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ON HYPERSEQUENCES OF AN ARBITRARY SEQUENCE AND THEIR WEIGHTED SUMS

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Abstract

We first study the hypersequences $\left(a_n^{(r)}\right)_{n\in\mathbb{N}_0}$, $r \in \mathbb{N}_0$, of an arbitrary sequence $(a_n)_{n\in\mathbb{N}_0}$. Then we apply the results to four different sequences, namely the constant sequence $a_n = 1$, $n \in \mathbb{N}_0$, the characteristic function of $\{1\}$, the Fibonacci sequence and the Lucas sequence. In the last two cases we obtain some new results on the hyperfibonacci and hyperlucas numbers. After that we investigate weighted sums of the type $\sum_{k=0}^{n} k^\ell a_k^{(r)}$, $\ell \in \mathbb{N}_0$, and derive a recurrence relation and its solution. This solution depends on the expression $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(k+n+1)^\ell$. We derive some old and new properties of the generalized expression $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(kx+y)^\ell$ and apply the results to the four sequences as above. In this way we obtain known and new formulas for the sums of powers of the first n consecutive positive integers and for weighted Fibonacci and Lucas sums.

1. Introduction

Let $(a_n)_{n\in\mathbb{N}_0}$ be an arbitrary sequence (of real or complex numbers). Then its *hypersequence of the rth generation* is defined recursively for all $n, r \in \mathbb{N}_0$ as

$$a_n^{(r)} := \sum_{k=0}^{n} a_k^{(r-1)}, \quad \text{and} \quad a_n^{(0)} := a_n.$$

For $r = 1$, we have $a_n^{(1)} = \sum_{k=0}^{n} a_k^{(0)} = \sum_{k=0}^{n} a_k$ and this is the sequence of partial sums of $(a_n)_{n\in\mathbb{N}_0}$; for $r = 2$, $a_n^{(2)} = \sum_{k=0}^{n} a_k^{(1)} = \sum_{k=0}^{n}\left(\sum_{j=0}^{k} a_j\right)$ is the sequence of partial sums of $(a_n^{(1)})_{n\in\mathbb{N}_0}$; and so on.

By this definition we obtain an array $(a_n^{(r)})$, where $r \in \mathbb{N}_0$ is the row and $n \in \mathbb{N}_0$ is the column of this array (see Table 1). It is a special case of the "tableau de sommes" of Lucas (see [19, pp. 7–12]) with equal initial values $a_0^{(1)} = a_0^{(2)} = \cdots = a_0^{(r)} = a_0$. The array has recently been studied by Dil and Mező [10].

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$r\backslash n$	0	1	2	3
0	a_0	a_1	a_2	a_3
1	a_0	a_0+a_1	$a_0+a_1+a_2$	$a_0+a_1+a_2+a_3$
2	a_0	$2a_0+a_1$	$3a_0+2a_1+a_2$	$4a_0+3a_1+2a_2+a_3$
3	a_0	$3a_0+a_1$	$6a_0+3a_1+a_2$	$10a_0+6a_1+3a_2+a_3$
4	a_0	$4a_0+a_1$	$10a_0+4a_1+a_2$	$20a_0+10a_1+4a_2+a_3$
5	a_0	$5a_0+a_1$	$15a_0+5a_1+a_2$	$35a_0+15a_1+5a_2+a_3$

Table 1: Terms of hypersequences $(a_n^{(r)})_{n\in\mathbb{N}_0}$

In this paper, we first mention a formula that gives the hypersequence of the rth generation of an arbitrary sequence $(a_n)_{n\in\mathbb{N}_0}$ as the partial sums of another sequence and apply the results to four special sequences, namely the constant sequence $a_n=1$, $n\in\mathbb{N}_0$, the characteristic function of $\{1\}$, the Fibonacci sequence, and the Lucas sequence. In the last two cases we obtain some new results on the hyperfibonacci and hyperlucas numbers.

Then, using Abel's summation by parts, we study the weighted sums of the type $\sum_{k=0}^{n} k^\ell a_k^{(r)}$, $\ell,n,r\in\mathbb{N}_0$, and derive a recurrence relation and its solution. This solution depends strongly on the expression $c_{\ell,m}(n):=\sum_{k=0}^{m}(-1)^k\binom{m}{k}(k+n+1)^\ell$ and shows that the weighted sums $\sum_{k=0}^{n} k^\ell a_k^{(r)}$ can be given as a linear combination of the terms $a_n^{(r+1)}, a_n^{(r+2)},\ldots,a_n^{(r+\ell+1)}$, $\ell\in\mathbb{N}_0$. We also derive some known and new properties of $c_{\ell,m}(n)$ and its generalization $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(kx+y)^\ell$, $x,y\in\mathbb{C}$.

Finally, we apply the results to the four special sequences defined above. We obtain some known and new formulas for the sums of powers of the first n consecutive positive integers, and for the weighted Fibonacci and Lucas sums, also known in the literature as "Ledin and Brousseau's summation problems", since Ledin and Brousseau [6, 18] started studying these sums in 1967. Since then several authors have developed different methods to study these sums. We first mention the finite difference approach for the more general sums $\sum_{k=0}^{\ell} k^\ell F_{k+s}$ by Brousseau [6] and the approach of Ledin [18], who showed that the weighted Fibonacci and Lucas sums can be expressed in the form $\sum_{k=0}^{\ell} k^\ell F_k = F_nP_1(\ell,n)+F_{n+1}P_2(\ell,n)+C(\ell)$ and $\sum_{k=0}^{\ell} k^\ell L_k = L_nP_1(\ell,n)+L_{n+1}P_2(\ell,n)+K(\ell)$, where $P_1(\ell,n)$ and $P_2(\ell,n)$ are polynomials in n of degree ℓ, and $C(\ell)$, $K(\ell)$ are two constants depending only on the degree ℓ. For these polynomials, Ledin [18] gave the expressions $P_i(\ell,n)=\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}M_{i,j}n^{\ell-j}$, $i=1,2$, where $M_{1,j}$ and $M_{2,j}$ are integer sequences, which were later determined by Zeitlin [26, Equations (6.1) and (6.2)] as $M_{i,j}=\sum_{k=0}^{j} k!\left\{{j \atop k}\right\}F_{k+i}$, $i=1,2$. Finally, we mention the recent works of Ollerton and Shannon [22, 25], Dresden [11], and Adegoke [1].

A completely different approach was given by Gauthier [12]. His method involves the differential operator $x(d/dx)$ and gives the same non-constant terms (depending on n) as our method, but the constant term has no obvious relation to the non-constant terms. In our method, the constant term is obtained by evaluating the non-constant terms at $n = 0$.

2. Hypersequences

The first expression of the next theorem is well-known (see, for example, [10, Proposition 2] for the special case $a_0^{(i)} = a_0$ for all $i \in \{1, 2, \ldots, k\}$), whereas the second one follows from the fact that $k \in \{0, 1, \ldots, n\}$ if and only if $n - k \in \{0, 1, \ldots, n\}$.

Theorem 1. *Let $(a_n)_{n\in\mathbb{N}_0}$ be an arbitrary sequence (of real or complex numbers) and $(a_n^{(r)})_{n\in\mathbb{N}_0}$, $r \in \mathbb{N}_0$, be its hypersequence of the rth generation as defined before. Then for all $r \in \mathbb{N}$ and $n \in \mathbb{N}_0$,*

$$a_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} a_k = \sum_{k=0}^{n} \binom{r+k-1}{k} a_{n-k}. \tag{2.1}$$

Throughout the paper, we follow the definition of the binomial coefficients $\binom{r}{k}$, $r \in \mathbb{R}$, $k \in \mathbb{Z}$, given in [14, Definition (5.1)], namely $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$ if $k \geq 0$, and $\binom{r}{k} = 0$ if $k < 0$.

The array $\left(a_n^{(r)}\right), r, n \in \mathbb{N}_0$, has the following property (see [10, Equation (1)]).

Theorem 2. *For all $r, n \in \mathbb{N}_0$, we have*

$$a_{n+1}^{(r+1)} = a_n^{(r+1)} + a_{n+1}^{(r)}. \tag{2.2}$$

This theorem says that $a_{n+1}^{(r+1)}$ is calculated by adding the value to its left and the value above it.

Remark 1. Equation (2.2) is a recurrence relation to calculate the $(n+1)$st column of the array, once the nth column is known, $n \in \mathbb{N}_0$. For example, by definition we have $a_0^{(r)} = a_0$. Hence, by Theorem 2 for $n = 0$ we have $a_1^{(r+1)} = a_1^{(r)} + a_0^{(r+1)} = a_1^{(r)} + a_0$ with the initial value $a_1^{(0)} = a_1$. The solution of this first-order recurrence relation is given by $a_1^{(r)} = a_1 + a_0 r$, $r \geq 0$. Similarly, $a_2^{(r+1)} = a_2^{(r)} + a_1^{(r+1)} = a_2^{(r)} + a_1 + a_0(r+1)$ with the initial value $a_2^{(0)} = a_2$. The solution of this first-order recurrence is given by $a_2^{(r)} = a_2 + a_1\binom{r}{1} + a_0\binom{r+1}{2}$, $r \geq 0$, and so on. Note that the second expression of (2.1) gives the general term of the nth column.

2.1. Examples

We now consider four examples.

Example 1. Let $a_n = 1$, $n \geq 0$, be the constant sequence 1 (the sequence A000012 in the OEIS [21]). Then we get the array $A_{Fe}(r, n)$ (see Table 2) called by Lucas (see [19, p. 83]) "le carré arithmétique de Fermat". According to (2.1) the general term of this array is given by

$$A_{Fe}(r,n) = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} = \binom{n+r}{r}, \quad r,n \in \mathbb{N}_0, \tag{2.3}$$

where we have used the well-known formula (see [14, Table 174, p. 174]),

$$\sum_{k=0}^{n} \binom{k+r-1}{r-1} = \binom{n+r}{r}. \tag{2.4}$$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11
2	1	3	6	10	15	21	28	36	45	55	66
3	1	4	10	20	35	56	84	120	165	220	286
4	1	5	15	35	70	126	210	330	495	715	1001
5	1	6	21	56	126	252	462	792	1287	2002	3003
6	1	7	28	84	210	462	924	1716	3003	5005	8008
7	1	8	36	120	330	792	1716	3432	6435	11440	19448
8	1	9	45	165	495	1287	3003	6435	12870	24310	43758

Table 2: "Le carré arithmétique de Fermat" $A_{Fe}(r,n)$

Example 2. Let $a_n = (n=1), n \in \mathbb{N}_0$, be the characteristic function of $\{1\}$, where (S) is Iverson's convention meaning 1 if the statement S is true and 0 if it is false. It is the sequence A063524 in [21], also called the "Dirichlet sequence" because it is the identity function for Dirichlet multiplication. In this case we obtain the array $A_{Di}(r,n)$ (see Table 3). According to (2.1) the general term is given by

$$A_{Di}(r,n) = \binom{n+r-2}{r-1}, \quad r,n \in \mathbb{N}_0. \tag{2.5}$$

Note that we have $A_{Fe}(r,n) = A_{Di}(r+1,n+1)$ for all $r,n \in \mathbb{N}_0$.

