

Traces in monoidal categories

Stephan Stolz and Peter Teichner

February 3, 2011

Abstract

The main result of this paper is the construction of a trace and a trace pairing for endomorphisms satisfying suitable conditions in a monoidal category. This construction is a common generalization of the trace for endomorphisms of dualizable objects in a balanced monoidal category and the trace of nuclear operators on a locally convex topological vector space with the approximation property.

Contents

1	Introduction	1
2	Motivation via field theories	5
3	Thickened morphisms and their traces	7
3.1	Thickened morphisms	9
3.2	The trace of a thickened morphism	11
4	Traces in various categories	12
4.1	The category of vector spaces	12
4.2	Thick morphisms with dualizable domain	14
4.3	Thick morphisms with semi-dualizable domain	16
4.4	The Category Ban of Banach spaces	18
4.5	A Category of topological vector spaces	19
4.6	The Riemannian bordism category	26
5	Properties of the trace pairing	27
5.1	The symmetry property of the trace pairing	28
5.2	Additivity of the trace pairing	28
5.3	Braided and balanced monoidal categories	31
5.4	Multiplicativity of the trace pairing	35

1 Introduction

The results of this paper provide an essential step in our proof that the partition function of a Euclidean field theory of dimension $2|1$ is an integral modular function [ST1]. While motivated by field theory, the two main results of this paper are the construction of traces and trace pairings for certain morphisms in a monoidal category.

To describe these results, let \mathcal{C} be a monoidal category with monoidal unit $\mathbb{1} \in \mathcal{C}$ [McL].

Question. What conditions on an endomorphism $f \in \mathcal{C}(X, X)$ allow us to construct a well-defined trace $\text{tr}(f) \in \mathcal{C}(\mathbb{1}, \mathbb{1})$ with the usual properties expected of a trace?

Our first main result, Theorem 1.6 below, provides an answer to this question. Our construction is a common generalization of the following two well-known classical cases:

1. If $X \in \mathcal{C}$ is a dualizable object (see Definition 4.5), then every endomorphism f has a well-defined trace [JSV, Proposition 3.1].
2. If $f: X \rightarrow X$ is a nuclear operator (see Definitions 4.17 and 4.20) on a topological vector space X (locally convex, complete, Hausdorff), then f has a well-defined trace provided X has the *approximation property*, i.e., the identity operator on X can be approximated by finite rank operators in the compact open topology [Li].

We note that in the category \mathbf{TV} of topological vector spaces (locally convex, complete, Hausdorff), equipped with the monoidal structure given by the projective tensor product (see section 4.5), an object $X \in \mathbf{TV}$ is dualizable if and only if X is finite dimensional. Hence extending the trace from endomorphisms of dualizable objects of \mathcal{C} to more general objects is analogous to extending the notion of trace from endomorphisms of finite dimensional vector spaces to those of infinite dimensional topological vector spaces. In fact, our answer will involve analogues of the notions *nuclear* and *approximation property* for general monoidal categories which we now describe.

The following notion is our analogue of a nuclear morphism.

Definition 1.1. A morphism $f: X \rightarrow Y$ in a monoidal category \mathcal{C} is *thick* if it can be factored in the form

$$X \cong \mathbb{1} \otimes X \xrightarrow{t \otimes \text{id}_X} Y \otimes Z \otimes X \xrightarrow{\text{id}_Y \otimes b} Y \otimes \mathbb{1} \cong Y \quad (1.2)$$

for morphisms $t: \mathbb{1} \rightarrow Y \otimes Z$, $b: Z \otimes X \rightarrow \mathbb{1}$.

As explained in the next section, the terminology is motivated by considering the bordism category. In the category \mathbf{Vect} of vector spaces, with monoidal structure given by the tensor product, a morphism $f: X \rightarrow Y$ is thick if and only if it has finite rank (see Theorem 4.1). In the category \mathbf{TV} a morphism is thick if and only if it is nuclear (see Theorem 4.19).

If $f: X \rightarrow X$ is a thick endomorphism with a factorization as above, we attempt to define its *categorical trace* $\text{tr}(f) \in \mathcal{C}(\mathbb{1}, \mathbb{1})$ to be the composition

$$\mathbb{1} \xrightarrow{t} X \otimes Z \xrightarrow{s_{X,Z}} Z \otimes X \xrightarrow{b} \mathbb{1} . \quad (1.3)$$

Here $s = s_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ for $X, Y \in \mathcal{C}$ is a natural family of isomorphisms. We don't assume that s satisfies the relations (5.8) required for the braiding isomorphism of a braided monoidal category. Apparently lacking an established name, we will refer to s as *switching isomorphism*. We would like to thank Mike Shulman for this suggestion.

For the monoidal category \mathbf{Vect} , equipped with the standard switching isomorphism $s: X \otimes Y \rightarrow Y \otimes X$, $x \otimes y \mapsto y \otimes x$, the categorical trace of a finite rank (i.e. thick) endomorphism $f: X \rightarrow X$ agrees with its classical trace (see Theorem 4.1). More generally, if X is a dualizable object of a monoidal category \mathcal{C} , the above definition agrees with the classical definition of the trace in that situation (Theorem 4.9). In

general, the above trace is *not well-defined*, since it might depend on the factorization of f given by the triple (Z, b, t) rather than just the morphism f . As we will see, this happens for example in the category \mathbf{Ban} of Banach spaces.

To understand the problem with defining $\mathrm{tr}(f)$, let us write $\widehat{\mathrm{tr}}(Z, t, b) \in \mathbf{C}(\mathbb{1}, \mathbb{1})$ for the composition (1.3), and $\Psi(Z, t, b) \in \mathbf{C}(X, Y)$ for the composition (1.2). There is an equivalence relation on these triples (see Definition 3.3) such that $\widehat{\mathrm{tr}}(Z, t, b)$ and $\Psi(Z, t, b)$ depend only on the equivalence class $[Z, t, b]$. In other words, there are well-defined maps

$$\widehat{\mathrm{tr}}: \widehat{\mathbf{C}}(X, X) \longrightarrow \mathbf{C}(\mathbb{1}, \mathbb{1}) \quad \Psi: \widehat{\mathbf{C}}(X, Y) \longrightarrow \mathbf{C}(X, Y)$$

where $\widehat{\mathbf{C}}(X, Y)$ denotes the set of equivalence classes of triples (Z, t, b) for fixed $X, Y \in \mathbf{C}$. We note that by construction the image of Ψ consists of the thick morphisms from X to Y . We will call elements of $\widehat{\mathbf{C}}(X, Y)$ *thickened morphisms*. If $\widehat{f} \in \widehat{\mathbf{C}}(X, Y)$ with $\Psi(\widehat{f}) = f \in \mathbf{C}(X, Y)$, we say that \widehat{f} is a *thickener* of f .

Using the notation $\mathbf{C}^{tk}(X, Y)$ for the set of thick morphisms from X to Y , it is clear that there is a well-defined trace map $\mathrm{tr}: \mathbf{C}^{tk}(X, X) \rightarrow \mathbf{C}(\mathbb{1}, \mathbb{1})$ making the diagram

$$\begin{array}{ccc} \mathbf{C}^{tk}(X, X) & \overset{\mathrm{tr}}{\dashrightarrow} & \mathbf{C}(\mathbb{1}, \mathbb{1}) \\ & \Psi \swarrow \quad \searrow \widehat{\mathrm{tr}} & \\ & \widehat{\mathbf{C}}(X, X) & \end{array}$$

commutative if and only if X has the following property:

Definition 1.4. An object X in a monoidal category \mathbf{C} with switching isomorphism s has the *trace property* if the map $\widehat{\mathrm{tr}}$ is constant on the fibers of Ψ .

For the category \mathbf{Ban} , we will show in Section 4.4 that the map Ψ can be identified with the homomorphism

$$\Phi: Y \otimes X' \longrightarrow \mathbf{Ban}(X, Y) \quad w \otimes f \mapsto (v \mapsto wf(v)) \quad (1.5)$$

where X' is the Banach space of continuous linear maps $f: X \rightarrow \mathbb{C}$ equipped with the operator norm. Operators in the image of Φ are referred to as *nuclear operators*, and hence a morphism in \mathbf{Ban} is thick if and only if it is nuclear. It is a classical result that the trace property for a Banach space X is equivalent to the injectivity of the map Ψ which in turn is equivalent to the *approximation property* for X : the identity operator of X can be approximated by finite rank operators in the compact-open topology, see e.g. [Ko, §43.2(7)]. Every Hilbert space has the approximation property, but deciding whether a Banach space has this property is surprisingly difficult. Grothendieck asked this question in the fifties, but the first example of a Banach space *without* the approximation property was found by Enflo only in 1973 [En]. Building on Enflo's work, Szankowski showed in 1981 that the Banach space of bounded operators on an (infinite dimensional) Hilbert space does not have the approximation property [Sz].

Theorem 1.6. *Let \mathbf{C} be a monoidal category equipped with a family of natural isomorphisms $s_{X,Y}: X \otimes Y \rightarrow Y \otimes X$. If $X \in \mathbf{C}$ is an object with the trace property, then any thick endomorphism $f: X \rightarrow X$ has a well-defined categorical trace $\mathrm{tr}(f) \in \mathbf{C}(\mathbb{1}, \mathbb{1})$. This compares to the two classical situations mentioned above as follows:*

- (i) If X is a dualizable object, then X has the trace property and any endomorphism f of X is thick. Moreover, the categorical trace of f agrees with its classical trace.
- (ii) In the category \mathbf{TV} of topological vector spaces (locally convex, complete, Hausdorff), a morphism is thick if and only if it is nuclear, and the approximation property of an object $X \in \mathbf{TV}$ implies the trace property. Moreover, if $f: X \rightarrow X$ is a nuclear endomorphism of an object with the approximation property, then the categorical trace of f agrees with its classical trace.

The first part sums up our discussion above. Statements (i) and (ii) appear as Theorem 4.9 resp. Theorem 4.19 below. We note that it would be interesting to find an object in \mathbf{TV} which has the trace property but not the approximation property.

To motivate our second main result, Theorem 1.9, we note that despite the categorical nature of the definition of the *trace property* for an object a monoidal category \mathbf{C} with switching isomorphism, it is *not functorial* in the sense that the image of some object $X \in \mathbf{C}$ with the trace property under a monoidal functor $F: \mathbf{C} \rightarrow \mathbf{D}$ compatible with the switching isomorphisms might *not* have the trace property. In particular, when the functor F is a field theory, then, as explained in the next section, this non-functoriality causes a problem for calculating the partition function of the field theory.

We circumvent this problem by replacing the *trace* by a closely related *trace pairing*

$$\mathrm{tr}: \mathbf{C}^{tk}(X, Y) \times \mathbf{C}^{tk}(Y, X) \longrightarrow \mathbf{C}(\mathbb{1}, \mathbb{1}) \quad (1.7)$$

for objects X, Y of a monoidal category \mathbf{C} with switching isomorphism $s_{X,Y}: X \otimes Y \cong Y \otimes X$. Unlike the trace map $\mathrm{tr}: \mathbf{C}^{tk}(X, X) \longrightarrow \mathbf{C}(\mathbb{1}, \mathbb{1})$ discussed above, which is only defined if X has the trace property, *no condition on X or Y* is needed to define the trace pairing $\mathrm{tr}(f, g)$ as follows. Let $\widehat{f} \in \widehat{\mathbf{C}}(X, Y)$, $\widehat{g} \in \widehat{\mathbf{C}}(Y, X)$ be thickeners of f resp. g (i.e., $\Psi(\widehat{f}) = f$ and $\Psi(\widehat{g}) = g$). We will show that elements of $\widehat{\mathbf{C}}(X, Y)$ can be pre-composed or post-composed with ordinary morphisms in \mathbf{C} (see Lemma 3.8). This composition gives elements $\widehat{f} \circ g$ and $f \circ \widehat{g}$ in $\widehat{\mathbf{C}}(Y, Y)$ which we will show to be equal in Lemma 3.10. Hence the trace pairing defined by

$$\mathrm{tr}(f, g) := \widehat{\mathrm{tr}}(\widehat{f} \circ g) = \widehat{\mathrm{tr}}(f \circ \widehat{g}) \in \mathbf{C}(\mathbb{1}, \mathbb{1})$$

is *independent* of the choice of \widehat{f} and \widehat{g} . We note that $\Psi(\widehat{f} \circ g) = \Psi(f \circ \widehat{g}) = f \circ g \in \mathbf{C}(Y, Y)$ and hence if Y has the trace property, then

$$\mathrm{tr}(f, g) = \mathrm{tr}(f \circ g) \quad \text{for } f \in \mathbf{C}^{tk}(X, Y), g \in \mathbf{C}^{tk}(Y, X). \quad (1.8)$$

In other words, the trace pairing $\mathrm{tr}(f, g)$ is a generalization of the ordinary trace of $f \circ g$, defined in situations where the usual trace might not be well-defined.

The trace pairing has the following properties that are analogous to properties one expects to hold for a trace. We note that the relationship (1.8) immediately implies these properties for our trace defined for thick endomorphism of objects satisfying the trace property.

Theorem 1.9. *Let \mathbf{C} be a monoidal category, equipped with a natural family of isomorphisms $s = s_{X,Y}: X \otimes Y \rightarrow Y \otimes X$. Then the trace pairing (1.7) has the following properties:*

1. $\mathrm{tr}(f, g) = \mathrm{tr}(g, f)$ for thick morphisms $f \in \mathbf{C}^{tk}(X, Y)$, $g \in \mathbf{C}^{tk}(Y, X)$.
2. If \mathbf{C} is an additive category with distributive monoidal structure (see Definition 5.3), then the trace pairing is a bilinear map;

3. $\text{tr}(f_1 \otimes f_2, g_1 \otimes g_2) = \text{tr}(f_1, g_1) \text{tr}(f_2, g_2)$ for $f_i \in \mathbf{C}^{tk}(X_i, Y_i)$, $g_i \in \mathbf{C}^{tk}(Y_i, X_i)$, provided s gives \mathbf{C} the structure of a symmetric monoidal category. More generally, this property holds if \mathbf{C} is a balanced monoidal category.

We recall that a balanced monoidal category is a braided monoidal category equipped with a natural family of isomorphisms $\theta = \theta_X: X \rightarrow X$ called *twist* satisfying a compatibility condition (see Definition 5.12). Symmetric monoidal categories are balanced monoidal category with $\theta \equiv \text{id}$. For a balanced monoidal category \mathbf{C} with braiding isomorphism $c_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ and twist $\theta_X: X \rightarrow X$, one defines the *switching isomorphism* $s_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ by $s_{X,Y} := (\text{id}_Y \otimes \theta_X) \circ c_{X,Y}$.

