COMPOSITION-THEORETIC SERIES AND FALSE THETA FUNCTIONS

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Abstract

Many natural partition-theoretic series can be equally readily interpreted as composition-theoretic series, but this viewpoint seems to have not been much employed in either theory. We consider some of the consequences of this viewpoint. As examples, we give results concerning the reciprocals of Ramanujan's theta functions and of the false theta functions of L. J. Rogers, and raise an array of questions related to these. Part of this study may be considered a natural dual of the truncated pentagonal number theorem of Andrews and Merca.

1. Introduction: Partitions, Compositions, and Theta Functions

Integer partitions are ubiquitous in mathematics and the physical sciences; they are connected to combinatorics, number theory, abstract algebra, statistical physics, and other fields [2]. In particular, partition theory is intertwined with the theory of modular forms central to contemporary number theory [4, 12].

While less widely studied in number theory, integer compositions are also ubiquitous in combinatorics and other fields; for instance, a deep study was given by

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P. A. MacMahon [10]. This paper, a follow-up work to [16, 17], shows that integer compositions can be useful to compute partition-theoretic statistics and other arithmetic functions. We begin with examples from [17] using Ramanujan's theta functions as partition generating functions; we then use the false theta functions of L. J. Rogers [15] to play a similar role to theta functions, to yield information about certain classes of restricted partition counting functions. Our hope is that this paper will motivate investigations of partition theorems from this point of view as a source of new results and interesting questions.

Let us give a sample result. In recent work [17], the second and third authors connect composition-theoretic infinite series to the reciprocals of Ramanujan's theta functions [5]. Consider the two theta functions

$$\psi(q) := \sum_{n\geq 0} q^{n(n+1)/2}, \qquad \varphi(q) \ := \ \sum_{n=-\infty}^{\infty} q^{n^2} \ , \tag{1}$$

with $|q| < 1$. It is noteworthy in the above series that the exponents of q are *polygonal numbers*: $n(n+1)/2$ is the nth triangular number, and n^2 is the nth perfect square, with $n \geq 1$.

For $k \geq 3$, let $P_k := \big\{n\big((k-2)n \pm (k-4)\big)/2 : n \in \mathbb{N}\big\}$ denote the set of *positive extended k-sided polygonal numbers*, which we will refer to as *k-gonal numbers*. Thus P_3 is the set of triangular numbers, P_4 is the squares, etc.

It is proved in [17] that the reciprocals of specializations of Ramanujan's theta functions have natural representations as q-series summed over compositions whose parts are k-gonal numbers; for example,

$$\frac{1}{\psi(q)} \ = \ \sum_{c\in\mathcal{C}_{P_3}} (-1)^{\ell(c)} q^{|c|}, \qquad \frac{1}{\varphi(q)} \ = \ \sum_{c\in\mathcal{C}_{P_4}} (-2)^{\ell(c)} q^{|c|}, \tag{2}$$

with $|q| < 1/2$ in the first case, and $|q| < 1/3$ in the second, and $\ell(c)$ the length of the composition c.

From their product forms [5], the reciprocal theta functions in (2) can be interpreted as partition generating functions. Thus, certain partition counting functions can be computed using compositions into k-gonal parts.

In this case, let $\mathrm{pod}(n)$ denote the number of *odd-distinct partitions* wherein odd parts may not be repeated, with size equal to n; the function $\mathrm{pod}(n)$ was studied in detail by Hirschhorn [8, Chapter 32] and Hirschhorn–Sellers [9]. Because $1/\psi(q) = \sum_{n\geq 0}(-1)^n \mathrm{pod}(n) q^n$ [9], it follows from the first identity in (2) that

$$\mathrm{pod}(n) \ = \ (-1)^n \sum_{\substack{c\in\mathcal{C}_{P_3} \\ |c|=n}} (-1)^{\ell(c)}, \tag{3}$$

where the right-hand sum is taken over the set of size-n compositions into triangular parts. Similar composition formulas for other partition statistics, such as the overpartition function $\overline{p}(n)$ derived from the formula for $1/\varphi(q)$ in (2), are proved in [17].

Here is another example proved in [17], that we will refer to later, which requires further notation be defined.

For $k \geq 5$, let $S_k := \{n \in \mathbb{N} : n \equiv 0, \pm 1 \pmod{k-2}\}$. We also require a modified length statistic ℓ_k^*, defined as follows. Define a subset $P_k^* \subset P_k$ of k-gonal numbers by

$$P_k^* := \left\{ n\big((k-2)n-(k-4)\big)/2 : n \in \mathbb{Z}, n \neq 0, n \equiv 0, 1 \,(\mathrm{mod}\ 4) \right\}. \tag{4}$$

Let $\ell_k^*(c)$ denote the number of parts $c_i \in \mathbb{N}$ in composition c such that $c_i \in P_k^*$. Finally, let $p_S(n) := \#\{\lambda \in \mathcal{P}_S : |\lambda| = n\}$. Then for $k \geq 6$ an *even* integer, the number of partitions of size n whose parts are in S_k is given by

$$p_{S_k}(n) = (-1)^n \sum_{\substack{c \in \mathcal{C}_{P_k} \\ |c|=n}} (-1)^{\ell(c)}, \tag{5}$$

where the summation is taken over size-n compositions into k-gonal parts. Similarly, for $k \geq 5$ an *odd* integer, the number of partitions of size n whose parts are in S_k is

$$p_{S_k}(n) = (-1)^n \sum_{\substack{c \in \mathcal{C}_{P_k} \\ |c|=n}} (-1)^{\ell_k^*(c)}. \tag{6}$$

