VAN KAMPEN'S EMBEDDING OBSTRUCTION IS INCOMPLETE FOR 2-COMPLEXES IN $\mathbb{R}^4$

MICHAEL H. FREEDMAN, VYACHESLAV KRUSHKAL AND PETER TEICHNER

1. Introduction

In 1933, anticipating formal cohomology theory, van Kampen [5] gave a slightly rough description of an obstruction $o(K) \in H^{2n}_{\mathbb{Z}/2}(K^*, \mathbb{Z})$ which vanishes if and only if an n-dimensional simplicial complex K admits a piecewise-linear embedding into $\mathbb{R}^{2n}$, $n \geq 3$. Here K^* is the deleted product of a complex K. The cohomology in question is the $\mathbb{Z}/2$-equivariant cohomology where $\mathbb{Z}/2$ acts on the space by exchanging the factors of K^* and acts on the coefficients by multiplication with $(-1)^n$. Many details were clarified by Shapiro [3] and Wu [6] in the 1950's. In 1991 Sarkaria [2] showed that for $n = 1$ this obstruction provides a necessary and sufficient condition in that dimension as well (and is thus identical to Kuratowski's subgraph condition). Sarkaria recently asked the first named author if it were possible that the vanishing of $o(K)$ might also imply embeddability for $n = 2$. It is the purpose of this paper to exhibit a simplicial 2-complex K with 14 vertices, 43 1-cells and 69 2-cells for which $o(K)$ is trivial but which does not admit an embedding into $\mathbb{R}^4$. If one considers relative settings $(K^n, L^{n-1}) \subset (D^{2n}, S^{2n-1})$ there is an analogous obstruction and a high dimensional theorem analogous to van Kampen's. But setting $n = 2$, $K^2 = D^2 \sqcup D^2 \sqcup D^2$ and $L^1 \subset S^3$ the Borromean rings gives an elementary (and well known) example where although the obstruction vanishes there is no relative embedding. Our task was to "unrelativize" this simple example.

In section 2 we recall van Kampen's obstruction (generalized to a relative setting) and give a modern proof that its vanishing implies the existence of a P.L. embedding into $\mathbb{R}^{2n}$, $n \geq 3$. Section 3 describes the example, proves the absence of a P.L. embedding and the vanishing of the obstruction. In section 4 we show that K does not embed, even topologically, into $\mathbb{R}^4$.

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2. Van Kampen's Obstruction

2.1. **The deleted product.** The deleted product K^* of a simplicial complex K is the subcomplex of $K \times K$ consisting of products of pairs of simplices having no vertex in common. Note that $\mathbb{Z}/2$ acts freely on K^* by exchanging the factors.

2.2. **Equivariant cohomology.** Let a group G act freely on a CW complex X, so that each element $g \in G$ acts by a cellular homeomorphism. Denote the n-skeleton of X by $X^{(n)}$. Then

$$\dots C_{n+1} \longrightarrow C_n \longrightarrow C_{n-1} \dots$$

is a chain complex of $\mathbb{Z}[G]$-modules, where $C_n = H_n(X^{(n)}, X^{(n-1)}; \mathbb{Z})$. Given a $\mathbb{Z}[G]$-module M, consider the G-equivariant cohomology groups:

$$H^q_G(X; M) := H_q(Hom_{\mathbb{Z}[G]}(C_*(X), M)).$$

If G acts freely on X and trivially on M then

$$H^q_G(X; M) \cong H^q_{ordinary}(X/G; M).$$

2.3. **The obstruction.** Let K be an n-dimensional simplicial complex ($n \geq 1$) and $f : K \longrightarrow \mathbb{R}^{2n}$ be a PL immersion (we can assume that for any two open cells σ and τ of K, $f(\sigma) \cap f(\tau) = \emptyset$ if $\dim(\sigma) + \dim(\tau) < 2n$, and the cells of the top dimension intersect each other transversely in at most a finite number of double points). $\mathbb{R}^{2n}$ is assumed to be endowed with a fixed orientation. Given such an f, following van Kampen [5] define the cochain o_f on K^* by $o_f(\sigma \times \tau) = f(\sigma) \cdot f(\tau) \in \mathbb{Z}$ (the intersection number of the n-simplices σ and τ). Precisely, $\sigma \times \tau$ should be read as an oriented generator of $H_{2n}(K^*, K^{*(2n-1)}; \mathbb{Z})$ corresponding to that top cell and the factors σ and τ should be oriented – there are two possible ways to do this – so that their product represents that generator.
The non-trivial element of $\mathbb{Z}/2$ acts on K^* freely by the involution $i(x, y) = (y, x)$ for $(x, y) \in K^*$ (which is a cellular map). One has

$$(o_f \circ i)(\sigma \times \tau) = o_f(\tau \times \sigma) = f(\tau) \cdot f(\sigma) = (-1)^n f(\sigma) \cdot f(\tau).$$