There are interesting examples of balanced monoidal categories that are not symmetric monoidal, e.g., categories of bimodules over a fixed von Neumann algebra (monoidal structure given by Connes fusion) or categories of modules over quantum groups. Traces in the latter are used to produce polynomial invariants for knots. Originally, we could only prove the multiplicative property of our trace pairing for symmetric monoidal categories. We are grateful to Gregor Masbaum for pointing out to us the classical definition of the trace of an endomorphism of a dualizable object in a balanced monoidal category which involves using the twist (see [JSV]).

The rest of this paper is organized as follows. In section 2 we explain the example that motivated this paper: we consider the d -dimensional Riemannian bordism category, explain what a *thick* morphism in that category is, and show that the partition function of a 2-dimensional Riemannian field theory can be expressed as the relative trace of the thick operators that a field theory associates to annuli. Section 2 is motivational and can be skipped by a reader who wants to see the precise definition of $\widehat{\mathbf{C}}(X, Y)$, the construction of $\widehat{\text{tr}}$ and a statement of the properties of $\widehat{\text{tr}}$ which are presented in section 3. In section 4 we discuss thick morphisms and their traces in various categories. In section 5 we prove the properties of $\widehat{\text{tr}}$ and deduce the corresponding properties of the trace pairing stated as Theorem 1.9 above.

Both authors were partially supported by NSF grants. The first author would like to thank the Max-Planck-Institut in Bonn for the stimulating atmosphere during his visit there during the Fall of 2009 and in July 2010.

2 Motivation via field theories

A well-known axiomatization of field theory is due to Graeme Segal [Se] who defines a field theory as a monoidal functor from a bordism category to the category \mathbf{TV} of topological vector spaces (locally convex, complete, Hausdorff). The precise definition of the bordism category depends on the type of field theory considered; for a d -dimensional *topological* field theory, the objects are closed $(d - 1)$ -dimensional manifolds and morphisms are d -dimensional bordisms (more precisely, equivalence classes of bordisms where we identify bordisms if they are diffeomorphic relative boundary). Composition is given by gluing of bordisms, and the monoidal structure is given by disjoint union.

For other types of field theories, the manifolds constituting the objects and morphisms in the bordism category come equipped with an appropriate geometric structure; e.g., a *conformal* structure for *conformal* field theories, a *Riemannian* metric for *Riemannian* field theories, or a *Euclidean* structure (= Riemannian metric with vanishing curvature tensor) for a *Euclidean* field theory. In these cases more care is needed

in the definition of the bordism category to ensure the existence of a well-defined composition and the existence of identity morphisms.

Let us consider the Riemannian bordism category $d\text{-RBord}$. The objects of $d\text{-RBord}$ are closed Riemannian $(d-1)$ -manifolds. A morphism from X to Y is a d -dimensional Riemannian bordism Σ from X to Y , that is, a Riemannian d -manifold Σ with boundary and an isometry $X \amalg Y \rightarrow \partial\Sigma$. More precisely, a morphism is an equivalence class of Riemannian bordisms, where two bordisms Σ, Σ' are considered equivalent if there is an isometry $\Sigma \rightarrow \Sigma'$ compatible with the boundary identifications. In order to have a well-defined composition by gluing Riemannian bordisms we require that all metrics are product metrics near the boundary. To ensure the existence of identity morphisms, we enlarge the set of morphisms from X to Y by also including all isometries $X \rightarrow Y$. Pre- or post-composition of a bordism with an isometry is the given bordism with boundary identification modified by the isometry. In particular, the identity isometry $Y \rightarrow Y$ provides the identity morphism for Y as object of the Riemannian bordism category $d\text{-RBord}$.

A more sophisticated way to deal with the issues addressed above was developed in our paper [ST2]. There we don't require the metrics on the bordisms to be a product metric near the boundary; rather, we have more sophisticated objects consisting of a closed $(d-1)$ -manifold equipped with a Riemannian collar. Also, it is technically advantageous not to mix Riemannian bordisms and isometries. This is achieved in that paper by constructing a suitable double category (or equivalently, a category internal to categories), whose vertical (resp. horizontal) morphisms are isometries (resp. bordisms) between closed $(d-1)$ -manifolds equipped with collars, and whose 2-morphisms are isometries of these bordisms. When using the results of the current paper in [ST1], we translate between the approach here using categories versus the approach via internal categories used in [ST2].

Let E be d -dimensional Riemannian field theory, that is, a symmetric monoidal functor

$$E: d\text{-RBord} \longrightarrow \text{TV}.$$

For the bordism category $d\text{-RBord}$ the symmetric monoidal structure is given by disjoint union; for the category TV it is given by the projective tensor product. Let X be a closed Riemannian $(d-1)$ -manifold and Σ be a Riemannian bordism from X to itself. Let Σ_{gl} be the closed Riemannian manifold obtained by gluing the two boundary pieces (via the identity on X). Both Σ and Σ_{gl} are morphisms in $d\text{-RBord}$:

$$\Sigma: X \longrightarrow X \quad \Sigma_{gl}: \emptyset \longrightarrow \emptyset$$

We note that \emptyset is the monoidal unit in $d\text{-RBord}$, and hence the vector space $E(\emptyset)$ can be identified with \mathbb{C} , the monoidal unit in TV . In particular, $E(\Sigma_{gl}) \in \text{Hom}(E(\emptyset), E(\emptyset)) = \text{Hom}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$ is a complex number.

Question. How can we calculate $E(\Sigma_{gl}) \in \mathbb{C}$ in terms of the operator $E(\Sigma): E(X) \rightarrow E(X)$?

We would like to say that $E(\Sigma_{gl})$ is the *trace* of the operator $E(\Sigma)$, but to do so we need to check that the conditions guaranteeing a well-defined trace are met. For a *topological* field theory E this is easy: In the topological bordism category every object X is dualizable (see Definition 4.5), hence $E(X)$ is dualizable in TV which is equivalent to $\dim E(X) < \infty$. By contrast, for a *Euclidean* field theory the vector space $E(X)$ is typically infinite dimensional, and hence to make sense of the trace of the operator

$E(\Sigma)$ associated to a bordism Σ from X to itself, we need to check that the operator $E(\Sigma)$ is thick and that the vector space $E(X)$ has the trace property.

It is easy to prove (see Theorem 4.29) that every object X of the bordism category $d\text{-RBord}$ has the trace property and that among the morphisms of $d\text{-RBord}$ (consisting of Riemannian bordisms and isometries), exactly the bordisms are thick. The latter characterization by the way motivated the adjective ‘thick’, since we think of isometries as ‘infinitely thin’ Riemannian bordisms. It is straightforward to check that being *thick* is a *functorial property* in the sense that the thickness of Σ implies that $E(\Sigma)$ is thick. Unfortunately, as already mentioned in the introduction, the trace property is not functorial and we cannot conclude that $E(X)$ has the trace property.

Replacing the problematical trace by the well-behaved *trace pairing* leads to the following result. It is applied in [ST1] to prove the modularity and integrality of the partition function of a supersymmetric Euclidean field theory of dimension 2.

Theorem 2.1. *Suppose Σ_1 is a Riemannian bordism of dimension d from X to Y , and Σ_2 is a Riemannian bordism from Y to X . Let $\Sigma = \Sigma_1 \circ \Sigma_2$ be the bordism from Y to itself obtained by composing the bordisms Σ_1 and Σ_2 , and let Σ_{gl} be the closed Riemannian d -manifold obtained from Σ by identifying the two copies of Y that make up its boundary. If E is d -dimensional Riemannian field theory, then*

$$E(\Sigma_{gl}) = \text{tr}(E(\Sigma_2), E(\Sigma_1)).$$

Proof. By Theorem 4.29 the bordisms $\Sigma_1: X \rightarrow Y$ and $\Sigma_2: Y \rightarrow X$ are thick morphisms in $d\text{-RBord}$, and hence the morphism $\text{tr}(\Sigma_1, \Sigma_2): \emptyset \rightarrow \emptyset$ is defined. Moreover, every object $X \in d\text{-RBord}$ has the trace property (see Theorem 4.29) and hence

$$\text{tr}(\Sigma_1, \Sigma_2) = \text{tr}(\Sigma_2 \circ \Sigma_1) = \text{tr}(\Sigma)$$

We will show that $\text{tr}(\Sigma) = \Sigma_{gl}$ (see part (3) of Theorem 4.29). Then functoriality of the construction of the trace pairing implies

$$E(\Sigma_{gl}) = E(\text{tr}(\Sigma_1, \Sigma_2)) = \text{tr}(E(\Sigma_1), E(\Sigma_2))$$

□

3 Thickened morphisms and their traces

In this section we will define the set of *thickened morphisms* $\widehat{\mathcal{C}}(X, Y)$ and the trace $\widehat{\text{tr}}(\widehat{f}) \in \mathcal{C}(\mathbb{1}, \mathbb{1})$ of thickened endomorphisms $\widehat{f} \in \widehat{\mathcal{C}}(X, X)$ for a monoidal category \mathcal{C} equipped with a natural family of isomorphisms $s_{X, Y}: X \otimes Y \rightarrow Y \otimes X$.

We recall that a *monoidal category* is a category \mathcal{C} equipped with a functor

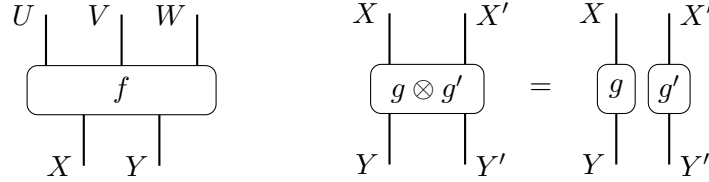
$$\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

called the *tensor product*, a distinguished element $\mathbb{1} \in \mathcal{C}$ and natural isomorphisms

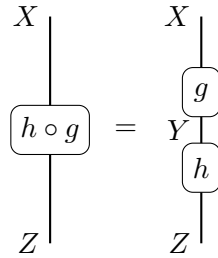
$$\begin{aligned} \alpha_{X, Y, Z}: (X \otimes Y) \otimes Z &\xrightarrow{\cong} X \otimes (Y \otimes Z) && \text{(associator)} \\ \ell_X: \mathbb{1} \otimes X &\xrightarrow{\cong} X && \text{(left unit constraint)} \\ r_X: X \otimes \mathbb{1} &\xrightarrow{\cong} X && \text{(right unit constraint)} \end{aligned}$$

for objects $X, Y, Z \in \mathcal{C}$. These natural isomorphisms are required to make two diagrams (known as the *associativity pentagon* and the *triangle for unit*) commutative, see [McL].

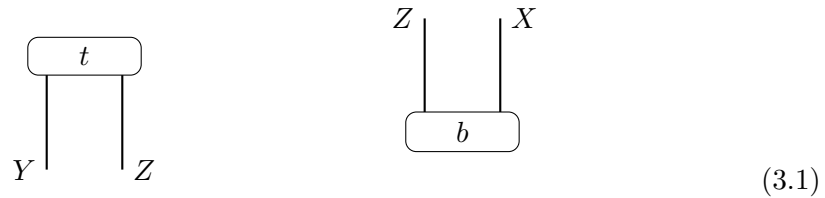
It is common to use diagrams to represent morphisms in \mathcal{C} (see for example [JS1]). The pictures



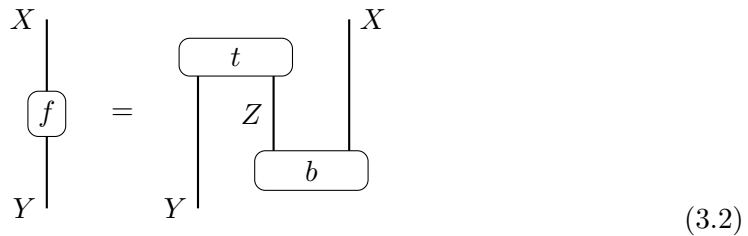
represent a morphism $f: U \otimes V \otimes W \rightarrow X \otimes Y$ and the tensor product of morphisms $g: X \rightarrow Y$ and $g': X' \rightarrow Y'$, respectively. The composition $h \circ g$ of morphisms $g: X \rightarrow Y$ and $h: Y \rightarrow Z$ is represented by the picture



With tensor products being represented by juxtaposition of pictures, the isomorphisms $X \cong X \otimes \mathbb{1} \cong \mathbb{1} \otimes X$ suggest to delete edges labeled by the monoidal unit $\mathbb{1}$ from our picture. E.g., the pictures



represent morphisms $t: \mathbb{1} \rightarrow Y \otimes Z$ and $b: Z \otimes X \rightarrow \mathbb{1}$, respectively. Rephrasing Definition 1.1 of the introduction in our pictorial notation, a morphism $f: X \rightarrow Y$ in \mathcal{C} is *thick* if it can be factored in the form



Here t stands for *top* and b for *bottom*. We will use the notation $\mathcal{C}^{tk}(X, Y) \subset \mathcal{C}(X, Y)$ for the subset of thick morphisms.

3.1 Thickened morphisms

It will be convenient for us to characterize the set of thick morphisms as the image of a map

$$\Psi: \widehat{\mathcal{C}}(X, Y) \longrightarrow \mathcal{C}(X, Y),$$

the domain of which we refer to as *thickened morphisms*.

Definition 3.3. Given objects $X, Y \in \mathcal{C}$, a *thickened morphism* from X to Y is an equivalence class of triples (Z, t, b) consisting of an object $Z \in \mathcal{C}$, and morphisms

$$t: \mathbb{1} \longrightarrow Y \otimes Z \quad b: Z \otimes X \longrightarrow \mathbb{1}$$

To describe the equivalence relation, it is useful to think of these triples as objects of a category, and to define a morphism from (Z, t, b) to (Z', t', b') to be a morphism $g \in \mathcal{C}(Z, Z')$ such that

$$t' = \begin{array}{c} \boxed{t} \\ | \quad | \\ Y \quad Z \\ | \quad | \\ \quad \boxed{g} \\ | \\ Z' \end{array} \quad \text{and} \quad b = \begin{array}{c} Z \quad X \\ | \quad | \\ \boxed{g} \\ | \\ Z' \\ \boxed{b'} \end{array} \quad (3.4)$$

Two triples (Z, t, b) and (Z', t', b') are *equivalent* if there is a sequence of triples (Z_i, t_i, b_i) for $i = 1, \dots, n$ with $(Z, t, b) = (Z_1, t_1, b_1)$ and $(Z', t', b') = (Z_n, t_n, b_n)$ and morphisms g_i between (Z_i, t_i, b_i) and $(Z_{i+1}, t_{i+1}, b_{i+1})$ (this means that g_i is either a morphism from (Z_i, t_i, b_i) to $(Z_{i+1}, t_{i+1}, b_{i+1})$ or from $(Z_{i+1}, t_{i+1}, b_{i+1})$ to (Z_i, t_i, b_i)). In other words, a thickened morphism is a path component of the category defined above. We write $\widehat{\mathcal{C}}(X, Y)$ for the thickened morphisms from X to Y .

