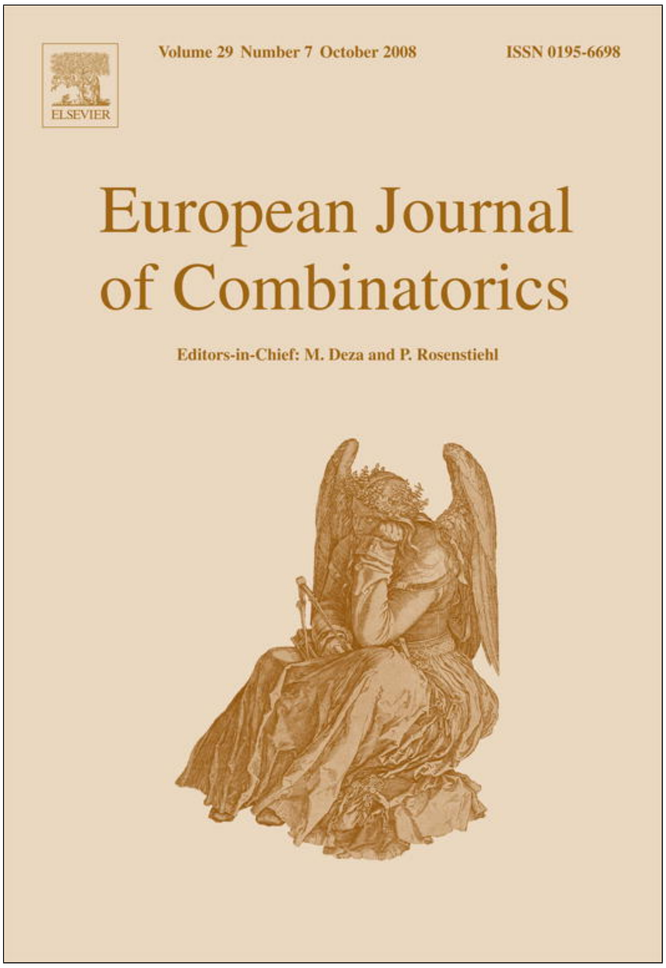
Volume 29 Number 7 October 2008
ISSN 0195-6698
ELSEVIER
European Journal of Combinatorics
Editors-in-Chief: M. Deza and P. Rosenstiehl

Available online at www.sciencedirect.com
ScienceDirect
European Journal of Combinatorics 29 (2008) 1535–1543
European Journal of Combinatorics
www.elsevier.com/locate/ejc

# Generalizations of Abel's and Hurwitz's identities

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Received 28 July 2007; accepted 21 December 2007
Available online 17 March 2008

**Abstract**

In 1826 N. Abel found a generalization of the binomial formula. In 1902 Abel's theorem was further generalized by A. Hurwitz. In this paper we describe constructions that provide infinitely many identities each being a generalization of a Hurwitz's identity. Moreover, we give combinatorial interpretations of all these identities as the forest volumes of certain directed graphs.
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## 1. Introduction

In 1826 N. Abel found the following surprising generalization of the binomial formula [1] (see also [2,10,11]):

**1.1** $(x + y)^n = \sum\{\binom{n}{k} y(y + kz)^{k-1}(x - kz)^{n-k} : k \in \{0, \dots, n\}\}$.

In 1902 Abel's theorem was further generalized by A. Hurwitz as follows (see also [10]). Let $V$ be a finite set and $x : V \to N$ a function identified by the set of pairs $x = \{(v, x(v)) : v \in V\}$, where $N$ is a commutative ring. For $A \subseteq V$, let $x(A) = \sum\{x(a) : a \in A\}$.

**1.2** [4] $(z + y)(z + y + x(V))^{|V|-1}) = \sum\{z(z + x(A))^{|A|-1} \cdot y(y + x(B))^{|B|-1} : A \subseteq V, B = V \setminus A\}$.

**1.3** [4] $(z + y + x(V))^{|V|} = \sum\{y(y + x(A))^{|A|-1} \cdot (z + x(B))^{|B|} : A \subseteq V, B = V \setminus A\}$.

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0195-6698/$ - see front matter Published by Elsevier Ltd
doi:10.1016/j.ejc.2007.12.003

Let $z = x - x(V)$. Then we have the following identity equivalent to that in **1.3**:

$$(x + y)^{|V|} = \sum\{y(y + x(A))^{|A|-1} \cdot (x - x(A))^{|B|} : A \subseteq V, B = V \setminus A\}.$$

Obviously, Abel's identity **1.1** is a particular case of the above identity when $x(v) = z$ for every $v \in V$.

