\pi^{-1}}(\sigma)}C_{\sigma(\pi w\pi^{-1})\sigma^{-1}}$$
$$= \psi(\sigma)((-1)^{\mathrm{inv}_w(\pi)}\cdot C_{\pi w\pi^{-1}}) = \psi(\sigma)(\psi(\pi)C_w).$$

This proves that ψ is an S_n-representation. Since ψ and ρ coincide on the Coxeter generators, the proof of Theorem 1.1.1 is completed.

□

We summarize

Corollary 2.1.2. *ρ and ψ are identical S_n-representations.*

2.2. **Second Proof.** In order to prove that ρ is an S_n-representation it suffices to verify the relations:

$$\rho(s)^2 = 1 \qquad (\forall s \in S)$$

$$\rho(s)\rho(t) = \rho(t)\rho(s) \qquad \text{if} \quad st = ts$$

$$\rho(s)\rho(t)\rho(s) = \rho(t)\rho(s)\rho(t) \qquad \text{if} \quad sts = tst.$$

We will prove the third relation. Verifying the other two relations is easier and left to the reader.

Let $s = (i, i+1)$ and $t = (i+1, i+2)$. For every permutation $\pi \in S_n$ let $\mathrm{Supp}(\pi) = \{i \in [n] \mid \pi(i) \neq i\}$. Denote the order orbit of an involution w under the conjugation action of $\langle s, t\rangle$ - the subgroup generated by s and t - by $O(w)$. Since w is an involution $O(w) \neq 2$; hence there are three options $O(w) = 1, 3, 6$.

Case (a). $O(w) = 1$. Then $sws = w$ and $twt = w$. Furthermore, in this case $\mathrm{Supp}(w) \cap \{i, i+1, i+2\} = \emptyset$, so that $\mathrm{sign}(s,;\ w) = \mathrm{sign}(t;\ w) = 1$; thus $\rho(s)\rho(t)\rho(s)C_w = \rho(t)\rho(s)\rho(t)C_w = C_w$.

Case (b). $O(w) = 3$ (this happens, for example, when $w = s$). Then there exists an element v in the orbit such that

$$v, tvt, stvts \quad \text{are distinct elements in the orbit,} \tag{5}$$

and

$$svs = v \quad \text{and} \quad t(stvts)t = stvts. \tag{6}$$

Thus

$$\rho(s) = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\rho(t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z \end{pmatrix},$$

where $x = \mathrm{sign}(s;\ v)$ and $z = \mathrm{sign}(t;\ stvts)$. $\rho(s)\rho(t)\rho(s) = \rho(t)\rho(s)\rho(t)$ holds if and only if $x = z$, which holds if and only if

$$s \in \mathrm{Des}(v) \Longleftrightarrow t \in \mathrm{Des}(stvts). \tag{7}$$

To prove this, observe that for every $w \in S_n$ and $s \in S$ the following holds :

(A) $sws = w$ and $s \notin \mathrm{Des}(w)$ if and only if $\mathrm{Supp}(w) \cap \mathrm{Supp}(s) = \emptyset$.

(B) $sws = w$ and $s \in \mathrm{Des}(w)$ if and only if $w = us$, where $\mathrm{Supp}(u) \cap \mathrm{Supp}(s) = \emptyset$.

Assuming $t \notin \mathrm{Des}(stvts)$ implies, by (6) and (A), that $\mathrm{Supp}(stvts) \cap \mathrm{Supp}(t) = \emptyset$. Hence

$$stvts(i+1) = i+1.$$

On the other hand, assuming $s \in \mathrm{Des}(v)$ implies, by (6) and (B), that there exists $u = vs$ with $i+1 \notin \mathrm{Supp}(u)$. Hence

$$stvts(i+1) = stusts(i+1) = i+2.$$

Contradiction. Similarly, assuming $s \notin \mathrm{Des}(v)$ and $t \in \mathrm{Des}(stvts)$ yields a contradiction (to verify this, replace v by $stvts$ and s by t). This completes the proof of Case (b).

