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Non-smoothable Four-manifolds with Infinite Cyclic Fundamental Group

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In [11], two of us constructed a closed oriented 4-dimensional manifold with fundamental group $\mathbb{Z}$ that does not split off $S^1 \times S^3$. In this note we show that this 4-manifold, and various others derived from it, do not admit smooth structures. Moreover, we find an infinite family of 4-manifolds with exactly the same properties. As a corollary, we obtain topologically slice knots that are not smoothly slice in any rational homology ball.

1 Introduction

The symmetric matrix

$$L := \begin{pmatrix} 1+t+t^2 & t+t^2 & 1+t & t \\ t+t^2 & 1+t+t^2 & t & 1+t \\ 1+t & t & 2 & 0 \\ t & 1+t & 0 & 2 \end{pmatrix}$$

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has determinant 1 and therefore describes a nonsingular form over $\mathbb{Z}[t]$, see [14, p.474]. Setting $t = x + x^{-1}$ and

$$\Lambda := \mathbb{Z}[x, x^{-1}] \ \ (= \text{the group ring } \mathbb{Z}[\mathbb{Z}]),$$

L extends to a form on Λ^4 which is hermitian with respect to the involution $\bar{x} = x^{-1}$. In [11], the second and fourth authors showed the following:

Theorem 1.1. L is not extended from the integers. □

Freedman and Quinn [8] proved that any nonsingular hermitian form on a finitely generated free Λ-module can be realized as the intersection form on π_2 of a closed oriented 4-manifold with fundamental group $\mathbb{Z}$. Moreover, in the odd case (as for the form L above) there are exactly two such 4-manifolds realizing a given form, one with nontrivial Kirby-Siebenmann invariant and hence not smoothable. It follows that there is a unique closed orientable 4-manifold M_L with $\pi_1 M_L = \mathbb{Z}$, intersection form L on $\pi_2 M_L$ and trivial Kirby-Siebenmann invariant (so M_L is smoothable after adding copies of $S^2 \times S^2$). See also Remark 4.1 for a more concrete construction.

Note that M_L was (before this paper was written) the only known closed orientable topological 4-manifold with fundamental group $\mathbb{Z}$ and trivial Kirby-Siebenmann invariant that is not the connected sum of $S^1 \times S^3$ with a simply connected 4-manifold. We shall give more examples in Section 5 below. Fintushel and Stern constructed a smooth example in [7] that is nonsplit in the smooth category but does split off $S^1 \times S^3$ topologically.

It remained an open problem during the last ten years as to whether M_L could be given a smooth structure. In this paper we will prove that M_L is *not* smoothable. Note that Donaldson's Theorem A, [4], does not apply directly, since the intersection form on $H_2 M_L$ is standard. However, we shall show that it does apply to most cyclic covers, and hence indirectly to M_L.

Theorem 1.2. Let M_n be the n-fold (cyclic) cover of M_L. For any $n \geq 1$, the intersection form L_n on $H_2 M_n$ is positive definite, odd and of rank $4n$. If $n = 1, 2$ this form represents the standard lattice in $\mathbb{R}^{4n}$ but for $n \geq 3$ it is not standard. In particular, none of the M_n admits a smooth structure. □

For $n \geq 3$, the last statement follows directly from [4] since smooth definite 4-manifolds can only have standard intersection forms. But if $M_1 (= M_L)$ or M_2 were smooth, then M_4 would be smooth as a common covering space.

Theorem 1.2 implies Theorem 1.1: if L was extended from the integers then it would be extended from its augmentation $\varepsilon(L) = L_1$, which is the standard form of rank 4. It is easy to see that this implies that all L_n would be standard as well.

Looking closely at the original proof of Theorem 1.1, namely that Lemma 4 in [11] holds for forms over the group ring $\Lambda_n := \mathbb{Z}[\mathbb{Z}/n]$ for all $n > 1$, one sees that the form L_n is not standard as a form with $\mathbb{Z}/n$ action for $n > 1$. This means that the $\mathbb{Z}/n$-equivariant intersection form on $H_2 M_n$ is not extended from the integers. Together with Theorem 1.2, this gives an interesting algebraic example for $n = 2$.

Theorem 1.2 suggests the following conjecture, which would reduce the realization problem for smooth 4-manifolds with fundamental group $\mathbb{Z}$ to the simply-connected case. As already conjectured in [11], the indefinite case should be purely algebraic and not require smoothness, whereas the definite case, just as for M_L, must use the smooth structure on the manifold.

Conjecture 1.3. If M is a closed smooth 4-manifold with fundamental group $\mathbb{Z}$, then its intersection form on $\pi_2 M$ is extended from the integers. □

The proof of Theorem 1.2 will be given in Section 2. In Section 3, we further analyze the structure of the forms L_n, obtaining some partial results on how they decompose. Section 4 contains an application of Theorem 1.2 to the study of topologically slice knots that are not smoothly slice:

Corollary 1.4. There exists a knot K in S^3 such that 0-surgery on K bounds a smooth 4-manifold W with fundamental group $\mathbb{Z}$ and such that the intersection form on $\pi_2 W$ is represented by L. Moreover, any knot with this property has trivial Alexander polynomial (and is thus topologically slice) but cannot bound a smooth disk in a rational homology ball. □

We will prove Corollary 1.4 in Section 4 and we will provide an explicit example of such a knot K

The remaining sections of the paper contain generalizations of our main results. In particular, in Section 5 we prove the following theorem. Our Corollary 1.4 continues to hold for all these hermitian forms $L(k)$.