In this paper we describe mechanisms that provide infinitely many identities each being a generalization of one of the above Hurwitz's identities. We use these mechanisms to obtain some such generalizations. As a by-product, we give combinatorial interpretations of all such identities as well. The "engine" of this mechanism is the relation between the so called forest volumes of graph-compositions and their bricks [8] (see also **3.1** below). We will see that the volume formula from [8] applied to very simple graph-compositions gives a natural generalization of Hurwitz's identities **1.2** and **1.3**. Namely, these generalizations correspond to the graph-compositions whose "frames" are the simple digraphs with two vertices and with either one or two arcs, and whose two bricks are a complete digraph and an arc empty graph. In particular, one of the mechanisms that we describe provides for every acyclic digraph $G$ a large variety of polynomial identities corresponding to a graph-composition whose frame is $G$.

Main notions, notation, and some simple observations are given in Section 2. In Section 3 we describe the above mentioned "engine" and some other necessary preliminaries. In Section 4 we give simple generalizations of the above Hurwitz's identities in order to elucidate the main idea of general mechanisms and the main proof arguments. In Section 5 we describe a mechanism that provides infinitely many identities that are generalizations of Hurwitz's identity **1.2** and that are related to the acyclic digraphs. We apply this mechanism to obtain a class of generalizations of Hurwitz's identity **1.2** related to the ditrees. We also describe a mechanism that provides infinitely many identities that are generalizations of Hurwitz's identity **1.3**.

The results of this paper were presented at the 8th International Conference on Graph Theory, Combinatorics, Algorithms and Applications, Kalamazoo, Michigan, June, 1996 (see also [9]).

## 2. Main notions, notation, and simple observations

We consider directed graphs. All graph-theoretical notions that are used but not defined here can be found in [3,12].

A *directed graph* or, simply, a *digraph* $G$ is a pair $(V, E)$, where $V$ is a finite non-empty set of elements (called *vertices* of $G$) and $E$ is a set of ordered pairs of different elements from $V$ (the elements of $E$ are called *arcs* of $G$). Let $V(G) = V$ and $E(G) = E$. If $E$ is the set of all ordered pairs of different elements from $V$, then $G$ is called a *complete digraph*.

A digraph is *acyclic* if it has no directed cycles.

A *source* (a *sink*) of a digraph $G$ is a vertex $v$ having no incoming (respectively, outgoing) arcs in $G$. Let $L(G)$ and $R(G)$ denote the sets of sources and sinks of $G$, respectively.

A digraph $F$ is a *subgraph* of $G$, written $F \subseteq G$, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. A digraph $F$ of $G$ is a *spanning subgraph* of $G$ if $V(F) = V(G)$ and $E(F) \subseteq E(G)$.

Two spanning subgraphs $F_1$ and $F_2$ of $G$ are *different* if $E(F_1) \neq E(F_2)$.

A *ditree* $T$ is a digraph with the properties:

(a1) $T$ has no directed cycles and

(a2) for every vertex $v$ in $V(T)$ except for one vertex, say $r$, there exists a unique arc $e_v = (v, t_v)$ starting at $v$.

The vertex $r$ is called the *root of* $T$, and $T$ is also called a *ditree rooted at* $r$. A *leaf* of a ditree $T$ is a vertex having no incoming arc in $T$. Clearly $R(T) = \{r\}$ and $L(T)$ is the set of leaves of $T$.

A *diforest* $F$ is a digraph such that every its component is a ditree.

A *spanning ditree* (*spanning diforest*) of a digraph $G$ is a spanning subgraph of $G$ which is a ditree (respectively, a diforest).

Let $\mathcal{T}_r(G)$ denote the set of different spanning ditrees of $G$ rooted at $r \in V(G)$. Similarly, let $\mathcal{F}(G)$ denote the set of different spanning diforests of $G$.

Let $d(v, G)$ and $d_{in}(v, G)$ denote the number of arcs in $G$ incident to $v$ and coming in $v$, respectively.

Let $x : V(G) \to N$ be a function, where $N$ is a commutative ring. This function can be described by the set of pairs $x = \{(v, x(v)) : v \in V(G), x(v) \in N\}$. We call $(G, x)$ a *weighted digraph*.

For a ditree $T$ rooted at $r$, let $\mathcal{T}(T, x) = \prod\{x(v)^{d(v,T)-1} : v \in V(T)\}$. Clearly

$$\mathcal{T}(T, x) = x(r)^{d_{in}(r,T)-1} \prod\{x(v)^{d_{in}(v,T)} : v \in V(T) \setminus r\}.$$

The *tree volume* of a digraph $G$ (or of $(G, x)$) *with respect to a given vertex* $r \in V(G)$ [7] is

$$\mathcal{T}_r(G, x) = \sum\{\mathcal{T}(T, x) : T \in \mathcal{T}_r(G)\}.$$

Clearly $\mathcal{T}_r(G, x)$ is a polynomial in variables $x(v)$, $v \in V(G)$.

Let $G^c$ be the digraph obtained from $G$ by adding a new vertex $c$ and the set of arcs $\{(v, c) : v \in V(G)\}$. For a function $x : V(G) \to N$, let $x^c : V(G^c) \to N$ be a function such that $x^c(c) = z \in N$ and $x$ is the restriction of $x^c$ on $V(G)$.