Let the non-trivial element of $\mathbb{Z}/2$ act on $\mathbb{Z}$ by multiplication by $(-1)^n$. Then $o_f \in Hom_{\mathbb{Z}[\mathbb{Z}/2]}(C_{2n}(K^*), \mathbb{Z})$ where $\mathbb{Z}$ is considered as a $\mathbb{Z}[\mathbb{Z}/2]-$module with the action of $\mathbb{Z}/2$ described above. Hence the obstruction $o(K)$ to the embedding of K in $\mathbb{R}^{2n}$ – the cohomology class of o_f – is an element of $H^{2n}_{\mathbb{Z}/2}(K^*; \mathbb{Z})$. Note that for n even the action of $\mathbb{Z}/2$ on $\mathbb{Z}$ is trivial, so $o(K) \in H^{2n}_{ordinary}(K^*/(\mathbb{Z}/2); \mathbb{Z})$.

Lemma 1. *The cohomology class $o(K)$ in $H^{2n}_{\mathbb{Z}/2}(K^*; \mathbb{Z})$ is independent of f.*

Proof. [3, Lemma 3.5]. The cochain o_f can be defined in a more general situation. Let M be a simplicial complex. Given a P.L. map $f : M \longrightarrow \mathbb{R}^N$, define o_f by $o_f(\sigma \times \tau) = f(\sigma) \cdot f(\tau)$ where $\dim(\sigma) + \dim(\tau) = N$. Then o_f is a cocycle and obeys the following naturality: given an inclusion $i : L \hookrightarrow K$, $o_{f\circ i} = i^{\#}(o_f)$. Here $i^{\#}$ denotes the cochain map induced by i on the deleted products.
Given two immersions f and g of K in $\mathbb{R}^{2n}$, we can assume that the set of vertices of $f(K)$ and $g(K)$ is in general position (if not, perturb f and g slightly without changing o_f and o_g). Consider f and g as maps defined on the subsets $K \times 0$ and $K \times 1$ of $K \times I$ and extend them by linearity to a map $F : K \times I \longrightarrow R^{2n}$. Denote the inclusions of K into $K \times 0$ and $K \times 1$ by i_0 and i_1 respectively, so that $f = F \circ i_0$ and $g = F \circ i_1$. Then $o_f = i_0^{\#}(o_F)$ and $o_g = i_1^{\#}(o_F)$. i_0 and i_1 are homotopic via a map which is injective at each time, hence it defines a $\mathbb{Z}/2$-equivariant homotopy on the deleted product, so o_f and o_g differ by an equivariant coboundary. □

2.4. **Elementary cochains.** Given disjoint simplices σ and ν of dimensions n respectively $n-1$ in K, define an *elementary* $(2n-1)$-*cochain* $u_{\sigma\nu}$ by $u_{\sigma\nu}(\sigma \times \nu) = u_{\sigma\nu}(\nu \times \sigma) = 1$, $u_{\sigma\nu}(\text{other } (2n-1)\text{-cells}) = 0$. These elementary cochains clearly generate $Hom_{\mathbb{Z}[\mathbb{Z}/2]}(C_{2n-1}(K^*); \mathbb{Z})$. Moreover, $\delta u_{\sigma\nu}(\sigma \times \omega)$ is equal to ± 1 if ω contains ν on its boundary (similarly for $\omega \times \sigma$) and 0 on all other $2n$-cells.
For a given immersion $f : K \to \mathbb{R}^{2n}$ van Kampen uses the following "finger move": introduce a small oriented P.L. n-sphere S_ν locally linking the simplex $f(\nu)$ which has exactly one intersection point with the image of each of the n-cells containing ν on its boundary and connect S_ν to $f(\sigma)$ by a thin tube, disjoint from $f(K)$ for $n \geq 2$ (see figure 2.1). This changes $f(\sigma) \cdot f(\omega)$ by ± 1 if $\nu \subset \partial\omega$ and leaves $f(\sigma) \cdot f(\omega)$ unchanged otherwise. In other words, if we denote by g the map obtained from f as a result of this modification, then $o_g = o_f \pm \delta u_{\sigma\nu}$. In fact, a more geometric proof of Lemma 1 can be based on the idea that any f and g may be connected by a 1-parameter family with finitely many non-general times in which o changes by $\pm\delta u_{\sigma\nu}$.

