PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 9, September 1995

6-DIMENSIONAL MANIFOLDS WITHOUT TOTALLY ALGEBRAIC HOMOLOGY

PETER TEICHNER

(Communicated by Thomas Goodwillie)

ABSTRACT. We construct 6-dimensional manifolds for which not all codimension 2 homology classes (with $\mathbb{Z}/2$-coefficients) are realized by algebraic subvarieties in any real algebraic structure on the manifold. It was known that such examples exist in dimension 11 and higher, and that dimension 6 is the best possible. We also give an elementary algebraic topological proof of a connection between codimension 2 submanifolds and vector bundles which was previously proven only by algebraic geometrical methods.

1. INTRODUCTION

Let M be a closed smooth manifold of dimension n. An old problem in topology, dating back to the development of homology theories, is the question which classes in $H_k(M;\mathbb{Z}/2)$ can be represented by k-dimensional submanifolds in M.

If $k \leq n/2$, then René Thom's famous work [4] implies that any k-dimensional homology class is represented by a submanifold. The same is true for $k = n-1$ because the Poincaré dual of a class $z \in H_{n-1}(M;\mathbb{Z}/2)$ can be expressed as the first Stiefel-Whitney class w_1 of a 1-dimensional vector bundle over M. Then z is represented by the submanifold of zeroes of an arbitrary transversal section of this bundle. Similarly, one realizes arbitrary classes $z \in H_{n-2}(M;\mathbb{Z}^t)$, where $t : \pi_1 M \to \{\pm 1\}$ is a twisting of the coefficient group $\mathbb{Z}$: The Poincaré dual of z can be written as the (twisted) Euler class of a 2-dimensional vector bundle E over M (with $w_1 E = t + w_1 M$), and then z is again represented by the submanifold of zeroes of an arbitrary transversal section of this bundle. Note that the mod 2 reduction of the Poincaré dual of z can be written as $w_2 E$, the second Stiefel-Whitney class of E.

Question. *What can be said bout classes in $H_{n-2}(M;\mathbb{Z}/2)$ which are not reductions of (twisted) integral classes?*

As far as our knowledge goes, only the following facts are known:

(1) [4] If $n \leq 5$, all classes in $H_{n-2}(M;\mathbb{Z}/2)$ are represented by submanifolds.

Received by the editors January 20, 1994.
1991 *Mathematics Subject Classification*. Primary 57R20, 57R40; Secondary 14C25.
The author was supported by a Feodor Lynen Fellowship from the Humboldt Foundation.

(2) [2] If $N \subseteq M$ is a codimension 2 submanifold, then there exists a real vector bundle E over M such that w_2E is Poincaré dual to N.
(3) [1] For every dimension $n \geq 11$ there exists a manifold M together with a class in $H^2(M; \mathbb{Z}/2)$ which is not w_2 of a vector bundle over M.

It is remarkable that the known proof of (2) in [2] uses a relative version of the Nash-Tognoli theorem to put a real algebraic structure on M such that N is a nonsingular subvariety. Then the authors use a certain Grothendieck formula to prove that a class in $H_{n-2}(M; \mathbb{Z}/2)$ is represented by a (possibly singular) algebraic subvariety (for some real algebraic structure on M) if and only if its Poincaré dual can be written as w_2E for some real vector bundle E over M.

In §2 of this paper we give a purely topological proof of fact (2) above. We will also sketch how it can be used to reprove fact (1).

In §3 we close the gap between facts (1) and (3) by constructing for every $n \geq 6$ several 2-sphere bundles over $(n-2)$-manifolds such that every 2-dimensional $\mathbb{Z}/2$-cohomology class which restricts to the generator in the fibre cannot be written as w_2E. This means that in our examples at most half of the classes in $H^2(M; \mathbb{Z}/2)$ can be written as w_2E. These n-manifolds are also orientable. They include the lowest dimensional examples of manifolds without *totally algebraic homology* in the following sense: Given a compact nonsingular affine algebraic variety X, let $H_k^{\mathrm{alg}}(X; \mathbb{Z}/2)$ denote the subgroup of homology classes represented by Zariski closed k-dimensional algebraic subvarieties of X. Then we have proved the following.

Theorem 1. *For each dimension $n \geq 6$ there exist compact oriented smooth n-manifolds M such that, for each nonsingular affine algebraic variety X diffeomorphic to M, the orders of the homology groups satisfy*

$$|H_{n-2}^{\mathrm{alg}}(X; \mathbb{Z}/2)| \leq \tfrac{1}{2}|H_{n-2}(M; \mathbb{Z}/2)|.$$

Moreover, one can choose M as above with $|H_{n-2}(M; \mathbb{Z}/2)|$ arbitrarily large.