The *forest volume* of $G$ (or of $(G, x)$) [7] is

$$\mathcal{F}(z, G, x) = \mathcal{T}_c(G^c, x^c).$$

We put $\mathcal{F}(z, G, x) = z^{-1}$ if $V(G) = \emptyset$.

The forest volume of a weighted digraph $(G, x)$ can be viewed as a generating function of spanning diforests of $G$ classified by their numbers of edges and degree sequences.

Obviously,

**2.1** *Let $T$ be a ditree and $ab \in E(T)$. Let $T_a$ be the maximal ditree such that $a$ is the root of $T_a$ and $T_a \subseteq T$ and let $T^a = T - T_a$. Then*

$$\mathcal{T}(T, x) = x(b)\mathcal{T}(T_a, x_a)\mathcal{T}(T^a, x^a),$$

*where $x_a$ and $x^a$ are restrictions of $x$ on $V(T_a)$ and $V(T^a)$, respectively.*

For a diforest $F$, let $\mathcal{F}(F, x) = \prod\{x(v)^{d_{in}(v,F)} : v \in V(F)\}$. Given $\mathcal{A} \subseteq \mathcal{F}(K)$, the *volume* of $\mathcal{A}$ [7] is

$$\mathcal{V}(\mathcal{A}, x) = \sum\{\mathcal{F}(F, x) : F \in \mathcal{A}\}.$$

From the definitions of $\mathcal{F}(z, G, x)$ and $\mathcal{V}(\mathcal{A}, x)$ we have:

**2.2** *Let $G$ be a digraph. Then $\mathcal{V}(\mathcal{F}(G), x) = \mathcal{F}(1, G, x)$.*

Given $A \subseteq V(G)$, let $\mathcal{F}(G, A) = \{F \in \mathcal{F}(G) : R(F) = A\}$, where, as above, $R(F)$ is the sets of sinks of $F$ (i.e. the set of the roots of the components in $F$). Let the digraph $G_A$ be obtained from $G$ by removing every arc going out of a vertex in $A$.

It is easy to see the following.

**2.3** *Let $G$ be a digraph and $A \in V(G)$. Then $\mathcal{F}(G, A) = \mathcal{F}(G_A)$.*

## 3. Forest volumes of graph-compositions

Let $G$ be a digraph with $V(G) = A$. Let $G_a,\ a \in A$, be disjoint digraphs with $V(G_a) = V_a$. We assume that $G$ and all $G_a$'s are disjoint. Let $B_a = \{(a, b) : b \in V_a\}$ and $B = \cup\{B_a : a \in A\}$.

The digraph $\Gamma$ is called the *$G$-composition of* $\{G_a : a \in V(G)\}$, written $\Gamma = G\{G_a : a \in V(G)\}$, if $V(\Gamma) = B$ and for two vertices $v_1 = a_1 b_1$ and $v_2 = a_2 b_2$ of $\Gamma$, $(v_1, v_2) \in E(\Gamma)$ if and only if either $a_1 \neq a_2$ and $(a_1, a_2) \in E(G)$ or $a_1 = a_2 = a$ and $(b_1, b_2) \in E(G_a)$. The graph $G$ is called the *frame* and the graphs $G_a$, $a \in V(G)$, are called the *bricks* of the $G$-composition $G\{G_a : a \in V(G)\}$.

Let $x : B \to N$ be a function, $x_a = x|_{B_a}$, $x(G_a) = \sum\{x(a, b) : b \in V_a\}$, and $x(\Gamma) = \sum\{x(a, b) : a \in A, b \in V_a\} = \sum\{x(G_a) : a \in A\}$.

Let $s : A \to N$ be a function such that $s(a) = x(G_a)$ for every $a \in A$. Put $d_a(G, s) = \sum\{s(v) : (a, v) \in E(G)\}$ for $a \in A$.

We need the following formula for the forest volumes of a graph-composition.

**3.1** [8] *Let $\Gamma = G\{G_a : a \in V(G)\}$. Then*

$$\mathcal{F}(z, \Gamma, x) = \mathcal{F}(z, G, s) \times \prod\{\mathcal{F}(z + d_a(G, s), G_a, x_a) : a \in V(G)\}.$$

We also need the following facts on the forest volume of a digraph.

The forest volume of a graph can be expressed as the determinant of a matrix.