Lemma 2. *Let c be any cochain representative of $o(K)$. Then there is an immersion g for which $o_g = c$.*

Proof. If f is any immersion of K into $\mathbb{R}^{2n}$, then $c = o_f + \delta u$ for some $(2n-1)$-cochain u. $c - o_f$ is a finite sum $\Sigma \pm \delta u_{\sigma_i \nu_i}$ for some pairs of disjoint cells σ_i, ν_i, since such elements generate the equivariant $2n$-coboundaries. Applying van Kampen's finger move described above to each pair σ_i, ν_i with the appropriate orientations of the spheres S_{ν_i}, produces a new immersion g for which $o_g = c$. □

Theorem 3. *A necessary and for $n \geq 3$ also sufficient condition that there exists an embedding of the n-dimensional complex K into $\mathbb{R}^{2n}$ is that $o(K) = 0$.*

The necessity follows from Lemma 1. Sufficiency follows from the following two lemmas:

Lemma 4. *Suppose $n \geq 3$ and that $f(\sigma)$ has a non-empty self-intersection for some n-simplex σ in K. Then there is an immersion $g : K \longrightarrow \mathbb{R}^{2n}$ differing from f only on σ such that $g(\sigma)$ has one self-intersection point less than $f(\sigma)$ and the rest of the singularities are unchanged.*

Proof. Let P be a point of self-intersection of $f(\sigma)$, γ a PL path homeomorphic to an interval in σ which joins the two preimage points of P and contains no singular points of f in its interior. Then $f(\gamma)$ is a PL circle which bounds for $n \geq 3$ a PL embedded (by general position) disk $W \cong D^2$ with interior in $\mathbb{R}^{2n} - f(K)$. Let $B \subset \sigma$ be a P.L. regular neighborhood of γ in σ. The complex $W \cup f(B)$ is clearly collapsable (first collapse $f(B)$ to $f(\gamma)$, then collapse W), so W admits a P.L. regular neighborhood $N \cong D^{2n}$ with $N \cap f(K - \sigma) = \emptyset$ and the preimage $f^{-1}(N \cap f(\sigma)) = B \subset \sigma$. Thus $\partial N \cap f(\sigma) = \partial N \cap f(B) = f(\partial B) \cong S^{n-1}$. By coning, $f(\partial B)$ bounds a PL n-disk B' in N. Now set g equal to f on $K - B$ and let g map B PL homeomorphically onto B'. Then g satisfies the conclusion of the lemma. (See figure 2.2) □

Lemma 5. *Suppose $n \geq 3$. If $f(\sigma) \cap f(\tau) \neq \emptyset$ for two n-simplexes σ and τ having a common vertex in K, then there is an immersion $g : K \longrightarrow \mathbb{R}^{2n}$ differing from f only on σ and τ such that the number of the intersection points between $g(\sigma)$ and $g(\tau)$ is one less than between $f(\sigma)$ and $f(\tau)$ and the rest of the singularities are unchanged.*

Proof. Let σ and τ have a common vertex V in K and $P \in f(\sigma) \cap f(\tau)$. Let γ and γ' be PL paths without self-intersections in σ and τ respectively joining V and the preimage points of P, such that γ and γ' contain no singular points of f in their interiors.
Then $f(\gamma)$ and $f(\gamma')$ form a PL circle which bounds for $n \geq 3$ a PL embedded disk $W \cong D^2$ with interior in $\mathbb{R}^{2n} - f(K)$. The arc $\gamma \cup \gamma'$ has a regular neighborhood B pinched at V so that $B \cong D^n \vee_V D^n$. As before use the collapsability of $W \cup f(B)$ to construct a regular neighborhood $N \cong D^{2n}$ of W pinched at $f(V)$ so that $N \cap f(K - (\sigma \cup \tau)) = \emptyset$ and $N \cap f(\sigma \cup \tau) = f(B)$. As before $\partial N \cap f(\sigma) = \partial N \cap f(B) = f(\text{"}\partial\text{"} B) \cong \partial D^n \vee_{f(V)} \partial D^n$. But $f(\text{"}\partial\text{"} B)$ also bounds B' – an embedded wedge of two PL n-disks in D^{2n}. Set g equal to f on $K - B$ and let g map B homeomorphically onto B'. (See figure 2.3) □