Example 3. Let $a_n := F_n$, $n \geq 0$, be the Fibonacci sequence (the sequence A000045 in [21]) defined recursively by

$$F_0 = 0,\ F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 0. \tag{2.6}$$

This sequence generates the array $F_n^{(r)}$ (see Table 4). According to (2.1) the general term is given by

$$F_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} F_k = \sum_{k=0}^{n} \binom{r+k-1}{k} F_{n-k}. \tag{2.7}$$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1	1	1	1
2	0	1	2	3	4	5	6	7	8	9	10	11
3	0	1	3	6	10	15	21	28	36	45	55	66
4	0	1	4	10	20	35	56	84	120	165	220	286
5	0	1	5	15	35	70	126	210	330	495	715	1001
6	0	1	6	21	56	126	252	462	792	1287	2002	3003
7	0	1	7	28	84	210	462	924	1716	3003	5005	8008
8	0	1	8	36	120	330	792	1716	3432	6435	11440	19448

Table 3: Dirichlet's array $A_{Di}(r,n)$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	1	2	3	5	8	13	21	34	55
1	0	1	2	4	7	12	20	33	54	88	143
2	0	1	3	7	14	26	46	79	133	221	364
3	0	1	4	11	25	51	97	176	309	530	894
4	0	1	5	16	41	92	189	365	674	1204	2098
5	0	1	6	22	63	155	344	709	1383	2587	4685
6	0	1	7	29	92	247	591	1300	2683	5270	9955
7	0	1	8	37	129	376	967	2267	5950	11220	21175
8	0	1	9	46	175	551	1518	3785	9735	20955	42130

Table 4: Hyperfibonacci numbers $F_n^{(r)}$

The sequence $\left(F_n^{(r)}\right)_{n\in\mathbb{N}_0}$ is known as the *hyperfibonacci sequence of the rth generation*. It was introduced in 2008 by Dil and Mező [10] by studying a symmetric algorithm for hyperharmonic, Fibonacci and other integer sequences. In 2014 Bahşi, Mező, and Solak [2] provided further refinements to this topic. A number of other authors have studied this sequence; we mention in particular Cristea, Martinjak, and Urbiha [9]. Besides the first equality of (2.7), they also gave three other different representations of $F_n^{(r)}$ (see [9, Theorem 3, Theorem 5, and Corollary 6]). Finally, alternative representations have been given by Belbachir and Belkhir [3, Theorem 8] and Komatsu and Szalay [16, Theorem 1].

We now derive a new representation of the hyperfibonacci numbers $F_n^{(r)}$.

Theorem 3. *Let $r, n \in \mathbb{N}_0$. Then we have*

$$F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k-1}{k} F_{2(r-k)}. \tag{2.8}$$

Proof. We use induction on $r \geq 0$. When $r = 0$, the right-hand side of (2.8) is equal to F_n and the left-hand side is equal to $F_n^{(0)}$. Therefore, the assertion is true for

$r=0$. Let us now assume that (2.8) is true for some $r \geq 0$. Then, by the induction assumption, and since $F_0^{(r)}=0$, $r \in \mathbb{N}_0$, by applying (2.4) we obtain

$$\begin{aligned} F_n^{(r+1)} = \sum_{k=1}^{n} F_k^{(r)} &= \sum_{k=1}^{n}\left(F_{k+2r} - \sum_{j=0}^{r-1}\binom{k+j-1}{j}F_{2(r-j)}\right) \\ &= \sum_{k=1}^{n} F_{k+2r} - \sum_{k=1}^{n}\left(\sum_{j=0}^{r-1}\binom{k+j-1}{j}F_{2(r-j)}\right) \\ &= \sum_{k=1}^{n+2r} F_k - \sum_{k=1}^{2r} F_k - \sum_{j=0}^{r-1}\left(\sum_{k=1}^{n}\binom{k+j-1}{j}\right)F_{2(r-j)} \\ &= F_{n+2r+2} - F_2 - (F_{2r+2} - F_2) - \sum_{j=0}^{r-1}\binom{n+j}{j+1}F_{2(r-j)}. \end{aligned}$$

With $j=k-1$, $k \in \{1,2,\ldots,r\}$, it follows that

$$\begin{aligned} F_n^{(r+1)} &= F_{n+2(r+1)} - \sum_{k=1}^{r}\binom{n+k-1}{k}F_{2(r-k+1)} - \binom{n+0-1}{0}F_{2(r+1-0)} \\ &= F_{n+2(r+1)} - \sum_{k=0}^{(r+1)-1}\binom{n+k-1}{k}F_{2(r+1-k)}, \end{aligned}$$

which is Equation (2.8) if r is replaced by $r+1$. □

By equating (2.7) and (2.8) we obtain the following result.

Corollary 1. *For all $r, n \in \mathbb{N}_0$, we have*

$$\begin{aligned} F_{n+2r} &= \sum_{k=0}^{n}\binom{n+r-1-k}{r-1}F_k + \sum_{k=0}^{r-1}\binom{n+k-1}{k}F_{2(r-k)} \\ &= \sum_{k=0}^{n}\binom{r+k-1}{k}F_{n-k} + \sum_{k=0}^{r-1}\binom{n+k-1}{k}F_{2(r-k)}. \end{aligned}$$

The Fibonacci sequence (F_n) satisfies the second-order linear homogeneous recurrence relation (2.6). Cristea, Martinjak, and Urbiha [9] proved by a combinatorial argument that the hyperfibonacci numbers of the rth generation also satisfy a second-order linear inhomogeneous recurrence relation.

Theorem 4. ([9, Lemma 2]) *For all $r \in \mathbb{N}_0$, we have*

$$F_0^{(r)} = 0,\ F_1^{(r)} = 1, \quad F_{n+2}^{(r)} = F_{n+1}^{(r)} + F_n^{(r)} + \binom{n+r}{r-1}, \quad n \geq 0. \tag{2.9}$$

In [9, p. 4] the authors noted that Theorem 4 provides an equivalent definition of the hyperfibonacci sequence. In contrast to the recurrence relation (2.2), where the hyperfibonacci numbers of two neighboring generations are involved, the recurrence relation (2.9) in Theorem 4 involves only the nth hyperfibonacci number of the rth generation and its two predecessors of the same rth generation.

Example 4. Let $a_n := L_n$, $n \geq 0$, be the Lucas sequence (the sequence A000032 in [21]) defined recursively by

$$L_0 = 2,\ L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n, \quad n \geq 0. \tag{2.10}$$

This sequence generates the array $L_n^{(r)}$ (see Table 5). According to (2.1) the general term is given by

$$L_n^{(r)} = \sum_{k=0}^{n} \binom{n+r-1-k}{r-1} L_k = \sum_{k=0}^{n} \binom{r+k-1}{k} L_{n-k}. \tag{2.11}$$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	2	1	3	4	7	11	18	29	47	76	123
1	2	3	6	10	17	28	46	75	122	198	321
2	2	5	11	21	38	66	112	187	309	507	828
3	2	7	18	39	77	143	255	442	751	1258	2086
4	2	9	27	66	143	286	541	983	1734	2992	5078
5	2	11	38	104	247	553	1074	2057	3791	6783	11861
6	2	13	51	155	402	955	2029	4086	7877	14660	26521
7	2	15	66	221	623	1578	3607	7693	15570	30230	56751
8	2	17	83	304	927	2505	6112	13805	29375	59605	116356

Table 5: Hyperlucas numbers $L_n^{(r)}$

The sequence $\left(L_n^{(r)}\right)_{n\in\mathbb{N}_0}$ is known as the *hyperlucas sequence of the rth generation*. It was first introduced in 2008 by Dil and Mező [10]. In 2014 Bahşi, Mező, and Solak provided more properties of the hyperlucas numbers, for example the first expression of (2.11) [2, Corollary 2.4].

As for the hyperfibonacci numbers $F_n^{(r)}$, we can give another representation for the hyperlucas numbers $L_n^{(r)}$.

Theorem 5. *Let $r, n \in \mathbb{N}_0$. Then we have*

$$L_n^{(r)} = L_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k}{k} L_{2(r-k)-1}. \tag{2.12}$$

Proof. We use induction on $r \geq 0$. When $r = 0$, the right-hand side of (2.12) is equal to L_n and the left-hand side is equal to $L_n^{(0)}$. Therefore, the assertion is true for $r = 0$. Let us now assume that (2.12) is true for some $r \geq 0$. Then, by the induction assumption and applying (2.4), we obtain

$$\begin{aligned}
L_n^{(r+1)} &= \sum_{k=0}^{n} L_k^{(r)} = \sum_{k=0}^{n}\Bigg(L_{k+2r} - \sum_{j=0}^{r-1}\binom{k+j}{j}L_{2(r-j)-1}\Bigg) \\
&= \sum_{k=0}^{n} L_{k+2r} - \sum_{k=0}^{n}\Bigg(\sum_{j=0}^{r-1}\binom{k+j}{j}L_{2(r-j)-1}\Bigg) \\
&= \sum_{k=0}^{n+2r} L_k - \sum_{k=0}^{2r-1} L_k - \sum_{j=0}^{r-1}\Bigg(\sum_{k=0}^{n}\binom{k+j}{j}\Bigg)L_{2(r-j)-1} \\
&= L_{n+2r+2} - L_1 - (L_{2r+1} - L_1) - \sum_{j=0}^{r-1}\binom{n+j+1}{j+1}L_{2(r-j)-1} \\
&= L_{n+2r+2} - L_{2r+1} - \sum_{k=1}^{r}\binom{n+k}{k}L_{2(r-k+1)-1}.
\end{aligned}$$

With $j = k-1$, $k \in \{1, 2, \ldots, r\}$, it follows that

$$\begin{aligned}
L_n^{(r+1)} &= L_{n+2(r+1)} - L_{2r+1} - \sum_{k=0}^{r}\binom{n+k}{k}L_{2(r-k+1)-1} + \binom{n}{0}L_{2(r+1-0)-1} \\
&= L_{n+2(r+1)} - \sum_{k=0}^{(r+1)-1}\binom{n+k}{k}L_{2(r+1-k)-1},
\end{aligned}$$

and this is Equation (2.12) for $r+1$ instead of r. □

By equating (2.11) and (2.12) we obtain the following result.

Corollary 2. *For all $r, n \in \mathbb{N}_0$, we have*

$$\begin{aligned}
L_{n+2r} &= \sum_{k=0}^{n}\binom{n+r-1-k}{r-1}L_k + \sum_{k=0}^{r-1}\binom{n+k}{k}L_{2(r-k)-1} \\
&= \sum_{k=0}^{n}\binom{r+k-1}{k}L_{n-k} + \sum_{k=0}^{r-1}\binom{n+k}{k}L_{2(r-k)-1}.
\end{aligned}$$

The Lucas sequence (L_n) satisfies the second-order linear homogeneous recurrence relation (2.10). The next theorem shows that the hyperlucas numbers of the rth generation also satisfy a second-order linear inhomogeneous recurrence relation.