Remark 3.5. The anonymous referee of this paper pointed out to us that $\widehat{\mathcal{C}}(X, Y)$ can be interpreted as a coend and that using properties of coends some of the proofs concerning $\widehat{\mathcal{C}}(X, Y)$ can be shortened.

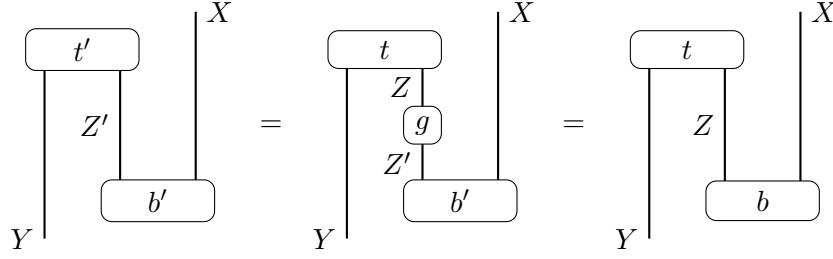
Lemma 3.6. *Given a triple (Z, t, b) as above, let $\Psi(Z, t, b) \in \mathcal{C}(X, Y)$ be the composition on the right hand side of equation (3.2). Then Ψ only depends on the equivalence class $[Z, t, b]$ of (Z, t, b) , i.e. the following map is well-defined:*

$$\Psi: \widehat{\mathcal{C}}(X, Y) \longrightarrow \mathcal{C}(X, Y) \quad [Z, t, b] \mapsto \Psi(Z, t, b)$$

We note that by construction the image of Ψ is equal to $\mathcal{C}^{tk}(X, Y)$, the set of thick morphisms from X to Y . As mentioned in the introduction, for $f \in \mathcal{C}(X, Y)$ we call any $\widehat{f} = [Z, t, b] \in \widehat{\mathcal{C}}(X, Y)$ with $\Psi(\widehat{f}) = f$ a *thickener* of f .

Remark 3.7. The difference between an *orientable* versus an *oriented* manifold is that the former is a property, whereas the latter means additional structure on the manifold. In a similar vein, being *thick* is a property of a morphism $f \in \mathcal{C}(X, Y)$, whereas a *thickener* \widehat{f} of a morphism f is additional structure on f . To make the analogy between these situations perfect, we were tempted to introduce the words *thickenable* or *thickable* into mathematical English. However, we finally decided against it, particularly because the thick-thin distinction for morphisms in the bordism category is just perfectly suited for the purpose.

Proof. Suppose that $g: Z \rightarrow Z'$ is an equivalence from (Z, t, b) to (Z', t', b') in the sense of Definition 3.3. Then the following diagram shows that $\Psi(Z', t', b') = \Psi(Z, t, b)$.



□

Thickened morphisms can be pre-composed or post-composed with ordinary morphisms to obtain again thickened morphisms as follows:

$$\begin{aligned} \widehat{\mathcal{C}}(Y, W) \times \mathcal{C}(X, Y) &\xrightarrow{\circ} \widehat{\mathcal{C}}(X, W) & ([Z, t, b], f) &\mapsto [Z, t, b \circ (\text{id}_Z \otimes f)] \\ \mathcal{C}(Y, W) \times \widehat{\mathcal{C}}(X, Y) &\xrightarrow{\circ} \widehat{\mathcal{C}}(X, W) & (f, [Z, t, b]) &\mapsto [Z, (f \otimes \text{id}_Z) \circ t, b] \end{aligned}$$

The proof of the following result is straightforward and we leave it to the reader.

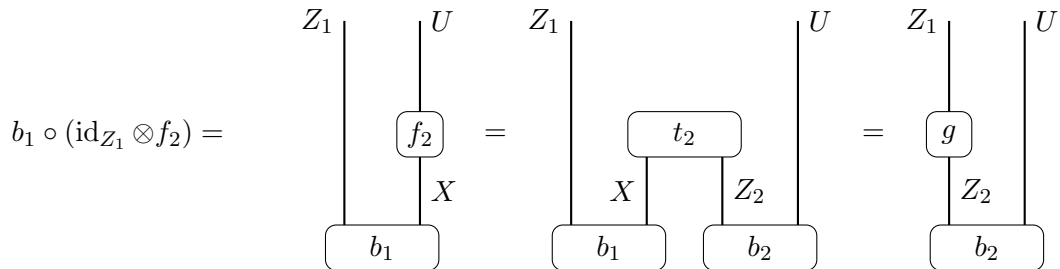
Lemma 3.8. *The composition of morphisms with thickened morphisms is well-defined and compatible with the usual composition via the map Ψ in the sense that the following diagrams commutes:*

$$\begin{array}{ccc} \widehat{\mathcal{C}}(Y, W) \times \mathcal{C}(X, Y) & \xrightarrow{\circ} & \widehat{\mathcal{C}}(X, W) \\ \downarrow \Psi \times \text{id} & & \downarrow \Psi \\ \mathcal{C}(Y, W) \times \mathcal{C}(X, Y) & \xrightarrow{\circ} & \mathcal{C}(X, W) \end{array} \quad \begin{array}{ccc} \mathcal{C}(Y, W) \times \widehat{\mathcal{C}}(X, Y) & \xrightarrow{\circ} & \widehat{\mathcal{C}}(X, W) \\ \downarrow \text{id} \times \Psi & & \downarrow \Psi \\ \mathcal{C}(Y, W) \times \mathcal{C}(X, Y) & \xrightarrow{\circ} & \mathcal{C}(X, W) \end{array}$$

Corollary 3.9. *If $f: X \rightarrow Y$ is a thick morphism in a monoidal category \mathcal{C} , then the compositions $f \circ g$ and $h \circ f$ are thick for any morphisms $g: W \rightarrow X$, $h: Y \rightarrow Z$.*

Lemma 3.10. *Let $\widehat{f}_1 \in \widehat{\mathcal{C}}(X, Y)$, $\widehat{f}_2 \in \widehat{\mathcal{C}}(U, X)$ and $f_i = \Psi(\widehat{f}_i)$. Then $\widehat{f}_1 \circ \widehat{f}_2 = f_1 \circ f_2 \in \widehat{\mathcal{C}}(U, Y)$.*

Proof. Let $\widehat{f}_i = [Z_i, t_i, b_i]$. Then $\widehat{f}_1 \circ \widehat{f}_2 = [Z_1, t_1, b_1 \circ (\text{id}_{Z_1} \otimes f_2)]$ and



Lemma 3.14. $\widehat{\text{tr}}(Z, t, b)$ depends only on the equivalence class of (Z, t, b) . In particular, the map (3.11) is well-defined.

Proof. Suppose that $g: Z \rightarrow Z'$ is an equivalence from (Z', t', b') to (Z, t, b) (see Definition 3.3). Then

$$\widehat{\text{tr}}(Z', t', b') = \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4} = \widehat{\text{tr}}(Z, t, b)$$

□

4 Traces in various categories

The goal in this section is to describe thick morphisms in various categories in classical terms and relate our *categorical trace* to classical notions of trace. At a more technical level, we describe the map $\Psi: \widehat{\mathcal{C}}(X, Y) \rightarrow \mathcal{C}(X, Y)$ and its image $\mathcal{C}^{tk}(X, Y)$ in these categories.

In the first subsection this is done for the category of vector spaces (not necessarily finite dimensional). In the second subsection we show that Ψ is a bijection if X is dualizable and hence *any* endomorphism of a dualizable object has a well-defined trace. Traces in categories for which all objects are dualizable are well-studied [JS2]. In the third subsection we introduce *semi-dualizable* objects and describe the map Ψ in more explicit terms. In a closed monoidal category every object is semi-dualizable. In the following subsection we apply these considerations to the category of Banach spaces. Next we discuss the category of topological vector spaces, and finally the Riemannian bordism category.

4.1 The category of vector spaces

Theorem 4.1. *Let \mathcal{C} be the monoidal category $\text{Vect}_{\mathbb{F}}$ of vector spaces (not necessarily finite dimensional) over a field \mathbb{F} equipped with the usual tensor product and the usual switching isomorphism $s_{X,Y}: X \otimes Y \rightarrow Y \otimes X$, $x \otimes y \mapsto y \otimes x$.*

1. *A morphism $f: X \rightarrow Y$ is thick if and only if it has finite rank.*
2. *The map $\Psi: \widehat{\mathcal{C}}(X, Y) \rightarrow \mathcal{C}(X, Y)$ is injective; in particular, every finite rank endomorphism $f: X \rightarrow X$ has a well-defined categorical trace $\text{tr}(f) \in \mathcal{C}(\mathbb{1}, \mathbb{1}) \cong \mathbb{F}$.*
3. *For a finite rank endomorphism f , its categorical trace $\text{tr}(f)$ agrees with its usual (classical) trace $\text{cl-tr}(f)$.*

For a vector space X , let X^* be the dual vector space, and define the morphism

$$\Phi: Y \otimes X^* \longrightarrow \mathbf{C}(X, Y) \quad \text{by} \quad y \otimes g \mapsto (x \mapsto yg(x)). \quad (4.2)$$

We note that the linear map $x \mapsto yg(x)$ has rank ≤ 1 . Since finite rank operators are finite sums of rank one operators, we see that the image Φ consists of the finite rank operators. The classical trace of the rank one operator $\Phi(y \otimes g)$ is defined to be $g(y)$; by linearity this extends to a well-defined trace for all finite rank operators.

The proof of Theorem 4.1 is based on the following lemma that shows that Ψ is equivalent to the map Φ .

Lemma 4.3. *There is a bijection $\alpha: Y \otimes X^* \rightarrow \widehat{\mathbf{C}}(X, Y)$ such that $\Psi \circ \alpha = \Phi$.*

In subsection 4.3, we will construct the map Φ and prove this lemma in the more general context where X is a semi-dualizable object of a monoidal category. For the proof of Theorem 4.1 we will need the following explicit description of α :

$$\alpha: Y \otimes X^* \longrightarrow \widehat{\mathbf{C}}(X, Y) \quad \tilde{t} \mapsto [X^*, t, \text{ev}]$$

Here $\text{ev}: X^* \otimes X \rightarrow \mathbb{C}$, $g \otimes v \mapsto g(v)$ is the evaluation map, and $t: \mathbb{C} \rightarrow Y \otimes X^*$ is the map characterized by $t(1) = \tilde{t}$.

Proof of Theorem 4.1. Statement (1) follows immediately from the Lemma. To prove part (2), it suffices to show that Φ is injective which is well known and elementary.

For the proof of part (3) let $f: X \rightarrow X$ be a thick morphism. We recall that its categorical trace $\text{tr}(f)$ is defined by $\text{tr}(f) = \widehat{\text{tr}}(\widehat{f})$ for a thickener $\widehat{f} \in \widehat{\mathbf{C}}(X, X)$. So $\Psi(\widehat{f}) = f$ and this is well-defined thanks to the injectivity of Ψ . Using that α is a bijection, there is a unique $\tilde{t} \in X \otimes X^*$ with $\alpha(\tilde{t}) = [X^*, t, \text{ev}] = \widehat{f}$. We note that $\widehat{\text{tr}}(\alpha(\tilde{t})) = \widehat{\text{tr}}([X^*, t, \text{ev}]) \in \mathbf{C}(\mathbb{F}, \mathbb{F}) = \mathbb{F}$ is the composition

$$\mathbb{F} \xrightarrow{\tilde{t}} X \otimes X^* \xrightarrow{s_{X, X^*}} X^* \otimes X \xrightarrow{\text{ev}} \mathbb{F}$$

In particular, if $\tilde{t} = y \otimes g \in X \otimes X^*$, then

$$1 \mapsto y \otimes g \mapsto g \otimes y \mapsto g(y)$$

and hence $\text{tr}(f) = g(y) = \text{cl-tr}(f)$ is the classical trace of the rank one operator $\Phi(\tilde{t}) = f$ given by $x \mapsto yg(x)$. Since both the categorical trace of $\alpha(\tilde{t})$ and the classical trace of $\Phi(\tilde{t})$ depend linearly on \tilde{t} , this finishes the proof. \square

Remark 4.4. We can replace the monoidal category of vector spaces by the monoidal category \mathbf{SVect} of super vector spaces. The objects of this category are just $\mathbb{Z}/2$ -graded vector spaces with the usual tensor product \otimes . The grading involution on $X \otimes Y$ is the tensor product $\epsilon_X \otimes \epsilon_Y$ of the grading involutions on X resp. Y . The switching isomorphism $X \otimes Y \cong Y \otimes X$ is given by $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$ for homogeneous elements $x \in X$, $y \in Y$ of degree $|x|, |y| \in \mathbb{Z}/2$. Then the statements above and their proof work for \mathbf{SVect} as well, except that the categorical trace of a finite rank endomorphism $T: X \rightarrow X$ is its *super trace* $\text{str}(T) := \text{cl-tr}(\epsilon_X T) = \text{tr}(T)$.

4.2 Thick morphisms with dualizable domain

As mentioned in the introduction, there is a well-known trace for endomorphisms of dualizable objects in a monoidal category (see e.g. [JSV, §3]). After recalling the definition of dualizable and the construction of that classical trace, we show in Theorem 4.9 that our categorical trace is a generalization.

Definition 4.5. An object X of a monoidal category \mathcal{C} is *(left) dualizable* if there is an object $X^\vee \in \mathcal{C}$ and morphisms $\text{ev}: X^\vee \otimes X \rightarrow \mathbb{1}$ (called *evaluation map*) and $\text{coev}: \mathbb{1} \rightarrow X \otimes X^\vee$ (*coevaluation map*) such that the following equations hold.

$$\begin{array}{ccc}
 \begin{array}{c} X \\ | \\ \text{coev} \\ | \quad | \\ X \quad X^\vee \\ | \quad | \\ \text{ev} \\ | \\ X \end{array} & = & \begin{array}{c} X \\ | \\ \text{id}_X \\ | \\ X \end{array} \\
 & & \begin{array}{c} X^\vee \\ | \\ \text{coev} \\ | \quad | \\ X^\vee \quad X \\ | \quad | \\ \text{ev} \\ | \\ X^\vee \end{array} & = & \begin{array}{c} X^\vee \\ | \\ \text{id}_{X^\vee} \\ | \\ X^\vee \end{array}
 \end{array} \tag{4.6}$$

The object X^\vee is unique up to natural isomorphism; we'll refer to it as the *object dual to X* .