Case (c). $O(w) = 6$ (this occurs, for example, when $w = (i,j)(i+1,k)$ with $j,k \neq i+2$). Then, for every element v in the orbit $tvt \neq v$ and $svs \neq v$. It follows that

$$\rho(s)\rho(t)\rho(s)C_w = C_{stswsts} = C_{tstwtst} = \rho(t)\rho(s)\rho(t)C_w.$$

This completes the proof of the third relation .

□

3. Characters

3.1. **Character Formula.** The following classical result follows from the work of Frobenius and Schur, see [10, §4] and [18, §7, Ex. 69].

Theorem 3.1.1. *Let G be a group, for which every complex representation is equivalent to a real representation. Then for every $w \in G$*

$$\sum_{\chi \in G^*} \chi(w) = \#\{u \in G \mid u^2 = w\},$$

where G^* denotes the set of the irreducible characters of G.

It is well known [17] that all complex representation of a Weyl group are equivalent to rational representations. In particular, Theorem 3.1.1 holds for $G = S_n$. One concludes

Corollary 3.1.2. *Let $\pi \in S_n$ be of cycle structure $1^{d_1}2^{d_2}\cdots n^{d_n}$. Then*

$$\sum_{\chi \in S_n{}^*} \chi(\pi) = \prod_{r=1}^{n} f(r, d_r),$$

where

$$f(r,d_r) = \begin{cases} 0 & \text{if } r \text{ is even and } d_r \text{ is odd} \\ 1 & \text{if } d_r = 0 \\ \binom{d_r}{2,\dots,2} \cdot r^{d_r/2} & \text{if } r \text{ and } d_r \text{ are even} \\ \sum_{k=0}^{\lfloor d_r/2 \rfloor} \binom{d_r}{d_r-2k,2,2,\dots,2} \cdot r^k & \text{if } r \text{ is odd.} \end{cases}$$

Proof. For every $A \subseteq [n]$ let $S_A := \{\pi \in S_n | \operatorname{Supp}(\pi) \subseteq A\}$ - the subgroup of permutations in S_n whose support is contained in A.

For every $\pi \in S_n$ and $1 \le r \le n$ let $A(\pi, r) \subseteq [n]$ be the subset of letters which appear in cycles of length r. In other words,

$$A(\pi, r) := \{i \in [n] \mid \forall j < r\ \pi^j(i) \ne i \text{ and } \pi^r(i) = i\}$$

For example, $A(\pi, 1)$ is the set of fixed points of π.

Denote by $\pi_{|r}$ the restriction of π to $A(\pi, r)$. Then $\pi_{|r}$ may be considered as a permutation in $S_{A(\pi,r)}$.

Observation 3.1.3. *For every $\pi \in S_n$*

$$\{u \in S_n \mid u^2 = \pi\} = \prod_{r=1}^{n} \{u \in S_{A(\pi,r)} \mid u^2 = \pi_{|r}\}.$$

Observation 3.1.4. *Let $\pi \in S_n$ be of cycle type $r^{n/r}$. Then*

$$\#\{u \in S_n \mid u^2 = \pi\} = \begin{cases} 0 & \text{if } r \text{ is even and } n/r \text{ is odd} \\ \binom{n/r}{2,\ldots,2} \cdot r^{\frac{n}{2r}} & \text{if } r \text{ is even and } n/r \text{ is even} \\ \sum_{k=0}^{\lfloor n/2r \rfloor} \binom{n/r}{n/r-2k,2,2,\ldots,2} \cdot r^k & \text{if } r \text{ is odd .} \end{cases}$$

Combining these observations with Theorem 3.1.1 implies Corollary 3.1.2.