Theorem 1.5. There are pairwise non-isomorphic unimodular, odd, hermitian forms $L(k)$, $k \in \mathbb{N}$, on Λ^4, with $L(1) = L$ and $\varepsilon(L(k))$ standard. None of these forms is extended from the integers, and none of the associated closed 4-manifolds (with infinite cyclic fundamental group and trivial Kirby–Siebenmann invariant) is smoothable. □

In the final Section 6 we show that the phenomenon described in Theorem 1.2 can be generalized to other fundamental groups. In particular, consider a finite aspherical 2-complex with fundamental group Γ, so this 2-complex represents a $K(\Gamma, 1)$. Given $1 \neq g \in \Gamma$, there is an embedding $i_g : \mathbb{Z} \hookrightarrow \Gamma$ induced by $1 \mapsto g$ because g has infinite order (otherwise a nontrivial finite cyclic group would have no higher homology). We will prove:

Theorem 1.6. There is a closed oriented 4-manifold M with $\pi_1(M) = \Gamma$ such that

(1) The intersection form λ on $\pi_2 M$ (modulo its radical) is given by extending the form L from $\mathbb{Z}[\mathbb{Z}]$ to $\mathbb{Z}[\Gamma]$ via i_g, but λ is not extended from the integers.

(2) If there exists an epimorphism $\varphi : \Gamma \to G$ to a finite group G such that $\varphi(g)$ has order ≥ 3 and $H_2(\operatorname{Ker} \varphi) = 0$, then M does not admit a smooth structure. In fact the cover M_φ of M corresponding to φ has a non-standard odd, positive definite intersection form. □

Taking $\Gamma = \mathbb{Z}$ we see that Theorem 1.2 can be logically seen as a special case of Theorem 1.6. Other classes of groups that satisfy all assumptions of Theorem 1.6 include knot groups, fundamental groups of non-orientable surfaces (except for the projective plane but including free groups) as well as almost all solvable Baumslag-Solitar groups, see Lemma 6.1. We leave it to the reader to formulate (and prove) the appropriate amalgamation of Theorems 1.5 and 1.6.

Remark 1.7. The assumption on the vanishing of $H_2(\operatorname{Ker} \varphi)$ in Theorem 1.6 is necessary for the last conclusion to hold. In fact, the intersection form on $H_2(M_\varphi)$ contains a metabolic form on $H_2(\operatorname{Ker} \varphi) \oplus H_2(\operatorname{Ker} \varphi)^*$ and hence can only be positive definite under our assumption.

2 Proof of Theorem 1.2

First observe that the signature σ and the Euler characteristic χ are multiplicative in finite covers and that they are both equal to 4 for $M_1 = M_L$. But since any finite cover satisfies $\pi_1 M_n = \mathbb{Z}$, it actually follows that

$$\sigma(M_n) = 4n = \chi(M_n) = \operatorname{rank} H_2 M_n$$

and hence the intersection form L_n on H_2M_n is positive definite. The intersection form

$$L_1 = \begin{pmatrix} 7 & 6 & 3 & 2 \\ 6 & 7 & 2 & 3 \\ 3 & 2 & 2 & 0 \\ 2 & 3 & 0 & 2 \end{pmatrix}$$

on H_2M_L (obtained from L via the augmentation $\varepsilon : \mathbb{Z}[x, x^{-1}] \to \mathbb{Z}$, $x \mapsto 1$) is odd and hence $w_2(M_L) \neq 0$ because $H^2(M_L; \mathbb{Z}/2) \cong \mathrm{Hom}(H_2M_L, \mathbb{Z}/2)$ by the universal coefficient theorem. Since $H^2(\mathbb{Z}; \mathbb{Z}/2) = 0$ it follows that the second Stiefel–Whitney class is nontrivial on the universal covering of M_1. Therefore, it must also be nontrivial on M_n and hence L_n is odd.

As a consequence L_1 and L_2 are odd definite unimodular forms of ranks 4 and 8, respectively, and hence are standard. It remains to show that L_n is nonstandard for $n \geq 3$. We shall use the following easy criterion. The converse of this criterion was proven to hold by Elkies [5]. His criterion was used to give a geometrically easier argument for Donaldson's Theorem A via the Seiberg–Witten equations.

Lemma 2.1. Let $w \in V$ be a *characteristic vector* for a unimodular form on V, i.e.

$$(w, v) \equiv (v, v) \quad \mod 2 \quad \forall\, v \in V.$$

If the form is standard, then $|w|^2 := (w, w) \geq \operatorname{rank} V$. □

Proof. Let e_i be an orthonormal basis of V, i.e. $(e_i, e_j) = \delta_{ij}$. Then $w = \sum e_i$ is a characteristic vector with $|w|^2 = \operatorname{rank} V$ and it suffices to show that any other characteristic vector has larger norm. But this follows from the inequality

$$|w + 2v|^2 = |w|^2 + 4(v, w) + 4|v|^2 \geq |w|^2$$

which is a consequence of $(v, w) = \sum v_i$ and $|v|^2 = \sum v_i^2$ when $v = \sum v_i \cdot e_i$. ■

It therefore suffices to show that there is a characteristic vector w for L_n whose L_n-norm satisfies

$$|w|^2 < 4n = \operatorname{rank} L_n.$$

We first explain an easy way to think about the forms L_n. As in Section 1, set

$$\Lambda := \mathbb{Z}[x, x^{-1}] = \mathbb{Z}[\mathbb{Z}] \qquad \text{and} \qquad \Lambda_n := \Lambda/(x^n - 1) = \mathbb{Z}[\mathbb{Z}/n].$$

The natural projection $\Lambda \to \Lambda_n$ converts our matrix L into a matrix $L_{/n}$ over Λ_n of rank 4.