**3.2** [8] *Let $G$ be a directed graph with possible parallel arcs and with no loops, $V(G) = \{v_1, \ldots, v_n\}$, $x : V(G) \to N$, and $c(v_i, v_j)$ denote the number of arcs from $v_i$ to $v_j$ in $G$. Let $M(G, x)$ be the $n \times n$ matrix $\{m_{ij}\}$, where $m_{ij} = -x(v_j)c(v_i, v_j)$ for $i \neq j$ and $m_{ii} = -\sum\{m_{ij} : j \neq i\}$. Then*

$$\mathcal{F}(z, G, x) = z^{-1}\det(zI + M(G, x)),\ \textit{and so}\ \mathcal{V}(\mathcal{F}(G), x) = \det(I + M(G, x)).$$

From **3.2** we have, in particular, the following three claims.

**3.3** [8] *Let $G$ be an acyclic digraph and $d_v(G, x) = \sum\{x(u) : (v, u) \in E(G)\}$. Then*

$$\mathcal{F}(z, G, x) = z^{-1} \prod\{z + d_v(G, x) : v \in V(G)\}.$$

**3.4** *Let $C$ be a directed cycle. Then*

$$\mathcal{F}(z, C, x) = z^{-1} \left(\prod\{z + x(v) : v \in V(C)\} - \prod\{x(v) : v \in V(C)\}\right).$$

**3.5** [8] *Let $K^0$ and $K^1$ denote the arc empty and the complete digraphs with $n$ vertices, respectively. Then*

$$\mathcal{F}(z, K^\epsilon, x) = (z + \epsilon x(K^\epsilon))^{n-1}, \quad \textit{where}\ \epsilon \in \{0, 1\}.$$

## 4. Some generalizations of Hurwitz's identities

In order to illustrate the main idea of the mechanisms of generating polynomial identities, we first describe and prove a simple generalization of Hurwitz's identity **1.2**.

Let $U$ and $V$ be disjoint non-empty sets, $x = \{(v, x(v)) : v \in V, x(v) \in N\}$, and $y = \{(u, y(u)) : u \in U, y(v) \in N\}$, and so $x$ and $y$ are functions from $V$ and $U$ to $N$, respectively. Let $\mathcal{P}$ denote the set of all functions $\alpha : V \to U \cup c$, where $c \notin U \cup V$. Let $P_u(\alpha) = \{v : \alpha(v) = u\}$, and so $\{P_u(\alpha) : u \in U \cup c\}$ is a partition of $V$. If $w \in U \cup c$ is such that $\alpha(v) \neq w$ for every $v \in V$, then $P_w(\alpha) = \emptyset$. We recall that $\mathcal{F}(z, D, x) = z^{-1}$ if $V(D) = \emptyset$.

If $\dot{V}$ is a digraph with the vertex set $V$ and $X \subseteq V$, then let $\dot{X}$ be the subgraph of $\dot{V}$ induced by $X$. If $D \subseteq G$ and $\sigma : V(G) \to N$, we write simply $\mathcal{F}(z, D, \sigma)$ and $T(D, \sigma)$ instead of $\mathcal{F}(z, D, \sigma|_{V(D)})$ and $T(D, \sigma|_{V(D)})$.

**4.1** *Let $\dot{U}$ be the digraph with the vertex set $U$ and with no arcs, $\dot{V}$ be a digraph with the vertex set $V$, and $G$ the digraph with exactly two vertices $u'$, $v'$ and exactly one arc $(v', u')$. Let $\Gamma = G\{G_{u'}, G_{v'}\}$, where $G_{u'} = \dot{U}$ and $G_{v'} = \dot{V}$.* [Clearly, $V(\Gamma) = U \cup V$.] *Then*

$$(z + y(U))z^{|U|-1}\mathcal{F}(z + y(U), \dot{V}, x) = \mathcal{F}(z, \Gamma, x \cup y)$$
$$= \sum \left\{\mathcal{F}(z, \dot{P}_c(\alpha), x) \prod\{zy(u)\mathcal{F}(y(u), \dot{P}_u(\alpha), x) : u \in U\} : \alpha \in \mathcal{P}\right\}.$$

**Proof** (*using* **3.1**, **3.3**, *and* **3.5**). We will establish our identity by finding $\mathcal{F}(z, \Gamma, x \cup y)$ in two different ways.

(**p1**) We first find $\mathcal{F}(z, \Gamma, x \cup y)$, by using the volume formula of **3.1**:

$$\mathcal{F}(z, \Gamma, x \cup y) = \mathcal{F}(z, G, s)\mathcal{F}(z + d_{u'}(G, s), \dot{U}, y)\mathcal{F}(z + d_{v'}(G, s), \dot{V}, x), \tag{4.1}$$

where $s = \{(u', s(u')), (v', s(v'))\}$, $s(u') = y(U)$, and $s(v') = x(V)$.