Theorem 6. *For all $r \in \mathbb{N}_0$, we have*

$$L_0^{(r)} = 2,\ L_1^{(r)} = 2r+1, \quad L_{n+2}^{(r)} = L_{n+1}^{(r)} + L_n^{(r)} + \frac{n+2r}{n+2}\cdot\binom{n+r}{r-1}, \quad n \geq 0. \tag{2.13}$$

Proof. We show that $L_n^{(r)} = \sum_{k=0}^{n} \binom{r+k-1}{k} L_{n-k}$ satisfies the recurrence relation (2.13). The first initial value is equal to

$$L_0^{(r)} = \sum_{k=0}^{0} \binom{r+k-1}{k} L_{0-k} = \binom{r-1}{0} L_0 = L_0 = 2,$$

while the second one is equal to

$$L_1^{(r)} = \sum_{k=0}^{1} \binom{r+k-1}{k} L_{1-k} = \binom{r-1}{0} L_1 + \binom{r}{1} L_0 = 2r+1.$$

Let l_n^r be defined as follows: $l_n^r := L_{n+2}^{(r)} - L_{n+1}^{(r)} - L_n^{(r)}$. Then

$$\begin{aligned}
l_n^r &= \sum_{k=0}^{n+2} \binom{r+k-1}{k} L_{n+2-k} - \sum_{k=0}^{n+1} \binom{r+k-1}{k} L_{n+1-k} - \sum_{k=0}^{n} \binom{r+k-1}{k} L_{n-k} \\
&= \sum_{k=0}^{n} \binom{r+k-1}{k} (L_{n+2-k} - L_{n+1-k} - L_{n-k}) \\
&\quad + \binom{r+n+1-1}{n+1} L_1 + \binom{r+n+2-1}{n+2} L_0 - \binom{r+n+1-1}{n+1} L_0 \\
&= 2\binom{r+n+1}{n+2} - \binom{r+n}{n+1} = \frac{2(r+n+1)!}{(n+2)!(r-1)!} - \frac{(r+n)!}{(n+1)!(r-1)!} \\
&= \frac{n+2r}{n+2} \cdot \binom{n+r}{r-1}
\end{aligned}$$

because $L_{n+2-k} - L_{n+1-k} - L_{n-k} = 0$. This completes the proof. □

3. Weighted Sums of the Type $\sum_{k=0}^{n} k^{\ell} a_k^{(r)}$, $\ell, n, r \in \mathbb{N}_0$

Let $t_\ell(n) := \sum_{k=0}^{n} k^\ell a_k$, $\ell \in \mathbb{N}_0$, or more generally, for $r \geq 0$, $t_\ell^{(r)}(n) := \sum_{k=0}^{n} k^\ell a_k^{(r)}$. In this section, we will study these sums and show that $t_\ell^{(r)}(n)$ can be expressed as a linear combination of $a_n^{(r+1)}, a_n^{(r+2)}, \ldots, a_n^{(r+\ell+1)}$. Obviously, $t_\ell^{(0)}(n) = t_\ell(n)$, and

$$t_0(n) = \sum_{k=0}^{n} k^0 a_k = \sum_{k=0}^{n} a_k = a_n^{(1)}. \tag{3.1}$$

Furthermore, by (2.1) for $r = 2$ we have

$$a_n^{(2)} = \sum_{k=0}^{n} (n+1-k) a_k = (n+1) \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} k a_k.$$

Solving for $t_1(n) = \sum_{k=0}^{n} k a_k$ we obtain

$$t_1(n) = \sum_{k=0}^{n} k a_k = (n+1)a_n^{(1)} - a_n^{(2)}. \tag{3.2}$$

Similarly, by (2.1) for $r = 3$ and since $\binom{n+2-k}{2} = \frac{1}{2}\big((n+2)(n+1) - (2n+3)k + k^2\big)$, we have

$$a_n^{(3)} = \sum_{k=0}^{n} \binom{n+2-k}{2} a_k = \frac{(n+2)(n+1)}{2} \sum_{k=0}^{n} a_k - \frac{2n+3}{2} \sum_{k=0}^{n} k a_k + \frac{1}{2} \sum_{k=0}^{n} k^2 a_k.$$

Solving for $t_2(n) = \sum_{k=0}^{n} k^2 a_k$ and using (3.2), we obtain

$$t_2(n) = \sum_{k=0}^{n} k^2 a_k = (n+1)^2 a_n^{(1)} - (2n+3)a_n^{(2)} + 2a_n^{(3)}. \tag{3.3}$$

Similarly, for $r = 4$, we obtain

$$t_3(n) = \sum_{k=0}^{n} k^3 a_k = (n+1)^3 a_n^{(1)} - (3n^2 + 9n + 7)a_n^{(2)} + 6(n+2)a_n^{(3)} - 6a_n^{(4)}, \tag{3.4}$$

and so on.

The last formulas can be written in matrix form as follows:

$$\begin{pmatrix} t_0(n) \\ t_1(n) \\ t_2(n) \\ t_3(n) \\ \vdots \end{pmatrix} = C(n) \cdot \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \\ a_n^{(3)} \\ a_n^{(4)} \\ \vdots \end{pmatrix},$$

where $C(n)$ is the infinite matrix given by

$$C(n) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ n+1 & -1 & 0 & 0 & \cdots \\ (n+1)^2 & -(2n+3) & 2 & 0 & \cdots \\ (n+1)^3 & -(3n^2+9n+7) & 6n+12 & -6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From these few cases, it is possible to guess some patterns for $\ell = 0, 1, 2, \ldots$.

- The matrix $C(n)$ is a lower triangular matrix.
- $t_\ell(n)$ contains $\ell + 1$ terms.
- The coefficients in $t_\ell(n)$ alternate in signs.

- The coefficient of $a_n^{(1)}$ in $t_\ell(n)$ is $(n+1)^\ell$ and that of $a_n^{(\ell+1)}$ is $(-1)^\ell \ell!$.
- Let $\ell \geq 0$. Then the sum of the coefficients of $t_\ell(n)$ is equal to n^ℓ. This implies that for each $\ell \geq 0$, the sum of the ℓth row of $C(n)$ is equal to n^ℓ.

We will prove these and other properties of the matrix entries of $C(n)$ later. In addition, and this turns out to be very important, Equations (3.1), (3.2), (3.3), and (3.4) can be seen as special cases of Abel's summation by parts (see, for example, [24, Theorem 3.41]), which we state for convenience.

Theorem 7. (Abel's Summation by Parts)
If $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}$ are two arbitrary sequences (of complex numbers), then we have the identity

$$\sum_{k=0}^{n} a_k b_k = a_n^{(1)} b_{n+1} + \sum_{k=0}^{n} a_k^{(1)} (b_k - b_{k+1}).$$

In fact, if we choose $b_k := k$, $k \in \{0, 1, \ldots, n\}$, then by Theorem 7 we obtain Equation (3.2):

$$\sum_{k=0}^{n} k a_k = (n+1)a_n^{(1)} + \sum_{k=0}^{n} (-1) a_k^{(1)} = (n+1)a_n^{(1)} - a_n^{(2)}.$$

In general, using $a_n^{(r)}$ instead of a_n, by Theorem 7 we obtain

$$\sum_{k=0}^{n} k a_k^{(r)} = (n+1)a_n^{(r+1)} - \sum_{k=0}^{n} a_k^{(r+1)} = (n+1)a_n^{(r+1)} - a_n^{(r+2)}.$$

Similarly, if we choose $b_k := k^2$, $k \in \{0, 1, \ldots, n\}$, and use (3.2), then we obtain

$$\begin{aligned}
\sum_{k=0}^{n} k^2 a_k &= (n+1)^2 a_n^{(1)} + \sum_{k=0}^{n} a_k^{(1)} \left(k^2 - (k+1)^2\right) = (n+1)^2 a_n^{(1)} - \sum_{k=0}^{n} a_k^{(1)} (2k+1) \\
&= (n+1)^2 a_n^{(1)} - 2\sum_{k=0}^{n} k a_k^{(1)} - \sum_{k=0}^{n} a_k^{(1)} \\
&= (n+1)^2 a_n^{(1)} - 2 \cdot \left((n+1)a_n^{(2)} - a_n^{(3)}\right) - a_n^{(2)}
\end{aligned}$$

and this is Identity (3.3) after simplification. Again, using $a_n^{(r)}$ instead of a_n, by Theorem 7 we obtain

$$\sum_{k=0}^{n} k^2 a_k^{(r)} = (n+1)^2 a_n^{(r+1)} - (2n+3)a_n^{(r+2)} + 2a_n^{(r+3)}.$$

In general, we have the following theorem.

Theorem 8. *For all $\ell, r, n \in \mathbb{N}_0$, we have*

$$t_\ell^{(r)}(n) = (n+1)^\ell a_n^{(r+1)} - \sum_{j=0}^{\ell-1} \binom{\ell}{j} t_j^{(r+1)}(n) \tag{3.5}$$

with the initial value $t_0^{(r)}(n) = \sum_{k=0}^{n} k^0 a_k^{(r)} = a_n^{(r+1)}$.

Proof. By Theorem 7 we obtain for $b_k := k^\ell$ and $a_k^{(r)}$ instead of a_k

$$t_\ell^{(r)}(n) = \sum_{k=0}^{n} k^\ell a_k^{(r)} = (n+1)^\ell a_n^{(r+1)} + \sum_{k=0}^{n} a_k^{(r+1)} \big(k^\ell - (k+1)^\ell\big).$$

The binomial theorem implies that $k^\ell - (k+1)^\ell = k^\ell - \sum_{j=0}^{\ell} \binom{\ell}{j} k^j = -\sum_{j=0}^{\ell-1} \binom{\ell}{j} k^j$. Consequently,

$$\begin{aligned} t_\ell^{(r)}(n) &= (n+1)^\ell a_n^{(r+1)} + \sum_{k=0}^{n} a_k^{(r+1)} \bigg(-\sum_{j=0}^{\ell-1} \binom{\ell}{j} k^j \bigg) \\ &= (n+1)^\ell a_n^{(r+1)} - \sum_{j=0}^{\ell-1} \binom{\ell}{j} \sum_{k=0}^{n} k^j a_k^{(r+1)} \end{aligned}$$

by changing the order of summation. This is the assertion, since by definition we have $\sum_{k=0}^{n} k^j a_k^{(r+1)} = t_j^{(r+1)}(n)$. □

We note that $t_\ell^{(r)}(n)$ as given in Theorem 8 is calculated recursively (with respect to ℓ), i.e., starting with $t_0^{(r)}(n) = a_n^{(r+1)}$ we obtain for

$$\ell = 1: \quad t_1^{(r)}(n) = (n+1)a_n^{(r+1)} - t_0^{(r+1)}(n) = (n+1)a_n^{(r+1)} - a_n^{(r+2)},$$

$$\begin{aligned} \ell = 2: \quad t_2^{(r)}(n) &= (n+1)^2 a_n^{(r+1)} - \tbinom{2}{0} t_0^{(r+1)}(n) - \tbinom{2}{1} t_1^{(r+1)}(n) \\ &= (n+1)^2 a_n^{(r+1)} - a_n^{(r+2)} - 2 \cdot \big((n+1)a_n^{(r+2)} - a_n^{(r+3)}\big) \\ &= (n+1)^2 a_n^{(r+1)} - (2n+3)a_n^{(r+2)} + 2a_n^{(r+3)}, \end{aligned}$$

and so on.