Lemma 2.2. The matrix $L_{/n}$ represents the $\mathbb{Z}/n$-equivariant intersection form on H_2M_n with values in Λ_n. The ordinary $\mathbb{Z}$-valued intersection form L_n on H_2M_n is given by the coefficient of the identity (in the ring Λ_n) of this form. □

Proof. Since the higher cohomology groups of $\mathbb{Z}$ vanish, the Künneth spectral sequence reduces to the (second) isomorphism

$$H_2(M_n) \cong H_2(M_1;\Lambda_n) \cong H_2(M_1;\Lambda) \otimes_\Lambda \Lambda_n \cong \pi_2 M_1 \otimes_\Lambda \Lambda_n \cong \Lambda_n^4.$$

The second statement of the Lemma follows from the well known expression of $\mathbb{Z}[G]$-valued intersections on a G-cover X

$$\langle a,b\rangle = \sum_{g\in G} (\bar{g}a, b)\cdot g \quad \forall\, a,b \in H_2X$$

in terms of ordinary intersection numbers $(\bar{g}a, b)$ of translates $\bar{g}a$ with b. ■

To fix notation, let V_n be the free Λ_n-module of rank 4 (with underlying free abelian group of rank $4n$). Then $L_{/n}$ defines a pairing $\langle\,,\rangle$ on V_n with values in Λ_n and L_n is the coefficient of the identity element in $\mathbb{Z}/n$ of this pairing. We write

$$(v,v') := L_n(v,v') = \langle v,v'\rangle_1 \quad \text{and hence} \quad |v|^2 := (v,v) = \langle v,v\rangle_1.$$

Let $\{e_1,e_2,e_3,e_4\}$ be the Λ_n-basis for V_n in which we have written the matrix L above. Then one gets a $\mathbb{Z}$-basis E for V_n by multiplying e_i with x^j, where x is a generator of $\mathbb{Z}/n$ and $j = 1,\dots,n$. One can compute the norms of these basis vectors (noting that $|x^je_i|^2 = \langle x^je_i, x^je_i\rangle_1 = x^j\cdot x^{-j}\langle e_i,e_i\rangle_1 = \langle e_i,e_i\rangle_1 = |e_i|^2$) by looking at the matrix L above. In particular

$$|x^je_i|^2 = |e_i|^2 = \begin{cases} 3,\ 5 \text{ or } 7 & \text{for } i = 1,2 \\ 2 & \text{for } i = 3,4. \end{cases}$$

The three possibilities in the first case (in which we are computing the coefficient of the identity element in $\langle e_1,e_1\rangle = \langle e_2,e_2\rangle = 1+t+t^2 = 3+x+x^{-1}+x^2+x^{-2} \in \Lambda_n$) correspond to $n \geq 3$, $n = 2$ (when $x^2 = 1$) and $n = 1$ (when $x = 1$). In any case $|e_i|^2$ is always odd for $i = 1,2$ and even for $i = 3,4$. This will be used in the lemma below.

Let

$$N := 1 + x + x^2 + \cdots + x^{n-1}$$

be the norm element in Λ_n. It satisfies the identity $N \cdot r = N \cdot \varepsilon(r)$ for all $r \in \Lambda_n$, where

$$\varepsilon : \Lambda_n \to \mathbb{Z}\,, \quad x \mapsto 1$$

is the augmentation map. It follows that for any $v, e \in V_n$ we have $\langle N \cdot v, e\rangle = N\langle v, e\rangle = N\varepsilon\langle v, e\rangle$ in Λ_n, and therefore

$$(N \cdot v, e) = \langle N \cdot v, e\rangle_1 = (N\varepsilon\langle v, e\rangle)_1 = \varepsilon\langle v, e\rangle \quad \in \mathbb{Z}.$$

Lemma 2.3. The vector

$$w := N \cdot (e_3 + e_4) \quad \in V_n$$

is characteristic for L_n and satisfies $|w|^2 = 4n$. □

Proof. We need to check that $(w, e) \equiv (e, e) \pmod 2$ for all basis vectors $e = x^j e_i$ in the basis E of V_n. We already calculated the right hand side, and for the left hand side we have

$$(w, x^j e_i) = \varepsilon\langle e_3 + e_4, e_i\rangle = \begin{cases} 5 & \text{for } i = 1, 2 \\ 2 & \text{for } i = 3, 4 \end{cases}$$

by referring to the intersection matrix $L_1 = \varepsilon(L)$ displayed above. These numbers are indeed odd for $i = 1, 2$ and even for $i = 3, 4$, as required. Finally, we calculate the norm

$$|w|^2 = (w, w) = \varepsilon(N\langle e_3 + e_4, e_3 + e_4\rangle) = \varepsilon(N)\varepsilon(2 + 2) = 4n. \qquad \blacksquare$$

Now consider the characteristic vector $w_1 := w - 2e_1$ and calculate

$$|w_1|^2 = |w|^2 - 4\left((w, e_1) - |e_1|^2\right) = 4n - 4\left(5 - |e_1|^2\right).$$

For $n = 2$ we have $|e_1|^2 = 5$, and so $|w_1|^2 = 4n = 8$, consistent with L_2 being standard. But for $n \geq 3$ we have $|e_1|^2 = 3$, and hence $|w_1|^2 = 4n - 8$, which means that L_n cannot be standard.