In Section 2 we introduce the notation and background theorems that will be used throughout the paper. In Section 3 we employ the basic theorem relating partition-theoretic and composition-theoretic viewpoints to produce an array of examples involving recursions which the reader should be able to imitate easily in other contexts. In Section 4 we consider classic number theoretic functions such as representations by squares. In Section 5 we apply this viewpoint more thoroughly to unrestricted partitions and to a specific closely related function, the false theta function $\Psi(-q^2, q)$, producing several theorems concerning the behavior of its coefficients and raising a number of questions and conjectures we invite the reader to explore.

2. Background and Notation

Let $\mathbb{N}$ denote the set of *natural numbers* (positive integers). Let $\mathcal{P}$ denote the set of *integer partitions*, unordered finite sums of natural numbers including the empty partition $\emptyset \in \mathcal{P}$ (see e.g. [1]). For nonempty $\lambda \in \mathcal{P}$, we notate $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. Let $S \subseteq \mathbb{N}$, and let $\mathcal{P}_S$ denote the set of partitions whose parts lie in S; we consider $\emptyset \in \mathcal{P}_S$ for all $S \subseteq \mathbb{N}$. For $\lambda \in \mathcal{P}$, let $|\lambda| \geq 0$ denote the *size* (sum of parts), let $\ell(\lambda) \geq 0$ denote the *length* (number of parts), and let

$m_i = m_i(\lambda) \geq 0$ be the *multiplicity* (frequency) of $i \in \mathbb{N}$ as a part of partition λ. Let $\mathcal{C}$ denote the set of *integer compositions*, which are ordered finite sums of natural numbers. We extend the partition-theoretic terminology and notation defined above to compositions, with the same meanings, i.e., for $c \in \mathcal{C}$ we write $c = (c_1, c_2, \ldots, c_r)$ with $c_i \geq 1$, let $|c|$ denote the sum of the parts, let $\ell(c)$ denote the number of parts, etc. Let $\mathcal{C}_S$ denote compositions whose parts all lie in $S \subseteq \mathbb{N}$, thus $\mathcal{C} = \mathcal{C}_{\mathbb{N}}$; take the empty composition $\emptyset \in \mathcal{C}_S$ for all S.

Throughout we denote by $(q)_\infty$ the product

$$(q)_\infty = \prod_{k=1}^{\infty}(1 - q^k).$$

The primary theorem which relates partitions to compositions for the purpose of this study is the following, the proof of which is an exercise in the application of the geometric series formula.

Theorem 1. *For $S \subseteq \mathbb{N}$, $\chi\colon S \to \mathbb{C}$ an arbitrary function, define the function $p_{S,\chi}\colon\ \mathbb{Z}_{\geq 0} \to \mathbb{C}$ by the coefficients of the power series*

$$\sum_{n \geq 0} p_{S,\chi}(n) q^n = \frac{1}{1 - \sum_{j \in S} \chi(j) q^j},$$

noting $p_{S,\chi}(0) = 1$. Then for $n \geq 0$, the following statements are equivalent to the former:

$$\text{(i)}\quad p_{S,\chi}(n) = \sum_{\substack{c \in \mathcal{C}_S \\ |c| = n}} \prod_{c_i \in \mathcal{C}} \chi(c_i),$$

$$\text{(ii)}\quad p_{S,\chi}(n) = \sum_{\substack{j \in S \\ j \leq n}} \chi(j) p_{S,\chi}(n - j).$$

In the case where $\chi\colon S \to \{1, -1\}$, which is the case for many of the functions we study in this paper, clause (i) becomes

$$p_{S,\chi}(n) = \sum_{\substack{c \in \mathcal{C}_S \\ |c| = n}} (-1)^{\ell'_{S,\chi}(c)},$$

where $\ell'_{S,\chi}(c)$ is the number of parts $c_i \in S$ of composition $c \in \mathcal{C}_S$ such that $\chi(c_i) = -1$.

Clause (i) of the theorem can be stated in terms of power series (or more applicably generating functions), in which form it appears as the $a_0 = 1$ case of Lemma 9 from [17] which we state here.

Lemma 1. *For $a_i \in \mathbb{C}$, let $g(q) = 1 + \sum_{n\geq 1} a_n q^n$ be analytic on $\{q \in \mathbb{C} : |q| < 1\}$ and set $G(q) = 1/g(q)$. Then on the domain of analyticity of $G(q)$, we have*

$$G(q) = 1 + \sum_{n=0}^{\infty} b_n q^n$$

with the coefficient b_n given as a sum over compositions $c \in \mathcal{C}$ of size n:

$$b_n = \sum_{\substack{c\in\mathcal{C}\\|c|=n}} (-1)^{\ell(c)} a_1^{m_1} a_2^{m_2} \cdots a_n^{m_n}.$$

3. Extensions with Recursive Formulas

In this section, we record extensions of formulas proved in [17], such as those noted in the Introduction.