Example 4.7. A finite dimensional vector space X is a dualizable object in the category $\text{Vect}_{\mathbb{F}}$: we take X^\vee to be the vector space dual to X , ev to be the usual evaluation map, and define

$$\text{coev}: \mathbb{F} \longrightarrow X \otimes X^\vee \quad \text{by} \quad 1 \mapsto \sum_i e_i \otimes e^i$$

where $\{e_i\}$ is a basis of X and $\{e^i\}$ is the dual basis of X^\vee . It is not hard to show that a vector space X is dualizable if and only if it is finite dimensional.

Definition 4.8. Let \mathcal{C} be a monoidal category with switching isomorphism s . Let $f: X \rightarrow X$ be an endomorphism of a dualizable object $X \in \mathcal{C}$. Then the *classical trace* $\text{cl-tr}(f) \in \mathcal{C}(\mathbb{1}, \mathbb{1})$ is defined by

$$\text{cl-tr}(f) := \begin{array}{c} \text{coev} \\ | \quad | \\ X \quad X^\vee \\ | \\ f \\ | \\ X^\vee \quad X \\ | \\ \text{ev} \end{array}$$

This definition can be found for example in section 3 of [JSV] for balanced monoidal categories (see Definition 5.12 for the definition of a balanced monoidal category and how the switching isomorphism is determined by the braiding and the twist of the balanced monoidal category). In fact, the construction in [JSV] is more general: they associate to a morphism $f: A \otimes X \rightarrow B \otimes X$ a trace in $\mathcal{C}(A, B)$ if X is dualizable; specializing to $A = B = \mathbb{1}$ gives the *classical trace* described above.

Theorem 4.9. *Let \mathcal{C} be a monoidal category with switching isomorphism s , and let X be a dualizable object of \mathcal{C} . Then*

1. *The map $\Psi: \widehat{\mathcal{C}}(X, Y) \rightarrow \mathcal{C}(X, Y)$ is a bijection. In particular, any morphism with domain X is thick, and any endomorphism $f: X \rightarrow X$ has a well-defined categorical trace $\text{tr}(f) \in \mathcal{C}(\mathbb{1}, \mathbb{1})$.*
2. *The categorical trace of f is equal to its classical trace $\text{cl-tr}(f)$ defined above.*

Proof. To prove part (1), it suffices to show that the map

$$\Phi: \mathcal{C}(X, Y) \longrightarrow \widehat{\mathcal{C}}(X, Y) \quad f \mapsto [X^\vee, (f \otimes \text{id}_{X^\vee}) \circ \text{coev}, \text{ev}]$$

is an inverse to Ψ . To prove this, let $f: X \rightarrow Y$ be a morphism. Then

$$\Psi(\Phi(f)) = \Psi([X^\vee, f \circ \text{coev}, \text{ev}]) = \begin{array}{c} X \\ \text{coev} \\ \begin{array}{c} X \quad X^\vee \\ | \quad | \\ f \quad \text{ev} \\ | \\ Y \end{array} \end{array} = f$$

Now let $[Z, t, b] \in \widehat{\mathcal{C}}(X, Y)$. Then

$$[Z, t, b] \xrightarrow{\Psi} \begin{array}{c} X \\ t \\ | \\ Z \\ | \\ Y \end{array} \quad b \quad \xrightarrow{\Phi} [X^\vee, t', b']$$

where

$$t' = \begin{array}{c} t \\ | \\ Z \\ | \\ Y \end{array} \quad \text{coev} \quad \begin{array}{c} X \\ | \\ b \\ | \\ X^\vee \end{array} \quad b' = \begin{array}{c} X^\vee \quad X \\ | \quad | \\ \text{ev} \end{array}$$

To show that (Z, t', b') is equivalent to (Z, t, b) , let $g: Z \rightarrow X^\vee$ be the composition

$$g = \begin{array}{c} Z \\ | \\ \text{coev} \\ | \\ b \\ | \\ X^\vee \end{array}$$

Then $t' = g \circ t$ and $b = b' \circ g$, i.e., g is an equivalence from (Z, t, b) to (X^\vee, t', b') (see Definition 3.3).

To prove part (2) we recall that by definition $\text{tr}(f) = \widehat{\text{tr}}(\widehat{f}) \in \mathbf{C}(\mathbb{1}, \mathbb{1})$, where $\widehat{f} = [Z, t, b] \in \widehat{\mathbf{C}}(X, X)$ is any thickener of f (see Definition 3.3 and Equation (3.12)). The proof of part (1) shows that for a dualizable object X we can always choose $[Z, t, b] = [X^\vee, (f \otimes \text{id}_{X^\vee}) \circ \text{coev}, \text{ev}]$. Comparing $\widehat{\text{tr}}(Z, t, b)$ (Equation (3.12)) and $\text{cl-tr}(f)$ (Definition 4.8), we see

$$\widehat{\text{tr}}(X^\vee, (f \otimes \text{id}_{X^\vee}) \circ \text{coev}, \text{ev}) = \text{cl-tr}(f)$$

□

4.3 Thick morphisms with semi-dualizable domain

For a vector space X , the dual vector space X^* is *not* the dual of X in the categorical sense of Definition 4.5 unless $\dim X < \infty$. Similarly, if X is a Banach space, the Banach space X' of continuous linear maps $g: X \rightarrow \mathbb{C}$ equipped with the operator norm is not the categorical dual of X unless $\dim X < \infty$. In both cases, there is an obvious evaluation map, but no coevaluation map satisfying the required properties. However, X^* resp. X' are duals of X in $\text{Vect}_{\mathbb{F}}$ resp. Ban in the following sense:

Definition 4.10. An object X of a monoidal category \mathbf{C} is *semi-dualizable* if there is an object $X^\vee \in \mathbf{C}$ and a morphism $\text{ev}: X^\vee \otimes X \rightarrow \mathbb{1}$ such that for all objects $Z \in \mathbf{C}$ the map

$$\gamma: \mathbf{C}(Z, X^\vee) \longrightarrow \mathbf{C}(Z \otimes X, \mathbb{1}) \quad (4.11)$$

given by

$$\begin{array}{ccc} \begin{array}{c} Z \\ | \\ \boxed{f} \\ | \\ X^\vee \end{array} & \mapsto & \begin{array}{c} Z \quad X \\ | \quad | \\ \boxed{f} \quad | \\ | \quad | \\ \boxed{\text{ev}} \end{array} \end{array}$$

is a bijection. The object X^\vee is unique up to natural isomorphism; we'll refer to it as the *object dual to X* .

If X is a dualizable object, then it is semi-dualizable, since composition with with the coevaluation map provides an inverse to the map (4.11). Examples of monoidal categories where every object is semi-dualizable include the category Vect and the category of Banach spaces. These categories are examples of closed monoidal categories, and we have the following general result:

Proposition 4.12. *Every object in a closed monoidal category is semi-dualizable.*

Proof. We recall that a monoidal category \mathbf{C} is *closed* if for any $X \in \mathbf{C}$ the functor $\mathbf{C} \rightarrow \mathbf{C}$, $Z \mapsto Z \otimes X$ has a right adjoint; i.e., if there is a functor

$$\underline{\mathbf{C}}(X, _): \mathbf{C} \longrightarrow \mathbf{C}$$

and natural bijections

$$\mathbf{C}(Z, \underline{\mathbf{C}}(X, Y)) \cong \mathbf{C}(Z \otimes X, Y) \quad Z, X, Y \in \mathbf{C} \quad (4.13)$$

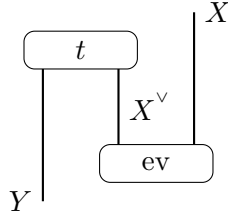
If X is an object of a closed monoidal category \mathbf{C} , we define $X^\vee := \underline{\mathbf{C}}(X, \mathbb{1})$ and let $\text{ev}: X^\vee \otimes X \rightarrow \mathbb{1}$ be the morphism corresponding to the identity morphism of the object X^\vee via the adjunction bijection above. Then the naturality of the adjunction bijections implies that the map (4.11) can be identified with the adjunction bijection. \square

Another example of a closed monoidal category is the category of *bornological vector spaces* [Me]. As also discussed in [Me, p. 9], the symmetric monoidal category \mathbf{TV} of topological vector spaces (locally convex, complete, Hausdorff) with the projective tensor product is *not an example* of a closed monoidal category (no matter which topology on the space of continuous linear maps is used). Still, some topological vector spaces are semi-dualizable (e.g., Banach spaces), and so it seemed preferable to state our results for semi-dualizable objects rather than objects in closed monoidal categories.

Let X be a semi-dualizable object of a monoidal category \mathbf{C} with dual X^\vee and evaluation map $\text{ev}: X^\vee \otimes X \rightarrow \mathbb{1}$. We define the map

$$\Phi: \mathbf{C}(\mathbb{1}, Y \otimes X^\vee) \longrightarrow \mathbf{C}(X, Y) \quad (4.14)$$

by sending $t: \mathbb{1} \rightarrow Y \otimes X^\vee$ to the composition



For the category $\mathbf{C} = \mathbf{Vect}_{\mathbb{F}}$ we can identify $\mathbf{C}(\mathbb{1}, Y \otimes X^\vee)$ with $Y \otimes X^\vee$ by evaluating $t: \mathbb{F} \rightarrow Y \otimes X^\vee$ at $1 \in \mathbb{F} = \mathbb{1}$. With this identification, the map Φ agrees with the map (4.2).

Proposition 4.15. *Let X be a semi-dualizable object of a monoidal category \mathbf{C} . Then for any object $Y \in \mathbf{C}$ the map*

$$\alpha: \mathbf{C}(\mathbb{1}, Y \otimes X^\vee) \longrightarrow \widehat{\mathbf{C}}(X, Y) \quad t \mapsto [X^\vee, t, \text{ev}]$$

is a natural bijection which makes the diagram

$$\begin{array}{ccc} \mathbf{C}(\mathbb{1}, Y \otimes X^\vee) & \xrightarrow[\alpha]{\cong} & \widehat{\mathbf{C}}(X, Y) \\ & \searrow \Phi & \swarrow \Psi \\ & \mathbf{C}(X, Y) & \end{array}$$

commutative.

Proof. We claim that the map

$$\beta: \widehat{\mathbf{C}}(X, Y) \longrightarrow \mathbf{C}(\mathbb{1}, Y \otimes X^\vee) \quad [Z, t, b] \mapsto \left(\mathbb{1} \xrightarrow{t} Y \otimes Z \xrightarrow{\text{id}_Y \otimes b^\#} Y \otimes X^\vee \right)$$

is an inverse to α , where $b^\#: Z \rightarrow X^\vee$ is the morphism corresponding to $b \in \mathbf{C}(Z \otimes X, \mathbb{1})$ via the bijection (4.11). We have

$$\beta(\alpha(t)) = \beta([X^\vee, t, \text{ev}]) = (\text{id}_Y \otimes \text{ev}^\#) \circ t = t \in \mathbf{C}(\mathbb{1}, Y \otimes X^\vee),$$

since by definition the evaluation map $\text{ev}: X^\vee \otimes X \rightarrow \mathbb{1}$ corresponds to the identity on X^\vee via the adjunction (4.13).

Composing the other way around we obtain

$$\alpha(\beta([Z, t, b]) = [X^\vee, t', \text{ev}],$$

where t' is the composition

$$\mathbb{1} \xrightarrow{t} Y \otimes Z \xrightarrow{\text{id}_Y \otimes b^\#} Y \otimes X^\vee$$

It follows from our definition of the equivalence relation (3.4) that the triple (X^\vee, t', ev) is equivalent to the triple (Z, t, b') , where b' is the composition

$$Z \otimes X \xrightarrow{b^\# \otimes \text{id}_X} X^\vee \otimes X \xrightarrow{\text{ev}} \mathbb{1}$$

This composition is b by construction of $b^\#$ and hence $\alpha(\beta([Z, t, b]) = [X^\vee, t', \text{ev}] = [Z, t, b]$. \square

4.4 The Category Ban of Banach spaces

Let X be a Banach space and $f: X \rightarrow X$ a continuous linear map. There are classical conditions (f is *nuclear* and X has the *approximation property*, see Definition 4.17) which guarantee that f has a well-defined (classical) trace which we again denote by $\text{cl-tr}(f) \in \mathbb{C}$. For example any Hilbert space H has the approximation property and a continuous linear map $f: H \rightarrow H$ is nuclear if and only if it is trace class. The main result of this subsection is Theorem 4.18 which shows that these classical conditions imply that f has a well-defined categorical trace and that the categorical trace of f agrees with its classical trace. Before stating this result, we review the (projective) tensor product of Banach spaces and define the notions *nuclear* and *approximation property*.

The category **Ban** of Banach spaces is a monoidal category whose monoidal structure is given by the *projective tensor product*, defined as follows. For Banach spaces X, Y , the *projective norm* on the algebraic tensor product $X \otimes_{\text{alg}} Y$ is given by

$$\|z\| := \inf \left\{ \sum \|x_i\| \cdot \|y_i\| \mid z = \sum x_i \otimes y_i \in X \otimes Y \right\}$$

where the infimum is taken over all ways of expressing $z \in X \otimes_{\text{alg}} Y$ as a finite sum of elementary tensors. Then the *projective tensor product* $X \otimes Y$ is defined to be the completion of $X \otimes_{\text{alg}} Y$ with respect to the projective norm.

We note that the morphism set $\mathbf{Ban}(X \otimes Y, Z)$ is in bijective correspondence to the set of continuous bilinear maps $X \times Y \rightarrow Z$. This bijection is given by sending $g: X \otimes Y \rightarrow Z$ to the composition $g \circ \chi$, where $\chi: X \times Y \rightarrow X \otimes Y$ is given by $(x, y) \mapsto x \otimes y$. In particular, if X^\vee is the Banach space dual to X , we have a morphism

$$\text{ev}: X^\vee \otimes X \longrightarrow \mathbb{C} \quad \text{determined by} \quad g \otimes x \mapsto g(x) \quad (4.16)$$

called the *evaluation map*.

Definition 4.17. A continuous linear map between Banach spaces is *nuclear* if f is in the image of the map

$$\Phi: Y \otimes X^\vee \longrightarrow \text{Ban}(X, Y) \quad \text{determined by} \quad y \otimes g \mapsto (x \mapsto yg(x))$$

A Banach space X has the *approximation property* if the identity of X can be approximated by finite rank operators with respect to the compact-open topology.

