CLASSIFICATION OF CLOSED TOPOLOGICAL 4-MANIFOLDS

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The big breakthrough in the classification of topological 4-manifolds certainly was Freedman's proof of the topological dics embedding theorem for poly-finite or -cyclic fundamental groups in 1982 [1]. It implies the topological s-cobordism and surgery theorems for the same class of fundamental groups. Freedman deduced that the intersection form on H_2 and the Kirby-Siebenmann invariant classify simply connected closed 4-manifolds up to homeomorphism and that every unimodular symmetric bilinear form is realized as the intersection form of such a manifold.

In this note we are going to describe some results in the nonsimply connected case. We have to stick to the above class of fundamental groups because the main problem of whether the topological disc embedding theorem holds in general is still unsolved. (It is nowadays known as the *A-B-slice problem.*)

In 1988, Hambleton and Kreck [3] proved a word by word generalization of Freedman's classification result to the case of a cyclic fundamental group of odd order. They had to introduce several new techniques into the field because a priori one has to work with the equivariant intersection form on π_2 to even obtain a homotopy classification. This difficulty can be understood very well in the case of an infinite cyclic fundamental group. In this case, a straightforward generalization of the techniques from the simply connected case lead to the result that the orientation character, the Kirby-Siebenmann invariant and the equivariant intersection form on π_2 classify such 4-manifolds and that any unimodular hermitian form over the group ring is realizable. This result was proven by Freedman and Quinn [2, Ch.10.7] and independently by Kreck [8] around 1983. It is however an open problem whether (in the oriented case) the form on π_2 is determined by the form on H_2. Algebraically, this is the question whether any unimodular hermitian form over the group ring of $\mathbb{Z}$ is induced from the trivial group.

Let me now come back to finite cyclic fundamental groups. The following example shows that already for order 2 the intersection form on π_2 does not classify the homotopy type: The two orientation preserving free involutions on $S^2 \times S^2$, given by $(x, y) \mapsto (-x, -y)$ respectivley $(x, (y_1, y_2, y_3)) \mapsto (-x, (-y_1, y_2, y_3))$ lead to the same equivariant intersection form but to two different quotient manifolds, one being a spin manifold the other being nonspin.

This means that instead of the odd-even (or type I-II) distinction known from the simply connected case, one needs to have three different w_2*-types* I, II respectively III: The universal covering is nonspin, the manifold is spin respectively the manifold is nonspin but the universal covering is spin.

With this definition, Hambleton and Kreck [4] proved that oriented 4-manifolds with even cyclic fundamental group are classified by the w_2-type, the Kirby-Siebenmann invariant and the intersection form on H_2. Again, in the proof new techniques have to be introduced and it is considerably harder than the odd order case. Results for nonorientable manifolds are not known for general cyclic groups but the case of order 2 has been recently worked out in [6]. We give a simple set of invariants, namely the Euler characteristic, the Stiefel-Whitney number w_1^4, an Arf-invariant and the Kirby-Siebenmann invariant, which classify nonoriented manifolds with fundamental group of order two. Moreover, we give a complete list of such manifolds. Such a list cannot exist in the simply connected case because definite forms over the integers abound and are not classified.

But what is known about noncyclic fundamental groups? If one asks the weaker question of a classification up to connected sum with copies of $S^2 \times S^2$ then there is a well developed theory due to Kreck [8]. For example, two 4-manifolds with the same fundamental group and w_2-type I are stably homeomorphic if and only if they have the same Kirby-Siebenmann invariant, signature and π_1-*fundamental class*. This last invariant is the image of the fundamental class of the 4-manifold in $H_4(\pi_1)$ and it is welldefined up to outer automorphisms of π_1. Note that in this stable situation the smooth disc embedding theorem is (trivially) true and thus the theory also applies for the smooth case.