Equation (3.5) can also be written as

$$\begin{aligned} t_\ell^{(r)}(n) &= (n+1)^\ell a_n^{(r+1)} + \sum_{k=0}^{n} k^\ell a_k^{(r+1)} - \sum_{k=0}^{n} (k+1)^\ell a_k^{(r+1)} \\ &= (n+1)^\ell a_n^{(r+1)} + t_\ell^{(r+1)}(n) - \sum_{k=0}^{n} (k+1)^\ell a_k^{(r+1)}, \end{aligned}$$

or, equivalently,

$$t_\ell^{(r+1)}(n) - t_\ell^{(r)}(n) = \sum_{k=0}^{n-1} (k+1)^\ell a_k^{(r+1)}. \tag{3.6}$$

Furthermore, from (3.5) we get $\sum_{j=0}^{\ell-1} \binom{\ell}{j} t_j^{(r+1)}(n) = (n+1)^\ell a_n^{(r+1)} - t_\ell^{(r)}(n)$. Adding $t_\ell^{(r+1)}(n)$ to both sides of this equation and using (3.6) we obtain

$$\begin{aligned}\sum_{j=0}^{\ell} \binom{\ell}{j} t_j^{(r+1)}(n) &= (n+1)^\ell a_n^{(r+1)} + t_\ell^{(r+1)}(n) - t_\ell^{(r)}(n)\\ &= (n+1)^\ell a_n^{(r+1)} + \sum_{k=0}^{n-1} (k+1)^\ell a_k^{(r+1)},\end{aligned}$$

that is,

$$\sum_{j=0}^{\ell} \binom{\ell}{j} t_j^{(r+1)}(n) = \sum_{k=0}^{n} (k+1)^\ell a_k^{(r+1)}.$$

The next theorem gives the solution of the recurrence relation given in Theorem 8. This solution can be expressed by means of

$$c_{\ell,m}(n) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} (k+n+1)^\ell, \quad \ell, m, n \in \mathbb{N}_0,$$

and by the Stirling numbers of the second kind $\left\{ {j \atop m} \right\}$ defined as the number of ways of partitioning a set of j elements into exactly m nonempty subsets, $0 \le m \le j$.

Theorem 9. *Let $\ell, n, r \in \mathbb{N}_0$. Then the solution of the recurrence relation for $t_\ell^{(r)}(n)$ is given by*

$$t_\ell^{(r)}(n) = \sum_{k=0}^{n} k^\ell a_k^{(r)} = \sum_{m=0}^{\ell} c_{\ell,m}(n) a_n^{(r+m+1)}. \tag{3.7}$$

Alternatively,

$$t_\ell^{(r)}(n) = \sum_{j=0}^{\ell} \binom{\ell}{j} \left(\sum_{m=0}^{j} (-1)^m m! \left\{ {j \atop m} \right\} a_n^{(r+m+1)} \right) (n+1)^{\ell-j}. \tag{3.8}$$

In particular, for $r = 0$, we have the following result.

Corollary 3. *Let $\ell, n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}t_\ell(n) &= \sum_{m=0}^{\ell} c_{\ell,m}(n) a_n^{(m+1)}\\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} \left(\sum_{m=0}^{j} (-1)^m m! \left\{ {j \atop m} \right\} a_n^{(m+1)} \right) (n+1)^{\ell-j}.\end{aligned}$$

In order to prove Theorem 9, we need four propositions concerning the expression $c_{\ell,m}(n)$, $\ell, m, n \in \mathbb{N}_0$, which we will formulate for the slight generalization

$$p_{\ell,m}(x,y) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} (kx+y)^\ell, \quad x, y \in \mathbb{C}, \tag{3.9}$$

an expression also studied recently by Katsuura [15], Boyadzhiev [4], and Miceli [20]. In fact, for $x = 1$ and $y = n+1$, Equation (3.9) reduces to $c_{\ell,m}(n)$, that is,

$$c_{\ell,m}(n) = p_{\ell,m}(1, n+1) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (k+n+1)^\ell.$$

3.1. Properties of $p_{\ell,m}(x,y)$

Proposition 1. *Let $x, y \in \mathbb{C}$.*
(i) (Boyadzhiev [4, p. 254]) *Let $\ell, m \in \mathbb{N}_0$. Then we have*

$$p_{\ell,m}(x,y) = (-1)^m m! \sum_{j=m}^{\ell} \binom{\ell}{j} \left\{ {j \atop m} \right\} x^j y^{\ell-j}. \tag{3.10}$$

(ii) (Katsuura [15, pp. 275–276]) *Let $\ell, m \in \mathbb{N}_0$. Then we have*

$$p_{\ell,m}(x,y) = \begin{cases} 0, & \text{for } 0 \le \ell \le m-1, \\ (-1)^m x^m m!, & \text{for } \ell = m. \end{cases} \tag{3.11}$$

Notice that Equation (3.11) means that $P(x,y) := \big(p_{\ell,m}(x,y)\big)_{\ell,m=0,1,\ldots}$ is an infinite lower triangular matrix that depends both on x and y. Consequently, $C(n) := \big(c_{\ell,m}(n)\big)_{\ell,m=0,1,\ldots}$ is also an infinite lower triangular matrix that depends on n. Note also that the case $\ell = m$ in (3.11) is particularly noteworthy, since the right-hand side does not depend on y.

Remark 2. As noted by Boyadzhiev [4], Identity (3.11) is not entirely new. In fact, it is a special case of Euler's Finite Difference Theorem [23, p. 68], which states that if $f(t) = a_0 + a_1 t + \cdots + a_\ell t^\ell$ is a complex polynomial of degree ℓ and m a nonnegative integer, then

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} f(k) = \begin{cases} 0, & \text{for } 0 \le \ell \le m-1, \\ (-1)^m a_m m!, & \text{for } \ell = m. \end{cases} \tag{3.12}$$

Proposition 2. *Let $x, y \in \mathbb{C}$ and ℓ, $m \in \mathbb{N}_0$.*
(i) *If $y = x$, then*

$$p_{\ell,m}(x,x) = (-1)^m m! \left\{ {\ell+1 \atop m+1} \right\} x^\ell. \tag{3.13}$$

In particular, if $x = 1$, *then*

$$c_{\ell,m}(0) = p_{\ell,m}(1,1) = \sum_{k=0}^{m}(-1)^k \binom{m}{k}(k+1)^{\ell} = (-1)^m m! \left\{ {\ell+1 \atop m+1} \right\}. \tag{3.14}$$

(ii) *If* $x = 0$, *then*

$$p_{\ell,m}(0,y) = 0^{\ell} \cdot y^{\ell}. \tag{3.15}$$

In particular, $p_{\ell,m}(0,1) = 0^{\ell}$ *and* $p_{\ell,m}(0,0) = 0^{\ell} \cdot 0^m$.
(iii) *If* $y = 0$, *then*

$$p_{\ell,m}(x,0) = (-1)^m m! \left\{ {\ell \atop m} \right\} x^{\ell}. \tag{3.16}$$

In particular, if $x = 1$, *then*

$$p_{m+1,m}(1,0) = \sum_{k=0}^{m}(-1)^k \binom{m}{k} k^{m+1} = (-1)^m \frac{m}{2}(m+1)!. \tag{3.17}$$

(iv) *Let* $x = -1$ *and* $y = m$. *Then*

$$p_{\ell,m}(-1,m) = \sum_{k=0}^{m}(-1)^k \binom{m}{k}(m-k)^{\ell} = m! \left\{ {\ell \atop m} \right\}. \tag{3.18}$$

In particular, if $0 \le \ell \le m-1$, *then* $p_{\ell,m}(-1,m) = 0$; *if* $\ell = m$, *then* $p_{m,m}(-1,m) = m!$; *and if* $\ell = m+1$, *then* $p_{m+1,m}(-1,m) = \frac{m}{2}(m+1)!$.
(v) (Carlitz [7, (3.8)]) *Let* $x = 1$ *and* $y = n+1$. *Then we have*

$$c_{\ell,m}(n) = \sum_{k=0}^{m}(-1)^k \binom{m}{k}(k+n+1)^{\ell} = (-1)^m m! \left\{ {\ell+n+1 \atop m+n+1} \right\}_{n+1}. \tag{3.19}$$

Proof. (i) Let $y = x$. Then by applying (3.10), we obtain

$$p_{\ell,m}(x,x) = (-1)^m m! \sum_{j=m}^{\ell} \binom{\ell}{j} \left\{ {j \atop m} \right\} x^{\ell} = (-1)^m m! \left\{ {\ell+1 \atop m+1} \right\} x^{\ell},$$

since (see [14, Identity (6.15)])

$$\sum_{j=m}^{\ell} \binom{\ell}{j} \left\{ {j \atop m} \right\} = \left\{ {\ell+1 \atop m+1} \right\}. \tag{3.20}$$

This proves the assertion (3.13). Identity (3.14) follows immediately by setting $x = 1$ in (3.13).
(ii) Let $x = 0$. Then by (3.9) and using the well-known formula (see [14, p. 163])

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} = (1-1)^m = 0^m, \tag{3.21}$$

we get $p_{\ell,m}(0,y) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k y^\ell = 0^m \cdot y^\ell$. This proves the assertion (3.15).
(iii) Let $y = 0$. Then by (3.9) and using Identity (see [14, Identity (6.19)])

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} k^j = (-1)^m m! \left\{ {j \atop m} \right\},$$

we get $p_{\ell,m}(x,0) = x^\ell \sum_{k=0}^{m} (-1)^k \binom{m}{k} k^\ell = (-1)^m m! \left\{ {\ell \atop m} \right\} x^\ell$ and this proves (3.16).
Finally, (3.17) is obtained by applying (3.16) and noting that $\left\{ {m+1 \atop m} \right\} = \binom{m+1}{2}$ (see [4, p. 253]).
(iv) If $x = -1$ and $y = m$, then by (3.9)

$$p_{\ell,m}(-1,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^\ell = m! \left\{ {\ell \atop m} \right\}$$

(see [23, Identity (9.21)]) and this proves (3.18). If $0 \le \ell \le m-1$, then $p_{\ell,m}(-1,m) = 0$; if $\ell = m$, then $p_{\ell,m}(-1,m) = m! \left\{ {m \atop m} \right\} = m!$; and if $\ell = m+1$, then we get $p_{\ell,m}(-1,m) = m! \left\{ {m+1 \atop m} \right\} = m! \binom{m+1}{2} = \frac{m}{2}(m+1)!$ since $\left\{ {m+1 \atop m} \right\} = \binom{m+1}{2}$.
(v) Rewriting Equation (32) from [5] (with $n+1$ instead of r, $\ell+n+1$ instead of n, and $m+n+1$ instead of m), we obtain

$$\left\{ {\ell+n+1 \atop m+n+1} \right\}_{n+1} = \sum_{j=m}^{\ell} \binom{\ell}{j} \left\{ {j \atop m} \right\} (n+1)^{\ell-j}. \tag{3.22}$$

Substitution of $x = 1$ and $y = n+1$ into Equation (3.10) reveals that the right-hand side of (3.22) is equal to $\frac{p_{\ell,m}(1,n+1)}{(-1)^m m!} = \frac{c_{\ell,m}(n)}{(-1)^m m!}$, and this proves Identity (3.19). □

Remark 3. Note that $\left\{ {m \atop n} \right\}_r$ are the "r-Stirling numbers of the second kind" first introduced by Broder [5] as the number of set partitions of $\{1, 2, \ldots, m\}$ into n nonempty, unordered parts so that $1, 2, \ldots, r$ are in distinct parts. Identity (3.22) gives a relation between the r-Stirling numbers of the second kind and the ordinary Stirling numbers of the second kind. It was derived by Carlitz [7], who introduced a generalization of the Stirling numbers (of the first and second kind) called "weighted Stirling numbers of the first and second kind" [7, 8], which turned out to be equivalent to the r-Stirling numbers (of the first and second kind).