Theorem 4.18. *Let $X, Y \in \text{Ban}$.*

1. *A morphism $f: X \rightarrow Y$ is thick if and only if it is nuclear.*
2. *If X has the approximation property, it has the trace property.*
3. *If X has the approximation property and $f: X \rightarrow X$ is nuclear, then the categorical trace of f agrees with its classical trace.*

Proof. This result holds more generally in the category of topological vector spaces which we will prove in the following subsection (see Theorem 4.19). For the proofs of statements (2) and (3) we refer to that section. For the proof of statement (1) we recall that a morphism $f: X \rightarrow Y$ in the category $\mathbf{C} = \text{Ban}$ is thick if and only if it is in the image of the map $\Psi: \widehat{\mathbf{C}}(X, Y) \rightarrow \mathbf{C}(X, Y)$ and that it is *nuclear* if it is in the image of $\Phi: Y \otimes X^\vee = \mathbf{C}(\mathbb{1}, Y \otimes X^\vee) \rightarrow \mathbf{C}(X, Y)$. Hence part (1) is a corollary of Proposition 4.15 according to which these two maps are equivalent for any semi-dualizable object X of a monoidal category \mathbf{C} . It remains to show that every Banach space X is semi-dualizable. This follows from the fact that Ban is a closed category and Proposition 4.12. Alternatively, this is easy to prove directly by taking X^\vee to be the Banach space of continuous linear maps $X \rightarrow \mathbb{C}$ equipped with the operator norm and ev to be the usual evaluation map (4.16). To see that the map $\gamma: \mathbf{C}(Z, X^\vee) \rightarrow \mathbf{C}(Z \otimes X, \mathbb{1})$ is a bijection, we construct a map

$$\mathbf{C}(Z \otimes X, \mathbb{1}) \longrightarrow \mathbf{C}(Z, X^\vee) \quad \mu \mapsto \mu^\#,$$

where $\mu^\#: Z \rightarrow X^\vee$ is defined by $\mu^\#(z)(x) = \mu(z \otimes x)$. It is easy to check that $\mu^\#$ is a bounded operator:

$$\begin{aligned} \|\mu^\#(z)\| &= \sup_{\|x\| \leq 1} |\mu^\#(z)(x)| = \sup_{\|x\| \leq 1} |\mu(z \otimes x)| \\ &= \sup_{\|x\| \leq 1} \|\mu\| \|z \otimes x\| \leq \|\mu\| \|z\| \end{aligned}$$

We leave it to the reader to verify that this map is an inverse for γ . □

4.5 A Category of topological vector spaces

In this subsection we extend Theorem 4.18 from Banach spaces to the category TV whose objects are locally convex topological vector spaces which are Hausdorff and complete. We recall that the topology on a vector space X is required to be invariant under translations and dilations. In particular, it determines a uniform structure on X which in turn allows us to speak of Cauchy nets and hence completeness; see [Sch, section I.1] for details. The morphisms of TV are continuous linear maps, and the projective tensor product described below gives TV the structure of a symmetric monoidal category. It contains the category Ban of Banach spaces as a full subcategory.

Theorem 4.19. *Let X, Y be objects in the category \mathbf{TV} .*

1. *A morphism $f: X \rightarrow Y$ is thick if and only if it is nuclear.*
2. *If X has the approximation property, then it has the trace property.*
3. *If X has the approximation property, and $f: X \rightarrow X$ is nuclear, then the categorical trace of f agrees with its classical trace.*

Before proving this theorem, we define what a nuclear map between topological vector spaces is, define the approximation property for a topological vector space, and explain the classical trace of a nuclear endomorphism of a vector space with the approximation property.

Definition 4.20. A continuous linear map $f: X \rightarrow Y$ between topological vector spaces X, Y is *nuclear* if it factors in the form

$$X \xrightarrow{p} X_0 \xrightarrow{f_0} Y_0 \xrightarrow{j} Y, \quad (4.21)$$

where f_0 is a nuclear map between Banach spaces (see Definition 4.17) and p, j are continuous linear maps.

The definition of nuclearity in Schaefer's book ([Sch, p. 98]) is phrased differently. We give his more technical definition at the end of this section and show that a continuous linear map is nuclear in his sense if and only if it is nuclear in the sense of the above definition.

Approximation property. A topological vector space X has the *approximation property* if the identity of X is in the closure of the subspace of finite rank operators with respect to the compact-open topology [Sch, Chapter III, section 9] (our completeness assumption for topological vector spaces implies that uniform convergence on compact subsets is the same as uniform convergence on pre-compact subsets).

The classical trace for nuclear endomorphisms. Let f be a nuclear endomorphism of $X \in \mathbf{TV}$, and let $I_\nu: X \rightarrow X$ be a net of finite rank morphisms which converges to the identity on X in the compact-open topology. Then $f \circ I_\nu$ is a finite rank operator which has a (classical) trace $\text{cl-tr}(f \circ I_\nu)$. It can be proved that the limit $\lim_\nu \text{cl-tr}(f \circ I_\nu)$ exists [Li, Proof of Theorem 1] and is independent of the choice of the net I_ν (this also follows from our proof of Theorem 4.19). The classical trace of f is defined by

$$\text{cl-tr}(f) := \lim_\nu \text{cl-tr}(f \circ I_\nu).$$

Before proving this result, we recall the construction of the (completed) projective tensor product. In addition we review the classical definitions of *nuclear* morphisms between topological vector spaces, the *approximation property* for topological vector spaces and the construction of a trace for nuclear endomorphisms of topological vector spaces with the approximation property.

Projective tensor product. The projective tensor product of Banach spaces defined in the previous section extends to topological vector spaces as follows. For $X, Y \in \mathbf{TV}$ the *projective topology* on the algebraic tensor product $X \otimes_{\text{alg}} Y$ is the finest locally convex topology such that the canonical bilinear map

$$\chi: X \times Y \longrightarrow X \otimes_{\text{alg}} Y \quad (x, y) \mapsto x \otimes y$$

is continuous [Sch, p. 93]. The *projective tensor product* $X \otimes Y$ is the (locally convex, Hausdorff and complete) topological vector space obtained as the completion of $X \otimes_{alg} Y$ with respect to the projective topology. It can be shown that the morphisms $\text{TV}(X \otimes Y, Z)$ are in bijective correspondence to continuous bilinear maps $X \times Y \rightarrow Z$; this bijection is given by sending $f: X \otimes Y \rightarrow Z$ to $f \circ \chi$ [Sch, Chapter III, section 6.2].

Semi-norms. For checking the convergence of a sequence or continuity of a map between topological vector spaces, it is convenient to work with semi-norms. Let X be a topological vector space, and $U \subset X$ a convex circled 0-neighborhood (U is *circled* if $\lambda U \subset U$ for every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$). Then U determines a semi-norm

$$\|x\|_U := \inf\{\lambda \in \mathbb{R}_{>0} \mid x \in \lambda U\} \quad (4.22)$$

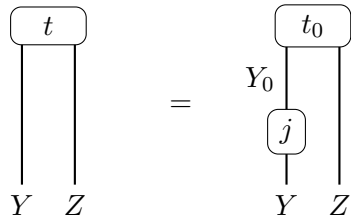
on X . Conversely, a collection of semi-norms determines a topology, namely the coarsest locally convex topology such that the given semi-norms are continuous maps. For example, if X is a Banach space with norm $\|\cdot\|$, we obtain the usual topology on X . As another example, the projective topology on the algebraic tensor product $X \otimes_{alg} Y$ is the topology determined by the family of semi-norms $\|\cdot\|_{U,V}$ parametrized by convex circled 0-neighborhoods $U \subset X$, $V \subset Y$ defined by

$$\|z\|_{U,V} := \inf\left\{\sum_{i=1}^n \|x_i\|_U \|y_i\|_V \mid z = \sum_{i=1}^n x_i \otimes y_i\right\},$$

where the infimum is taken over all ways of writing $z \in X \otimes_{alg} Y$ as a finite sum of elementary tensors. It follows from this description that the projective tensor product defined above is compatible with the projective tensor product of Banach spaces defined earlier (see [Sch, Chapter III, section 6.3]).

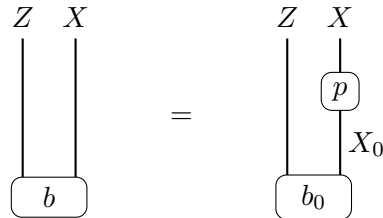
Our next goal is the proof of Theorem 4.19, for which we will use the following lemma.

Lemma 4.23. 1. Any morphism $t: \mathbb{C} \rightarrow Y \otimes Z$ in TV factors in the form



where Y_0 is a Banach space.

2. Any morphism $b: Z \otimes X \rightarrow \mathbb{C}$ factors in the form



where X_0 is a Banach space.

For the proof of this lemma, we will need the following two ways to construct Banach spaces from a topological vector space X :

1. Let U be a convex, circled neighborhood of $0 \in X$. Let X_U be the Banach space obtained from X by quotienting out the null space of the semi-norm $\|\cdot\|_U$ and by completing the resulting normed vector space. Let $p_U: X \rightarrow X_U$ be the evident map.
2. Let B be a convex, circled bounded subset of X . We recall that B is *bounded* if for each neighborhood U of $0 \in X$ there is some $\lambda \in \mathbb{C}$ such that $B \subset \lambda U$. Let X_B be the vector space $X_B := \bigcup_{n=1}^{\infty} nB$ equipped with the norm $\|x\|_B := \inf\{\lambda \in \mathbb{R}_{>0} \mid x \in \lambda B\}$. If B is closed in X , then X_B is complete (by our assumption that X is complete), and hence X_B is a Banach space ([Sch, Ch. III, §7; p. 97]). The inclusion map $j_B: X_B \rightarrow X$ is continuous thanks to the assumption that B is bounded.

Proof. To prove part (1) we use the fact (see e.g. Theorem 6.4 in Chapter III of [Sch]) that any element of the completed projective tensor product $Y \otimes Z$, in particular the element $t(1)$, can be written in the form

$$t(1) = \sum_{i=1}^{\infty} \lambda_i y_i \otimes z_i \quad \text{with} \quad y_i \rightarrow 0 \in Y, z_i \rightarrow 0 \in Z, \sum |\lambda_i| < \infty \quad (4.24)$$

Let $B' := \{y_i \mid i = 1, 2, \dots\} \cup \{0\}$, and let B be the closure of the *convex, circled hull* of B' (the convex circled hull of B' is the intersection of all convex circled subsets of W containing B'). We note that B' is bounded, hence its convex, circled hull is bounded, and hence B is bounded.

We define $j: Y_0 \rightarrow Y$ to be the map $j_B: Y_B \rightarrow Y$. To finish the proof of part (1), it suffices to show that $t(1)$ is in the image of the inclusion map $j_B \otimes \text{id}_Z: Y_B \otimes Z \hookrightarrow Y \otimes Z$. It is clear that each partial sum $\sum_{i=1}^n \lambda_i y_i \otimes z_i$ belongs to the algebraic tensor product $Y_B \otimes_{\text{alg}} Z$, and hence we need to show that the sequence of partial sums is a Cauchy sequence with respect to the semi-norms $\|\cdot\|_{B,V}$ on $Y_B \otimes_{\text{alg}} Z$ that define the projective topology (here V runs through all convex circled 0-neighborhoods $V \subset Z$). Since $y_i \in B$, it follows $\|y_i\|_B \leq 1$ and hence we have the estimate

$$\left\| \sum_{i=1}^n \lambda_i y_i \otimes z_i \right\|_{B,V} \leq \sum_{i=1}^n |\lambda_i| \|y_i\|_B \|z_i\|_V \leq \sum_{i=1}^n |\lambda_i| \|z_i\|_V$$

Since $z_i \rightarrow 0$, we have $\|z_i\|_V \leq 1$ for all but finitely many i , and this implies that the partial sums form a Cauchy sequence.

To prove part (2) we recall that the morphism $b: Z \otimes X \rightarrow \mathbb{C}$ corresponds to a continuous bilinear map $b': Z \times X \rightarrow \mathbb{C}$. The continuity of b' implies that there are convex, circled 0-neighborhoods $V \subset Z$, $U \subset X$ such that $|b'(z, x)| < 1$ for $z \in V$, $x \in U$. It follows that $|b'(z, x)| \leq \|z\|_V \|x\|_U$ for all $z \in Z$, $x \in X$. Hence b extends to a continuous bilinear map

$$\tilde{b}': Z \times X_U \longrightarrow \mathbb{C}$$

for the completion X_U of X . This corresponds to the desired morphism $b_0: Z \otimes X_U \rightarrow \mathbb{C}$; the property $b = b_0 \circ (\text{id}_Z \otimes p_U)$ is clear by construction. Defining the map $p: X \rightarrow X_0$ to be $p_U: X \rightarrow X_U$, we obtained the desired factorization of b . \square

Proof of Theorem 4.19. The factorization (4.21) shows that a nuclear morphism $f: X \rightarrow Y$ factors through a nuclear map $f_0: X_0 \rightarrow Y_0$ of Banach spaces. Then f_0 is thick by Theorem 4.18 and hence f is thick, since pre- or post-composition of a thick morphism with an any morphism is thick.

To prove the converse, assume that f is thick, i.e., that it can be factored in the form

$$f = (\text{id}_Y \otimes b)(t \otimes \text{id}_X) \quad t: \mathbb{1} \rightarrow Y \otimes Z \quad b: Z \otimes X \rightarrow \mathbb{1}.$$

Then using Lemma 4.23 to factorize t and b , we see that f can be further factored in the form

$$f = \begin{array}{c} \begin{array}{c} X \\ | \\ p \\ | \\ X_0 \\ | \\ b_0 \\ | \\ Z \\ | \\ t_0 \\ | \\ Y_0 \\ | \\ j \\ | \\ Y \end{array} \end{array}$$

This implies that f has the desired factorization $X \xrightarrow{p} X_0 \xrightarrow{f_0} Y_0 \xrightarrow{j} Y$, where

$$f_0 = \begin{array}{c} \begin{array}{c} X_0 \\ | \\ b_0 \\ | \\ Z \\ | \\ t_0 \\ | \\ Y_0 \end{array} \end{array}$$

It remains to show that f_0 is a nuclear map between Banach spaces. In the category of Banach spaces a morphisms is nuclear if and only if it is thick by Theorem 4.18. At first glance, it seems that the factorization of f_0 above shows that f_0 is thick. However, on second thought one realizes that we need to replace Z by a *Banach space* to make that argument. This can be done by using again our Lemma 4.23 to factorize t_0 and hence f_0 further in the form

$$f_0 = \begin{array}{c} \begin{array}{c} X_0 \\ | \\ b_0 \\ | \\ Z \\ | \\ j' \\ | \\ Z_0 \\ | \\ t'_0 \\ | \\ Y_0 \end{array} \end{array}$$

This shows that f_0 is a thick morphism in the category **Ban** and hence nuclear by Theorem 4.18.

The key for the proof of parts (2) and (3) of Theorem 4.19 will be the following lemma.