Here are two examples of Theorem 1; the first is well known [13].

Proposition 1. *The number of odd-distinct partitions of size n can be calculated by the recurrence*

$$\mathrm{pod}(n) \ = \ \sum_{\substack{j\in P_3\\ j\leq n}} (-1)^{j-1}\,\mathrm{pod}(n-j).$$

Proof. Set $S = P_3$ and $\chi = -1$ identically in Theorem 1. Thus the length statistic is $\ell'_{P_3,\chi}(c) = \ell(c)$, and the right-hand side of Theorem 1 (i) can be recognized as the right side of (3). The result follows by part (ii) of the theorem. □

Proposition 2. *For $k \geq 6$ an even integer, the number of partitions of size n whose parts are in S_k is given by*

$$p_{S_k}(n) = \sum_{\substack{j\in P_k\\ j\leq n}} (-1)^{j-1} p_{S_k}(n-j).$$

Proof. Set $S = P_k$ and $\chi = -1$ identically in Theorem 1. Then much as in the previous example, $\ell'_{P_k,\chi}(c) = \ell(c)$, and the right-hand side of Theorem 1 (i) can be recognized as the right side of (5). □

Remark 1. A similar formula holds for the case $k \geq 5$ an odd integer.

We note that taking $q \mapsto -q$ produces a dual theorem to Theorem 1. For simplicity we consider $\chi\colon S \to \{1, -1\}$. We require another modified length statistic. Let $\ell^*_{S,\chi}(c)$ denote the number of parts $c_i \in S$ of composition $c \in \mathcal{C}_S$ such that $(-1)^{c_i}\chi(c_i) = -1$.

Theorem 2. *For $S \subseteq \mathbb{N}$, $\chi\colon S \to \{1,-1\}$ an arbitrary function, define a function $p^*_{S,\chi}\colon\ \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ by the coefficients of the power series*

$$\sum_{n\geq 0} p^*_{S,\chi}(n)(-q)^n = \frac{1}{1-\sum_{j\in S}\chi(j)(-q)^j},$$

noting $p^*_{S,\chi}(0) = 1$. Then for $n \geq 1$, the following statements are equivalent:

(j) $p^*_{S,\chi}(n) = \sum_{\substack{c\in\mathcal{C}_S\\ |c|=n}} (-1)^{\ell^*_{S,\chi}(c)}$,

(ii) $p^*_{S,\chi}(n) = (-1)^n \sum_{\substack{j\in S\\ j\leq n}} (-1)^j \chi(j) p^*_{S,\chi}(n-j)$,

where the first summation is taken over size-n compositions whose parts are all in S, and $\ell^*_{S,\chi}(c)$ is the number of parts $c_i \in S$ of composition $c \in \mathcal{C}_S$ such that $(-1)^{c_i}\chi(c_i) = -1$.

Ramanujan's theta functions produced many examples in [17], such as those noted above. While these examples are (up to a trivial factor) modular forms, modularity does not play a role, and is certainly not a requirement.

Further examples of this type would yield families of similar theorems. Consider series where the nonzero coefficients are predictable, with the nonzero coefficients having exponents that belong to an easily identified set. Most immediately connected to these examples would be eta-quotients with clear partition-theoretic interpretations. Then one additionally interprets the reciprocal series as a generating function for compositions with weighted counts.

4. Some Polynomials and Classical Number Theoretic Functions

Here we use Lemma 1 to derive further examples of composition sums like those in the preceding sections. We now look at the reciprocals of polynomials as toy models.

Example 1. It is well known that

$$\frac{1}{1-q-q^2} = \sum_{n=0}^{\infty} F_{n+1} q^n,$$

where F_n is the nth Fibonacci number: $F_0 = 0, F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Using Lemma 1, we have

$$F_{n+1} = \sum_{\substack{c\in\mathcal{C}_{\{1,2\}}\\ |c|=n}} 1^{m_1}1^{m_2} = \sum_{\substack{c\in\mathcal{C}_{\{1,2\}}\\ |c|=n}} 1,$$

i.e. we have recovered the well-known result (see, e.g., [18, p. 109, Ex. 35 (c)]) that the number of compositions of size n into 1's and 2's is the $(n+1)$st Fibonacci number.

Example 2. A close relative of the Fibonacci sequence is the Pell sequence $\{r_n\}_{n=0}^{\infty}$: $r_0 = 0, r_1 = 1$, $r_{n+2} = 2r_{n+1} + r_n$ for $n \geq 0$. It can be easily shown that

$$\sum_{n=0}^{\infty} r_{n+1} q^n = \frac{1}{1 - 2q - q^2}.$$

Lemma 1 reveals that

$$r_{n+1} = \sum_{\substack{c \in \mathcal{C}_{\{1,2\}} \\ |c|=n}} 2^{m_1}.$$

Example 3. Let $p_k(n)$ denote the number of partitions of n into at most k parts. It is well known that

$$\sum_{n=0}^{\infty} p_k(n) q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

For any particular k, we can use Lemma 1 to get an explicit representation of $p_k(n)$ as a composition sum. For $k = 3$, we have

$$(1-q)(1-q^2)(1-q^3) = 1 - q - q^2 + q^4 + q^5 - q^6,$$

so

$$p_3(n) = \sum_{\substack{c \in \mathcal{C}_{\{1,2,4,5,6\}} \\ |c|=n}} (-1)^{m_4 + m_5}.$$

By the same reasoning,

$$p_4(n) = \sum_{\substack{c \in \mathcal{C}_{\{1,2,5,8,9,10\}} \\ |c|=n}} (-2)^{m_5} (-1)^{m_{10}}.$$

Of course, if one is willing to obtain a composition sum over *all* compositions of n, rather than over compositions with parts restricted to some subset of the natural numbers, then we need not restrict our attention to lacunary series and polynomials. We may consider taking the reciprocal of series with few or no zero coefficients, as in the next two examples.