Since $G$ has two vertices $u'$, $v'$ and exactly one arc $(v', u')$, we have $d_{v'}(G, s) = y(U)$ and, by **3.3**, $\mathcal{F}(z, G, s) = z + y(U)$. Therefore

$$\mathcal{F}(z + d_{v'}(G, s), \dot{V}, x) = \mathcal{F}(z + y(U), \dot{V}, x).$$

Since $\dot{U}$ has no arcs, by **3.5**, $\mathcal{F}(z + d_{u'}(G, s)), \dot{U}, y) = z^{|U|-1}$. Thus from (4.1) we have

$$\mathcal{F}(z, \Gamma, x \cup y) = (z + y(U))z^{|U|-1}\mathcal{F}(z + y(U), \dot{V}, x). \tag{4.2}$$

(**p2**) Now let us find $\mathcal{F}(z, \Gamma, x \cup y)$ in another way. Note that the digraph $\Gamma$ has no loops or parallel arcs. Let $w = x \cup y$ and $w^c = w \cup \{(c, z)\}$.

By the definition of the forest volume of a digraph,

$$\mathcal{F}(z, \Gamma, w) = \mathcal{T}_c(\Gamma^c, w^c) = \sum\{\mathcal{T}_c(T, w^c) : T \in \mathcal{T}_c(\Gamma^c)\}, \tag{4.3}$$

where $\Gamma^c = \Gamma \cup \{(\gamma, c) : \gamma \in V(\Gamma)\}$ and $\mathcal{T}_c(T, w^c) = \prod\{w(v)^{d(v,T)-1} : v \in V(T)\}$.

Since $\Gamma$ has no arc $(u, v)$ with $u \in U$, every spanning ditree $T$ of $\Gamma^c$ contains the arc set $(U, c) = \{(u, c) : u \in U\}$. Let $T' = T - (U, c)$. Clearly $T'$ is a spanning diforest of $\Gamma^c$, and so each component of $T'$ is a ditree. Let $T_u$ be the component of $T'$ containing $u \in U \cup c$. Since there is no arc going out of $u$ in $\Gamma^c - (U, c)$, clearly $u$ is the root of $T_u$. Then

$$\mathcal{T}_c(T, w^c) = \mathcal{T}_c(T_c, w^c) \prod\{zy(u)\mathcal{T}(T_u, w) : u \in U\}. \tag{4.4}$$

For $v \in V$, put $\alpha(T, v) = u$ if $v \in V(T_u)$. Then $\alpha(T) : V \to U \cup c$ is a function.
For $\alpha \in \mathcal{P}$, let $\mathcal{D}_\alpha = \{T \in \mathcal{T}_c(\Gamma^c) : \alpha(T) = \alpha\}$.
Obviously, $\mathcal{D}_\alpha = \{T \in \mathcal{T}_c(\Gamma^c) : V(T_u - u) = P_u(\alpha), u \in U \cup c\}$ and $\mathcal{D}_\alpha \neq \emptyset$.
Let $S(\Gamma, w^c, \alpha) = \sum\{\mathcal{T}_c(T, w^c) : T \in \mathcal{D}_\alpha\}$.

By (4.3),

$$\mathcal{F}(z, \Gamma, w) = \sum\{\mathcal{T}_c(T, w^c) : T \in \mathcal{T}_c(\Gamma^c)\} = \sum\{S(\Gamma, w^c, \alpha) : \alpha \in \mathcal{P}\}. \tag{4.5}$$

By (4.4),

$$S(\Gamma, w^c, \alpha) = \sum\{\mathcal{T}_c(T^c, w^c) \prod\{zy(u)\mathcal{T}(T_u, w) : u \in U\} : T \in \mathcal{D}_\alpha\}.$$

For $u \in U \cup c$, let $\ddot{P}_u(\alpha)$ denote the subgraph of $\Gamma^c$ induced by $P_u(\alpha) \cup u$, and as above, $\dot{P}_u(\alpha)$ the subgraph of $\Gamma^c$ (or, the same thing, of $\dot{V}$) induced by $P_u(\alpha)$ (possibly, $P_u(\alpha = \emptyset$ for some $u \in U \cup c$). Then

$$\begin{aligned} S(\Gamma, w^c, \alpha) &= \left(\sum\{\mathcal{T}_c(T, w^c) : T \in \mathcal{T}_c(\ddot{P}_c(\alpha)\}\right) \\ &\quad \times \prod\left\{zy(u) \sum\{\mathcal{T}_u(T, w) : T \in \mathcal{T}_u(\ddot{P}_u)\} : u \in U\right\} \\ &= \mathcal{F}(z, \dot{P}_c(\alpha), x) \prod\{zy(u)\mathcal{F}(y(u), \dot{P}_u(\alpha), x) : u \in U\}. \end{aligned}$$

Now the required identity follows from (4.2), (4.5), and the last equality. $\square$

Thus **4.1** provides infinitely many identities corresponding to digraphs $G_{v'} = \dot{V}$.