Lemma 4.25. For any $\widehat{f} \in \widehat{\mathbf{TV}}(Y, X)$ the map

$$\mathbf{TV}(X, Y) \longrightarrow \mathbf{TV}(\mathbb{1}, \mathbb{1}) = \mathbb{C} \quad g \mapsto \widehat{\mathrm{tr}}(\widehat{f} \circ g) \quad (4.26)$$

is continuous with respect to the compact open topology on $\mathbf{TV}(X, Y)$.

To prove part (2) of Theorem 4.19, assume that X has the approximation property and let $I_\nu: X \rightarrow X$ be a net of finite rank operators converging to the identity of X in the compact open topology. Then by the lemma, for any $\widehat{f} \in \widehat{\mathbf{TV}}(X, X)$, the net

$$\widehat{\mathrm{tr}}(f \circ \widehat{I}_\nu) = \widehat{\mathrm{tr}}(\widehat{f} \circ I_\nu) \quad \text{converges to} \quad \widehat{\mathrm{tr}}(\widehat{f} \circ \mathrm{id}_X) = \widehat{\mathrm{tr}}(\widehat{f}).$$

Here $\widehat{I}_\nu \in \widehat{\mathbf{TV}}(X, X)$ are thickeners of I_ν . They exist since every finite rank morphism is nuclear and hence thick by part (1). This implies the *trace condition* for X , since $\widehat{\mathrm{tr}}(\widehat{f}) = \lim_\nu \widehat{\mathrm{tr}}(f \circ \widehat{I}_\nu)$ depends only on f .

To prove part (3) let $f \in \mathbf{C}(X, X)$ be a nuclear endomorphism of a topological vector space X , and let I_ν be a net of finite rank operators converging to the identity of X in the compact open topology. By part (1) f is thick, i.e., there is a thickener $\widehat{f} \in \mathbf{C}(X, X)$. Then as discussed above, we have

$$\mathrm{cl}\text{-tr}(f) = \lim_\nu \mathrm{cl}\text{-tr}(f \circ I_\nu) \quad \text{and} \quad \mathrm{tr}(f) = \widehat{\mathrm{tr}}(\widehat{f}) = \lim_\nu \widehat{\mathrm{tr}}(f \circ \widehat{I}_\nu) = \lim_\nu \mathrm{tr}(f \circ I_\nu)$$

For the finite rank operator $f \circ I_\nu$ its classical trace $\mathrm{cl}\text{-tr}(f \circ I_\nu)$ and its categorical trace $\mathrm{tr}(f \circ I_\nu)$ agree by part (3) of Theorem 4.1. \square

Proof of Lemma 4.25. Let $\widehat{f} = [Z, t, b] \in \widehat{\mathbf{TV}}(Y, X)$. Then $\widehat{f} \circ g = [Z, t, b \circ (\mathrm{id}_Z \otimes g)]$ and hence $\widehat{\mathrm{tr}}(\widehat{f} \circ g)$ is given by the composition

$$\mathbb{C} \xrightarrow{t} X \otimes Z \xrightarrow{s_{X,Z}} Z \otimes X \xrightarrow{\mathrm{id}_Z \otimes g} Z \otimes Y \xrightarrow{b} \mathbb{C} \quad (4.27)$$

As in Equation (4.24) we write $t(1) \in X \otimes Z$ in the form

$$t(1) = \sum_{i=1}^{\infty} \lambda_i x_i \otimes z_i \quad \text{with} \quad x_i \rightarrow 0 \in X, z_i \rightarrow 0 \in Z, \sum_{i=1}^{\infty} |\lambda_i| < \infty$$

This implies

$$\widehat{\mathrm{tr}}(\widehat{f} \circ g) = \sum_{i=1}^{\infty} \lambda_i b'(z_i, g(x_i)),$$

where $b': Z \times Y \rightarrow \mathbb{C}$ is the continuous bilinear map corresponding to $b: Z \otimes Y \rightarrow \mathbb{C}$.

To show that the map $g \mapsto \widehat{\mathrm{tr}}(\widehat{f} \circ g)$ is continuous, we will construct for given $\epsilon > 0$ a compact subset $K \subset X$ and an open subset $U \subset Y$ such that for

$$g \in \mathcal{U}_{K,U} := \{g \in \mathbf{TV}(X, Y) \mid g(K) \subset U\} \subset \mathbf{TV}(X, Y)$$

we have $|\widehat{\mathrm{tr}}(\widehat{f} \circ g)| < \epsilon$. To construct U , we note that the continuity of b' implies that there are 0-neighborhoods $U \subset Y$, $V \subset Z$ such that $y \in U$, $z \in V$ implies $|b'(z, y)| < \epsilon$. Without loss of generality we can assume $z_i \in V$ for all i (by replacing z_i by cz_i and x_i by x_i/c for a sufficiently small number c) and $\sum |\lambda_i| = 1$ (by replacing λ_i by λ_i/s and x_i by sx_i for $s = \sum |\lambda_i|$). We define $K := \{x_i \mid i \in \mathbb{N}\} \cup \{0\} \subset X$. Then for $g \in \mathcal{U}_{K,U}$ we have

$$|\widehat{\mathrm{tr}}(\widehat{f} \circ g)| = \left| \sum \lambda_i b'(z_i, g(x_i)) \right| \leq \sum |\lambda_i| |b'(z_i, g(x_i))| < (\sum |\lambda_i|) \epsilon = \epsilon.$$

\square

Finally we compare our definition of a nuclear map between topological vector spaces (see Definition 4.17) with the more classical definition which can be found e.g. in [Sch, p. 98]. A continuous linear map $f: X \rightarrow Y$ between topological vector spaces X, Y is *nuclear in the classical sense* if there is a convex circled 0-neighborhood $U \subset X$, a closed, convex, circled, bounded subset $B \subset Y$ such that $f(U) \subset B$ and the induced map of Banach spaces $X_U \rightarrow Y_B$ is nuclear.

Lemma 4.28. *A morphism in TV is nuclear if and only if it is nuclear in the classical sense.*

Proof. If $f: X \rightarrow Y$ is nuclear in the classical sense, it factors in the form

$$X \xrightarrow{p_U} X_U \xrightarrow{f_0} Y_B \xrightarrow{j_B} Y,$$

where f_0 is nuclear. Hence f is nuclear.

Conversely, let us assume that f is nuclear, i.e., that it factors in the form

$$X \xrightarrow{p} X_0 \xrightarrow{f_0} Y_0 \xrightarrow{j} Y,$$

where f_0 is a nuclear map between Banach spaces. To show that f is nuclear in the classical sense, we will construct a convex circled 0-neighborhood $U \subset X$, and a closed, convex, circled, bounded subset $B \subset Y$ such that $f(U) \subset B$ and we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{p} & X_0 & \xrightarrow{f_0} & Y_0 & \xrightarrow{j} & Y \\ & \searrow & \uparrow & & \downarrow & \nearrow & \\ & & X_U & \xrightarrow{f'} & Y_B & & \end{array}$$

(Note: The diagram above is a simplified representation of the commutative diagram in the image. The original diagram has arrows from X to X_U labeled p_U, from X_U to X_0 labeled p, from X_0 to Y_0 labeled f_0, from Y_0 to Y_B labeled j_B, from X_U to Y_B labeled f', and from Y_B to Y labeled j.)

Then the induced map f' factors through the nuclear map f_0 , hence f' is nuclear and f is nuclear in the classical sense.

We define $U := p^{-1}(\mathring{B}_\delta)$, where $\mathring{B}_\delta \subset X_0$ is the open ball of radius δ around the origin in the Banach space X_0 , and $\delta > 0$ is chosen such that $\mathring{B}_\delta \subset f_0^{-1}(\mathring{B}_1)$. The continuity of p implies that U is open. Moreover, $p(U) \subset \mathring{B}_\delta \stackrel{\text{def}}{=} \delta \mathring{B}_1$ implies $\|p(x)\| < \delta$ for $x \in U$ which in turn implies the estimate $\|p(x)\| < \delta \|x\|_U$ for all $x \in X$. It follows that the map p factors through p_U .

We define $B := j(B_1) \subset Y$, where B_1 is the closed unit ball in Y_0 . This is a bounded subset of Y , since for any open subset $U \subset Y$ the preimage $j^{-1}(U)$ is an open 0-neighborhood of Y_0 and hence there is some $\epsilon > 0$ such that $B_\epsilon \subset j^{-1}(U)$. Then $\epsilon j(B_1) = j(B_\epsilon)$ is a subset of U . This implies $j(B_1) \subset \frac{1}{\epsilon}U$ and hence B is a bounded subset of Y (it is clear that B is closed, convex and circled). By construction of B we have the inequality $\|j(y)\|_B \leq \|y\|$ which implies that j factors through j_B . Also by construction, $f(U)$ is contained in B and hence f induces a morphism $f': X_U \rightarrow Y_B$. It follows that the outer edges of the diagram form a commutative square. The fact that j_B is a monomorphism and that the image of p_U is dense in X_U then imply that the middle square of the diagram above is commutative. \square

4.6 The Riemannian bordism category

We recall from section 2 that the objects of the d -dimensional Riemannian bordism category $d\text{-RBord}$ are closed $(d-1)$ -dimensional Riemannian manifolds. Given $X, Y \in d\text{-RBord}$, the set $d\text{-RBord}(X, Y)$ of morphisms from X to Y is the disjoint union of the set of isometries from X to Y and the set of Riemannian bordisms from X to Y (modulo isometry relative boundary), except that for $X = Y = \emptyset$, the identity isometry equals the empty bordism.

Theorem 4.29. 1. A morphism in $d\text{-RBord}$ is thick if and only if it is a bordism.

2. For any $X, Y \in d\text{-RBord}$ the map

$$\Psi: \widehat{d\text{-RBord}}(X, Y) \longrightarrow d\text{-RBord}(X, Y)$$

is injective. In particular, every object $X \in d\text{-RBord}$ has the trace property and every bordism Σ from X to X has a well-defined trace $\text{tr}(\Sigma) \in d\text{-RBord}(\emptyset, \emptyset)$.

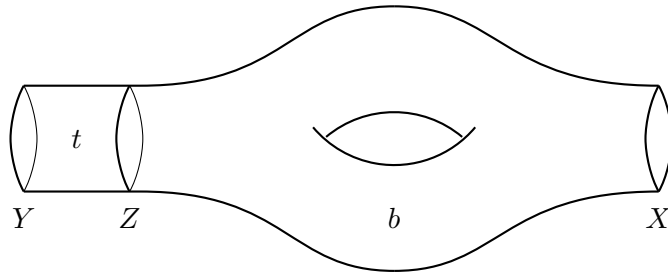
3. If Σ is a Riemannian bordism from X to X , then $\text{tr}(\Sigma) = \Sigma_{\text{gl}}$, the closed Riemannian manifold obtained by gluing the two copies of X in the boundary of Σ .

Proof. To prove part (1) suppose that $f: X \rightarrow Y$ is a thick morphism, i.e., it can be factored in the form

$$X \cong \emptyset \amalg X \xrightarrow{t \amalg \text{id}_X} Y \amalg Z \amalg X \xrightarrow{\text{id}_Y \amalg b} Y \amalg \emptyset \cong Y \quad (4.30)$$

We note that the morphisms $t: \emptyset \rightarrow Y \amalg Z$ and $b: Z \amalg X \rightarrow \emptyset$ must both be *bordisms* (the only case where say t could possibly be an isometry is $Y = Z = \emptyset$; however that isometry is the same morphism as the empty bordism). Hence the composition f is a bordism.

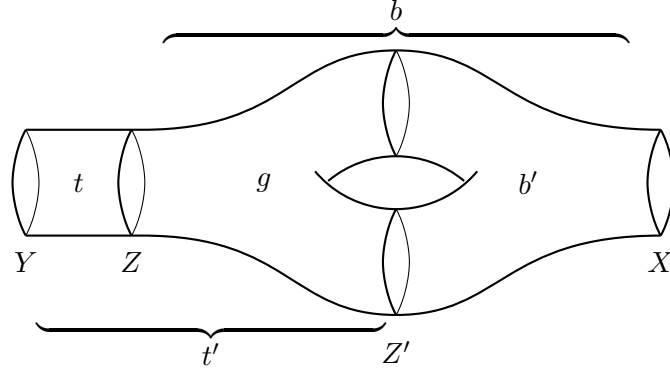
Conversely, assume that Σ is a bordism from X to Y . Then Σ can be decomposed as in the following picture:



Here $t = Y \times [0, \epsilon] \subset \Sigma$, $Z = Y \times \{\epsilon\}$ and $b = \Sigma \setminus (Y \times [0, \epsilon])$ are bordisms, where $\epsilon > 0$ is chosen suitably so that $Y \subset \Sigma$ has a neighborhood isometric to $Y \times [0, 2\epsilon]$ equipped with the product metric. Regarding t as a Riemannian bordism from \emptyset to $Y \amalg Z$, and similarly b as a Riemannian bordism from $Z \amalg X$ to \emptyset , it is clear from the construction that the composition (4.30) is Σ .

To show that Ψ is injective, let $[Z', t', b'] \in \widehat{d\text{-RBord}}(X, Y)$, and let $\Sigma = \Psi([Z', t', b']) \in d\text{-RBord}$ be the composition (1.2). In other words, we have a decomposition of the bordism Σ into two pieces t', b' which intersect along Z' . Now let (Z, t, b) be the triple constructed in the proof of part (1) above. By choosing ϵ small enough, we can assume

that $Z = Y \times \epsilon \subset \Sigma$ is in the interior of the bordism t' , and we obtain a decomposition of the bordism Σ as shown in the picture below.



Regarding g as a bordism from Z to Z' we see that

$$t' = (\text{id}_Y \amalg g) \circ t \quad \text{and} \quad b = b' \circ (g \amalg \text{id}_X),$$

which implies that the triple (Z, t, b) and (Z', t', b') are equivalent in the sense of Definition 3.3. If $[Z'', t'', b''] \in \widehat{d\text{-RBord}}$ is another element with $\Psi([Z'', t'', b'']) = \Sigma$, then by choosing ϵ small enough we conclude $[Z'', t'', b''] = [Z, t, b] = [Z', t', b']$.