Example 4. Consider $g(q) = \prod_{j=1}^{\infty}(1-q^j)^{24} = \sum_{n=0}^{\infty} \tau(n+1) q^n$, where $\tau(\cdot)$ is Ramanujan's tau-function [14]. Then

$$\frac{1}{g(q)} = \sum_{n=0}^{\infty} p^{(24)}(n) q^n,$$

where $p^{(k)}(n)$ is the number of k-colored partitions of n. We therefore have

$$p^{(24)}(n) = \sum_{\substack{c \in \mathcal{C} \\ |c|=n}} (-1)^{\ell(c)} \prod_{i=1}^{n} \tau(i+1)^{m_i}.$$

Example 5. In a similar spirit, if we let

$$g(q) = 1/\varphi(q)^4 = \sum_{n=0}^{\infty} (-1)^n \overline{p}^{(4)}(n) q^n,$$

where $\overline{p}^{(4)}(n)$ is the number of four-color overpartitions of n, and note that

$$\frac{1}{g(q)} = \sum_{n=0}^{\infty} r_4(n) q^n = \varphi(q)^4,$$

where $r_4(n)$ denotes the number of representations of n as a sum of four squares, we deduce that

$$r_4(n) = (-1)^n \sum_{\substack{c \in \mathcal{C} \\ |c|=n}} (-1)^{\ell(c)} \prod_{i=1}^{n} \overline{p}^{(4)}(i)^{m_i}.$$

5. Partitions and the False Theta Function $\Psi(-q^2, q)$

In this section we explore the utility of the composition-theoretic point of view in the study of more traditional combinatorial objects. Our starting point is the application of this point of view to the standard partition generating function

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q)_\infty} = \frac{1}{1 - q - q^2 + q^5 + q^7 - q^{12} - \dots}. \tag{7}$$

The latter expansion is the well-known pentagonal number theorem of Euler (see [1, p. 11, Cor. 1.7] or any other text on basic partition theory).

By Theorem 1, we have the following observation, which does not seem to have been widely employed in partition theory.

Corollary 1. *The number of partitions of n is equal to the number of compositions c of n into pentagonal numbers $\frac{k}{2}(3k+1)$, $k \in \mathbb{Z}$, $k \neq 0$, weighted by $(-1)^{\ell'(c)}$ where $\ell'(c)$ counts the number of parts of c from the set $\{5, 7, 22, 26, \dots\}$ produced by even k.*

There is something intriguing about this fact. After all, the number of unweighted compositions of n into pentagonal numbers, i.e., the coefficients of the series

$$\frac{1}{1 - q - q^2 - q^5 - q^7 - q^{12} - q^{15} - \dots},$$

clearly have exponential growth, being bounded below by the growth of the Fibonacci numbers which have exponential growth with base the golden ratio, $1.618\ldots^n$, and above by the number of compositions of n with unrestricted parts, 2^n. The actual rate of growth from numerical calculations is approximately 1.729918422^n, reaching this precision in about 50 terms.

Problem: State the exact value of the exponential base concisely. Is it even algebraic?

And yet the partitions, arising from weighting this count, have subexponential growth, being known to have main-term asymptotic [7]

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty.$$

Thus, if we list the positively and negatively weighted compositions in this set and match them, the "surviving" uncancelled compositions, all of positive weight, are a vanishing proportion of the entire set as n grows.

Indeed it is possible to map partitions into the "Fibonacci" subset of the pentagonal compositions consisting solely of 1s and 2s (and therefore being of positive weight): simply map the partition

$$\lambda = (\lambda_1, \ldots, \lambda_k) \quad \to \quad 1^{\lambda_1 - 2} 2 1^{\lambda_2 - 2} 2 1^{\lambda_3 - 2} 2 \ldots 1111,$$

where parts of size 1 are simply appended instead of subtracting 2. Then we know the remaining positively and negatively weighted compositions of n into pentagonal numbers are equal in number, and we can simply list and match them.

Problem: Can a more structured sign-cancelling almost-matching be produced, on weighted compositions of n into pentagonal numbers, in which the unmatched compositions have some more useful association with the original partition, and are positively weighted?

A sign of the delicacy with which the weighting is balanced, is that the behavior is extremely sensitive to changes in the weightings. For instance, the part 5 is weighted negatively. If we weight it positively, then we know that the weighted count of compositions increases, for now we are considering the coefficients of the generating function

$$\frac{1}{(q)_\infty - 2q^5} = \frac{1}{(q)_\infty \left(1 - \frac{2q^5}{(q)_\infty}\right)},$$

but this is precisely the partition generating function, times a geometric series with positive coefficients.