3 Intersection forms

In the previous section we showed that the intersection forms L_n on $V_n = H_2(M_n; \mathbb{Z})$ ($\cong \mathbb{Z}^{4n}$) are not standard for any $n \geq 3$, i.e. not equivalent to the diagonal form I_{4n}.

This section initiates a further investigation of the structure of these forms, and thus implicitly the topological structure of the 4-manifolds M_n.

In what follows, a free abelian group of finite rank with a positive definite unimodular form will be referred to simply as a *lattice*. By a classical theorem of Eichler (see [13, 6.4]), any lattice splits uniquely as an orthogonal sum of indecomposable lattices. There is a unique indecomposable lattice in each rank 1, 8, 12, 14 and 15, but after that the number grows dramatically with the rank.

Of special interest here are the lattices Γ_{4m} (of rank $4m$) which are indecomposable for $m \geq 3$. In particular Γ_8 (also known as E_8) and Γ_{12} (also known as D_{12}^+ as in Conway and Sloan's treatise [3]) are the smallest non-trivial indecomposable lattices. Explicitly, Γ_{4m} consists of all vectors in $\mathbb{R}^{4m}$ whose coordinates are all integers or all half-integers with an even integer sum. It is spanned by the vectors $v_i + v_j$ and $\frac{1}{2}(v_1 + \cdots + v_{4m})$ (for an orthonormal basis $v_1, \ldots, v_{4m}$).

Before stating any results, we recall some standard notions (and introduce some new ones) concerning a lattice V with inner product $(\ ,\ )$. The *norm* of $v \in V$ is the inner product (v, v), also denoted $|v|^2$. A vector $w \in V$ is *characteristic* if $(w, v) \equiv |v|^2 \mod 2$ for all $v \in V$. Among the characteristic vectors, those with the smallest norm will be called *minimal*. The total number $\mu(V)$ of minimal characteristic vectors in V is a useful invariant of V. It clearly multiplies under orthogonal sums, since a vector $(u, v) \in U \oplus V$ is (minimal) characteristic if and only if $u \in U$ and $v \in V$ are (minimal) characteristic. For example, we have $\mu(I_k) = 2^k$ (where I_k denotes the standard lattice) and $\mu(\Gamma_8) = 1$ (since Γ_8 is even).

Define the *defect* of the lattice to be

$$d(V) = \tfrac{1}{8}(\mathrm{rank}(V) - |w|^2)$$

for any minimal characteristic vector w in V. It is an integer, by a classical lemma of van der Blij (see [13, 5.2]), and clearly adds under orthogonal sums. For example $d(\Gamma_{4m}) = \lfloor m/2 \rfloor$. Indeed for odd m the minimal characteristic vectors are exactly the vectors $\pm 2v_i$ (and so $\mu(\Gamma_{4m}) = 8m$) while 0 is the unique minimal characteristic vector when m is even.

Elkies shows in [5] that the defect $d(V)$ is always nonnegative, and equal to 0 if and only if V is standard. It is therefore a measure of non-diagonalizability. In [6], he goes on to prove that there are only finitely many indecomposable lattices of defect one (including Γ_8 and Γ_{12}) and raises the question of whether this is also the case for larger defects. (This has been confirmed for defects 2 and 3 by Gaulter [10].)

Theorem 3.1. The defect of the lattice V_n $(= H_2(M_n;\mathbb{Z})$ with the form $L_n)$ satisfies the inequality $\lfloor n/3 \rfloor \leq d(V_n) < n/2$. □

Proof. The result is obvious for $n = 1$ or 2, so assume $n \geq 3$. We use the notation from the previous section, in particular, $e_1, \ldots, e_4$ is a basis for which the intersection form L is given by the matrix on page 1 and $w = N \cdot (e_3 + e_4)$. We consider the characteristic vector

$$w_0 := w - 2(1 + x^3 + \cdots + x^{3(\lfloor n/3 \rfloor - 1)})e_1$$

in V_n. This vector is of the general form

$$w_a := w - 2a(x)e_1$$

where $a(x) = a_0 + a_1 x + \cdots + a_k x^k$ is an integer polynomial. The norm of w_a can be written in terms of the quadratic expressions $a^i := a_0 a_i + a_1 a_{i+1} + \cdots + a_k a_{i-1}$ (i.e. the dot product of the coefficient vector $(a_0, a_1, \ldots, a_k)$ with its ith shift $(a_i, a_{i+1}, \ldots, a_{i-1})$) as follows:

$$\begin{aligned} |w_a|^2 &= 4n - 4\left(\varepsilon\langle e_3 + e_4, a(x)e_1\rangle - \left(a(x)a(x^{-1})\langle e_1, e_1\rangle\right)_1\right) \\ &= 4n - 4\left(5a(1) - 3a^0 - 2(a^1 + a^2)\right). \end{aligned}$$

For w_0, the coefficient vector consists of a sequence of $\lfloor n/3 \rfloor$ ones separated by and terminating with at least two zeros, and so $a(1) = a^0 = \lfloor n/3 \rfloor$ and $a^1 = a^2 = 0$. Thus $|w_0|^2 = 4n - 8\lfloor n/3 \rfloor$, and so by definition $d(V_n) \geq \lfloor n/3 \rfloor$.