□

3.2. **Proof of Theorem 1.1.2.** We shall compute the character of the representation ρ and compare it with Corollary 3.1.2. By Corollary 2.1.2,

$$\operatorname{Tr}(\rho(\pi)) = \sum_{w \in I_n \cap \operatorname{St}_n(\pi)} (-1)^{\operatorname{inv}_w(\pi)}, \tag{8}$$

where $\operatorname{St}_n(\pi)$ is the centralizer of π in S_n.

Observation 3.2.1. *Let $\pi \in S_n$, $w \in \operatorname{St}_n(\pi) \cap I_n$ and $a_1 \in \operatorname{Supp}(w)$. Then one of the following holds:*

1. *(a_1, a_2) is a cycle in w and $a_1, a_2 \notin \operatorname{Supp}(\pi)$.*
2. *$(a_1, a_{r+1})(a_2, a_{r+2}) \cdots (a_r, a_{2r})$ are cycles in w and $(a_1, a_2, \ldots, a_r)(a_{r+1}, \ldots, a_{2r})$ are cycles in π.*
3. *$(a_1, a_{m+1})(a_2, a_{m+2}) \cdots (a_m, a_{2m})$ are cycles in w and $(a_1, a_2, \ldots, a_{2m})$ is a cycle in π.*

It follows that

Corollary 3.2.2. *For every $\pi \in S_n$ and every $w \in \operatorname{St}_n(\pi) \cap I_n$ there is a unique decomposition*

$$w = w_1 \cdots w_r \qquad (\forall r, w_r \in C_{S_{A(\pi,r)}}(\pi_{|r}) \cap I_{S_{A(\pi,r)}}),$$

where $A(\pi,r)$, $\pi_{|r}$ and $S_{A(\pi,r)}$ are defined as in the proof of Corollary 3.1.2. Then

$$\mathrm{Inv}_w(\pi) = \uplus_{r=1}^{n} \mathrm{Inv}_{w_r}((\pi_{|r}),$$

disjoint union.

Hence, it suffices to prove that $\mathrm{Tr}(\rho(\pi))$ is equal to the right hand side of Corollary 3.1.2, for π of cycle type $r^{n/r}$. Without loss of generality we may assume that $\pi = (1,2,\ldots,r)(r+1,\ldots,2r)\cdots(n-r+1,n-r+2,\ldots,n)$.

Fact 3.2.3. (1) *For every r and i, $w = (i+1, i+r+1)(i+2, i+r+2)\cdots(i+r, i+2r)$ and $\pi = (i+1,i+2,\ldots,i+r)(i+r+1,\ldots,i+2r)$*

$$(-1)^{\mathrm{inv}_w(\pi)} = 1.$$

(2) *If $r = 2m$ is even, $w = (i+1, i+m+1)(i+2, i+m+2)\cdots(i+m, i+2m)$) and $\pi = (i+1, i+2,\ldots,i+r)$ then*

$$(-1)^{\mathrm{inv}_w(\pi)} = -1.$$

Lemma 3.2.4. *For every odd r and a permutation $\pi = (1,2,\ldots,r)(r+1,\ldots,2r)\cdots(n-r+1,n-r+2,\ldots,n)$*

$$\sum_{w\in I_n\cap \mathrm{St}_n(\pi)} (-1)^{\mathrm{inv}_w(\pi)} = \#\ I_n \cap \mathrm{St}_n(\pi) = \sum_{k=0}^{\lfloor n/2r\rfloor} \binom{n/r}{n/r-2k,2,2,\ldots,2}\cdot r^k.$$

Proof of Lemma 3.2.4. The first equality follows from Fact 3.2.3(1). The second equality follows from Observation 3.2.1(1)-(2).

