Schmidt's continued fractions

Here we give a quick summary of Asmus Schmidt's continued fraction algorithm [S1], its ergodic theory [S2], and further results of Hitoshi Nakada concerning these [N1], [N2], [N3]. Define the following matrices in $\mathrm{PGL}(2,\mathbb{Z}[i])$:

$$V_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix},\ V_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix},\ V_3 = \begin{pmatrix} 1-i & i \\ -i & 1+i \end{pmatrix},$$
$$E_1 = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & -1+i \\ 0 & i \end{pmatrix},\ E_3 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix}.$$

Note that

$$S^{-1}V_iS = V_{i+1},\ S^{-1}E_iS = V_{i+1},\ S^{-1}CS = C \text{ (indices modulo 3)}$$

where S elliptic of order three $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, and that

$$\tau \circ m \circ \tau = m^{-1}$$

for the Möbius transformations m induced by $\{V_i, E_i, C\}$ (here τ is complex conjugation).

In [S1] Schmidt uses infinite words in these letters to represent complex numbers as infinite products $z = \prod_n T_n, T_n \in \{V_i, E_i, C\}$ in two different ways. Let $M_N = \prod_{n=1}^{N} T_n$. We have *regular chains*

$$\det M_N = \pm 1 \Rightarrow T_{n+1} \in \{V_i, E_i, C\},\ \det M_N = \pm i \Rightarrow T_{n+1} \in \{V_i, C\},$$

representing z in the upper half-plane $\mathcal{I}$ (the model circle) and *dually regular chains*

$$\det M_N = \pm i \Rightarrow T_{n+1} \in \{V_i, E_i, C\},\ \det M_N = \pm 1 \Rightarrow T_{n+1} \in \{V_i, C\},$$

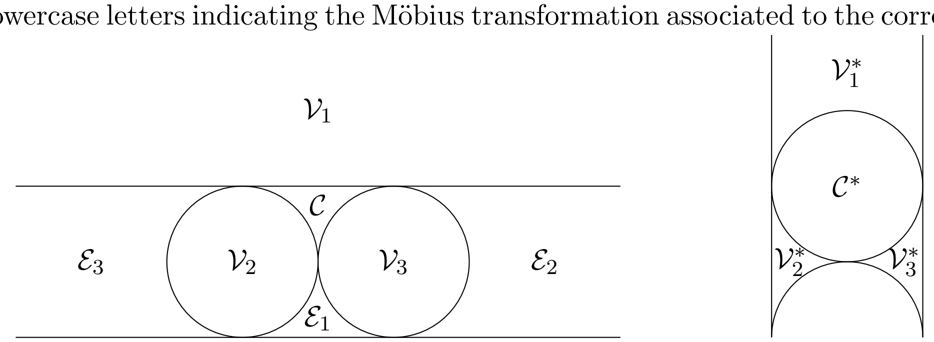
representing $z \in \{0 \le x \le 1, y \ge 0, |z - 1/2| \ge 1/2\} =: \mathcal{I}^*$ (the model triangle). The model circle is a disjoint union of four triangles and three circles, and the model triangle is a disjoint union of three triangles and one circle (pictured below):

$$\mathcal{I} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{C}$$
$$\mathcal{I}^* = \mathcal{V}_1^* \cup \mathcal{V}_2^* \cup \mathcal{V}_3^* \cup \mathcal{C}^*$$

where

$$\mathcal{V}_i = v_i(\mathcal{I}),\ \mathcal{E}_i = e_i(\mathcal{I}^*),\ \mathcal{C} = c(\mathcal{I}^*),\ \mathcal{V}_i^* = v_i(\mathcal{I}^*),\ \mathcal{C}^* = c(\mathcal{I}),$$

(lowercase letters indicating the Möbius transformation associated to the corresponding matrix).



By considering $z = \prod_n T_n$ we obtain rational approximations $p_i^{(N)}/q_i^{(N)}$ to z by

$$M_N \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} p_1^{(N)} & p_2^{(N)} & p_3^{(N)} \\ q_1^{(N)} & q_2^{(N)} & q_3^{(N)} \end{pmatrix}$$

(which are the orbits of $\infty, 0, 1$ under the partial products $m_N = t_1 \circ \cdots \circ t_N$).

The shift map T on $X = \mathcal{I} \cup \mathcal{I}^* = \{\text{chains, dual chains}\}$ maps X to itself via Möbius transformations, specifically (mapping $\mathcal{V}_i, \mathcal{C}^*$ onto $\mathcal{I}$ and $\mathcal{V}_i^*, \mathcal{E}_i, \mathcal{C}$ onto $\mathcal{I}^*$)

$$T(z) = \begin{cases} v_i^{-1}z & z \in \mathcal{V}_i \cup \mathcal{V}_i^* \\ e_i^{-1}z & z \in \mathcal{E}_i \\ c^{-1}z & z \in \mathcal{C} \cup \mathcal{C}^* \end{cases}.$$