We would like to point out another gap in our knowledge: What is the minimal dimension in which the Poincaré dual of a class w_2E cannot be represented by an embedding? This minimum must be bigger than 5, and in §4 we will give an example which shows that it is less than 10. This, together with the results of [2], implies the following.

Theorem 2. *There exists a compact nonsingular affine real algebraic variety X of dimension 9 such that a certain class in $H_7^{\mathrm{alg}}(X; \mathbb{Z}/2)$ cannot be represented by a smooth submanifold of X.*

I would like to thank my former advisor, Matthias Kreck, for bringing the above questions to my attention. They were raised in talks by J. Bochnak and W. Kucharz at the Max-Planck Institut in Bonn in the summer of 1992.

2. A topological proof of fact (2)

By crushing out the complement of a tubular neighborhood of a codimension 2 submanifold $N \subseteq M$, one gets a map $M \to MO(2)$, the Thom space of the universal bundle γ_2 over $BO(2)$. The Poincaré dual to N is then just the pullback of the Thom class $u \in H^2(MO(2); \mathbb{Z}/2)$.

This is René Thom's basic translation of codimension 2 bordism classes in $M \times I$ and homotopy classes of maps $M \to MO(2)$. It means that if we can construct a map $\eta : MO(2) \to BSO$ which induces an isomorphism on $H^2(\cdot\,; \mathbb{Z}/2)$, then we have proved fact (2) in a universal manner. To find η, we consider the cofibration sequence

$$BO(1) \to BO(2) \xrightarrow{p} MO(2)$$

which gives a way of constructing maps out of $MO(2)$: Take a map out of $BO(2)$ and a null homotopy of its restriction to $BO(1)$. This is exactly the way η will be defined: Let L be the nontrivial line bundle over $BO(2)$. Then the difference in the H-space structure on BSO (given by the Whitney sum)

$$\gamma_2 - L : BO(2) \to BSO$$

restricts to the trivial bundle over $BO(1)$, and we define $\eta : MO(2) \to BSO$ by choosing any null homotopy. Now the above cofibration sequence gives an exact sequence

$$H^1(BO(2)) \xrightarrow{\cong} H^1(BO(1)) \xrightarrow{0} H^2(MO(2)) \xrightarrow{p^*} H^2(BO(2)) \to H^2(BO(1)),$$

which shows that $p^*(u) = w_2(\gamma_2)$. Since p^* is injective, in order to prove $\eta^*(w_2\gamma) = u$, it suffices to check that $w_2(\gamma_2 - L) = w_2(\gamma_2)$. But this follows from the product formula for Stiefel-Whitney classes [3]. □

Sketch of proof of fact (1). It is not hard to check that our map η induces an isomorphism on $\mathbb{Z}[\frac{1}{2}]$-cohomology up to dimension 7 and an isomorphism on $\mathbb{Z}/2$-cohomology up to dimension 5. Since $MO(2)$ and BSO are both simply connected, this implies that η is a 5-equivalence. Therefore, a class in $H^2(M^n; \mathbb{Z}/2)$, for $n \leq 5$, comes from an embedding if and only if it can be written as w_2E.

But BSO has no homotopy groups between dimension 3 and 7 except $\pi_4 = \mathbb{Z}$. Therefore, there is a single obstruction for lifting a map $X \to K(\mathbb{Z}/2, 2)$ (X any complex of dimension ≤ 8) over $w_2\gamma : BSO \to K(\mathbb{Z}/2, 2)$. This k-invariant lies in $H^5(K(\mathbb{Z}/2, 2); \mathbb{Z})$ and can be identified as the Bockstein applied to ι_2^2.

This means that a class $z \in H^2(X; \mathbb{Z}/2)$ can be written as w_2E if and only if z^2 is the reduction of an integer class.

(Note the equation $(w_2E)^2 = p_1E \pmod 2$, where p_1 is the first Pontrjagin class [3].) For a manifold M of dimension ≤ 5 every such class z^2 is the reduction of an integer class. This is obvious in all cases except when M is 5-dimensional and nonorientable. But then it follows from the equation $Sq^1(z^2) = 0$. □

3. Certain 2-sphere-bundles

We recall from the last section that the equation $(w_2E)^2 = p_1E \pmod 2$ implies that a class $z \in H^2(X; \mathbb{Z}/2)$ does not equal w_2E if z^2 is not the reduction of an integer class. Therefore, the following lemma will be very useful.