□

Lemma 3.2.5. *For every even r and π as above*

$$\sum_{w\in I_n\cap \mathrm{St}_n(\pi)} (-1)^{\mathrm{inv}_w(\pi)} = \begin{cases} 0 & \text{if } n/r \text{ is odd} \\ \binom{n/r}{2,\ldots,2}\cdot r^{\frac{n}{2r}} & \text{if } n/r \text{ is even.}\end{cases}$$

Proof of Lemma 3.2.4. Consider the following two set of elements in $I_n \cap \mathrm{St}_n(\pi)$

$$\{w \in \mathrm{St}_n(\pi)\ :\ (k-1)r+1, (k-1)r+2,\ldots,kr \notin \mathrm{Supp}(w)\},$$

and

$$\{w \in \mathrm{St}_n(\pi)\ :\ (i, \frac{r}{2}+i) \in \mathrm{Pair}(w) \text{ for } (k-1)r < i \le (k-1)r + r/2\}.$$

Clearly, these two sets are of the same cardinality. By Fact 3.2.3(2), their signed contribution is the opposite. We are left with involutions in $\mathrm{St}_n(\pi)$, for which all 2-cycles are of second type in Observation 3.2.1. Lemma 3.2.5 follows.

□

Lemmas 3.2.4 and 3.2.5 complete the proof of Theorem 1.1.2.

□

4. The Hecke Algebra Case

4.1. **A Combinatorial Lemma.** Recall Definition 1.2.1. In order to prove Theorem 1.2.2 we need the following combinatorial interpretation of the involutive length $\hat{\ell}$.

Lemma 4.1.1. *Let $w \in S_n$ be an involution of cycle type $2^k 1^{n-2k}$. Then*

$$\hat{\ell}(w) := \sum_{i \in \mathrm{Supp}(w)} i + \frac{1}{2}\,\mathrm{inv}(w_{|\mathrm{Supp}(w)}) - \binom{2k+1}{2} - \frac{k}{2}. \tag{9}$$

Proof. Denote the right hand side of (9) by $f(w)$. Let u and $v = s_i u s_i$ be involutions in S_n with $\hat{\ell}(v) = \hat{\ell}(u) + 1$. Then $|\{i, i+1\} \cap \mathrm{Supp}(u)| > 0$. If $|\{i, i+1\} \cap \mathrm{Supp}(u)| = 1$ then $|\sum\limits_{i \in \mathrm{Supp}(v)} i - \sum\limits_{i \in \mathrm{Supp}(u)} i| = \pm 1$ and $\mathrm{inv}(v_{|\mathrm{Supp}(v)}) = \mathrm{inv}(u_{|\mathrm{Supp}(u)})$. If $|\{i, i+1\} \cap \mathrm{Supp}(u)| = 2$ then $\sum\limits_{i \in \mathrm{Supp}(v)} i = \sum\limits_{i \in \mathrm{Supp}(u)} i$ and $|\,\mathrm{inv}(v_{|\mathrm{Supp}(v)}) - \mathrm{inv}(u_{|\mathrm{Supp}(u)})| \in \{0, \pm 2\}$. Thus in both cases $|f(v) - f(u)| \le 1$. This proves, by induction on $\hat{\ell}$, that for every involution w, $f(w) \le \hat{\ell}(w)$.

On the other hand, if w is an involution with $f(w) > 0$ then either $\sum\limits_{i \in \mathrm{Supp}(w)} i > \binom{2k+1}{2}$ or $\sum\limits_{i \in \mathrm{Supp}(w)} i = \binom{2k+1}{2}$ and $\mathrm{inv}(w_{|\mathrm{Supp}(w)}) > k$. In the first case there exists $i+1 \in \mathrm{Supp}(w)$, such that $i \notin \mathrm{Supp}(w)$. Then $f(s_i w s_i) = f(w) - 1$. In the second case there exists an i and $k > j > i+1$ such that one of the following holds: (i, j) and $(i+1, k) \in \mathrm{Pair}(w)$, then $f(s_{i+1} w s_{i+1}) = f(w) - 1$; or : (i, k) and $(i+1, j) \in \mathrm{Pair}(w)$, then $f(s_i w s_i) = f(w) - 1$. We conclude that for every involution w, $\hat{\ell}(w) \le f(w)$.