The shift $T : X \to X$ is shown to be ergodic ([S2], theorem 5.1) with respect to the following probability measure

$$\tilde{f}(z) = \begin{cases} \frac{1}{2\pi^2}(h(z) + h(sz) + h(s^2z)) & z = x + yi \in \mathcal{I} \\ \frac{1}{2\pi}\frac{1}{y^2} & z = x + yi \in \mathcal{I}^* \end{cases}$$

where

$$h(z) = \frac{1}{xy} - \frac{1}{x^2}\arctan\left(\frac{x}{y}\right).$$

By inducing to $X \setminus \cup_i (\mathcal{V}_i \cup \mathcal{V}_i^*)$ Schmidt gives "faster" convergents $\hat{p}_\alpha^{(n)}/\hat{q}_\alpha^{(n)}$ and a sequence of exponents e_n (1 for E_i, C, and the return time k for V_i^k). He then gives results analagous to those of simple continued fractions via the pointwise ergodic theorem, including the arithmetic and geometric mean of the exponentswhich exist for almost every z ([S2] theorem 5.3)

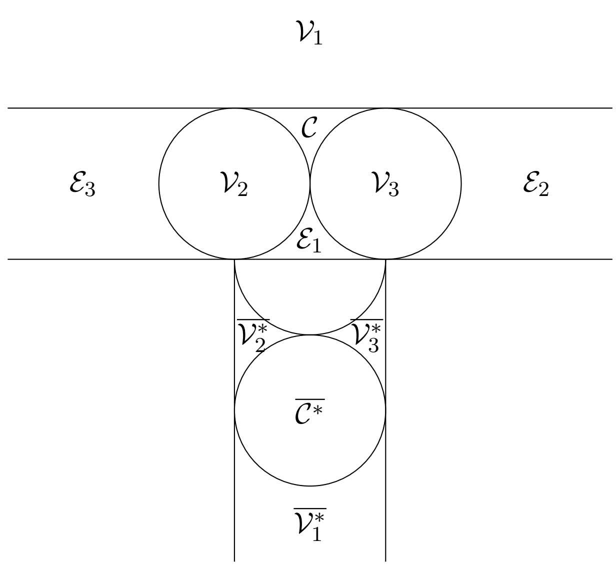
$$\lim_{n\to\infty}\left(\prod_{i=1}^n e_i\right)^{1/n} = 1.26\ldots, \quad \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n e_i = 1.6667\ldots.$$

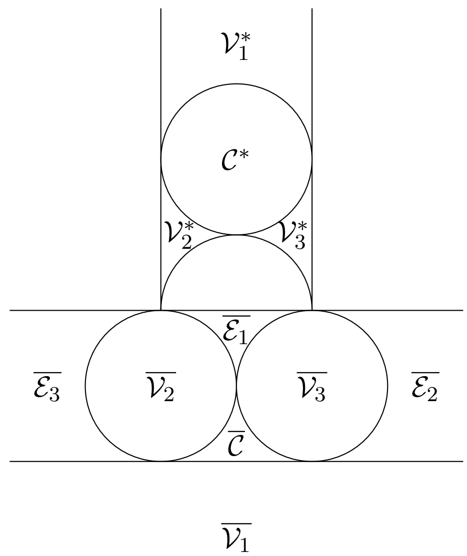
In [N1], Nakada constructs an invertible extension of T on a space of geodesics in two copies of three-dimensional hyperbolic space. In one copy we take geodesics from $\overline{\mathcal{I}^*}$ to $\mathcal{I}$ and in the other the geodesics from $\overline{\mathcal{I}}$ to $\mathcal{I}^*$ where the overline indicates complex conjugation. The regions are pictured below. The extension acts as Schmidt's T depending on the second coordinate. Nakada doesn't provide a second proof of ergodicity, but quotes Schmidt's result. Also in [N1], results about the the density of Gaussian rationals p/q that appear as convergents and satisfy $|z - p/q| < c/|q|^2$ are obtained. For instance ([N1], theorem 7.3), for almost every $z \in X$ and $0 < c < \frac{1}{1+1\sqrt{2}}$ it holds that

$$\lim_{N\to\infty}\frac{1}{N}\#\{p/q \in \mathbb{Q}(i) : p/q = p_i^{(n)}/q_i^{(n)},\ 1 \le n \le N,\ i = 1,2,3,\ |z - p/q| < c/|q|^2\} = \frac{c^2}{\pi}.$$

In [N2], main theorem, Nakada describes the rate of convergence of Schmidt's convergents. Namely for almost every z

$$\lim_{n\to\infty}\frac{1}{n}\log|q_i^{(n)}| = \frac{E}{\pi}, \quad \lim_{n\to\infty}\frac{1}{n}\log\left|z - \frac{p_i^{(n)}}{q_i^{(n)}}\right| = -\frac{2E}{\pi}, \quad E = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}.$$





For reference, the relationship between the super-apollonian Möbius generators $\{\mathfrak{s}_i, \mathfrak{s}_i^\perp\}$ and Schmidt's $\{v_i, e_i, c\}$ are

$$\mathfrak{s}_1 = c^2 \circ \tau,\ \mathfrak{s}_2 = e_1^2 \circ \tau,\ \mathfrak{s}_3 = e_2^2 \circ \tau,\ \mathfrak{s}_4 = e_3^2 \circ \tau,$$

$$\mathfrak{s}_1^\perp = 1 \circ \tau,\ \mathfrak{s}_2^\perp = v_1^2 \circ \tau,\ \mathfrak{s}_3^\perp = v_2^2 \circ \tau,\ \mathfrak{s}_4^\perp = v_3^2 \circ \tau.$$

References

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