For the proof of part (3), we decompose as in the proof of (1) the bordism Σ into two pieces t and b by cutting it along the one-codimensional submanifold $Z = X \times \{\epsilon\}$ (here $Y = X$ since Σ is an *endomorphism*). We regard t as a bordism from \emptyset to $X \amalg Z$ and b as a bordism from $Z \amalg X$ to \emptyset . Then $\text{tr}(\Sigma)$ is given by the composition

$$\emptyset \xrightarrow{t} X \amalg Z \xrightarrow{s_{X,Z}} Z \amalg X \xrightarrow{b} \emptyset.$$

which geometrically means to glue the two bordisms along X and Z . Gluing first along Z we obtain the Riemannian manifold Σ , then gluing along X we get the closed Riemannian manifold Σ_{gl} . \square

5 Properties of the trace pairing

The goal of this section is the proof of our main theorem 1.9 according to which our trace pairing is symmetric, additive and multiplicative. There are three subsections devoted to the proof of these three properties, plus a subsection on braided monoidal and balanced monoidal categories needed for the multiplicative property. Each proof will be based on first proving the following analogous properties for the trace $\widehat{\text{tr}}(\widehat{f})$ of thickened endomorphisms $\widehat{f} \in \widehat{\mathcal{C}}(X, X)$:

Theorem 5.1. *Let \mathcal{C} be a monoidal category, equipped with a natural family of isomorphisms $s = s_{X,Y}: X \otimes Y \rightarrow Y \otimes X$. Then the trace map*

$$\widehat{\text{tr}}: \widehat{\mathcal{C}}(X, X) \longrightarrow \mathcal{C}(\mathbb{1}, \mathbb{1})$$

has the following properties:

1. (*symmetry*) $\widehat{\text{tr}}(\widehat{f} \circ g) = \widehat{\text{tr}}(g \circ \widehat{f})$ for $\widehat{f} \in \widehat{\mathcal{C}}(X, Y)$, $g \in \mathcal{C}(Y, X)$;

1. There is a zero object $0 \in \mathbf{C}$ (an object which is terminal and initial);
2. finite products and coproducts exist;
3. Given a finite collection of objects, the canonical morphism from their coproduct to their product (given by identity maps on coordinates) is an isomorphism.
4. Any morphism from $X \rightarrow Y$ has an additive inverse under the canonical addition (defined below) on $\mathbf{C}(X, Y)$.

We remark that the requirement (1) is redundant since an initial object is a coproduct of the empty family of objects, and a terminal object is the product of the same family; the map from the initial to the terminal object is an isomorphism by (3), thus making these objects zero objects.

If \mathbf{C} is an additive category, we will use the notation $X_1 \oplus \cdots \oplus X_n$ for the coproduct of objects $X_i \in \mathbf{C}$ (the term *direct sum* is used for finite coproducts in additive categories). Any map

$$f: X = X_1 \oplus \cdots \oplus X_n \longrightarrow Y = Y_1 \oplus \cdots \oplus Y_m,$$

amounts to an $m \times n$ matrix of morphisms $f_{ij} \in \mathbf{C}(X_j, Y_i)$, $1 \leq i \leq m$, $1 \leq j \leq n$, by identifying Y via property (3) with the *product* of the Y_i 's. In particular, there are morphisms

$$\Delta := \begin{pmatrix} \text{id} \\ \text{id} \end{pmatrix} : X \longrightarrow X \oplus X \quad \nabla := (\text{id} \ \text{id}) : Y \oplus Y \longrightarrow Y$$

referred to as *diagonal map* and *fold map*, respectively. For $f, g \in \mathbf{C}(X, Y)$ their sum $f + g$ is defined to be the composition

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y$$

This addition gives $\mathbf{C}(X, Y)$ the structure of an abelian group. Abusing notation we write $0: X \rightarrow Y$ for the additive unit which is given by the unique morphism that factors through the zero object. Identifying morphisms between direct sums with matrices, their composition corresponds to multiplication of the corresponding matrices.

Definition 5.3. A monoidal structure \otimes on an additive category \mathbf{C} is *distributive* if for any objects $X, Y \in \mathbf{C}$ the functors

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ Z & \mapsto & Z \otimes X \end{array} \quad \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ Z & \mapsto & Y \otimes Z \end{array}$$

preserve coproducts. In particular, for objects $Z_1, Z_2 \in \mathbf{C}$ we have canonical distributivity isomorphisms

$$(Z_1 \otimes X) \oplus (Z_2 \otimes X) \cong (Z_1 \oplus Z_2) \otimes X \quad \text{and} \quad (Y \otimes Z_1) \oplus (Y \otimes Z_2) \cong Y \otimes (Z_1 \oplus Z_2).$$

In addition, these functors send the initial object $0 \in \mathbf{C}$ (thought of as the coproduct of the empty family of objects of \mathbf{C}) to an initial object of \mathbf{C} ; in others words, there are canonical isomorphisms $0 \otimes X \cong 0$ and $Y \otimes 0 \cong 0$.

Definition 5.4. Given triples (Z_1, t_1, b_1) and (Z_2, t_2, b_2) representing elements of $\widehat{\mathbf{C}}(X, Y)$ we define their *sum* by

$$(Z_1, t_1, b_1) + (Z_2, t_2, b_2) := (Z_1 \oplus Z_2, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, (b_1 \ b_2))$$

Here $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ is a morphism from $\mathbb{1}$ to $(Y \otimes Z_1) \oplus (Y \otimes Z_2) \cong Y \otimes (Z_1 \oplus Z_2)$ and $\begin{pmatrix} b_1 & b_2 \end{pmatrix}$ is a morphism from $(Z_1 \otimes X) \oplus (Z_2 \otimes X) \cong (Z_1 \oplus Z_2) \otimes X$ to $\mathbb{1}$. The addition of triples induces a well-defined addition

$$\widehat{\mathcal{C}}(X, Y) \times \widehat{\mathcal{C}}(X, Y) \xrightarrow{+} \widehat{\mathcal{C}}(X, Y)$$

since, if $g_i: Z_i \rightarrow Z'_i$ is an equivalence from (Z_i, t_i, b_i) to (Z'_i, t'_i, b'_i) , then $g_1 \oplus g_2: Z_1 \oplus Z_2 \rightarrow Z'_1 \oplus Z'_2$ is an equivalence from $(Z_1, t_1, b_1) + (Z_2, t_2, b_2)$ to $(Z'_1, t'_1, b'_1) + (Z'_2, t'_2, b'_2)$.

Lemma 5.5. *The above addition gives $\widehat{\mathcal{C}}(X, Y)$ the structure of an abelian group.*

Proof. We claim:

1. The additive unit is represented by any triple (Z, t, b) with $t = 0$ or $b = 0$;
2. the additive inverse of $[Z, t, b]$ is represented by $(Z, -t, b)$ or $(Z, t, -b)$ (here $-t, -b$ are the additive inverses to the morphisms t resp. b whose existence is guaranteed by axiom (4) in the definition of an additive category).

It follows from the description of addition that $[0, 0, 0]$ is an additive unit in $\widehat{\mathcal{C}}(X, Y)$. It is easy to check that $0: Z \rightarrow 0$ is an equivalence from $(Z, t, 0)$ to $(0, 0, 0)$ and that $0: 0 \rightarrow Z$ is an equivalence from $(0, 0, 0)$ to $(Z, 0, b)$, which proves the first claim.

The diagonal map $\Delta: Z \rightarrow Z \oplus Z$ is an equivalence from $(Z, t, 0)$ to $(Z, t, b) + (Z, t, -b) = (Z \oplus Z, \begin{pmatrix} t \\ t \end{pmatrix}, (b - b))$. Similarly, the fold map $\nabla: Z \oplus Z \rightarrow Z$ is an equivalence from $(Z, t, b) + (Z, -t, b) = (Z \oplus Z, \begin{pmatrix} t \\ -t \end{pmatrix}, (b b))$ to $(Z, 0, b)$. This proves the second claim. \square

Lemma 5.6. *The map $\Psi: \widehat{\mathcal{C}}(X, Y) \rightarrow \mathcal{C}(X, Y)$ is a homomorphism.*

Proof. Using the notation from the definition above, we want to show that $\Psi(Z, t, b) = \Psi(Z_1, t_1, b_1) + \Psi(Z_2, t_2, b_2)$. We recall (see Equation (1.2) of the introduction and the paragraph following it) that $\Psi(Z, t, b) \in \mathcal{C}(X, Y)$ is given by the composition

$$Y \cong Y \otimes \mathbb{1} \xleftarrow{\text{id}_Y \otimes b} Y \otimes Z \otimes X \xleftarrow{t \otimes \text{id}_X} \mathbb{1} \otimes X \cong X$$

Here we write the arrows from right to left in order that composition corresponds to matrix multiplication. Identifying $Y \otimes Z \otimes X$ with $(Y \otimes Z_1 \otimes X) \oplus (Y \otimes Z_2 \otimes X)$, these maps are given by the following matrices:

$$\text{id}_Y \otimes b = \begin{pmatrix} \text{id}_Y \otimes b_1 & \text{id}_Y \otimes b_2 \end{pmatrix} \quad t \otimes \text{id}_X = \begin{pmatrix} t_1 \otimes \text{id}_X \\ t_2 \otimes \text{id}_X \end{pmatrix}$$

Hence $\Psi(Z, t, b)$ is given by the matrix product

$$\begin{aligned} & \begin{pmatrix} \text{id}_Y \otimes b_1 & \text{id}_Y \otimes b_2 \end{pmatrix} \begin{pmatrix} t_1 \otimes \text{id}_X \\ t_2 \otimes \text{id}_X \end{pmatrix} \\ &= (\text{id}_Y \otimes b_1) \circ (t_1 \otimes \text{id}_X) + (\text{id}_Y \otimes b_2) \circ (t_2 \otimes \text{id}_X) \\ &= \Psi(Z_1, t_1, b_1) + \Psi(Z_2, t_2, b_2) \end{aligned}$$

\square

Proof of part (2) of Theorem 5.1. We recall from the definition of $\widehat{\text{tr}}$ (see Equation (3.11)) that $\widehat{\text{tr}}(Z, t, b) \in \mathbb{C}(\mathbb{1}, \mathbb{1})$ is given by the composition

$$\mathbb{1} \xleftarrow{b} Z \otimes X \xleftarrow{s_{X,Z}} X \otimes Z \xleftarrow{t} \mathbb{1}$$

Identifying $Z \otimes X$ with $(Z_1 \otimes X) \oplus (Z_2 \otimes X)$, and $X \otimes Z$ with $(X \otimes Z_1) \oplus (X \otimes Z_2)$, this composition is given by the following matrix product:

$$\begin{aligned} & (b_1 \quad b_2) \begin{pmatrix} s_{X,Z_1} & 0 \\ 0 & s_{X,Z_2} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \\ &= b_1 \circ s_{X,Z_1} \circ t_1 + b_2 \circ s_{X,Z_2} \circ t_2 \\ &= \widehat{\text{tr}}(Z_1, t_1, b_1) + \widehat{\text{tr}}(Z_2, t_2, b_2) \end{aligned}$$

□

To show that the additivity of $\widehat{\text{tr}}$ implies that the pairing $\text{tr}(f, g)$ is linear in f and g , we will need the following fact.

Lemma 5.7. *The composition map $\widehat{\mathbb{C}}(X, Y) \times \widehat{\mathbb{C}}(Y, X) \longrightarrow \widehat{\mathbb{C}}(Y, Y)$ is bilinear.*

The proof of this lemma is analogous to the previous two proofs, and so we leave it to the reader.

Proof of part (2) of Theorem 1.9. Let $f_1, f_2 \in \mathbb{C}(X, Y)$, $\widehat{f}_1, \widehat{f}_2 \in \widehat{\mathbb{C}}(X, Y)$ with $\Psi(\widehat{f}_i) = f_i$, and $g \in \mathbb{C}^{tk}(Y, X)$. Then by Lemma 5.6 $\Psi(\widehat{f}_1 + \widehat{f}_2) = f_1 + f_2$, and hence

$$\begin{aligned} \text{tr}(f_1 + f_2, g) &= \widehat{\text{tr}}((\widehat{f}_1 + \widehat{f}_2) \circ g) = \text{tr}(\widehat{f}_1 \circ g + \widehat{f}_2 \circ g) \\ &= \widehat{\text{tr}}(\widehat{f}_1 \circ g) + \widehat{\text{tr}}(\widehat{f}_2 \circ g) = \text{tr}(f_1, g) + \text{tr}(f_2, g) \end{aligned}$$

□

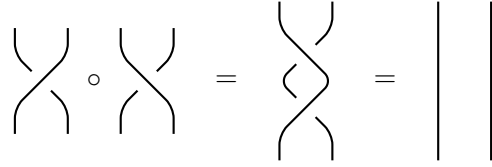
5.3 Braided and balanced monoidal categories

We recall from [JS2, Definition 2.1] that a braided monoidal category is a monoidal category equipped with a natural family of isomorphisms $c = c_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ such that

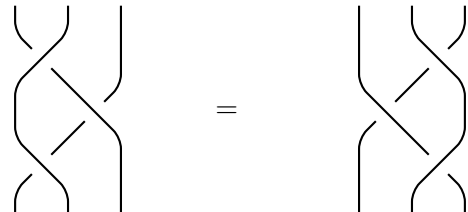
$$\begin{array}{ccc} \begin{array}{c} X \quad Y \quad Z \\ | \quad | \quad | \\ \boxed{c_{X,Y \otimes Z}} \\ | \quad | \quad | \\ Y \quad Z \quad X \end{array} & = & \begin{array}{c} X \quad Y \quad Z \\ | \quad | \quad | \\ \boxed{c_{X,Y}} \\ | \quad | \quad | \\ Y \quad Z \quad X \\ \boxed{c_{X,Z}} \end{array} \end{array} \quad \text{and} \quad \begin{array}{ccc} \begin{array}{c} X \quad Y \quad Z \\ | \quad | \quad | \\ \boxed{c_{X \otimes Y, Z}} \\ | \quad | \quad | \\ Z \quad X \quad Y \end{array} & = & \begin{array}{c} X \quad Y \quad Z \\ | \quad | \quad | \\ \boxed{c_{Y,Z}} \\ | \quad | \quad | \\ Z \quad X \quad Y \\ \boxed{c_{X,Z}} \end{array} \end{array} \quad (5.8)$$

There is a close relationship between braided monoidal categories and the braid groups B_n . We recall that the elements of B_n are *braids with n strands*, consisting of isotopy classes of n non-intersecting piecewise smooth curves $\gamma_i: [0, 1] \rightarrow \mathbb{R}^3$, $i = 1, \dots, n$ with endpoints at $\{1, \dots, n\} \times \{0\} \times \{0, 1\}$. One requires that the z -coordinate of each curve

is strictly decreasing (so that strands are “going down”). Composition of braids is defined by concatenation of their strands; e.g., in B_2 we have the composition

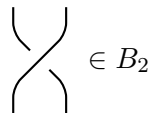


The last equality follows from the obvious isotopy in \mathbb{R}^3 , also known as the second *Reidemeister move*. The three Reidemeister moves are used to understand isotopies of arcs in \mathbb{R}^3 via their projections to the plane \mathbb{R}^2 . Generically, the isotopy projects to an isotopy in the plane but there are three codimension one singularities where this does not happen: a cusp, a tangency and a triple point (of immersed arcs in the plane). The corresponding ‘moves’ on the planar projection are the Reidemeister moves; for example, in the above figure one can see a tangency in the middle of moving the braid in the center to the (trivial) braid on the right. A cusp singularity cannot arise for braids but a triple point can: the resulting (third Reidemeister move) isotopies are also known as *braid relations*. A typical example would be the following isotopy (with a triple point in the middle):


(5.9)

Thinking of the groups B_n as categories with one object, we can form their disjoint union to obtain the *braid category* $\mathbf{B} := \coprod_{n=0}^{\infty} B_n$. The category \mathbf{B} can be equipped with the structure of a braided monoidal category [JS2, Example 2.1]; in fact, \mathbf{B} is equivalent to the *free braided monoidal category* generated by one object (this is a special case of Theorem 2.5 of [JS2]).