Proposition 3. *Let $x, y \in \mathbb{C}$ and ℓ, $m \in \mathbb{N}_0$.*
(i) *If $m = 0$, then*

$$p_{\ell,0}(x,y) = y^\ell. \tag{3.23}$$

In particular, if $\ell = 0$*, then* $p_{0,0}(x,y) = y^0 = 1$.
(ii) *If* $m = 1$*, then*

$$p_{\ell,1}(x,y) = y^\ell - (x+y)^\ell. \tag{3.24}$$

(iii) *If* $m = 2$*, then*

$$p_{\ell,2}(x,y) = y^\ell - 2(x+y)^\ell + (2x+y)^\ell. \tag{3.25}$$

(iv) *If* $m = \ell - 1$*, then*

$$p_{\ell,\ell-1}(x,y) = (-1)^{\ell-1}\ell! \cdot x^{\ell-1}\Big(y + \frac{\ell-1}{2}x\Big). \tag{3.26}$$

(v) *If* $m = \ell$*, then*

$$p_{\ell,\ell}(x,y) = (-1)^\ell \ell! x^\ell. \tag{3.27}$$

(vi) *If* $\ell = 0$*, then*

$$p_{0,m}(x,y) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} = 0^m. \tag{3.28}$$

(vii) *If* $\ell \in \mathbb{N}_0$*, then the* ℓ*th row sum is given by*

$$\sum_{m=0}^{\ell} p_{\ell,m}(x,y) = (y-x)^\ell. \tag{3.29}$$

Proof. (i) Inserting $m = 0$ into (3.9) we obtain (3.23).
(ii) Identity (3.24) follows from (3.9) by substituting $m = 1$.
(iii) Similarly, if we replace $m = 2$ in (3.9), we get (3.25).
(iv) By (3.10) for $m = \ell - 1$ and noting that $\left\{ {\ell \atop \ell-1} \right\} = \binom{\ell}{2}$, we get

$$\begin{aligned} p_{\ell,\ell-1}(x,y) &= (-1)^{\ell-1}(\ell-1)!\left(\binom{\ell}{\ell-1}\left\{ {\ell-1 \atop \ell-1} \right\} x^{\ell-1}y + \binom{\ell}{\ell}\left\{ {\ell \atop \ell-1} \right\} x^\ell\right) \\ &= (-1)^{\ell-1}(\ell-1)!\ell x^{\ell-1}\Big(y + \frac{\ell-1}{2}x\Big). \end{aligned}$$

This proves (3.26).
(v) Equation (3.27) is exactly the second expression of Identity (3.11).
(vi) Equation (3.28) is exactly Identity (3.21).
(vii) Equation (3.10) and the binomial theorem (noting that $\left\{ {j \atop m} \right\} = 0$ for $j < m$) imply that

$$\begin{aligned} \sum_{m=0}^{\ell} p_{\ell,m}(x,y) &= \sum_{m=0}^{\ell}\left((-1)^m m! \sum_{j=m}^{\ell}\binom{\ell}{j}\left\{ {j \atop m} \right\} x^j y^{\ell-j}\right) \\ &= \sum_{j=0}^{\ell}\binom{\ell}{j}\Big(\sum_{m=0}^{\ell}(-1)^m m!\left\{ {j \atop m} \right\}\Big) x^j y^{\ell-j} \\ &= \sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^j x^j y^{\ell-j} = (y-x)^\ell, \end{aligned}$$

where we have used the following identity

$$\sum_{m=0}^{\ell}(-1)^m m!\left\{ {j \atop m} \right\} = (-1)^j, \quad j \le \ell,$$

a special case ($x = -1$, j instead of n and noting that $(-1)^{\underline{k}} = (-1)^k k!$) of [14, (6.10)]. □

Remark 4. If $y = x$ is inserted into (3.29), then the following resut is obtained: $\sum_{m=0}^{\ell} p_{\ell,m}(x,x) = 0^\ell$. In particular, if $x = 1$, then by using (3.13) and (3.14) we get

$$\sum_{m=0}^{\ell} c_{\ell,m}(0) = \sum_{m=0}^{\ell}(-1)^m m!\left\{ {\ell+1 \atop m+1} \right\} = 0^\ell. \tag{3.30}$$

Remark 5. Equations (i)-(iii) in Proposition 3 can also be obtained by noting that by (3.9) the sequence $(p_{\ell,m}(x,y))_{m\in\mathbb{N}_0}$ is the binomial transform of $\big((mx+y)^\ell\big)_{m\in\mathbb{N}_0}$, and that $\big((mx+y)^\ell\big)_{m\in\mathbb{N}_0}$ is the inverse binomial transform of $(p_{\ell,m}(x,y))_{m\in\mathbb{N}_0}$. That is (see [14, (5.48)])

$$(mx+y)^\ell = \sum_{k=0}^{m}(-1)^k \binom{m}{k} p_{\ell,k}(x,y).$$

In fact, for $m = 0$ we obtain $y^\ell = \sum_{k=0}^{0}(-1)^k\binom{0}{k}p_{\ell,k}(x,y) = p_{\ell,0}(x,y)$, and this is Equation (3.23). For $m = 1$ we obtain

$$(x+y)^\ell = \sum_{k=0}^{1}(-1)^k\binom{1}{k}p_{\ell,k}(x,y) = p_{\ell,0}(x,y) - p_{\ell,1}(x,y).$$

Solving for $p_{\ell,1}(x,y)$ and using (3.23), we obtain $p_{\ell,1}(x,y) = y^\ell - (x+y)^\ell$, and this is Equation (3.24). Similarly, for $m = 2$, we get

$$(2x+y)^\ell = \sum_{k=0}^{2}(-1)^k\binom{2}{k}p_{\ell,k}(x,y) = p_{\ell,0}(x,y) - 2p_{\ell,1}(x,y) + p_{\ell,2}(x,y).$$

Solving for $p_{\ell,2}(x,y)$ and using (3.23) and (3.24) we obtain

$$p_{\ell,1}(x,y) = y^\ell - 2(x+y)^\ell + (2x+y)^\ell,$$

and this is Equation (3.25).

The first few entries of the matrix $P(x,y)$ are:

$$P(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ y & -x & 0 & 0 & \cdots \\ y^2 & -x(x+2y) & 2x^2 & 0 & \cdots \\ y^3 & -x(x^2+3xy+3y^2) & 2x^2(3x+3y) & -6x^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proposition 4. *Let $x, y \in \mathbb{C}$.*
(i) *For all $\ell \in \mathbb{N}_0$, we have*

$$p_{\ell,0}(x,y) = y^\ell, \quad p_{\ell,m+1}(x,y) = p_{\ell,m}(x,y) - p_{\ell,m}(x,x+y), \quad m \geq 0. \tag{3.31}$$

(ii) *For all ℓ, $m \in \mathbb{N}_0$, we have*

$$p_{\ell,m}(x,x+y) = \sum_{j=m}^{\ell} \binom{\ell}{j} p_{j,m}(x,y) x^{\ell-j}. \tag{3.32}$$

(iii) *For all ℓ, $m \in \mathbb{N}_0$, we have*

$$p_{\ell,m+1}(x,y) = -\sum_{j=m}^{\ell-1} \binom{\ell}{j} p_{j,m}(x,y) x^{\ell-j}. \tag{3.33}$$

(iv) *For all $m \in \mathbb{N}_0$, we have*

$$p_{0,m}(x,y) = 0^m, \quad p_{\ell+1,m}(x,y) = y p_{\ell,m}(x,y) - mx p_{\ell,m-1}(x,x+y), \quad \ell \geq 0. \tag{3.34}$$

(v) *For all $\ell, m \in \mathbb{N}_0$, we have $p_{0,0}(x,y) = 1$ and*

$$p_{\ell+1,m+1}(x,y) = \big((m+1)x + y\big) p_{\ell,m+1}(x,y) - (m+1)x p_{\ell,m}(x,y). \tag{3.35}$$

(vi) *Let $\ell \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then*

$$p_{\ell-1,m}(x,y) = \frac{1}{\ell} \cdot \frac{\partial}{\partial y} p_{\ell,m}(x,y) \tag{3.36}$$

and, conversely,
(vii) *for all $\ell, m \in \mathbb{N}$, we have*

$$p_{\ell,m}(x,y) = (mx+y) p_{\ell-1,m}(x,y) - mx p_{\ell-1,m-1}(x,y). \tag{3.37}$$

Proof. (i) The initial value is given by (3.23). Furthermore, by (3.9) and using the recurrence relation of the binomial coefficients, noting that $\binom{m}{m+1} = 0$, we have

$$\begin{aligned}
p_{\ell,m+1}(x,y) &= \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} (kx+y)^\ell \\
&= \sum_{k=0}^{m+1} (-1)^k \binom{m}{k} (kx+y)^\ell + \sum_{k=0}^{m+1} (-1)^k \binom{m}{k-1} (kx+y)^\ell \\
&= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (kx+y)^\ell - \sum_{k=0}^{m+1} (-1)^{k-1} \binom{m}{k-1} \Big((k-1+1)x+y\Big)^\ell.
\end{aligned}$$