This has a consequence in terms of compositions:

Proposition 3. *The weighted count of compositions of n into pentagonal parts with an odd number of 5s is nonpositive for all n, and negative for $n \geq 5$. Hence, the coefficients of q^n for $n \geq 5$ in*

$$\frac{-q^5}{(q)_\infty}\left(\frac{1}{(q)_\infty - 2q^5}\right)$$

are all negative.

Proof. The first claim follows immediately from the fact that changing signs on the part 5 increases the coefficients. The second claim only requires that we establish that this is in fact the generating function for such compositions.

Write $A = -(q)_\infty + 1 + q^5$. Then the generating function for such compositions, taking first one part size 5, then three, then five, etc. is

$$\begin{aligned}
(-q^5)(\binom{1}{1} + \binom{2}{1}A + \binom{3}{1}A^2 + \dots) + (-q^{15})(\binom{3}{3} + \binom{4}{3}A + \binom{5}{3}A^2 + \dots) + \dots \\
= (-q^5)\frac{1}{(1-A)^2} + (-q^{15})\frac{1}{(1-A)^4} + \dots \\
= \frac{-q^5}{(1-A)^2}\left(\frac{1}{1 - \frac{q^{10}}{(1-A)^2}}\right) = \frac{-q^5}{(q)_\infty}\left(\frac{1}{(q)_\infty - 2q^5}\right).
\end{aligned}$$

$\square$

However, not only do we have an increase, but in fact the rate of growth in the coefficients is now exponential.

Theorem 3. *For*

$$\sum_{n=0}^{\infty} c(n)q^n = \frac{1}{(q)_\infty - 2q^5},$$

the coefficients satisfy $c(n) > F_n$, the n-th Fibonacci number.

Proof. We rewrite the generating function in the following fashion:

$$\begin{aligned}
\frac{1}{(q)_\infty - 2q^5} &= \frac{1}{1 - q - q^2 - q^5 + q^7 - q^{12} - q^{15} + \dots} \\
&= \left(\frac{1}{1-q-q^2}\right)\left(\frac{1}{1 - \frac{q^5}{1-q-q^2} + \frac{q^7}{1-q-q^2} - \frac{q^{12}}{1-q-q^2} - \dots}\right).
\end{aligned}$$

We now observe that the coefficients of

$$\frac{q^5 - q^7}{1-q-q^2} = \sum_{n=0}^{\infty}(F_{n-5} - F_{n-7})q^n = q^5 + \sum_{n=6}^{\infty} F_{n-6}q^n$$

are all clearly positive, and the same holds for

$$\frac{q^{12}+q^{15}-q^{22}-q^{26}}{1-q-q^2}.$$

The signs in the pentagonal number theorem are periodic with period 4 in their nonzero appearances, so every grouping of four terms afterward will exhibit the same positivity. That is, the denominator in the second term is

$$1-\left(\frac{q^5-q^7}{1-q-q^2}+\frac{q^{12}+q^{15}-q^{22}-q^{26}}{1-q-q^2}+\frac{q^{35}+q^{40}-q^{51}-q^{57}}{1-q-q^2}+\dots\right).$$

Every fraction appearing, and all those subsequent, have positive coefficients within the parentheses since the coefficient in each is always of the form

$$(F_n - F_{n-i_2}) + (F_{n-i_1} - F_{n-i_3}) \quad , \quad 1 < i_1 < i_2 < i_3,$$

and F_n is an increasing function of n for $n > 1$.

Hence this generating function consists of the Fibonacci series generating function, times a power series with positive coefficients, and its coefficients are bounded below by F_n and its exponential growth. □

The same logic holds if the sign on the 7 is changed, though now the lower bound sequence is the "dying rabbit" version of the Fibonacci problem,

$$\frac{1}{1-q-q^2+q^5}.$$

This is described as OEIS sequence A023435 [11], where the interpretation of the sequence as "number of compositions of n into 1s, 2s, and 3s, with compositions having different placements of 2s identified" means that the sequence grows faster than the number of compositions into 1s and 3s, which is the Narayana's cows sequence A000930, known to have exponential asymptotic $1.46\dots^n$.

In general, any increase in the weighting on any term (i.e., subtracting a term) leads to an increase in the coefficients above $p(n)$, and this can be combinatorially explained by Proposition 3. It is less obvious that the resulting growth is exponential, and certainly a combinatorial argument that this is the case, like those given above, seems to be considerably less accessible. Yet this behavior seems to hold even if the decrease is

1. on a non-pentagonal number;

2. on a pentagonal number already weighted positively;

3. of less than 2, i.e. not merely a sign change.

In other words, by considering these compositions, we have found the following conjecture, for which we would hope for possibly an analytic and ideally a combinatorial explanation.

Conjecture 1. Given any $a > 0$ and $j \in \mathbb{N}$, the coefficients of the function

$$\frac{1}{(q)_\infty - aq^j} = \frac{1}{(q)_\infty \left(1 - \frac{aq^j}{(q)_\infty}\right)}$$

grow exponentially.