Proof of Theorem 3. If the obstruction vanishes, then by lemma 2, there is an immersion f for which $o_f = 0$, in other words, $f(\sigma)\cdot f(\tau) = 0$ for any two disjoint n-cells σ and τ in K, hence $f(\sigma) \cap f(\tau)$ can be divided into pairs of intersection points of opposite signs. These singularities can be resolved using the standard Whitney trick (see e.g. [1]). Note that the term "opposite signs" does not depend on the orientations of $f(\sigma)$ or $f(\tau)$
The remaining singularities are all of the form described in Lemmas 4 and 5, so by an inductive application of these lemmas, an embedding $K \hookrightarrow \mathbb{R}^{2n}$ can be produced. □

2.5. **The relative case.** Consider now the relative embedding problem: Given a subcomplex L^{n-1} of K^n and a piecewise-linear embedding $\phi : L \hookrightarrow S^{2n-1}$, is there a P.L. embedding $f : K \hookrightarrow D^{2n}$ extending ϕ? As in the absolute case, an immersion $f : K \to D^{2n}$ extending ϕ gives rise to a cochain o_f whose cohomology class in $H^{2n}_{\mathbb{Z}/2}(K^*, (K \times L \cup L \times K) \cap K^*; \mathbb{Z})$ will be denoted by $o(K, L)$. Clearly, $o(K, \emptyset) = o(K) \in H^{2n}_{\mathbb{Z}/2}(K^*; \mathbb{Z})$.
Note that the finger move cannot be performed on the $(n-1)$-cells ν of L. But the elementary cochains $u_{\sigma\nu}$ for such ν are zero in the relative cochain group, so the proofs of lemmas 1, 2, remain valid in the relative setting.

3. The 2-dimensional Example

3.1. **The Construction.** Let G_7 (respectively G'_7) be the complete graph on 7 vertices $v_1, \dots, v_7$ (respectively on $v'_1, \dots, v'_7$). Let $G = G_7 \cup G'_7 \cup$edge $v_1v'_1$. Denote the triangle on the vertices v_a, v_b, v_c by $\Delta_{abc}$and on v'_a, v'_b, v'_c by Δ'_{abc}.
Set $E = \Delta_{123} \cup \Delta'_{123} \cup$edge $v_1v'_1$ and let L be the PL loop

$$v_1v_2v_3v_1v'_1v'_2v'_3v'_1v_1v_3v_2v_1v'_1v'_3v'_2v'_1v_1,$$

i.e. a representative of the commutator of the two generators of $\pi_1(E, v_1)$. Define K to be the 2-complex obtained by attaching 2-cells to G as follows:

1. 34 2-simplices to all triangles Δ_{abc} in G_7, except Δ_{123} (denote them by $\overline{\Delta}_{abc}$),

2. 34 2-simplices to all triangles Δ'_{abc} in G'_7 except Δ'_{123} (denote them by $\overline{\Delta}'_{abc}$),

3. a 2-cell (16-gon) to the loop L (denote it by $\overline{L}$).

Note that G_7 with the 2-cells (1) attached forms the 2-skeleton of the standard 6-simplex Δ^6 on vertices $v_1, \dots, v_7$, with $\overline{\Delta}_{123}$ removed. (Similarly for G'_7.)

3.2. K **does not PL-embed into** $\mathbb{R}^4$**.** To prove that K cannot be PL-embedded into $\mathbb{R}^4$, we shall use $\mathbb{Z}/2$-coefficients. Van Kampen [5] proved that the 2-skeleton T of a 6-simplex Δ^6 on vertices $v_1, \dots, v_7$ cannot be embedded into $\mathbb{R}^4$. He described an immersion of T into $\mathbb{R}^4$ with one self-intersection point (so that $\overline{\Delta}_{123}$ and $\overline{\Delta}_{456}$ intersect at one point and all other pairs of disjoint 2-cells in T are disjoint in the image), and proved that this singularity represents the non-trivial obstruction to embeddability in $H^4(T^*/(\mathbb{Z}/2); \mathbb{Z}/2)$ (where T^* is the deleted product of T) by showing that the number (mod 2) of 4-cells of T^* on which the cochain o_f is nonzero is preserved under finger moves.