The first term on the right-hand side is $p_{\ell,m}(x,y)$, while the second term is equal to $p_{\ell,m}(x,x+y)$ by means of the transformation $k' = k-1$ and noting that by definition we have $\binom{m}{-1} = 0$, $m \geq 0$. This proves the recurrence relation (3.31).
(ii) From the binomial theorem we have $(kx+x+y)^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j}(kx+y)^j x^{\ell-j}$ and so by changing the order of summation we obtain

$$\begin{aligned} p_{\ell,m}(x,x+y) &= \sum_{k=0}^{m}(-1)^k \binom{m}{k}(kx+x+y)^\ell \\ &= \sum_{k=0}^{m}(-1)^k \binom{m}{k}\left(\sum_{j=0}^{\ell}\binom{\ell}{j}(kx+y)^j x^{\ell-j}\right) \\ &= \sum_{j=0}^{\ell}\binom{\ell}{j}x^{\ell-j}\left(\sum_{k=0}^{m}(-1)^k\binom{m}{k}(kx+y)^j\right) \\ &= \sum_{j=0}^{\ell}\binom{\ell}{j}p_{j,m}(x,y)x^{\ell-j}. \end{aligned}$$

This is Equation (3.32), since by (3.11) we have $p_{j,m}(x,y) = 0$ for $j < m$.
(iii) Equation (3.33) follows immediately from (i) and (ii).
(iv) The initial value is given by (3.28). Furthermore, by definition, we have

$$p_{\ell+1,m}(x,y) = x\sum_{k=0}^{m}(-1)^k k\binom{m}{k}(kx+y)^\ell + y\sum_{k=0}^{m}(-1)^k\binom{m}{k}(kx+y)^\ell.$$

By (3.9) the second term on the right-hand side is equal to $yp_{\ell,m}(x,y)$. By using the formula $k\binom{m}{k} = m\binom{m-1}{k-1}$ the first term on the right-hand side is equal to

$$\begin{aligned} mx\sum_{k=0}^{m}(-1)^k\binom{m-1}{k-1}(kx+y)^\ell &= mx\sum_{k=1}^{m}(-1)^{k-1+1}\binom{m-1}{k-1}\big((k-1+1)x+y\big)^\ell \\ &= -mx\sum_{k'=0}^{m-1}(-1)^{k'}\binom{m-1}{k'}(k'x+x+y)^\ell, \end{aligned}$$

and the last expression is by definition equal to $-mxp_{\ell,m-1}(x,x+y)$, where $k' = k-1$ and $\binom{m-1}{-1} = 0$, $m > 0$. This proves (3.34).
(v) By Equation (3.34), using $m+1$ instead of m, we obtain

$$p_{\ell+1,m+1}(x,y) = yp_{\ell,m+1}(x,y) - (m+1)xp_{\ell,m}(x,x+y),$$

and by (3.31) it follows that $p_{\ell,m}(x,x+y) = p_{\ell,m}(x,y) - p_{\ell,m+1}(x,y)$. Inserting this expression into the above equation we get (3.35).
(vi) The partial derivative of $p_{\ell,m}(x,y)$ with respect to y is given by

$$\frac{\partial}{\partial y}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(kx+y)^\ell = \ell\cdot\sum_{k=0}^{m}(-1)^k\binom{m}{k}(kx+y)^{\ell-1} = \ell\cdot p_{\ell-1,m}(x,y),$$

and this proves (3.36).
(vii) Equation (3.37) is precisely Equation (3.35) for ℓ instead of $\ell+1$ and m instead of $m+1$. The proof of the proposition is complete. □

For $x = 1$ and $y = n+1$ it follows from (3.36) and (3.37) the corollary.

Corollary 4. *Let $\ell, m \in \mathbb{N}$. Then*

$$c_{\ell-1,m}(n) = \frac{1}{\ell} \cdot \frac{d}{dn} c_{\ell,m}(n), \quad c_{\ell,m}(n) = (m+n+1)c_{\ell-1,m}(n) - mc_{\ell-1,m-1}(n). \tag{3.38}$$

Remark 6. Equation (3.36) shows how to calculate the entries of the $(\ell-1)$st row of the matrix $P(x, y)$, once the entries of the ℓth row are known, and conversely, Equation (3.37) shows how to calculate the entries of the ℓth row, if the entries of the $(\ell-1)$st row are known. The same holds by Corollary 4 for the entries of the matrix $C(n)$. Note also that the second equation of (3.38) together with $c_{0,0}(n) = 1$ is a recurrence relation for $c_{\ell,m}(n)$.

3.2. Proof of Theorem 9

We are now able to prove Theorem 9.

Proof of Theorem 9. We need to show that $\sum_{m=0}^{\ell} c_{\ell,m}(n)a_n^{(r+m+1)}$ solves the recurrence relation (3.5). The proof is by induction on $\ell \geq 0$. By Equation (3.28), we have $c_{0,m}(n) = 0^m$. Consequently, $t_0^{(r)}(n) = \sum_{m=0}^{n} 0^m \cdot a_n^{(r+m+1)} = a_n^{r+1}$, and this is exactly the initial value of (3.5). Let $\sum_{m=0}^{j} c_{j,m}(n)a_n^{(r+m+1)}$ be a solution of the recurrence relation (3.5) for $j \in \{0, 1, \ldots, \ell-1\}$. Then $t_j^{(r+1)}(n) = \sum_{m=0}^{j} c_{j,m}(n)a_n^{(r+1+m+1)}$. We now consider the right-hand side of (3.5), namely $(n+1)^\ell a_n^{(r+1)} - \sum_{j=0}^{\ell-1} \binom{\ell}{j} t_j^{(r+1)}(n)$. Substituting the previous equation into this expression, we obtain

$$(n+1)^\ell a_n^{(r+1)} - \sum_{j=0}^{\ell-1} \binom{\ell}{j} \sum_{m=0}^{j} c_{j,m}(n)a_n^{(r+1+m+1)}. \tag{3.39}$$

Let DS be the double sum of the above term. Then

$$\begin{aligned} DS = {} & \tbinom{\ell}{0} c_{0,0}(n)a_n^{(r+2)} + \tbinom{\ell}{1}\Big(c_{1,0}(n)a_n^{(r+2)} + c_{1,1}(n)a_n^{(r+3)}\Big) \\ & + \tbinom{\ell}{2}\Big(c_{2,0}(n)a_n^{(r+2)} + c_{2,1}(n)a_n^{(r+3)} + c_{2,2}(n)a_n^{(r+4)}\Big) + \cdots \\ & + \tbinom{\ell}{\ell-1}\Big(c_{\ell-1,0}(n)a_n^{(r+2)} + c_{\ell-1,1}(n)a_n^{(r+3)} + \cdots + c_{\ell-1,\ell-1}(n)a_n^{(r+\ell+1)}\Big). \end{aligned}$$

Collecting the terms with the same factor $a_n^{(r+2)}, \dots, a_n^{(r+\ell+1)}$, respectively, we obtain

$$\begin{aligned} DS = &\Big(\tbinom{\ell}{0}c_{0,0}(n) + \tbinom{\ell}{1}c_{1,0}(n) + \cdots + \tbinom{\ell}{\ell-1}c_{\ell-1,0}(n)\Big)a_n^{(r+2)} \\ &+ \Big(\tbinom{\ell}{1}c_{1,1}(n) + \tbinom{\ell}{2}c_{2,1}(n) + \cdots + \tbinom{\ell}{\ell-1}c_{\ell-1,1}(n)\Big)a_n^{(r+3)} + \cdots \\ &+ \Big(\tbinom{\ell}{\ell-2}c_{\ell-2,\ell-2}(n) + \tbinom{\ell}{\ell-1}c_{\ell-1,\ell-2}(n)\Big)a_n^{(r+\ell)} + \tbinom{\ell}{\ell-1}c_{\ell-1,\ell-1}(n)a_n^{(r+\ell+1)}. \end{aligned}$$

That is,

$$\begin{aligned} DS &= \sum_{j=0}^{\ell-1} \tbinom{\ell}{j}c_{j,0}(n)a_n^{(r+2)} + \sum_{j=1}^{\ell-1} \tbinom{\ell}{j}c_{j,1}(n)a_n^{(r+3)} + \cdots \\ &\quad + \sum_{j=\ell-2}^{\ell-1} \tbinom{\ell}{j}c_{j,\ell-2}(n)a_n^{(r+\ell)} + \sum_{j=\ell 1}^{\ell-1} \tbinom{\ell}{j}c_{j,\ell-1}(n)a_n^{(r+\ell+1)} \\ &= -c_{\ell,1}(n)a_n^{(r+2)} - c_{\ell,2}(n)a_n^{(r+3)} - \cdots - c_{\ell,\ell-1}(n)a_n^{(r+\ell)} - c_{\ell,\ell}(n)a_n^{(r+\ell+1)}, \end{aligned}$$

where the last equality follows from (3.33) for $x = 1$ and $y = n + 1$. Substituting this term into (3.39), and since by (3.23) for $y = n + 1$ we have $(n + 1)^\ell = c_{\ell,0}(n)$, it follows that

$$\begin{aligned} (n+1)^\ell a_n^{(r+1)} - DS &= c_{\ell,0}(n)a_n^{(r+1)} + c_{\ell,1}(n)a_n^{(r+2)} + c_{\ell,2}(n)a_n^{(r+3)} + \cdots \\ &\quad + c_{\ell,\ell-1}(n)a_n^{(r+\ell)} + c_{\ell,\ell}(n)a_n^{(r+\ell+1)} \\ &= \sum_{m=0}^{\ell} c_{\ell,m}(n)a_n^{(r+m+1)} = t_\ell^{(r)}(n). \end{aligned}$$

Hence, $\sum_{m=0}^{j} c_{j,m}(n)a_n^{(r+m+1)}$ is a solution of (3.5) also for $j = \ell$. Altogether, we have shown that $\sum_{m=0}^{\ell} c_{\ell,m}(n)a_n^{(r+m+1)}$ is the (unique) solution of the recurrence relation (3.5) and this proves (3.7).