From **4.1** we have, in particular:

**4.2** *Let $U$ and $V$ be disjoint sets, $x : V \to N$, and $y : U \to N$. Then*

$$\begin{aligned} &(z + y(U))z^{|U|-1}(z + y(U) + x(V))^{|V|-1} \\ &\quad = \sum\left\{(z + x(P_c(\alpha))^{|P_c(\alpha)|-1} \prod\{zy(u)(y(u) + x(P_u(\alpha))^{|P_u(\alpha)|-1}) : u \in U\}\right. \\ &\qquad \left. : \alpha \in \mathcal{P}\right\}. \end{aligned}$$

**Proof** (*using* **3.5** *and* **4.1**). Let $\dot{V}$ in **4.1** be a complete digraph. Let, as in **4.1**, $\dot{P}_u(\alpha)$ denote the subgraph of $\dot{V}$ induced by $P_u(\alpha)$, $u \in U \cup c$. Since $\dot{V}$ is a complete digraph, clearly each $\dot{P}_u(\alpha)$ is also a complete digraph. Therefore by **3.5**,

$$\mathcal{F}(z, \dot{V}, x) = (z + x(V))^{|V|-1} \quad \text{and} \quad \mathcal{F}(z, \dot{P}_u(\alpha), x) = (z + x(P_u(\alpha))^{|P_u(\alpha)|-1}$$
$$\text{for } u \in U \cup c.$$

Now the required identity follows from **4.1**. $\square$

Hurwitz's identity **1.2** is a particular case of **4.2** when $U = \{u\}$ with $P_c(\alpha) = A$, and $P_u(\alpha) = B$.

From **2.3** and **4.1** we have:

**4.3** *Let $G$ be a digraph, $U \subseteq V(G)$, $x : V(G) \to N$, and $\mathcal{P}$ the set of all functions from $U$ to $V(G - U)$. Then*

$$\begin{aligned} x(U)\mathcal{F}(x(U), G - U, x) &= \mathcal{V}(\mathcal{F}(G, U), x) \\ &= \sum\left\{\prod\{x(u)\mathcal{F}(x(u), P_u(\alpha), x) : u \in U\} : \alpha \in \mathcal{P}\right\}. \end{aligned}$$

From **4.3** we have, in particular, the following identity.

**4.4** *Let $U$ and $V$ be disjoint non-empty sets, $K$ the complete graph with the vertex set $U \cup V$, $x : U \cup V \to N$, and $\mathcal{P}$ the set of functions from $V$ to $U$. Then*

$$\begin{aligned} x(U)(x(U \cup V)^{|V|-1} &= \mathcal{V}(\mathcal{F}(K, U), x) = \mathcal{V}(\mathcal{F}(K_U), x) \\ &= \sum \left\{ \prod \{x(u)(x(u) + x(P_u(\alpha))^{|P_u(\alpha)|-1} : u \in U\} : \alpha \in \mathcal{P} \right\}. \end{aligned}$$

Now we will describe a generalization of Hurwitz's identity **1.3**.

**4.5** *Let $\dot{U}$ be the digraph with the vertex set $U$ and with no arcs, $\dot{V}$ a digraph with the vertex set $V$, and $G$ the complete digraph on two vertices $u'$ and $v'$. Let $\Gamma = G\{G_{u'}, G_{v'}\}$, where $G_{u'} = \dot{U}$ and $G_{v'} = \dot{V}$. Then*

$$\begin{aligned} &(z + x(V) + y(U))(z + x(V))^{|U|-1} \mathcal{F}(z + y(U), \dot{V}, x) = \mathcal{F}(z, \Gamma, x \cup y) \\ &\quad = \sum \left\{ \mathcal{F}(z, \dot{P}_c(\alpha), x) \prod \{(z + x(V \setminus P_u(\alpha)) y(u) (\mathcal{F}(y(u), \dot{P}_u(\alpha), x) : u \in U\} \right. \\ &\qquad \left. : \alpha \in \mathcal{P} \right\}. \end{aligned}$$

The left side of the above identity follows from the volume formula in **3.1** applied to the graph-composition $\Gamma = G\{G_{u'}, G_{v'}\}$ in **4.5**. On the other hand, we know that $\mathcal{F}(z, \Gamma, x) = \mathcal{T}_c(\Gamma^c, x^c)$. Thus the right side of the identity in **4.5** can be obtained by classifying the ditrees of $\Gamma^c$ by the functions $\alpha$ from $V$ to $U \cup c$ that they induce (like in the proof of **4.1**), by considering every ditree $T$ in each class, and by applying the formula in **2.1** to every arc $uv$ in $T$ such that $u \in U$ and $v \in (V \setminus P_u(\alpha)) \cup c$.