In the opposite direction, decreasing a weight on any term by any amount has a more complicated behavior than merely decreasing the resulting coefficients. Instead, it seems to lead to terms that have a negative exponential contribution of slow period, varying in sign. For instance, the coefficients of

$$\frac{1}{(q)_\infty + 2q^{12}}$$

appear to change in sign approximately every 25 terms (with this length very slowly growing). However, the absolute value of coefficients empirically exhibits exponential growth (base approximately 1.3956). We are led to the following conjecture.

Conjecture 2. Given any $a > 0$ and $j \in \mathbb{N}$, the coefficients of the function

$$\frac{1}{(q)_\infty + aq^j} = \frac{1}{(q)_\infty \left(1 + \frac{aq^j}{(q)_\infty}\right)}$$

alternate in sign, with the maximum of absolute value of the first n terms growing exponentially.

Analysis of these conjectures along the lines of the earlier proofs suggests that it is of combinatorial interest to consider the reciprocals of truncations of the pentagonal number series of the form

$$\frac{1}{1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots \pm q^{(k/2)(3k+1)}}$$

for some finite range, the final sign depending on the usual sign of the final pentagonal number. (One notes that this is different from

$$\frac{1}{(1-q)(1-q^2)\dots(1-q^k)},$$

which truncates the product. That is the generating function for partitions into a limited number of parts, discussed in the previous section.)

This draws an interesting connection to a study initiated by George Andrews and Mircea Merca [3] on the truncated pentagonal number recurrence for the partition function, which in a way could be considered the dual of this object. We give sufficient context for this paper here.

An immediate consequence of Euler's equation (7) is the well-known recurrence for the partition numbers,

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \cdots = 0, \quad n > 0, \quad p(0) = 1.$$

Since $p(n)$ is a positive, increasing function, it follows that for $n \geq 5$,

$$p(n) - p(n-1) - p(n-2) < 0,$$

since one must add a positive amount to get to 0. (Since $p(n-5) \geq p(n-12)$, $p(n-7) \geq p(n-15)$, and so on for all following pairs.)

Likewise for $n \geq 12$,

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) > 0,$$

since now one must add a negative amount to get to 0, and so forth every 2 terms.

On the other hand, this logic does not as easily apply for the intermediate values. It can be seen without much difficulty that $p(n) - p(n-1)$ is the number of partitions that contain no 1, and hence this difference is always positive and counts a describable set; what about

$$p(n) - p(n-1) - p(n-2) + p(n-5)?$$

Andrews and Merca answered this question completely, showing the following theorem.

Theorem 4 (Andrews and Merca, [3]). *These sums alternate in sign (e.g., $p(n) - p(n-1) - p(n-2) + p(n-5) < 0$ for $n \geq 7$), and furthermore, when we have taken $2k$ terms, the absolute value of the difference counts the set M_k of partitions in which k is the smallest part not appearing, and there are more parts larger than k than there are less than k.*

The generating function being analyzed by Andrews and Merca is a polynomial over $(q)_\infty$. For instance, the generating function for the first four terms of the pentagonal number recurrence, $p(n) - p(n-1) - p(n-2) + p(n-5)$, is

$$\frac{1 - q - q^2 + q^5}{(q)_\infty}.$$

Their main result then states that this will eventually have entirely negative coefficients.

Returning to the object of our consideration, we go further than changing a single term and ask, what if we change two signs in opposite directions? Or an infinitude?

An object of combinatorial interest in the q-series literature is the *false theta function*

$$\Psi(a,b) := \sum_{n=0}^{\infty} a^{\binom{n+1}{2}} b^{\binom{n}{2}} - \sum_{n=-\infty}^{-1} a^{\binom{n+1}{2}} b^{\binom{n}{2}}.$$

In particular, we have the following four special cases of this function related to the pentagonal numbers, where we give $(q)_\infty$ for comparison.

$$\begin{aligned}
(q)_\infty &= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \dots \\
\Psi(-q^2, q) &= 1 - q - q^2 + q^5 - q^7 + q^{12} + q^{15} - q^{22} + q^{26} - \dots \\
\Psi(-q^2, -q) &= 1 + q - q^2 - q^5 + q^7 + q^{12} - q^{15} - q^{22} + q^{26} + \dots \\
\Psi(q^2, q) &= 1 - q + q^2 - q^5 + q^7 - q^{12} + q^{15} - q^{22} + q^{26} - \dots \\
\Psi(q^2, -q) &= 1 + q + q^2 + q^5 - q^7 - q^{12} - q^{15} - q^{22} + q^{26} + \dots
\end{aligned}$$

It is an exercise in expanding each series by the residue of the indices mod 4 to see that signs of each are periodic mod 8 or a divisor thereof.

For three of these series, we have the "reduced coefficients" behavior.

$$\begin{aligned}
\frac{1}{\Psi(-q^2,-q)} &= \frac{1}{(q)_\infty}\left(\frac{1}{1+\frac{2}{(q)_\infty}(q - q^5 + q^{12} - q^{22} + \dots)}\right) \\
\frac{1}{\Psi(q^2,q)} &= \frac{1}{(q)_\infty}\left(\frac{1}{1+\frac{2}{(q)_\infty}(q^2 - q^5 + q^{15} - q^{22} + \dots)}\right) \\
\frac{1}{\Psi(q^2,-q)} &= \frac{1}{(q)_\infty}\left(\frac{1}{1+\frac{2}{(q)_\infty}(q + q^2 - q^7 - q^{22} + \dots)}\right)
\end{aligned}$$

The denominators on the right-hand sides are all clearly of positive coefficients. For instance,

$$\begin{aligned}
&\frac{q - q^5 + q^{12} - q^{22}}{(q)_\infty} \\
&\qquad = \sum_{n=0}^{\infty} \left((p(n-1) - p(n-5)) + (p(n-12) - p(n-22)) \right) q^n
\end{aligned}$$

and

$$\frac{q+q^2-q^7-q^{22}}{(q)_\infty} = \sum_{n=0}^{\infty}\left((p(n-1)-p(n-7))+(p(n-2)-p(n-22))\right)q^n.$$

These are positive generating functions since $p(n)$ is increasing.