Lemma 1. *Let E be a 3-dimensional vector bundle over a space X with sphere bundle SE.*

(a) *There exists a class $z \in H^2(SE; \mathbb{Z}/2)$ which restricts to the generator of $H^2(\text{fibre-}S^2; \mathbb{Z}/2)$ if and only if $w_3E = 0$.*

(b) *Assume that* w_2E *is not the reduction of a class in* $H^2(X;\mathbb{Z}^{w_1E})$. *Then any class* z *as in* (a) *has the property that* z^2 *is not the reduction of a class in* $H^4(SE;\mathbb{Z})$.

Proof. Part (a) follows from the Leray-Serre spectral sequence for the fibration $S^2 \to SE \to X$ since the differential (or *transgression*)

$$d_3 : H^2(S^2;\mathbb{Z}/2) \to H^3(X;\mathbb{Z}/2)$$

takes the generator to w_3E.

Now take any class $z \in H^2(SE;\mathbb{Z}/2)$ as in (a) and note that it is mapped to the Thom class $u_E \in H^3(DE, SE;\mathbb{Z}/2)$ in the long exact sequence of the disk-sphere bundle pair (DE, SE). Applying Sq^2 to these elements maps z to z^2 and u_E to $w_2E \cup u_E$. We have the following commutative diagram (where the right-hand maps are the Thom isomorphisms given by the cup-product with the Thom classes):

$$\begin{tikzcd}
H^4(SE;\mathbb{Z}/2) \arrow[r, "\delta"] & H^5(DE, SE;\mathbb{Z}/2) & H^2(X;\mathbb{Z}/2) \arrow[l, "\cong"'] \\
H^4(SE;\mathbb{Z}) \arrow[r, "\delta"] \arrow[u, "r_2"] & H^5(DE, SE;\mathbb{Z}) \arrow[u, "r_2"] & H^2(X;\mathbb{Z}^{w_1E}) \arrow[l, "\cong"'] \arrow[u, "r_2"]
\end{tikzcd}$$

We know that in the upper row z^2 is mapped to w_2E. Therefore, if z^2 was in the image of the reduction map r_2, so would be w_2E. But this is excluded by our assumption. □

We are now left with the task of finding for each $n \geq 4$ an n-dimensional manifold M together with a 3-dimensional vector bundle E with $w_3E = 0$ and w_2E not coming from $H^2(M;\mathbb{Z}^{w_1E})$. Since we want the corresponding 2-sphere bundle SE to be oriented, we also need to satisfy $w_1E = w_1M$. Undoubtfully, the most difficult case is in dimension 4. In fact, having found a 4-dimensional example, one can get the higher dimensional examples just by crossing with a k-sphere. It is easy to check that all the conditions are hereby preserved and that one gets manifolds of the form $SE^6 \times S^k$. We are thus finished using the following lemma.

Lemma 2. *Let* M *be any closed* 4*-manifold with fundamental group* $\mathbb{Z}/4$. *Then there exists a* 3*-dimensional vector bundle* E *over* M *with* $w_3E = 0, w_1E = w_1M$, *and* w_2E *not coming from* $H^2(M;\mathbb{Z}^{w_1E})$.

Such a 4-manifold can, for example, be constructed by doubling the thickening of a 2-complex consisting of a 2-disk which is attached to a circle by a map of degree 4. This is the same as doing surgery on the circle in the product of a lens space $L^3(\mathbb{Z}/4)$ with S^1.

Proof of Lemma 2. The coefficient sequence $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2$ and Poincaré duality give the commutative diagram

$$\begin{tikzcd}
H_2(M;\mathbb{Z}) \arrow[r, "r_2"] & H_2(M;\mathbb{Z}/2) \arrow[r, "\delta"] & H_1(M;\mathbb{Z}) \arrow[r, "\cdot 2"] & H_1(M;\mathbb{Z}) \\
H^2(M;\mathbb{Z}^{w_1M}) \arrow[r, "r_2"] \arrow[u, "\cong"] & H^2(M;\mathbb{Z}/2) \arrow[u, "\cong"] & &
\end{tikzcd}$$

Since $H_1(M;\mathbb{Z})\cong\mathbb{Z}/4$, there is a class $z\in H^2(M;\mathbb{Z}/2)$ which does not come from $H^2(M;\mathbb{Z}^{w_1M})$. This is our candidate for w_2E. In fact, since the fibre of the map

$$w_1\gamma_3\times w_2\gamma_3 : BO(3)\to K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,2)$$

is 3-connected (it is $B\mathrm{Spin}(3)=\mathbb{HP}^\infty$), we can lift the pair (w_1M, z) to $BO(3)$, i.e. get a 3-dimensional vector bundle E over M with $w_1E=w_1M$ and $w_2E=z$, because all obstructions for such a lifting lie in $H^i(M;*)$ for $i\geq 5$ and thus vanish.