To prove (3.8), we note that, by (3.10), for $x = 1$, $y = n + 1$ we have

$$c_{\ell,m}(n) = p_{\ell,m}(1, n+1) = (-1)^m m! \sum_{j=m}^{\ell} \binom{\ell}{j} \left\{ {j \atop m} \right\} (n+1)^{\ell-j}$$

and thus,

$$\begin{aligned} \sum_{m=0}^{\ell} c_{\ell,m}(n)a_n^{(r+m+1)} &= \sum_{m=0}^{\ell} \Bigg(\sum_{j=m}^{\ell} \binom{\ell}{j} (-1)^m m! \left\{ {j \atop m} \right\} (n+1)^{\ell-j} \Bigg) a_n^{(r+m+1)} \\ &= \sum_{m=0}^{\ell} \Bigg(\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^m m! \left\{ {j \atop m} \right\} (n+1)^{\ell-j} \Bigg) a_n^{(r+m+1)} \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} \Bigg(\sum_{m=0}^{\ell} (-1)^m m! \left\{ {j \atop m} \right\} a_n^{(r+m+1)} \Bigg) (n+1)^{\ell-j}. \end{aligned}$$

This is the alternative representation (3.8) since $\left\{ {j \atop m} \right\} = 0$ for $j < m$. □

4. Application to the Four Examples

Example 1. Let $a_n = 1$, $n \geq 0$. Then by (2.3) we have $a_n^{(r)} = \binom{n+r}{r}$ and therefore $a_n^{(r+m+1)} = \binom{n+r+m+1}{n}$. By (3.7), (3.8) and (3.19) it follows that

$$\begin{aligned}\sum_{k=0}^{n} k^{\ell} \binom{k+r}{r} &= \sum_{m=0}^{\ell} c_{\ell,m}(n) \binom{n+r+m+1}{n} \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} \left(\sum_{m=0}^{j} (-1)^m m! \left\{ {j \atop m} \right\} \binom{n+r+m+1}{n} \right) (n+1)^{\ell-j} \\ &= \sum_{m=0}^{\ell} (-1)^m m! \left\{ {\ell+n+1 \atop m+n+1} \right\}_{n+1} \binom{n+r+m+1}{n}.\end{aligned}$$

The special case $r = 0$ gives the following representations for the sums of powers of the first n consecutive positive integers:

$$\begin{aligned}\sum_{k=0}^{n} k^{\ell} &= \sum_{m=0}^{\ell} c_{\ell,m}(n) \binom{n+m+1}{n} \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} \left(\sum_{m=0}^{j} (-1)^m m! \left\{ {j \atop m} \right\} \binom{n+m+1}{n} \right) (n+1)^{\ell-j}\end{aligned}$$

or, alternatively,

$$\sum_{k=0}^{n} k^{\ell} = \sum_{m=0}^{\ell} (-1)^m m! \left\{ {\ell+n+1 \atop m+n+1} \right\}_{n+1} \binom{n+m+1}{n}.$$

Example 2. Let a_n be the characteristic function of $\{1\}$, already defined at p. 4. Then by (2.5) we have $a_n^{(r)} = \binom{n+r-2}{n-1}$ and $a_n^{(r+m+1)} = \binom{n+r+m-1}{r+m}$. By (3.7), (3.8) and (3.19) it follows that

$$\begin{aligned}\sum_{k=0}^{n} k^{\ell} \binom{k+r-2}{k-1} &= \sum_{m=0}^{\ell} c_{\ell,m}(n) \binom{n+r+m-1}{r+m} \\ &= \sum_{j=0}^{\ell} \binom{\ell}{j} \left(\sum_{m=0}^{j} (-1)^m m! \left\{ {j \atop m} \right\} \binom{n+r+m-1}{r+m} \right) (n+1)^{\ell-j} \\ &= \sum_{m=0}^{\ell} (-1)^m m! \left\{ {\ell+n+1 \atop m+n+1} \right\}_{n+1} \binom{n+r+m-1}{r+m},\end{aligned} \tag{4.1}$$

For $r = 0$ we have $\binom{k-2}{k-1} = (k = 1)$ and the left-hand side of (4.1) is 1. Hence, the second identity of (4.1) gives

$$1 = \sum_{j=0}^{\ell} \binom{\ell}{j} \left(\sum_{m=0}^{j} (-1)^m m! \left\{ {j \atop m} \right\} \binom{n+m-1}{m} \right) (n+1)^{\ell-j}. \tag{4.2}$$

By using the following identity

$$n^m = \sum_{k=0}^{m} (-1)^{m-k} k! \left\{ {m \atop k} \right\} \binom{n+k-1}{k} \tag{4.3}$$

(see [14, Identity (6.12)]), we get another representation of the sums of powers of the first n consecutive positive integers.

Corollary 5. *For all $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, the following identity holds:*

$$\sum_{k=1}^{n} k^m = \sum_{k=0}^{m} (-1)^{m-k} k! \left\{ {m \atop k} \right\} \binom{n+k}{k+1}. \tag{4.4}$$

Proof. From (4.3) and by changing the order of summation, we get

$$\begin{aligned}\sum_{k=1}^{n} k^m &= \sum_{k=1}^{n} \sum_{j=0}^{m} (-1)^{m-j} j! \left\{ {m \atop j} \right\} \binom{k+j-1}{j} \\ &= \sum_{j=0}^{m} (-1)^{m-j} j! \left\{ {m \atop j} \right\} \sum_{k=1}^{n} \binom{k+j-1}{j}.\end{aligned}$$

This proves the corollary (writing k instead of j on the right-hand side), since the second sum in the last equation is equal to $\binom{n+j}{j+1}$ by (2.4). □

Note that Identity (4.4) is different from the formula given in [4, p. 265]:

$$\sum_{k=1}^{n} k^m = \sum_{k=0}^{m} \binom{n+1}{k+1} \left\{ {m \atop k} \right\} k!.$$

Example 3. Let $a_n := F_n$, $n \geq 0$, be the Fibonacci sequence. Then by (2.8) we have $F_n^{(r)} = F_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k-1}{k} F_{2(r-k)}$ and

$$F_n^{(r+m+1)} = F_{n+2(r+m+1)} - \sum_{k=0}^{r+m} \binom{n+k-1}{k} F_{2(r+m+1-k)}.$$

By (3.7) we have

$$\sum_{k=0}^{n} k^\ell F_k^{(r)} = \sum_{m=0}^{\ell} c_{\ell,m}(n) F_n^{(r+m+1)}.$$

For $r = 0$ this means that

$$\sum_{k=0}^{n} k^\ell F_k = \sum_{m=0}^{\ell} c_{\ell,m}(n) F_{n+2(m+1)} - \sum_{m=0}^{\ell} c_{\ell,m}(n) \sum_{k=0}^{m} \binom{n+k-1}{k} F_{2(m+1-k)}. \quad (4.5)$$

We will prove later that the double sum on the right-hand side of (4.5) does not depend on n and it is equal to the first sum taken at $n = 0$. That is,

$$\sum_{m=0}^{\ell} c_{\ell,m}(n) \sum_{k=0}^{m} \binom{n+k-1}{k} F_{2(m+1-k)} = \sum_{m=0}^{\ell} c_{\ell,m}(0) F_{2(m+1)}. \quad (4.6)$$

We can therefore state the following theorem for weighted Fibonacci sums.

Theorem 10. *For all ℓ, $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n} k^\ell F_k = \sum_{m=0}^{\ell} c_{\ell,m}(n) F_{n+2(m+1)} - \sum_{m=0}^{\ell} c_{\ell,m}(0) F_{2(m+1)}. \quad (4.7)$$

Moreover, since by (3.14) we have $c_{\ell,m}(0) = (-1)^m m! \left\{ {\ell+1 \atop m+1} \right\}$, we conjecture that the constant term in (4.7) can be expressed as follows.

Conjecture 1. For all $\ell \in \mathbb{N}_0$, we have

$$\sum_{m=0}^{\ell} (-1)^m m! \left\{ {\ell+1 \atop m+1} \right\} F_{2(m+1)} = (-1)^\ell \sum_{m=0}^{\ell} m! \left\{ {\ell \atop m} \right\} F_{m+2}.$$

Note that the unsigned sum on the right-hand side (the sequence A000557 in [21]) is exactly the sequence $M_{2,j}$ (see [26, Equation (6.2)] for ℓ instead of j).

Example 4. Let $a_n := L_n$, $n \geq 0$, be the Lucas sequence. Then by (2.12) we have $L_n^{(r)} = L_{n+2r} - \sum_{k=0}^{r-1} \binom{n+k}{k} L_{2(r-k)-1}$ and

$$L_n^{(r+m+1)} = L_{n+2(r+m+1)} - \sum_{k=0}^{r+m} \binom{n+k}{k} L_{2(r+m+1-k)-1}.$$

By (3.7) we have

$$\sum_{k=0}^{n} k^\ell L_k^{(r)} = \sum_{m=0}^{\ell} c_{\ell,m}(n) L_n^{(r+m+1)}.$$

For $r = 0$ this means that

$$\sum_{k=0}^{n} k^\ell L_k = \sum_{m=0}^{\ell} c_{\ell,m}(n) L_{n+2(m+1)} - \sum_{m=0}^{\ell} c_{\ell,m}(n) \sum_{k=0}^{m} \binom{n+k}{k} L_{2(m+1-k)-1}. \quad (4.8)$$

As in the case of the Fibonacci numbers we will prove that the double sum on the right-hand side of (4.8) does not depend on n and is given by the first sum taken at $n = 0$. That is,

$$\sum_{m=0}^{\ell} c_{\ell,m}(n) \sum_{k=0}^{m} \binom{n+k}{k} L_{2(m+1-k)-1} = \sum_{m=0}^{\ell} c_{\ell,m}(0) \sum_{k=0}^{m} L_{2(m+1-k)-1},$$

and this is equal to $\sum_{m=0}^{\ell} c_{\ell,m}(0) L_{2(m+1)} - L_0 \cdot 0^{\ell}$ by the well-known formula $\sum_{k=0}^{m} L_{2(m+1-k)-1} = \sum_{k=0}^{m} L_{2k+1} = L_{2(m+1)} - L_0$ and (3.30). Therefore, we can state the following theorem for weighted Lucas sums.

Theorem 11. *For all ℓ, $n \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{n} k^{\ell} L_k = \sum_{m=0}^{\ell} c_{\ell,m}(n) L_{n+2(m+1)} - \sum_{m=0}^{\ell} c_{\ell,m}(0) L_{2(m+1)} + L_0 \cdot 0^{\ell}. \tag{4.9}$$

Furthermore, since by (3.14) we have $c_{\ell,m}(0) = (-1)^m m! \left\{ {\ell+1 \atop m+1} \right\}$, we conjecture that the constant term in (4.9) can be expressed as follows.

Conjecture 2. For all $\ell \in \mathbb{N}_0$, we have

$$\sum_{m=0}^{\ell} (-1)^m m! \left\{ {\ell+1 \atop m+1} \right\} L_{2(m+1)} = (-1)^{\ell} \sum_{m=0}^{\ell} m! \left\{ {\ell \atop m} \right\} L_{m+2}.$$

Note that multiplying the term on the right-hand side by (-1) results in the sequence A263968 in [21].

In the next subsection we will prove Theorem 10 together with Theorem 11.

4.1. Proof of Theorems 10 and 11

In order to prove Theorem 10 and Theorem 11 we need two lemmas.

Lemma 1. *For all m, n, $k \in \mathbb{N}_0$, we have*

$$\binom{n+m-k-1}{m} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n+m-j-1}{m-j}. \tag{4.10}$$

Identity (4.10) is a version of Identity 3.49 in [13, p. 28] for j instead of k, k instead of n, $n-1$ instead of r and $n+m-1$ instead of x.