If, in particular, $\dot{V}$ is a complete graph, then from **3.5** and **4.5** we have:

**4.6** *Let $U$ and $V$ be disjoint sets, $x = \{(v, x(v)) : v \in V, x(v) \in N\}$ and $y = \{(u, y(u)) : u \in U, y(u) \in N\}$. Then*

$$\begin{aligned} (z + y(U) + x(V))^{|V|} = \sum \Big\{ &(z + x(P_c(\alpha))^{|P_c(\alpha)|-1} \prod \{y(u)(y(u) + x(P_u(\alpha))^{|P_u(\alpha)|-1}) \\ &\times (z + x(V \setminus P_u(\alpha)) : u \in U\} : \alpha \in \mathcal{P} \Big\}. \end{aligned}$$

Hurwitz's identity **1.3** is a particular case of **4.6** when $U = \{u\}$, $y(u) = y$, $P_u(\alpha) = A$, and $P_c(\alpha) = B$, and so $V \setminus P_u(\alpha) = P_c(\alpha) = B$.

## 5. Generating polynomial identities

Now we will describe further generalizations of **4.1** and **4.5** (which provide reacher mechanisms for generating identities) using the volume formula **3.1** for a graph-composition $\Gamma = G\{G_a : a \in V(G)\}$.

In this section we will use the following notation. As above, $L = L(G)$ and $R = R(G)$ denote the set of sources and sinks of $G$, respectively. Let $G$ be a digraph with $V(G) = A$ and $\{G_a : a \in A\}$ be a collection of disjoint digraphs. Let $V(G_a) = V_a$ and $B_a = \{(a, b) : b \in V_a\}$. Since there is a natural bijection from $V_a$ to $B_a$ for every $a \in A$, we can assume that $V_a = B_a$ and $V(\Gamma) = V = \cup\{V_a : a \in A\}$.

Let $\Gamma^c$ be the graph obtained from $\Gamma$ by adding a new vertex $c$ and the set of new arcs $\{(\gamma, c) : \gamma \in V(\Gamma)\}$, and so $\Gamma^c = \Gamma \cup \{(\gamma, c) : \gamma \in V(\Gamma)\}$. Let $N_a = \{d \in A : (a, d) \in E(G)\}$ and $V^a = \cup\{V_d : d \in N_a\}$. Given $a \in A$, let $\mathcal{P}_a$ denote the set of functions $\alpha_a : V_a \to V^a \cup c$. Let $P_u(\alpha_a) = \{v \in V_a : \alpha_a(v) = u\}$. Clearly $\{P_u(\alpha_a) : u \in V^a \cup c\}$ is a partition of $V_a$. As above, $\dot{P}_u(\alpha_a)$ is the subgraph of $G_a$ induced by $P_u(\alpha_a)$ and $P_u(\alpha_a) = \emptyset$ if $\alpha_a(v) \neq u$ for every $v \in V_a$. We also recall that $\mathcal{F}(z, G, x) = z^{-1}$ if $V(G) = \emptyset$.

By using **3.1** and **3.3**, one can prove the following.

**5.1** *Let $x : V \to N$, $x(c) = z$, and $s : A \to N$ be functions such that $s(a) = x(V_a)$. Let $G$ be an acyclic digraph with $V(G) = A$, $G_a$ a digraph with $V(G_a) = V_a$ for every $a \in A$, and $\Gamma = G\{G_a : a \in V(G)\}$. Then*

$$\begin{aligned} &z^{-1} \prod\{z + d_v(G, s) : v \in V(G)\} \times \prod\{\mathcal{F}(z + d_a(G, s), G_a, x_a) : a \in V(G)\} \\ &\quad = \mathcal{F}(z, \Gamma, x) = z^{-1} \prod\{z\mathcal{F}(z, G_a, x) : a \in R\} \times \sum \Big\{\prod\{x(u)\mathcal{F}(x(u), \dot{P}_u(\alpha_a), x) \\ &\qquad : u \in V^a \cup c, a \in A \setminus R\} : \alpha_a \in \mathcal{P}_a, a \in A \setminus R\Big\}. \end{aligned}$$

The proof of this identity is by the interpretation of both sides as the forest volume of $G\{G_a : a \in V(G)\}$ and uses arguments similar to those in the proof of **4.1**, where the digraph $G$ (of two vertices and one arc) is replaced by an arbitrary acyclic digraph.

Since $G$ is an arbitrary acyclic digraph and each brick $G_a$ is an arbitrary digraph, theorem **5.1** provides a mechanism (an algorithm) for generating polynomial identities. We can obtain a large variety of identities by considering in **5.1** various specific digraphs $G$ and the $G_a$'s. For example, a large variety of identities correspond to graph-compositions whose frame $G$ is a ditree (or a specific ditree, for instance, a dipath, a distar, etc. — see the next theorem). Obviously, an induced subgraph of a complete (an arc empty) digraph is also complete (respectively, arc empty). We know the formula for the forest volume of these graphs (see **3.5**) and these formulas have exactly one parameter (the number of vertices). Therefore if each $G_a$ in the above construction is either complete or arc empty, then **3.5** and **5.1** provide a variety of explicit identities corresponding to different frames.