To obtain the upper bound, note that $d(V_n) \leq \frac{1}{8}\,\mathrm{rank}(V) = n/2$, and the inequality is strict because the form on V_n is odd. ■

Corollary 3.2. If V_n splits off an I_k summand for some k, then $k \leq 4n - 8\lfloor n/3 \rfloor$. □

This is immediate from the theorem and the additivity of the defect. Observe that the bound $4n - 8\lfloor n/3 \rfloor$ is of the order of $4n/3 = \mathrm{rank}(V_n)/3$ for large n, and so this result roughly states that at least two-thirds of the form V_n cannot be diagonalized.

Unfortunately this gives no new information for $n < 6$ beyond that contained in Theorem 1.2. Special arguments can be used, however, to further restrict the forms for small n. Indeed for $n = 3$ or 4 they can be identified precisely, using Conway and Sloan's root-system labeling (also see Conway's lovely little book [2, pp. 53–58]).

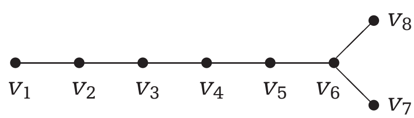
Theorem 3.3. V_3 and V_4 are the unique odd indecomposable lattices in dimensions 12 and 16. In particular, we have $V_3 \cong \Gamma_{12}$ and $V_4 \cong D_8^2[(12)]$. □

Proof. The lattices of rank 12 are Γ_{12}, $\Gamma_8 \oplus I_4$ and I_{12}, and so by Theorem 1.2 it suffices to eliminate $\Gamma_8 \oplus I_4$. But

$$\mu(\Gamma_8 \oplus I_4) = 1 \cdot 2^4 = 16 \quad \text{while} \quad \mu(\Gamma_{12}) = 24$$

and so one need only produce 17 or more minimal characteristic vectors in V_3. In fact the complete list is $\pm(w - 2x^i e)$ for $i = 0, 1, 2$ and $e = e_1$, e_2, $e_1 + e_4$ or $e_2 + e_3$, as is readily verified. (An explicit isomorphism $V_3 \cong \Gamma_{12}$ is easily deduced from this.)

For V_4 we take a different tact, focusing on vectors of norm 2. A theorem of Witt asserts that the norm 2 vectors in any lattice span a sub-lattice isomorphic to a direct sum of root lattices A_n, D_n and E_n. From the classification in [3, §16.4] of lattices of rank 16, it suffices to find a copy of $D_8 \oplus D_8$ in V_4, where D_8 is the lattice whose intersection matrix with respect to a suitable basis $v_1 \ldots, v_8$ of norm 2 vectors is given by the corresponding Dynkin diagram:



Here two nodes are joined by an edge or not according to whether their associated vectors have inner product 1 or 0. One easily checks that in V_4 the vectors

$$v_1 = x^2 e_4, \quad v_2 = x^2(-e_1 + e_2 + e_3), \quad v_3 = x^2 e_3, \quad v_4 = x^2 e_1 - e_2, \quad v_5 = e_1 - e_2$$

$$v_6 = e_3, \quad v_7 = -e_1 + e_2 + e_3 - e_4, \quad v_8 = w - (e_1 + e_2 + x^2(e_3 + e_4))$$

span a copy of D_8, and the vectors $w_i = xv_i$ (for $i = 1, \ldots, 8$) span an orthogonal copy of D_8. ■

4 Alexander polynomial one knots that are not slice

In this section we shall prove Corollary 1.4, starting with the second part. Let K be a knot in S^3 such that the 3-manifold N obtained by 0-surgery on K bounds a smooth 4-manifold W with fundamental group $\mathbb{Z}$ and such that the intersection form on $\pi_2 W$ is represented by L.

Since the intersection form on $H_2(W; \Lambda) \cong \pi_2 W$ is non-singular it follows immediately that $\Delta_K(t) = 1$, hence K is topologically slice. Moreover, $H_2(W) \cong \pi_2 W \otimes_\Lambda \mathbb{Z}$ is torsionfree and therefore the same is true for $H_1(W,N) \cong H^3(W)$. This, together with the non-singularity of $L \otimes_\Lambda \mathbb{Z}$ implies that the induced map $H_1(N) \to H_1(W)$ is an isomorphism.