More generally, if \mathbf{C}_k is the free braided monoidal category generated by k objects, there is a braided monoidal functor $F: \mathbf{C}_k \rightarrow \mathbf{B}$ which sends n -fold tensor products of the generating objects to the object $n \in \mathbf{B}$ (whose automorphism group is B_n); on morphisms, F sends the structure maps $\alpha_{X,Y,Z}$, ℓ_X and r_X (see beginning of Section 3) to the identity, and the braiding isomorphisms c_{X_i, X_j} for generating objects X_i, X_j to the braid



This suggests to represent $c_{X,Y}$ by an overcrossing and $c_{Y,X}^{-1}$ by an undercrossing in the pictorial representations of morphisms in braided monoidal categories, a convention

broadly used in the literature [JS1] that we will adopt. In other words, one defines:

$$\begin{array}{c} X & Y \\ | & | \\ \diagdown & / \\ | & | \\ Y & X \end{array} := \boxed{c_{X,Y}} \quad \text{and} \quad \begin{array}{c} X & Y \\ | & | \\ / & \diagdown \\ | & | \\ Y & X \end{array} := \boxed{c_{Y,X}^{-1}}$$

The key result, [JS2, Cor. 2.6], is that for any objects $X, Y \in \mathbf{C}_k$ the map

$$F: \mathbf{C}_k(X, Y) \rightarrow \mathbf{B}(F(X), F(Y))$$

is a bijection. One step in the argument is to realize that the Yang-Baxter relations 5.9 follow from the relations 5.8 together with the naturality of the braiding isomorphism c . An immediate consequence is the following statement which we will refer to below.

Proposition 5.10. (Joyal-Street) *Let $f, g: X_1 \otimes \cdots \otimes X_k \rightarrow X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)}$ be morphisms of a braided monoidal category \mathbf{C} which are in the image of the tautological functor $T: \mathbf{C}_k \rightarrow \mathbf{C}$ which sends the i -th generating object of the free braided monoidal category \mathbf{C}_k to X_i . If $f = T(\tilde{f})$, $g = T(\tilde{g})$ and the associated braids $F(\tilde{f}), F(\tilde{g}) \in B_k$ agree, then $f = g$.*

We note that a morphism f is in the image of T if and only if it can be written as a composition of the isomorphisms α, ℓ, r, c and their inverses. For such a composition it is easy to read off the braid $F(\tilde{f})$: in the pictorial representation of f we simply ignore all associators and units and replace each occurrence of the braiding isomorphism c (respectively its inverse) by an overcrossing (respectively and undercrossing); then $F(\tilde{f})$ is the resulting braid. For example, for

$$f = \begin{array}{c} X & Y & Z \\ | & | & | \\ \boxed{c_{X,Y}^{-1}} & & \\ | & | & | \\ \boxed{c_{X,Z}} & & \\ | & | & | \\ \boxed{c_{Y,Z}} & & \\ | & | & | \\ Z & Y & X \end{array} \quad F(\tilde{f}) = \begin{array}{c} \diagdown & / & | \\ | & | & | \\ / & \diagdown & | \\ | & | & | \end{array}$$

Another result that we will need for the proof of the multiplicativity property are the following isotopy relations. Roughly speaking, they say that if the unit $\mathbb{1} \in \mathbf{C}$ is involved, then the isotopy does not need to be ‘relative boundary’ as in the previous pictures.

Lemma 5.11. *Let V, W be objects of a braided monoidal category \mathbf{C} , and let $f: V \rightarrow \mathbb{1}$ and $g: \mathbb{1} \rightarrow V$ be morphisms in \mathbf{C} . Then there are the following relations:*

$$\begin{array}{c} V & W \\ | & | \\ \diagdown & / \\ | & | \\ W & \end{array} \quad = \quad \begin{array}{c} V & W \\ | & | \\ \boxed{f} & \\ | & | \\ W & \end{array} \quad = \quad \begin{array}{c} V & W \\ | & | \\ / & \diagdown \\ | & | \\ W & \end{array}$$

Proof.

Here the first and last equality come from interpreting compositions involving the box f with no output (= morphism with range $\mathbb{1}$). The second equality is the naturality of the braiding isomorphism, and the third equality is a compatibility between the unit constraints r_W, ℓ_W and the braiding isomorphism which is a consequence of the relations (5.8) [JS2, Prop. 2.1].

This proves the first equality; the proofs of the other equalities are analogous. \square

Definition 5.12. A *balanced monoidal category* is a braided monoidal category \mathcal{C} together with a natural family of isomorphisms $\theta_X: X \xrightarrow{\cong} X$, called *twists*, parametrized by the objects $X \in \mathcal{C}$. We note that the naturality means that for any morphism $g: X \rightarrow Y$ we have $g \circ \theta_X = \theta_Y \circ g$. In addition it is required that $\theta_X = \text{id}_X$ for $X = \mathbb{1}$, and for any objects $X, Y \in \mathcal{C}$ the twist $\theta_{X \otimes Y}$ is determined by the following equation:

If $\theta_X = \text{id}_X$ for *all* objects $X \in \mathcal{C}$, this equality reduces to the requirement $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ for the braiding isomorphism. This is the additional requirement in a *symmetric monoidal category*, so symmetric monoidal categories can be thought of as balanced monoidal categories with $\theta_X = \text{id}_X$ for all objects X .

For a balanced monoidal category \mathcal{C} with braiding isomorphism c and twist θ , we define the switching isomorphism $s = s_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ by $s_{X,Y} := (\text{id}_Y \otimes \theta_X) \circ c_{X,Y}$; pictorially:

$$(5.14)$$

5.4 Multiplicativity of the trace pairing

In this subsection we will prove the multiplicativity of $\widehat{\text{tr}}$ and derive from it the multiplicativity of the trace pairing (part (3) of Theorem 1.9). Our first task is to define a tensor product of thickened morphisms.

Definition 5.15. Let $\widehat{f}_i = [Z_i, t_i, b_i] \in \widehat{\mathcal{C}}(X_i, Y_i)$ for $i = 1, 2$. If \mathcal{C} is a braided monoidal category, we define

$$\widehat{f}_1 \otimes \widehat{f}_2 := [Z, t, b] \in \widehat{\mathcal{C}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

where $Z = Z_1 \otimes Z_2$, and

$$t = \begin{array}{c} \boxed{t_1} \quad \boxed{t_2} \\ | \quad | \\ Y_1 \quad Y_2 \quad Z_1 \quad Z_2 \end{array} \quad b = \begin{array}{c} Z_1 \quad Z_2 \quad X_1 \quad X_2 \\ | \quad | \\ \boxed{b_1} \quad \boxed{b_2} \end{array}$$

We will leave it to the reader to check that this tensor product is well-defined. A crucial property of this tensor product for thickened morphisms is its compatibility with the tensor product for morphisms:

Lemma 5.16. *The map $\Psi: \widehat{\mathcal{C}}(X, Y) \rightarrow \mathcal{C}(X, Y)$ is multiplicative, i.e., $\Psi(\widehat{f}_1 \otimes \widehat{f}_2) = \Psi(\widehat{f}_1) \otimes \Psi(\widehat{f}_2)$.*

Proof.

$$\Psi(\widehat{f}_1 \otimes \widehat{f}_2) = \Psi([Z, t, b]) = \begin{array}{c} X \\ | \\ \boxed{t} \\ | \\ Y \end{array} \begin{array}{c} Z \\ | \\ \boxed{b} \\ | \\ X \end{array} = \begin{array}{c} X_1 \quad X_2 \\ | \quad | \\ \boxed{t_1} \quad \boxed{t_2} \\ | \quad | \\ Y_1 \quad Y_2 \end{array} \begin{array}{c} \boxed{b_1} \quad \boxed{b_2} \\ | \quad | \\ X_1 \quad X_2 \end{array}$$

$$= \Psi([Z_1, t_1, b_1]) \otimes \Psi([Z_2, t_2, b_2]) = \Psi(\widehat{f}_1) \otimes \Psi(\widehat{f}_2)$$

Here the fourth equality is a consequence of Lemma 5.11 (applied to $g = t_2: \mathbb{1} \rightarrow W = Y_2 \otimes Z_2, V = X_1$). The fifth equality follows from Proposition 5.10 and the obvious isotopy. The sixth equality is again a consequence of Lemma 5.11 (applied to $f = b_1: W = Z_1 \otimes X_1 \rightarrow \mathbb{1}, V = Y_2$). \square

Proof of part (3) of Theorem 5.1. Let $\widehat{f}_i = [Z_i, t_i, b_i] \in \widehat{\mathcal{C}}(X_i, X_i), i = 1, 2$. Then

$$\widehat{\text{tr}}(\widehat{f}_1 \otimes \widehat{f}_2) = \widehat{\text{tr}}(Z, t, b) =$$

$$(5.17)$$

where $Z := Z_1 \otimes Z_2, X := X_1 \otimes X_2$, and t, b are as in Definition 5.15. We note that $\theta_X = \theta_{X_1 \otimes X_2}$ can be expressed in terms of $\theta_1 := \theta_{X_1}$ and $\theta_2 := \theta_{X_2}$ as in Equation 5.13 and that the braiding isomorphism $c_{X,Z}$ can be expressed graphically as

$$c_{X,Z} = c_{X_1 \otimes X_2, Z_1 \otimes Z_2} =$$

It follows that

$$\widehat{\text{tr}}(\widehat{f}_1 \otimes \widehat{f}_2) =$$

$$= \widehat{\text{tr}}(\widehat{f}_1) \otimes \widehat{\text{tr}}(\widehat{f}_2)$$

Here the first equality is obtained from equation (5.17) by combining the pictures for the morphisms t , $c_{X,Z}$, θ_X and b that $\widehat{\text{tr}}(\widehat{f}_1 \otimes \widehat{f}_2)$ is a composition of. The second equality follows from the naturality of the braiding isomorphism (see picture (3.13)). The third equality is a consequence of Proposition 5.10 and an isotopy moving the strand with label Z_2 to the right. The fourth equality is a consequence of an isotopy moving the strand with label X_2 left and down. The fifth equality follows from Lemma 5.11 with $g = t_1$ and the relations (5.8). The sixth equality is a consequence of Proposition 5.10 and an isotopy which moves the strands labeled X_1 and Z_1 to the left, and the strand labeled X_2 to the right.

□

Proof of part (3) of Theorem 1.9. Let $f_i \in C^{tk}(X_i, Y_i)$, $g_i \in C^{tk}(Y_i, X_i)$ for $i = 1, 2$, and let $\widehat{f}_i \in \widehat{C}(X_i, Y_i)$ with $\Psi(\widehat{f}_i) = f_i$. Then according to Lemma 5.16, the map

$$\Psi: \widehat{C}(X_1 \otimes X_2, Y_1 \otimes Y_2) \longrightarrow C(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

sends $\widehat{f}_1 \otimes \widehat{f}_2$ to $f_1 \otimes f_2$ and hence

$$\begin{aligned} \operatorname{tr}(f_1 \otimes f_2, g_1 \otimes g_2) &= \widehat{\operatorname{tr}}\left(\left(\widehat{f}_1 \otimes \widehat{f}_2\right) \circ \left(g_1 \otimes g_2\right)\right) = \widehat{\operatorname{tr}}\left(\left(\widehat{f}_1 \circ g_1\right) \otimes \left(\widehat{f}_2 \circ g_2\right)\right) \\ &= \widehat{\operatorname{tr}}\left(\widehat{f}_1 \circ g_1\right) \widehat{\operatorname{tr}}\left(\widehat{f}_2 \circ g_2\right) = \operatorname{tr}\left(f_1, g_1\right) \operatorname{tr}\left(f_2, g_2\right) \end{aligned}$$

Here the third equality uses the multiplicative property of $\widehat{\operatorname{tr}}$, i.e., statement (3) of Theorem 5.1. \square

References

- [En] P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. 130 (1973), 309–317
- [JS1] A. Joyal and R. Street, *The geometry of tensor calculus I*, Adv. Math. 88 (1991), 55–112.
- [JS2] A. Joyal and R. Street, *Braided tensor categories*, Adv. Math. 102 (1993), no. 1, 20–78.
- [JSV] A. Joyal, R. Street and D. Verity, *Traced monoidal categories*, Math. Proc. Cambridge Philos. Soc. 119 (1996), no. 3, 447–468.
- [Ko] G. Köthe, *Topological vector spaces. II.*, Grundlehren der Mathematischen Wissenschaften, 237. Springer-Verlag, New York-Berlin, 1979. xii+331 pp.
- [Li] G. L. Litvinov, *Approximation properties of locally convex spaces and the problem of uniqueness of the trace of a linear operator*, Teor. Funktsii. Funktsional. Anal. i Prilozhen. No. 39 (1983), 73–87. English translation in: Selected translations. Selecta Math. Soviet. 11 (1992), no. 1, 25–40
- [McL] S. Mac Lane, *Categories for the working mathematician*, Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998. xii+314 pp.
- [Me] R. Meyer, *Local and analytic cyclic homology*, EMS Tracts in Mathematics, 3. European Mathematical Society (EMS), Zürich, 2007. viii+360 pp.
- [Sch] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*, Second edition. Graduate Texts in Mathematics, 3. Springer-Verlag, New York, 1999. xii+346 pp.
- [Se] G. Segal, *The definition of conformal field theory*. Topology, geometry and quantum field theory, 423 – 577, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.
- [ST1] S. Stolz and P. Teichner, *Super symmetric field theories and integral modular forms*, preprint
- [ST2] S. Stolz and P. Teichner, *Super symmetric field theories and generalized cohomology*, preprint
- [Sz] A. Szankowski, *$B(\mathcal{H})$ does not have the approximation property*, Acta Math. 147 (1981), no. 1-2, 89–108.