□

4.2. **Proof of Theorem 1.2.2.** The proof consists of two parts. In the first part we prove that ρ_q is an $H_n(q)$-representation by verifying the defining relations along the lines of the second proof of Theorem 1.1.1. In the second part we apply Lusztig's version of Tits' deformation theorem to prove that ρ_q is a Gelfand model.

Part 1: Proof of Theorem 1.2.2(1). First, consider the braid relation $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. To verify this relation observe that there are six possible types of orbits of an involution w under conjugation by $\langle s_i, s_{i+1} \rangle$ - the subgroup in S_n generated by s_i and s_{i+1}. These orbits differ by the action of w on the letters $i, i+1, i+2$:

1. $i, i+1, i+2 \notin \mathrm{Supp}(w)$.
2. Exactly one of the letters $i, i+1, i+2$ is in $\mathrm{Supp}(w)$.
3. Exactly two of the letters $i, i+1, i+2$ are in $\mathrm{Supp}(w)$ and these two letters form a 2-cycle in w.
4. Exactly two of the letters $i, i+1, i+2$ are in $\mathrm{Supp}(w)$ and these two letters do not form a 2-cycle in w.

5. $i, i+1, i+2 \in \mathrm{Supp}(w)$ and two of these letters form a 2-cycle in w.
6. $i, i+1, i+2 \in \mathrm{Supp}(w)$ and no two of these letters form 2-cycle in w.

Note that an orbit of the first type is of order one; orbits of the second, third and fifth type are of order three; and orbits of the fourth and sixth type are of order six. Moreover, by Lemma 4.1.1, orbits of same order form isomorphic intervals in the weak involutive order (see Definition 1.2.1). In particular, all orbits of order six have a representative w of minimal involutive length, such that the orbit has the form :

$$\begin{tikzcd}[column sep=small] & s_i s_{i+1} s_i w s_i s_{i+1} s_i \arrow[dl] \arrow[dr] & \\ s_i s_{i+1} w s_{i+1} s_i \arrow[d] & & s_{i+1} s_i w s_i s_{i+1} \arrow[d] \\ s_{i+1} w s_{+1} i \arrow[dr] & & s_i w s_i \arrow[dl] \\ & w & \end{tikzcd} \tag{10}$$

All orbits of order three are of a linear form :

$$w \prec_L s_i w s_i \prec_L s_{i+1} s_i w s_i s_{i+1}, \tag{11}$$

or

$$w \prec_L s_{i+1} w s_{i+1} \prec_L s_i s_{i+1} w s_{i+1} s_i. \tag{12}$$

Thus the analysis is reduced into three cases.

Case (a). An orbit of order six. By (3) and (10), the representation matrices of the generators with respect to the ordered basis C_w, $C_{s_i w s_i}$, $C_{s_{i+1} s_i w s_i s_{i+1}}$, $C_{s_i s_{i+1} s_i w s_i s_{i+1} s_i}$, $C_{si+1 w s_{i+1}}$, $C_{s_i si+1 w s_{i+1} s_i}$ are :

$$\rho_q(T_i) = \begin{pmatrix} 1-q & q & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & q & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q & q \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\rho_q(T_{i+1}) = \begin{pmatrix} 1-q & 0 & 0 & 0 & q & 0 \\ 0 & 1-q & q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 1-q \end{pmatrix}$$

It is easy to verify that indeed

$$\rho_q(T_i)\rho_q(T_{i+1})\rho_q(T_i) = \rho_q(T_{i+1})\rho_q(T_i)\rho_q(T_{i+1}).$$

Case (b). An orbit of order three. Without loss of generality, the orbit is of type (11), (the analysis of type (12) is analogous). Then $s_{i+1}ws_{i+1} = w$ and $s_i(s_{i+1}s_iws_is_{i+1})s_i = s_{i+1}s_iws_is_{i+1}$. By (7), $s_{i+1} \in \mathrm{Des}(w)$ if and only if $s_i \in \mathrm{Des}(s_{i+1}s_iws_is_{i+1})$, see proof of Proposition **??**.