Now assume that K bounds a smooth disk D in a rational homology ball B. Then let $C = B \setminus \nu D$ and $X = S^3 \setminus \nu K$. Note that X and C have the rational homology of a circle. Consider the *smooth* 4-manifold

$$M := W \cup_\partial C.$$

We have inclusion-induced isomorphisms

$$\mathbb{Z} \cong H_1(X) \xrightarrow{\cong} H_1(N) \xrightarrow{\cong} H_1(W) \xrightarrow{\cong} H_1(M)/\{\text{torsion}\} \xleftarrow{\cong} H_1(C)/\{\text{torsion}\}.$$

Given $n \in \mathbb{N}$ we denote the covers corresponding to the homomorphism $\mathbb{Z} \to \mathbb{Z}/n$ by the subscript n. Now let n be a prime power larger than 2 such that $H_*(C)$ has no n-torsion. It follows immediately from [9, Proof of Lemma 2.3] (cf. also [1, p. 184]) that X_n and C_n are still rational homology circles and that the projection maps $X_n \to X$ and $C_n \to C$ give isomorphisms of rational homology.

It is well-known that $H_1(N_n) \cong H_1(X_n)$. By Poincaré duality it now follows that N_n is a rational homology $S^1 \times S^2$. Using $H_1(N) \xrightarrow{\cong} H_1(W)$ it follows from the above discussion and from the Meyer–Vietoris sequence corresponding to $M_n = W_n \cup_{N_n} C_n$ that the inclusion maps induce an isomorphism

$$H_2(W_n)/\{\text{torsion}\} \xrightarrow{\cong} H_2(M_n)/\{\text{torsion}\}.$$

Since $\pi_1(W) = \mathbb{Z}$ we can apply Lemma 2.2 to conclude that the intersection form on W_n (and hence on the smooth 4-manifold M_n) is given by L_n. But since L_n is positive and non-standard by Theorem 1.2 we can again use Donaldson's theorem A to see that the M_n cannot exist smoothly and hence K cannot be smoothly slice to begin with. This concludes the proof of the second part of Corollary 1.4.

To prove the first part of Corollary 1.4, it suffices to construct a smooth 4-manifold W with the required properties and check that ∂W is 0-surgery on a knot. Start with one 0- and one 1-handle (which we draw as a dotted unknot U in S^3). Then W_J is obtained by attaching four 2-handles along a framed link J in $S^1 \times D^2$, the complement of U. The link J is chosen so that

1. J is the unlink if one ignores U,
2. J has trivial linking numbers with U and
3. J represents the matrix L'.

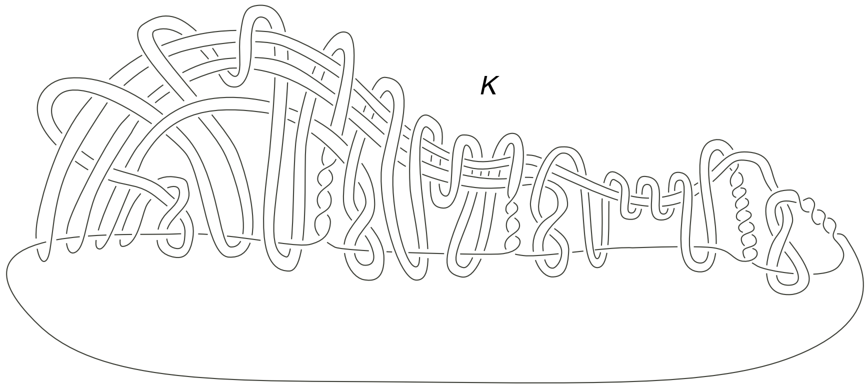


Figure 1 A knot as in Corollary 1.4.

Here L' is obtained from L by row and column operations that have the effect of changing $\varepsilon(L)$ to the identity matrix $\varepsilon(L')$. Note that property (2) implies that W_J is homotopy-equivalent to $S^1 \vee 4\, S^2$ and a $\mathbb{Z}[\mathbb{Z}]$-basis of $\pi_2 W_J$ is given by the cores of the four 2-handles together with null-homotopies of J in the complement of U. These four null-homotopies read off an intersection matrix depending on the crossing changes necessary. Property (3) requires that this matrix is given by L', in particular J is 1-framed (when ignoring U).

It is easy to see that any matrix can be realized by a link J with properties (1) and (2). The last step is to prove that ∂W_J is 0-surgery on a knot: This uses property (1) and the fact that J is 1-framed. As a consequence one can blow down J completely without changing the boundary! If one draws a picture with four disjointly embedded disks bounding J and punctured by U, then the blow down procedure puts a full twist into all strands that go through each of these four disks. Therefore, U turns into a knot K (and the framing has turned from a dot to a zero when studying the boundary alone). This is our non-smoothly slice knot with trivial Alexander polynomial; in fact, it is a whole family of such knots.

One example of such a knot K is shown in Figure 1. It is constructed from the unknot by four finger moves. The self linking and twisting of the fingers produce the diagonal entries of L', while the linking between fingers produce the off-diagonal entries. To see this, blow up four $+1$ curves (the link J in the discussion above) to unhook each of the finger tips. This transforms K into an unknot U. Now pull J back along the fingers so that U appears as the round unknot while the components of J follow the original fingers, clasping at both ends.

To calculate the associated intersection form, first introduce the notation

$$q_n = 2x^n - (x^{n+1} + x^{n-1}) \quad \text{and} \quad c_n = nq_0 - (q_1 + q_{-1})$$

so the matrix L' can be written as the sum of the identity matrix with a matrix in 2×2 block form in which each block has all entries equal, namely to c_1 and c_3 for the two diagonal blocks, and $-c_2$ for both off diagonal blocks.