Let S denote the sphere formed by the four 2-cells disjoint from Δ_{123}: $\overline{\Delta}_{456}, \overline{\Delta}_{457}$, $\overline{\Delta}_{567}, \overline{\Delta}_{647}$. This is the dual tetrahedron to Δ_{123} in the 6-simplex. Similarly, let S' be the dual tetrahedron to Δ'_{123}).

Lemma 6. *If $K - (16 - gon)$ is PL embedded into S^4 then* $link_{mod2}(\Delta_{123}, S) = link_{mod2}(\Delta'_{123}, S') = 1$ *and* $link\ (\Delta_{123}, S') = link(\Delta'_{123}, S) = 0$.

Proof. (see figure 3.1) Let D be an (immersed) PL 2-disk in S^4 bounded by Δ_{123} and transverse to K. Then we have an immersed complex T (on vertices $v_1, \dots, v_7$) such that $T - D$ is embedded (hence the only possible intersections detected by the obstruction are those of D with S). Applying the finger move described in section 2.4 several times to the disk D and edges v_4v_5, v_5v_6, v_4v_6, we can achieve the situation when $D \cdot \overline{\Delta}_{457} = D \cdot \overline{\Delta}_{567} = D \cdot \overline{\Delta}_{467} = 0$. The introduced intersections of D with with the cells other than S don't have an impact on the obstruction since intersections of adjacent 2-cells are not counted. Since the embedding obstruction for T is non-zero, $D \cdot S = D \cdot \overline{\Delta}_{456} = 1$ mod 2. Hence link(Δ_{123}, S)$=D \cdot S = 1$ mod 2.

The second equation link$(\Delta_{123}, S') = 0$ follows since $\overline{\Delta}_{124} \cup \overline{\Delta}_{134} \cup \overline{\Delta}_{234}$ is a disk in $S^4 - S'$ bounded by Δ_{123}. The same arguments apply to Δ'_{123}. □

Lemma 7. *For any PL-embedding $f : K \to \mathbb{R}^4$ the inclusion map $f(E) \overset{i}{\hookrightarrow} S^4 - f(S \cup S')$ induces a monomorphism on fundamental groups.*

Proof. For a group H and a subgroup U of H, denote by $[H, U]_2$ the group generated by all elements of H of the form $huh^{-1}u^{-1}v^2$ for $h \in$

$H; u, v \in U$. Define the *mod 2- lower central series of a group H* by:

$$H_0 = H, \quad H_{n+1} = [H, H_n]_2, \quad H_{\omega_2} = \bigcap_{n=0}^{\infty} H_n.$$

By Alexander duality and Lemma 6 the hypothesis of Stallings Theorem [4] are satisfied for the map $i_* : \pi_1(f(E)) \to \pi_1(S^4 - f(S \cup S'))$, i.e. i_* induces an isomorphism on $H_1(.;\mathbb{Z}/2)$ and an epimorphism on $H_2(.;\mathbb{Z}/2)$. Therefore, $i_\#$ in the diagram below is injective.

$$\begin{array}{ccc} \pi_1(f(E)) & \xrightarrow{i_*} & \pi_1(S^4 - f(S \cup S')) \\ \downarrow & & \downarrow \\ \pi_1(f(E))/\pi_1(f(E))_{\omega_2} & \xrightarrow{i_\#} & \pi_1(S^4 - f(S \cup S'))/\pi_1(S^4 - f(S \cup S'))_{\omega_2} \end{array}$$

Since $\pi_1(f(E))$ is free, $\pi_1(f(E))_{\omega_2}$ is trivial [4, Theorem 6.3], so i_* is injective. $\square$

Since L is not null-homotopic in E, this lemma shows that $\overline{L}$, the 16-gon attached to L (in the definition of K), cannot be embedded (even mapped) into $S^4 - (S \cup S')$, so K is not PL embeddable into S^4.

3.3. **The obstruction $o(K)$ vanishes.** There is a standard immersion of K into S^4 (obtained from two disjoint immersions of T described in 3.2 by removing 2-cells $\overline{\Delta}_{123}$ and $\overline{\Delta}'_{123}$ and attaching the edge v_1v_1' and then $\overline{L}$ to L) so that $K - \overline{L}$ is embedded. Since L is homologically trivial in $S^4 - (S \cup S')$, link(L, S)=link$(L, S') = 0$. Hence $\overline{L} \cdot S = 0$, but $\overline{L}\cdot$ (a particular 2-cell in the triangulation of S) may be not zero. (Similarly for S'). Using van Kampen's trick from section 2.4, as in section 3.2 we shall modify this situation to get $\overline{L} \cdot \overline{\Delta}_{457} = \overline{L} \cdot \overline{\Delta}_{567} = \overline{L} \cdot \overline{\Delta}_{467} = 0$. But for the fourth 2-cell in S we have $\overline{L} \cdot \overline{\Delta}_{456} = \overline{L} \cdot S = 0$.
The same procedure can be applied to $\overline{L}$ and S' to get an immersion of K for which the obstruction vanishes. $\square$

Note that a similar example can be built using any non-trivial loop L in the commutator subgroup of $\pi_1(E)$.