This bundle E satisfies all the desired properties except that we have to show $w_3E=0$. First note the equation $w_3=w_1w_2+Sq^1w_2$ [3] and the fact that in our situation $Sq^1: H^2(M;\mathbb{Z}/2)\to H^3(M;\mathbb{Z}/2)$ is trivial: It is the composition [3]

$$H^2(M;\mathbb{Z}/2)\xrightarrow{\beta} H^3(M;\mathbb{Z})\xrightarrow{r_2} H^3(M;\mathbb{Z}/2),$$

which must be zero since $H^3(M;\mathbb{Z})\cong H_1(M;\mathbb{Z}^{w_1M})\cong H_1(\mathbb{Z}/4;\mathbb{Z}^{w_1M})$ is either $\mathbb{Z}/4$ or 0 depending on whether M is orientable or not.

This means that we are done if M is orientable and if not, we have to show that $w_1E\cup w_2E=0$: By Poincaré duality (and the fact that $H^3(M;\mathbb{Z}/2)=\mathbb{Z}/2$), multiplication by $w_1M=w_1E$ gives an isomorphism $H^3(M/\mathbb{Z}/2)\to H^4(M;\mathbb{Z}/2)$. This forces our equation to hold because we know that $(w_1M)^2=0$ since this already holds in the cohomology of the fundamental group $\mathbb{Z}/4$. □

Remark. It is clear from the above proof that the assumption on the fundamental group of the 4-manifold can be weakened considerably. But we did not see a reason for figuring out the most general case because already the examples given abound.

4. The 9-dimensional example

The (unstable) cohomology operation $T: H^2(X;\mathbb{Z}/2)\to H^9(X;\mathbb{Z}/2)$ defined by

$$T(x):=(Sq^2Sq^1x)x+(Sq^1x)^3+(Sq^1x)x^3$$

is trivial for $X=MO(2)$ because (using $Sq^iu=w_iu$ [3]) one computes

$$T(u)=(w_1^3u+w_1w_2u)u^2+(w_1u)^3+w_1u^4=w_1(w_2u^3+u^4)=0.$$

However, for $X=BSO(3)$ the Wu-formula [3] shows that this operation is nontrivial since

$$T(w_2)=(Sq^2w_3)w_2^2+w_3^3+w_3w_2^3=w_3^3\neq 0.$$

Now we can pick any 9-manifold M with a map E to $BSO(3)$ which evaluates nontrivially on $T(w_2)$, i.e., $\langle E_*[M], T(w_2)\rangle\neq 0$. This pair (M, E) exists since by [4] the (nonoriented) bordism group $\mathcal{N}_9(BSO(3))$ maps onto $H_9(BSO(3);\mathbb{Z}/2)$. It is clear that the Poincaré dual of w_2E cannot be represented by an embedding into M. □

Remark. We have checked that dimension 9 is the smallest one in which such an argument (with $\mathbb{Z}/2$-cohomology) can work. However, we have also found a 6-manifold with a map to BSO which cannot lift over any 2-equivalence $MO(2)\to BSO$. Unfortunately, this does not exactly answer the question posed at the end of the Introduction.

References

1. R. Benedetti and M. Dedó, *Counterexamples to representing homology classes by real algebraic subvarieties up to homeomorphism*, Compositio Math. **53** (1984), 143–151.
2. J. Bochnak and W. Kucharz, *Algebraic cycles and approximation theorems in real algebraic geometry*, Trans. Amer. Math. Soc. **337** (1993), 463–472.
3. J. Milnor and J. Stasheff, *Characteristic classes*, Ann. of Math. Stud., no. 76, Princeton Univ. Press, Princeton, NJ, 1974.
4. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.

University of Mainz, Fachbereich, Mathematik, 55099 Mainz, Germany
Current address: Institut des Hautes Études Scientifique, 35, Route de Chartres, 91440 Bures-Sur-Yvette, France
E-mail address: teichner@umamza.mathematik.uni-mainz.de