Now orient U and all the components of J counterclockwise and label each arc of J with a power of x in a natural way: Start with $1'$s right before the clasps, and then proceed along J (following the orientation) multiplying by x (or x^{-1}) each time J links U positively (or negatively). Locally, each finger then has two oppositely oriented strands that are labeled by adjacent powers of x, say x^m and x^{m-1}, which we abbreviate by simply labeling the finger at that point with the higher exponent m. Also any crossing between fingers can be given a sign, namely, the sign of the crossing between the strands with the higher exponents of x. With these conventions, a positive finger crossing in which the i-th finger with label m passes over the jth with labels n will contribute q_{m-n} to the ijth entry of L', and an analogous negative crossing contributes $-q_{m-n}$. Self crossings and twisting of the ith finger contribute in a similar way to the diagonal entries. The calculation is then straightforward.

Remark 4.1. Note that any knot K whose 0-surgery N bounds a 4-manifold W as in Corollary 1.4 has trivial Alexander polynomial. Therefore, K is topologically $\mathbb{Z}$-slice and hence there is a topological 4-manifold C that is a homotopy circle and has boundary N. To construct our non-smoothable 4-manifold in the title slightly more concretely, we only have to prove that

$$M_L := W_J \cup_\partial C$$

has trivial Kirby-Siebenmann. This follows from the additivity of the (relative) Kirby-Siebenmann and the fact that K has trivial Arf invariant: ∂C also bounds a spin manifold W' with signature zero and

$$KS(W \cup_\partial C) = KS(C, \partial C) = KS(W' \cup_\partial C) = 0$$

because on spin manifolds the Kirby-Siebenmann invariant equals the signature divided by 8.

5 More forms

Recall that our basic form $L : \Lambda^4 \times \Lambda^4 \to \Lambda$ (where $\Lambda = \mathbb{Z}[x, x^{-1}]$) was obtained by substituting $x + x^{-1}$ for t in the matrix displayed at the beginning of this paper. In this section, we investigate the unimodular forms

$$L(a) := \begin{pmatrix} 1+a+a^2 & a+a^2 & 1+a & a \\ a+a^2 & 1+a+a^2 & a & 1+a \\ 1+a & a & 2 & 0 \\ a & 1+a & 0 & 2 \end{pmatrix}$$

obtained by substituting other elements $a \in \Lambda$ for t. (We shall always use the same notation for a matrix and its associated form.) Note that $L(a)$ is hermitian with respect to the involution $x \mapsto x^{-1}$ if and only if $a = \bar{a}$, where $\bar{a}$ denotes the image of a under this involution (called the "conjugate" of a). So we assume that $a = \bar{a}$ and can formulate the precise version of Theorem 1.5 in the introduction.

Theorem 5.1. Define inductively $b_1 = 1$ and $b_{k+1} = 4b_k + 1$, and set

$$L(k) := L(x^{b_k} + x^{-b_k}).$$

Then the forms $L(k)$ for $k = 1, 2, \ldots$ are pairwise non-isomorphic. None of these forms is extended from the integers, and none of their associated 4-manifolds is smoothable. $\square$

As with $L = L(1)$ the proof requires a study of some related forms. First some notation. For any $a = \sum a_i x^i \in \Lambda$, recall that $\varepsilon(a) = \sum a_i \in \mathbb{Z}$ and $(a)_1 = a_0$. Thus $\varepsilon : \Lambda \to \mathbb{Z}$ is the usual augmentation map, and $(a) \mapsto (a)_1$ defines a linear (nonmultiplicative) projection $\pi : \Lambda \to \mathbb{Z}$. We consider also the corresponding maps $\Lambda_n \to \mathbb{Z}$, denoted by the same names for any $n \in \mathbb{N}$, where $\Lambda_n = \Lambda/(x^n - 1)$.

Now for any (self-conjugate) $a \in \Lambda$ and any n, the matrix $L(a)$ can be viewed as a matrix over Λ_n in which case it is denoted by $L(a)_{/n}$, as is the associated form $\Lambda_n^4 \times \Lambda_n^4 \to \Lambda_n$. These forms in turn induce integral forms $L_n(a) : \Lambda_n^4 \times \Lambda_n^4 \to \mathbb{Z}$ of rank $4n$ (since $\Lambda_n^4 \cong \mathbb{Z}^{4n}$) by composition with π, that is, $L_n(a)(v, w) = (L(a)_{/n}(v, w))_1$.

Note that $L(a)_{/1} = L_1(a) = \varepsilon(L(a)) = \varepsilon(L(a)_{/n})$ for all n.

Lemma 5.2. For any $a \in \Lambda$ with $\bar{a} = a$, the forms $L_n(a)$ are positive definite, odd and of rank $4n$. In particular the forms $L_1(a)$ and $L_2(a)$ are standard. Furthermore, if $L(a)_{/n}$ is extended from the integers, then $L_n(a)$ is standard. $\square$

Proof. We first show that $L_1(a)$ is standard. The $(1,1)$-entry of its defining matrix is $1+\varepsilon(a)+\varepsilon(a)^2$ which is clearly odd for any a. By the classification of odd forms of rank 4, it follows that $L_1(a)$ is standard if and only if it is positive definite. But this can be checked over $\mathbb{Q}$. First observe (following Quebbemann's definition [14, p. 474]) that

$$L(a) = \begin{pmatrix} \frac{1}{2}(I+B(a)^2) & B(a) \\ B(a) & 2I \end{pmatrix} \qquad \text{where} \qquad B(a) := \begin{pmatrix} 1+a & a \\ a & 1+a \end{pmatrix}.$$