Given the above, by (3), the representation matrices of the generators with respect to the ordered basis $w \prec_L s_iws_i \prec_L s_{i+1}s_iws_is_{i+1}$ are

$$\rho_q(T_i) = \begin{pmatrix} 1-q & q & 0 \\ 1 & 0 & 0 \\ 0 & 0 & x \end{pmatrix}$$

and

$$\rho_q(T_{i+1}) = \begin{pmatrix} x & 0 & 0 \\ 0 & 1-q & q \\ 0 & 1 & 0 \end{pmatrix},$$

where $x \in \{1, -q\}$. These matrices satisfy the braid relation.

Case (c). An orbit of order one. Then $s_iws_i = w$ and $s_{i+1}ws_{i+1} = w$ and $s_i, s_{i+1} \notin \mathrm{Des}(w)$. By (3), $\rho_q(T_i)\rho_q(T_{i+1})\rho_q(T_i)C_w = \rho_q(T_{i+1})\rho_q(T_i)\rho_q(T_{i+1})C_w = C_w$, completing the proof of the third relation. The proof of the first two relations is easier and left to the reader.

Part 2: Proof of Theorem 1.2.2(2). Consider the Hecke algebra $H_n(q)$ as the algebra spanned by $\{T_v |\ v \in S_n\}$ over $\mathbb{Q}[q^{1/2}]$ with the multiplication rules

$$T_vT_u = T_{vu} \qquad \ell(vu) = \ell(v) + \ell(u)$$

and

$$(T_s + q)(T_s - 1) = 0 \qquad (\forall s \in S).$$

By Lusztig version of Tits' deformation theorem [13, Theorem 3.1], the group algebra of S_n over $\mathbb{Q}[q^{1/2}]$ is embedded in $H_n(q)$. In particular, every element $w \in S_n$ may be expressed as a linear combination

$$w = \sum_{v \in S_n} m_{v,w}(q^{1/2})T_v,$$

where $m_{v,w}$ is a rational function in $q^{1/2}$.

It follows that ρ_q may be considered as an S_n representation, via

$$\rho_q(w) := \sum_{v \in S_n} m_{v,w}(q^{1/2})\rho_q(T_v).$$

The resulting character $\rho_q(w)$ is thus a rational function in $q^{1/2}$. By discreteness of the S_n character values, the character is constant in "small" generic neighborhoods and thus constant whenever it is defined.

On the other hand, by Theorem 1.1.2, $\rho_{|q=1} = \rho$ is a model for the group algebra of S_n. This completes the proof.

□

4.3. **Proof of Proposition 1.2.3.** Let SYT_n be the set of all standard Young tableaux of order n, and let $\mathrm{SYT}(\lambda) \subseteq \mathrm{SYT}_n$ be the subset of standard Young tableaux of shape λ. For each partition λ of n, fix a standard Young tableau $P_\lambda \in \mathrm{SYT}(\lambda)$. By [15, Theorem 4], the value of the irreducible $H_n(q)$-character, χ_q^λ, at T_μ is

$$\chi_q^\lambda(T_\mu)) = \sum_{\{w \mapsto (P_\lambda, Q) \mid \ w \text{ is } \mu\text{-unimodal and } Q \in \mathrm{SYT}(\lambda)\}} (-q)^{\mathrm{des}(w)},$$

where the sum runs over all permutations $w \in S_n$ which are mapped under RSK correspondence to (P_λ, Q) for some $Q \in \mathrm{SYT}(\lambda)$. By [18, Lemma 7.23.1], the descent set of $w \in S_n$, which is mapped under RSK to (P_λ, Q), is determined by Q. Hence