More generally, in order to obtain varieties of explicit identities (depending on a finite number of parameters), we may well consider (instead of the class of complete digraphs) any class that is closed under the digraph operation of taking an induced subgraph, provided an explicit forest volume formula for graphs of this class can be found (in terms of a finite set of parameters). For example, natural classes of digraphs to consider are the classes of complete bipartite and, moreover, $k$-partite digraphs. In [8] we describe a more general class of this nature (the class of so called *totally decomposable digraphs*). Moreover, we gave a simple algorithm for finding the explicit forest volume formulas (based on [5,6]) for digraphs of this class. These formulas contain a finite number of parameters. The classes of complete and complete $k$-partite digraphs are subclasses of this class. It follows that there are a great variety of graphs that can be used as non-trivial bricks in **5.1** to generate explicit identities (and to do it by computer). Thus **5.1** describes a mechanism that provides infinitely many identities that are generalizations of Hurwitz's identity **1.2** and are related to the acyclic digraphs.

Here is a specification of **5.1** when the frame $G$ of the graph-decomposition $\Gamma = G\{G_a : a \in V(G)\}$ is a ditree $T$ with $V(T) = A$ and root $r$, and so $L = L(T)$ is the set of leaves of $T$ and $R(T) = \{r\}$. Put $p(u) = v$ if $(u, v) \in E(T)$. Since $T$ is a ditree, clearly $p$ is a function from $A \setminus r$ to $A$ which is called the *pointer* of $T$.

From **3.3** and **5.1** we have, in particular:

**5.2** *Suppose that all assumptions of* **5.1** *are satisfied. Suppose also that $G$ is a ditree $T$ as above. Then*

$$\prod\{z+s(a) : a \in A \setminus (L \cup r)\} \times \prod\{\mathcal{F}(z+s(p(a)), G_a, x) : a \in A\} = \mathcal{F}(z, \Gamma, x)$$
$$= \mathcal{F}(z, G_r, x) \sum \left\{ \prod\{x(u)\mathcal{F}(x(u), \dot{P}_u(\alpha_a), x) : u \in V_{p(a)} \cup c, a \in A \setminus r\} \right.$$
$$\left. : \alpha_a \in \mathcal{P}_a, a \in A \setminus r \right\}.$$

Now we will describe one possible further generalization of **4.5** based on a graph-composition $\Gamma = G\{G_a : a \in V(G)\}$ whose frame $G$ is not just a complete graph on two vertices as in **4.5**. Let $T$ be a ditree as in **5.2**, $p$ the pointer of $T$, and $B \subseteq L(T)$. Let $G = T \cup \{(r, b) : b \in B\}$ be the frame of our graph-decomposition $\Gamma$. Let $\mathcal{P} = \bigotimes\{\mathcal{P}_a : a \in A - r\}$ and $\alpha = \{\alpha_a : a \in A - r\} \in \mathcal{P}$. For $b \in B$ let $bTr = v_k, v_{k-1} \ldots, v_0$ be the path in $T$ from $v_k = b$ to $v_0 = r$. Put $\sigma_\alpha(u) = v$ if $u \in V_b$, $v \in V_r$, and $v = \alpha_1\alpha_2 \cdots \alpha_k(u)$, where $\alpha_i := \alpha_{v_i}$. Let $V_B = \{V_b : b \in B\}$ and $\sigma_\alpha^{-1}(v) = \{u \in V_B : \sigma_\alpha(u) = v\}$ for $v \in V_r$ (possibly, $\sigma_\alpha^{-1}(v) = \emptyset$ for some $v \in V_r$).

**5.3** *Suppose that all assumptions of* **5.1** *are satisfied except that $G$ is a graph defined above. Suppose also that $E(G_r) = \emptyset$. Then*

$$\mathcal{F}(z, G, s) \times \prod\{\mathcal{F}(z+d_a(G, s), G_a, x_a) : a \in V(G)\} = \mathcal{F}(z, \Gamma, x)$$
$$= z^{-1} \sum \left\{ \prod\{(z + x(V_B \setminus \sigma_\alpha^{-1}(v)) : v \in V_r\} \prod\{x(u)\mathcal{F}(x(u), \dot{P}_u(\alpha_a), x) \right.$$
$$\left. : u \in V_{p(a)} \cup c, a \in A \setminus r\} : \alpha = \{\alpha_a \in \mathcal{P}_a, a \in A \setminus r\} \in \mathcal{P} \right\}.$$

The proof of **5.3** uses **2.1** and is a combination of the proofs of **4.5** and **5.1**. The forest volume $\mathcal{F}(z, G, s)$ can be obtained using the determinant formula **3.2**. For example, if $G$ is a directed cycle, then $\mathcal{F}(z, G, s)$ is given by **3.4**.

Obviously, **5.3** with $B = \emptyset$ is a particular case of **5.2** when $E(G_r) = \emptyset$, and **4.5** is a particular case of **5.3** when $T$ has one arc $(v', u')$ and $B = \{v'\}$.

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