4. K does not embed topologically into S^4

A PL *almost-embedding* of K is a PL immersion such that the pairs of disjoint cells in the triangulation of K (i.e. cells which have no vertices in common) do not intersect in the image. It is clear that van Kampen's obstruction vanishes if K can be almost-embedded into S^4.

A topological embedding of K can be approximated arbitrarily closely by PL almost-embeddings: K is a finite complex, so the distances between images of disjoint cells in S^4 are bounded below by some $\epsilon > 0$. Then any $\epsilon/3-$ PL approximation (which is an immersion) of the given embedding will be a PL almost-embedding.

Lemma 8. *Suppose $K - (16-gon)$ is topologically embedded into S^4. Then $link_{mod2}(\Delta_{123}, S) = link_{mod2}(\Delta'_{123}, S') = 1$ and $link(\Delta_{123}, S') = link(\Delta'_{123}, S) = 0$.*

Proof. First note that the proof of the second part of this lemma is identical to that in lemma 6. Now let α be the (mod 2) 1-cycle in $S^4 - S$ determined by Δ_{123} and a the homology class of α in $H_1(S^4 - S; \mathbb{Z}/2)$.

Suppose $\text{link}_{mod2}(\Delta_{123}, S) = 0$. Then $a = 0$ and $\alpha = \partial\beta$ for some 2-chain β in $S^4 - S$. Since the support of β is compact, $\epsilon > 0$ can be chosen sufficiently small, so that a 2ϵ-neighborhood N of S is disjoint from the support of β and the ϵ-PL-approximations of the given embedding of K are almost-embeddings.

Let $\overline{K}$ be an ϵ-PL-approximation of the given embedding of K (hence a PL almost-embedding), $\overline{S}$ the image of S ($\overline{S} \subset N$) and $\overline{\alpha}$ the image of Δ_{123}. By general position we can assume that $\overline{\alpha}$ is an embedding. Observe that α is homotopic in $S^4 - N$ to $\overline{\alpha}$.

The support of β lies in $S^4 - N$, so α (hence $\overline{\alpha}$) is null-homologous (mod 2) in $S^4 - \overline{S}$. Then $\overline{\alpha} = \partial\overline{\beta}$ for some PL 2-chain $\overline{\beta}$. We have $(\overline{\beta} \cdot \overline{S})_{mod2} = \text{link}_{mod2}(\overline{\alpha}, \overline{S}) = 0$.
Now consider a PL (immersed) 2-disk D in S^4 bounded by $\overline{\alpha}$ and transverse to $\overline{K}$. D is homologous to $\overline{\beta}$ in S^4, so $(D \cdot \overline{S})_{mod2} = (\overline{\beta} \cdot \overline{S})_{mod2} = 0$. As in the proof of Lemma 6 and section 3.3, D may be modified to D' with $(D' \cdot \overline{\Delta}_{567})_{mod2} = (D' \cdot \overline{\Delta}_{467})_{mod2} = (D' \cdot \overline{\Delta}_{457})_{mod2} = (D' \cdot \overline{\Delta}_{456})_{mod2} = 0$. Restricting, we have a P.L. almost embedding of (T (on vertices $v_1, \ldots, v_7$) $- \overline{\Delta}_{123}$) and an extension over $T = (T - \overline{\Delta}_{123}) \cup D'$ for which the obstruction in $H^4(T^*/(\mathbb{Z}/2); \mathbb{Z}/2)$ vanishes. This contradicts van Kampen's result (see section 3.2) that this obstruction is nonzero. Therefore, $\text{link}_{mod2}(\Delta_{123}, S) = 1$ and similarly $\text{link}_{mod2}(\Delta'_{123}, S') = 1$. $\square$

Hence the statement and the proof of lemma 7 remain true in the topological case, so K is not even topologically embeddable into S^4.

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University of California, San Diego, La Jolla, CA, 92093, USA