There arises the obvious question in which situations one can cancel the additional $S^2 \times S^2$ summands. If the fundamental group is finite, Hambleton and Kreck [5] prove the following beautiful result: Two stably homeomorphic 4-manifolds with the same Euler characteristic become homeomorphic after adding just a single $S^2 \times S^2$. In other words, one can cancel all $S^2 \times S^2$ summands except (possibly) one. Note that in general one cannot cancel this last summand because the original 4-manifolds might have had different definite intersection forms on H_2. Also note that this cancellation theorem uses the topological s-cobordism theorem.

By constructing special examples of 4-manifolds with small H_2 in every stable homeomorphism class, these techniques were applied in my PhD thesis [9] to obtain new classification results. They can be easiest formulated for spin manifolds though results exist in all w_2-types as long as the manifolds are oriented. The above problem with the last $S^2 \times S^2$ summand forces the assumption of an indefinite intersection form on H_2. But under this assumption, I proved that the Euler characteristic, the signature and a *sym*-invariant classify spin 4-manifolds with a finite fundamental group π_1 which satisfies one of the following properties:

- π_1 has cyclic Sylow subgroups.
- π_1 has 4-periodic cohomology.
- $\pi_1 = Sl_2(p), p$ a prime number.

Here the sym-invariant takes only the two values 0 or 1 and it is 0 if and only if the π_1-equivariant intersection form on π_2 can be written as $q + q^*$ for some π_1-equivariant form q.

A very similar result was obtained by Hambleton and Kreck in [4] for all odd order groups. They need a stronger indefiniteness assumption, namely that the difference between the rank and the absolute value of the signature of the intersection form on H_2 is bigger than a certain constant which only depends on the fundamental group. Under this assumption, a closed 4-manifold with odd order fundamental group is classified by the w_2-type, the Kirby-Siebenmann invariant, the Euler characteristic, the signature and the π_1-*fundamental class.*

Let me finish with the remark that for infinite fundamental groups I know of very few results (with the exception of the infinite cyclic case described above). There are some classification results up to s-cobordism from which I would like to mention a theorem of Hillman in [7]: Let F and B be closed hyperbolic surfaces.

Then a closed 4-manifold M is topologically s-cobordant to the total space of an F-bundle over B if and only if $\pi_1 M$ is an extension of $\pi_1 B$ by $\pi_1 F$ and the Euler characteristic of M is the product of the Euler characteristics of F and B.

References

[1] M. Freedman. *The Topology of 4-dimensional Manifolds.* J. of Diff. Geom. 17 (1982) 357-453.

[2] M. Freedman, F. Quinn. *Topology of 4-Manifolds.* Princeton Mathematical Series 39, Princeton University Press 1990.

[3] I. Hambleton, M. Kreck. *On the Classification of Topological Four-Manifolds with Finite Fundamental Group.* Math. Annalen 280 (1988) 85-104.

[4] I. Hambleton, M. Kreck. *Cancellation, Elliptic Surfaces and the Topology of certain Four-Manifolds.* Preprint of the Max-Planck Institut für Mathematik Bonn 1991.

[5] I. Hambleton, M. Kreck. *Cancellation of Hyperbolic Forms and Topological Four-Manifolds.* Preprint of the Max-Planck Institut für Mathematik Bonn 1991.

[6] I. Hambleton, M. Kreck, P. Teichner. *Nonorientable 4-Manifolds with Fundamental Group of Order 2.* Preprint 1992. To appear in the Transactions of the A. M. S.

[7] J. Hillman. *On 4-manifolds homotopy equivalent to surface bundles over surfaces.* Top. and its Appl. 40 (1991) 275-286.

[8] M. Kreck. *Surgery and Duality.* To appear as a book in the Vieweg-Verlag, Wiesbaden. As a preprint of the Johannes-Gutenberg-Universität Mainz 1985 available under the title: *An Extension of Results of Browder, Novikov and Wall about Surgery on Compact Manifolds.*

[9] P. Teichner. *Topological Four-Manifolds with Finite Fundamental Group.* PhD Thesis at the University of Mainz, Germany, March 1992.

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