Remark 2. The connection with the rank function mentioned earlier is that the number $N(0,n)$ of partitions with rank 0, i.e. with largest part equal to the nuumber of parts, has generating function

$$\frac{\Psi(-q^2,-q)-1}{(q)_\infty}.$$

By connection with the expansion given above, we have the recurrence

$$(-1)N(0,n) = p(n) - 2(p(n-1)-p(n-5)+p(n-12)-p(n-22)\dots),$$

which is different from the form immediately following from the direct expansion of the power series, and displays more explicitly the combinatorial fact that $N(0,n)$ and $p(n)$ have the same parity.

The remaining function, $\frac{1}{\Psi(-q^2,q)}$, has different and potentially more interesting behavior. We may write it as

$$\frac{1}{\Psi(-q^2,q)} = \frac{1}{(q)_\infty}\left(\frac{1}{1-\frac{2}{(q)_\infty}(q^7-q^{12}-q^{15}+q^{22}+\dots)}\right).$$

Now the term beneath the negative sign in the denominator is by no means obviously positive. However, the series appears to have the expected "increased coefficients" behavior: the coefficients are positive, larger then $p(n)$ after $n=7$, and quickly close in on exponential growth at a rate of approximately 1.5362^n.

We now observe the connection with Andrews and Merca's result on the truncated pentagonal number theorem. The series

$$\frac{q^7-q^{12}-q^{15}+q^{22}}{(q)_\infty}$$

is the generating function for

$$p(n-7)-p(n-12)-p(n-15)+p(n-22),$$

and this is exactly the difference between the pentagonal number recurrence for the partition function truncated at the 8th term (after $p(n-22)$), and after the fourth

$(p(n-5))$. Hence, letting $\mathcal{M}_k$ be the generating function for the count of M_k, we have

$$\frac{q^7 - q^{12} - q^{15} + q^{22}}{(q)_\infty} = \mathcal{M}_4 - \mathcal{M}_2.$$

We would hope that this would always have nonpositive coefficients, as it does initially, or, under the sign, that $\mathcal{M}_2 - \mathcal{M}_4$ would always have positive coefficients, and the same would hold for $\mathcal{M}_{4k-2} - \mathcal{M}_{4k}$ in general. Alas, it does not.

Remark 3. Interestingly, empirical calculation is at first misleading here. The coefficients of $\mathcal{M}_2 - \mathcal{M}_4$ grow quite steadily for 1000 terms, and reach their maximum at $n = 1068$, thereafter quickly becoming negative of large absolute value.

We show this descent in the following theorem, which generalizes the clause on the eventual negativity of $p(n) - p(n-1) - p(n-2) + p(n-5)$ from Andrews and Merca's result.

Theorem 5. *Let $1 \leq a < b < c$, $a, b, c \in \mathbb{N}$. Then the coefficients of*

$$\frac{1 - q^a - q^b + q^c}{(q)_\infty} = \sum_{n=0}^{\infty} (p(n) - p(n-a) - p(n-b) + p(n-c))q^n$$

are positive if $c \leq a + b$, and eventually at least some if not all are negative of increasing absolute value if $c > a + b$.

Proof. Suppose $c = a + b$. Then

$$\frac{1 - q^a - q^b + q^c}{(q)_\infty} = \frac{(1-q^a)(1-q^b)}{(q)_\infty}$$

and this is simply the generating function for partitions in which the parts a and b do not appear.

Suppose $c < a + b$. Then write

$$\begin{aligned}\frac{1 - q^a - q^b + q^c}{(q)_\infty} = \frac{(1-q^a)(1-q^b)}{(q)_\infty} &+ \frac{q^c - q^{a+b}}{(q)_\infty} \\ &= \frac{(1-q^a)(1-q^b)}{(q)_\infty} + q^c \frac{1 - q^{a+b-c}}{(q)_\infty}\end{aligned}$$

and this is the generating function for partitions of two types, one in which parts a and b do not appear, and another in which one part c is guaranteed to appear but the part $a + b - c$ does not (does not otherwise, if $c = a + b - c$).

Finally suppose $c > a + b$. Then write

$$\begin{aligned}F(q) = \frac{1 - q^a - q^b + q^c}{(q)_\infty} = \frac{(1-q^a)(1-q^b)}{(q)_\infty} &+ \frac{q^c - q^{a+b}}{(q)_\infty} \\ &= \frac{(1-q^a)(1-q^b)}{(q)_\infty} - q^{a+b} \frac{1 - q^{c-a-b}}{(q)_\infty}.\end{aligned}$$

Intuitively, the restriction that 2 parts not appear is much more restrictive than the restriction that 1 part not appear, but we may be precise.