Lemma 2. *For all k, $\ell \in \mathbb{N}_0$, we have*

$$(k+1)^{\ell} = \sum_{j=0}^{k} \binom{k}{j} j! \left\{ {\ell+1 \atop j+1} \right\}. \tag{4.11}$$

Proof. By applying (3.20) and noting that $\left\{ {n \atop j} \right\} = 0$ for $n < j$, it follows that $\left\{ {\ell+1 \atop j+1} \right\} = \sum_{n=0}^{\ell} \binom{\ell}{n} \left\{ {n \atop j} \right\}$. Hence, $\binom{k}{j} j! \left\{ {\ell+1 \atop j+1} \right\} = \binom{k}{j} j! \sum_{n=0}^{\ell} \binom{\ell}{n} \left\{ {n \atop j} \right\}$. Summing up over j from 0 to k and changing the order of summation, we get

$$\sum_{j=0}^{k} \binom{k}{j} j! \left\{ {\ell+1 \atop j+1} \right\} = \sum_{j=0}^{k} \binom{k}{j} j! \sum_{n=0}^{\ell} \binom{\ell}{n} \left\{ {n \atop j} \right\} = \sum_{n=0}^{\ell} \binom{\ell}{n} \sum_{j=0}^{k} \binom{k}{j} j! \left\{ {n \atop j} \right\}.$$

Since $k^n = \sum_{j=0}^{k} \binom{k}{j} j! \left\{ {n \atop j} \right\}$ (see [14, (6.10)]) we obtain Identity (4.11) by using the binomial theorem. □

Now we prove Theorem 10 (and Theorem 11 as well).

Proof of Theorem 10. We have to prove Identity (4.6). First we consider the double sum on the left-hand side of this equation which we refer to as LHS for short:

$$\begin{aligned} LHS = {} & c_{\ell,0}(n)\tbinom{n-1}{0}F_2 + c_{\ell,1}(n)\Big(\tbinom{n-1}{0}F_4 + \tbinom{n}{1}F_2\Big) \\ & + c_{\ell,2}(n)\Big(\tbinom{n-1}{0}F_6 + \tbinom{n}{1}F_4 + \tbinom{n+1}{2}F_2\Big) + \cdots \\ & + c_{\ell,\ell}(n)\Big(\tbinom{n-1}{0}F_{2(\ell+1)} + \tbinom{n}{1}F_{2\ell} + \cdots + \tbinom{n+\ell-1}{\ell}F_2\Big). \end{aligned}$$

Collecting the coefficients of the same Fibonacci numbers we obtain

$$\begin{aligned} LHS = {} & \Big(c_{\ell,0}(n)\tbinom{n-1}{0} + c_{\ell,1}(n)\tbinom{n}{1} + c_{\ell,2}(n)\tbinom{n+1}{2} + \cdots + c_{\ell,\ell}(n)\tbinom{n+\ell-1}{\ell}\Big)F_2 \\ & + \Big(c_{\ell,1}(n)\tbinom{n-1}{0} + c_{\ell,2}(n)\tbinom{n}{1} + c_{\ell,3}(n)\tbinom{n+1}{2} + \cdots + c_{\ell,\ell}(n)\tbinom{n+\ell-2}{\ell-1}\Big)F_4 + \cdots \\ & + \Big(c_{\ell,\ell-1}(n)\tbinom{n-1}{0} + c_{\ell,\ell}(n)\tbinom{n}{1}\Big)F_{2\ell} + c_{\ell,\ell}(n)\tbinom{n-1}{0}F_{2(\ell+1)}, \end{aligned}$$

that is,

$$LHS = \sum_{k=0}^{\ell} \sum_{m=k}^{\ell} c_{\ell,m}(n) \binom{n+m-k-1}{m-k} F_{2(k+1)} = \sum_{k=0}^{\ell} c_{\ell,k}(0) F_{2(k+1)}.$$

We must therefore show that the following applies for all $0 \le k \le \ell$:

$$\sum_{m=k}^{\ell} c_{\ell,m}(n) \binom{n+m-k-1}{m-k} = c_{\ell,k}(0) = (-1)^k k! \left\{ {\ell+1 \atop k+1} \right\}, \tag{4.12}$$

where the second identity follows from (3.14).

We will prove (4.12) by induction on $k \ge 0$. The assertion for $k = 0$ follows from the first identity of (4.2) since $(-1)^0 0! \left\{ {\ell+1 \atop 1} \right\} = (\ell + 1 > 0) = (\ell \ge 0) = 1$. Now assume that the assertion (4.12) is true for $0, 1, \ldots, k-1$, and consider the sequence

$a_n = (n = j+1), n \in \mathbb{N}_0$, $j \geq 0$, the characteristic function of $\{j+1\}$, $j \geq 0$. Then by (2.1) we have $a_n^{(r)} = \binom{n+r-1-(j+1)}{r-1} = \binom{n+r-j-2}{n-j-1}$ and $a_n^{(r+m+1)} = \binom{n+r+m-j-1}{r+m}$. For $r = 0$, Theorem 9 implies that $\sum_{k=0}^{n} k^\ell \binom{k-j-2}{k-j-1} = \sum_{m=0}^{\ell} c_{\ell,m}(n)\binom{n+m-j-1}{m}$. The left-hand side of this equation is equal to $(j+1)^\ell$, because $\binom{k-j-2}{k-j-1} = (k = j+1)$. Therefore, using k instead of j, for all $k \geq 0$, we get the remarkable formula

$$(k+1)^\ell = \sum_{m=0}^{\ell} c_{\ell,m}(n)\binom{n+m-k-1}{m}, \tag{4.13}$$

which contains the first identity of (4.1) (for $r = 0$ and $k = 0$) as a special case.

By Lemma 1, changing the order of summation and by the induction assumption, we obtain

$$\begin{aligned}(k+1)^\ell &= \sum_{m=0}^{\ell} c_{\ell,m}(n)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{n+m-j-1}{m-j}\\ &= \sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{m=j}^{\ell} c_{\ell,m}(n)\binom{n+m-j-1}{m-j}\\ &= \sum_{m=0}^{\ell} c_{\ell,m}(n)\binom{n+m-1}{m} - \binom{k}{1}\sum_{m=1}^{\ell} c_{\ell,m}(n)\binom{n+m-2}{m-1} + \cdots\\ &\quad + (-1)^k\sum_{m=k}^{\ell} c_{\ell,m}(n)\binom{n+m-k-1}{m-k}.\end{aligned}$$

Solving for the last sum on the right-hand side of the above equation and by the induction assumption we get

$$\begin{aligned}\sum_{m=k}^{\ell} c_{\ell,m}(n)\binom{n+m-k-1}{m-k} &= (-1)^k\cdot\Big((k+1)^\ell - \sum_{j=0}^{k-1}(-1)^j\tbinom{k}{j}c_{\ell,j}(0)\Big)\\ &= (-1)^k\cdot\Big((k+1)^\ell - \sum_{j=0}^{k}(-1)^j\tbinom{k}{j}c_{\ell,j}(0)\Big) + c_{\ell,k}(0),\end{aligned}$$

and the last expression is equal to $c_{\ell,k}(0)$. In fact, by Lemma 2 and noting that by (3.15) we have $(-1)^j\binom{k}{j}c_{\ell,j}(0) = (-1)^j\binom{k}{j}(-1)^j j!\left\{{\ell+1 \atop j+1}\right\} = \binom{k}{j}j!\left\{{\ell+1 \atop j+1}\right\}$, the term in parentheses is equal to 0. Therefore, the assertion (4.12) is also true for k and this proves Theorem 10 and Theorem 11 as well, since no properties of the Fibonacci numbers were used in the proof. □

5. Conclusion

In this paper, we first studied the hypersequences $\left(a_n^{(r)}\right)_{n\in\mathbb{N}_0}$, $r \in \mathbb{N}_0$, of an arbitrary sequence $(a_n)_{n\in\mathbb{N}_0}$. Then we applied the results to four different sequences, namely the constant sequence $a_n = 1$, $n \in \mathbb{N}_0$, the characteristic function of $\{1\}$, the Fibonacci sequence and the Lucas sequence. In the last two cases we obtained some new results on the hyperfibonacci and hyperlucas numbers.

We then studied the weighted sums of the type $\sum_{k=0}^{n} k^\ell a_k^{(r)}$, $\ell, n, r \in \mathbb{N}_0$, and derived a recurrence relation and its solution. It turned out that this solution depends strongly on the expression $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(k+n+1)^\ell$. We have derived some known and new properties of this expression and its generalization $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(kx+y)^\ell$, $x, y \in \mathbb{C}$, and applied the results to the four sequences as above. In this way we obtained old and new formulas for the sums of powers of the first n consecutive positive integers and for weighted Fibonacci and Lucas sums.

As noted by Koshy [17, p. 354] (see also Remark 6), knowing the coefficients of $\sum_{k=0}^{n} k^\ell F_k$ and $\sum_{k=0}^{n} k^\ell L_k$, one obtains by the first formula of (3.38) the coefficients of $\sum_{k=0}^{n} k^{\ell-1} F_k$ and $\sum_{k=0}^{n} k^{\ell-1} L_k$, respectively. However, it is not clear how to calculate the non-constant term.

The second formula of (3.38) solves the inverse problem, namely knowing the coefficients of $\sum_{k=0}^{n} k^{\ell-1} F_k$ and $\sum_{k=0}^{n} k^{\ell-1} L_k$, how to determine the coefficients of $\sum_{k=0}^{n} k^\ell F_k$ and $\sum_{k=0}^{n} k^\ell L_k$, respectively. This formula is a complete answer to the statement made by Koshy [17, p. 354]:

> "On the other hand, if we could use the coefficients of $S(m-1)$ to determine those in $S(m)$ it would be a tremendous advantage in the study of weighted Fibonacci and Lucas sums. The same would hold for $T(m)$."

Note that Koshy employs the following definitions: $S(m) := \sum_{k=0}^{n} k^m F_k$ and $T(m) := \sum_{k=0}^{n} k^m L_k$. For example, let $\ell = 3$. Then (see [17, (29.13) and (29.14)])

$$\sum_{k=0}^{n} k^3 F_k = (n+1)^3 F_{n+2} - (3n^2+9n+7)F_{n+4} + (6n+12)F_{n+6} - 6F_{n+8} + (F_3 - 6F_5 + 6F_7),$$

where $F_3 - 6F_5 + 6F_7 = 50$. Using the second formula from (3.38) we have

$$\begin{aligned}
c_{4,0}(n) &= (n+1)c_{3,0}(n) = (n+1)^4 \\
c_{4,1}(n) &= (n+2)c_{3,1}(n) - c_{3,0}(n) = -(4n^3+18n^2+28n+15) \\
c_{4,2}(n) &= (n+3)c_{3,2}(n) - 2c_{3,1}(n) = 12n^2+48n+50 \\
c_{4,3}(n) &= (n+4)c_{3,3}(n) - 3c_{3,2}(n) = -(24n+60) \\
c_{4,4}(n) &= (n+5)c_{3,4}(n) - 4c_{3,3}(n) = 24.
\end{aligned}$$

So by (4.7) for $\ell = 4$, we obtain

$$\sum_{k=0}^{n} k^4 F_k = (n+1)^4 F_{n+2} - (4n^3 + 18n^2 + 28n + 15)F_{n+4} + (12n^2 + 48n + 50)F_{n+6} - (24n+60)F_{n+8} + 24F_{n+10} - (F_2 - 15F_4 + 50F_6 - 60F_8 + 24F_{10})$$

and this is exactly the formula given in [17, (29.18)], since the last term in parentheses on the right-hand side is equal to 416.

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