$$\mathrm{Tr}\ \rho_q(T_\mu) = \sum_\lambda \chi_q^\lambda(T_\mu)) = \sum_\lambda \sum_{\{w \mapsto (P_\lambda, Q) \mid \ w \text{ is } \mu\text{-unimodal and } Q \in \mathrm{SYT}(\lambda)\}} (-q)^{\mathrm{des}(w)}$$

$$= \sum_\lambda \sum_{\{w \mapsto (Q, Q) \mid \ w \text{ is } \mu\text{-unimodal and } Q \in \mathrm{SYT}(\lambda)\}} (-q)^{\mathrm{des}(w)}$$

$$= \sum_{\{w \mapsto (Q,Q) \mid \ Q \in \mathrm{SYT}_n \text{ and } w \text{ is } \mu\text{-unimodal}\}} (-q)^{\mathrm{des}(w)} = \sum_{\{w \in I_n \mid \ w \text{ is } \mu\text{-unimodal}\}} (-q)^{\mathrm{des}(w)}.$$

The last equality follows from the well known property of the RSK correspondence: $w \mapsto (P, Q)$ if and only if $w^{-1} \mapsto (Q, P)$ [18, Theorem 7.13.1]. Thus w is an involution if and only if $w \mapsto (Q, Q)$ for some $Q \in \mathrm{SYT}_n$.

$\square$

5. Remarks and Questions

5.1. **Classical Weyl Groups.** Let B_n be the Weyl group of type B, S^B - the set of simple reflections in B_n, I_n^B - the set of involutions in D_n and $V_n^B := \mathrm{span}_{\mathbb{Q}}\{C_w \mid w \in I_n^B\}$ be a vector space over $\mathbb{Q}$ spanned by the involutions.

Define a map $\rho^B : S^B \mapsto GL(V_n)$ by

$$\rho^B(s)C_w = \mathrm{sign}(s;\ w) \cdot C_{sws} \qquad (\forall s \in S^B, w \in I_n^B) \tag{13}$$

where, for $s = s_0$ the sign is

$$\mathrm{sign}(s_0;\ w) := \begin{cases} -1 & \text{if } sws = w \text{ and } s_0 \in \mathrm{Des}(w) \\ 1 & \text{otherwise.} \end{cases} \tag{14}$$

For $s \neq s_0$ the sign is

$$\mathrm{sign}(s;\ w) := \begin{cases} -1 & \text{if } sws = w \text{ and } s \in \mathrm{Des}(|w|) \\ 1 & \text{otherwise.} \end{cases} \tag{15}$$

Here $w = (|w(1)|, \ldots, |w(n)|) \in S_n$.

Theorem 5.1.1. *ρ^B is a Gelfand model for B_n.*

A proof will be given elsewhere.

Models for classical Weyl groups of type D_n for odd n were constructed in [4, 3]. These constructions fail for even n. A natural question is whether there exists a signed conjugation (or a representation of type $\rho_s C_w = a_{s,w} C_w + b_{s,w} C_{sws}$) which gives a model for D_{2n}. It is also desired to find representation matrices for the models of the Hecke algebras of types B and D which specialize at $q = 1$ to models of the corresponding group algebra.

We conclude with the following questions regarding an arbitrary Coxeter group W.

Question 5.1.2. *Find a signed conjugation which gives a Gelfand model for W; Find a representation of the form $\rho_s C_w = a_{s,w} C_w + b_{s,w} C_{sws}$, which gives a Gelfand model for the Hecke algebra of W.*

Question 5.1.3. *Find a character formula for the Gelfand model of the Hecke algebra of W.*

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Department of Mathematics, Bar-Ilan University, Ramat Gan 52900, Israel
E-mail address: radin@math.biu.ac.il

Department of Applied Mathematics, Massachusetts Institute of Technology, MA 02139, USA
E-mail address: apost@math.mit.edu

Department of Mathematics, Bar-Ilan University, Ramat Gan 52900, Israel
E-mail address: yuvalr@math.biu.ac.il