If the coefficients of $F(q)$ are all positive, then clearly so are the coefficients of

$$\frac{F(q)}{(1-q^a)(1-q^b)(1-q^{c-a-b})} = \frac{1}{(1-q^{c-a-b})(q)_\infty} - \frac{q^c}{(1-q^a)(1-q^b)(q)_\infty}.$$

But the coefficients of the first term are

$$\sum_{k=0}^{\infty} p(n-k(c-a-b)),$$

and the coefficients of the second term are the double sum

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} p(n-c-ka-jb).$$

Well-known results on numerical semigroups yield that $c+ka+jb$ will eventually represent all large numbers of the form $c+m\cdot\gcd(a,b)$, and will do so for sufficiently large numbers an increasing number of times. Thus, for all but a fixed finite set of early terms in the singly-indexed sum, there will be a corresponding larger term subtracted.

For the remaining finite set, we may select a sufficiently large number of early terms in the second sum represented further multiple times. Since the partition function is subexponential, it holds that for any fixed K and ϵ, it is eventually true for all sufficiently large n that

$$p(n-K) > (1-\epsilon)p(n).$$

Thus, selecting say $\epsilon = 0.5$, and selecting a growing number of further multiply-represented terms in the latter sum as were not immediately cancelled in the first, we find that for sufficiently large n the terms subtracted overwhelm the first series, and in fact do so by growing amounts.

Hence at least some of the coefficients of $F(q)$ originally must have eventually become negative, with at least some of them of increasing absolute value.

$\square$

Remark 4. This is clearly a crude analysis compared to more elegant asymptotics tools, but it can be constructed with mostly combinatorial arguments plus basic facts about the growth of $p(n)$.

So attempting to prove the positivity of the right-hand factor in the denominator of

$$\frac{1}{\Psi(-q^2,q)} = \frac{1}{(q)_\infty}\left(\frac{1}{1-\frac{2}{(q)_\infty}(q^7-q^{12}-q^{15}-q^{22}+\dots)}\right)$$

by grouping the terms as per the truncated pentagonal number theorem fails to result in obviously positive coefficients - each such group does contribute positive coefficients for a long time (later groups for increasingly long times) but eventually turns negative.

The following is thus an open question that would seem to be of interest concerning false theta functions.

Conjecture 3. The coefficients of $H(q)$ in $\frac{1}{\Psi(-q^2,q)} = \frac{1}{(q)_\infty(1-H(q))}$ are strictly positive. The coefficients of $\frac{1}{\Psi(-q^2,q)}$ grow exponentially.

A natural followup if the conjecture is shown would be to find the rate of growth.

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References

[1] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, vol. 2, Addison–Wesley, Reading, Mass., 1976. Reissued, Cambridge University Press, Cambridge, 1998.

[2] G. E. Andrews, *q-series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS Regional Conference Series in Mathematics, no. 66, American Mathematical Society, Providence, R.I., 1986.

[3] G. E. Andrews and M. Merca, The truncated pentagonal number theorem, *J. Comb. Theory, Ser. A* **119** (2012), 1639–1643

[4] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, 2nd ed., Graduate Texts in Mathematics, Vol. 41, Springer, New York, 1990.

[5] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, Student Mathematical Library, Vol. 34. American Mathematical Society, Providence, R.I., 2006.

[6] F. J. Dyson, Some guesses in the theory of partitions, *Eureka (Cambridge)* **8** (1944), 10–15

[7] G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc. (2)* **17** (1918), 75–115.

[8] M. D. Hirschhorn, *The Power of q: a Personal Journey*, Developments in Mathematics 49, Springer, New York, 2017.

[9] M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of partitions with odd parts distinct, *Ramanujan J.* **22** (2010), 273–284.

[10] P. A. MacMahon, *Combinatory Analysis*, vol. II, Cambridge University Press, Cambridge, 1916; Reissued (with volumes I and II bound in one volume), AMS Chelsea, Providence, R.I., 2001.

[11] OEIS Foundation Inc. (2023), The On-line Encyclopedia of Integer Sequences, Published electronically at https://oeis.org/A023435.

[12] K. Ono, *The Web of Modularity: Arithmetic of the coefficients of Modular Forms and q-series*, Conference Board of the Mathematical Sciences No. 102, American Mathematical Society, Providence, R.I., 2004.

[13] S. Radu and J. A. Sellers, Congruence properties modulo 5 and 7 for the pod function *Int. J. Number Theory* **7** (2011), 2249–2259.

[14] S. Ramanujan, On certain arithmetical functions, *Trans. Camb. Philos. Soc.* **22** (1916), 159–184.

[15] L. J. Rogers, On two theorems of combinatory analysis and some allied identities, *Proc. London Math. Soc.* **16** (1917), 315–336.

[16] R. Schneider and A. V. Sills, Combinatorial formulas for arithmetic density, *Integers* **22** (2022), #A63, 7 pp.

[17] R. Schneider and A. V. Sills, Composition-theoretic series in partition theory, *Ramanujan J.*, to appear.

[18] R. P. Stanley, *Enumerative Combinatorics*, Volume 1, 2nd. ed., Cambridge University Press, Cambridge, 2012.