Now a quick calculation gives

$$PL(a)\bar{P}^t = D \qquad \text{where} \qquad P = \begin{pmatrix} I & -\frac{1}{2}B(a) \\ 0 & I \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \frac{1}{2}I & 0 \\ 0 & 2I \end{pmatrix}$$

and so applying the augmentation map $\varepsilon : \mathbb{Q}[x, x^{-1}] \to \mathbb{Q}$ to this equation we see that $L_1(a)$ is positive definite over $\mathbb{Q}$. (This also shows that $L(a)$ considered as a form on $(\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}])^4$ is extended from a form defined on $\mathbb{Z}[\frac{1}{2}]^4$.) The argument given at the beginning of Section 2 now yields all but the last statement of the lemma.

Now assume that $L(a)_{/n}$ is extended from the integers. Then it must be extended from its augmentation $\varepsilon(L(a)_{/n})$, which we have just seen is standard. Hence $L_n(a)$ is standard. ■

Lemma 5.3. Let

$$a = a_0 + \sum_{\ell=1}^{m} a_\ell (x^\ell + x^{-\ell}).$$

If $a_0^2 + 2\sum a_\ell^2 < a_0 + 4\sum a_\ell$ (where all sums are from 1 to m), then $L_n(a)$ is not standard for any $n > 4m$. □

Note that this lemma applies to all elements a for which all $a_i \in \{0,1\}$ and $a_\ell \neq 0$ for at least one $\ell > 0$, and in particular to $a = x^k + x^{-k}$ for $k > 0$.

Proof. First observe that all exponents in powers of x appearing in the matrix $L(a)$ are between $-2m$ and $2m$, and so the condition $n > 4m$ precludes any cancelation when passing to the quotient $L(a)_{/n}$. For notational convenience, write $(u, v) = L_n(a)(u, v)$ for the inner product and $|v|^2 = (v, v)$ for the associated norm.

We now proceed as in the proof of Theorem 1.2. Let $N = 1 + x + \cdots + x^{n-1}$ and $w = N(e_3 + e_4)$. We will show that $w_1 = w - 2e_1$ is characteristic of norm $< 4n$.

The argument that w is characteristic, which of course implies that w_1 is as well, is exactly the same as the argument in Section 2. We need only observe that for any $j = 0, \dots, n-1$, the inner products $(x^j e_i, x^j e_i)$ are odd for $i = 1, 2$ and even for $i = 3, 4$ (in particular equal to

$$(1 + a + a^2)_1 = 1 + a_0 + a_0^2 + 2 \sum a_\ell^2$$

in the first case, using the condition $n > 4m$, and $(2)_1 = 2$ in the second) and that the same is true of the inner products $(w, x^j e^i) = \varepsilon(L(a)_{/n}(e_3 + e_4, e_i))$ (which equal

$$\varepsilon(1 + 2a) = 1 + 2\varepsilon(a) = 1 + 2(a_0 + 2 \sum a_\ell)$$

in the first case and $\varepsilon(2) = 2$ in the second).

The norm $|w_1|^2 = |w|^2 - 4((w, e_1) - |e_1|^2)$ which equals

$$4n - 4\left((1 + a_0 + a_0 + 4 \sum a_\ell) - (1 + a_0 + a_0^2 + 2 \sum a_\ell^2)\right)$$

by the calculations above. The condition on a is exactly what is needed to show that this is less than $4n$. ■

To prove Theorem 5.1, we first show that the form $L(k) = L(x^{b_k} + x^{-b_k})$ is not isomorphic to any $L(j)$ for $j < k$. Indeed, Lemma 5.3 shows on the one hand that the form $L_n(j)$ is non-standard for $n > 4b_j$, which implies by Lemma 5.2 that $L(j)_{/b_k}$ is not extended from the integers since $b_k > 4b_j$. On the other hand, the matrix $L(k)$ has entries in $\mathbb{Z}[x^{b_k}, x^{-b_k}]$, whence $L(k)_{/b_k}$ has integral entries and so is extended from the integers.

Next we observe that $L(k)$ is not extended from the integers. For if it were, then each $L(k)_{/n}$ would be as well, which would imply by Lemma 5.2 that all the forms $L_n(k)$ would be standard, contradicting Lemma 5.3.

The final statement in the theorem follows from Lemma 5.3, exactly as in the proof of Theorem 1.2.

Remark 5.4. The argument in the previous paragraph shows more generally that if $L : \Lambda^n \times \Lambda^n \to \Lambda$ is any form extended from the integers with L_1 standard, then all the integral forms L_n are standard. It is conceivable that the converse holds as well. This would imply Conjecture 1.3 in the definite case.

6 More general fundamental groups

In this section we want to prove Theorem 1.6 from the introduction. But first we recall a class of groups to which the theorem actually applies. Let $k \in \mathbb{Z}$ and consider the solvable Baumslag-Solitar groups

$$\Gamma_k := \langle a, b | aba^{-1} = b^k \rangle.$$

Note that $\Gamma_0 = \mathbb{Z}, \Gamma_1 = \mathbb{Z}^2$ and that for $k \neq 0$ we have a semi-direct product decomposition (where a generates the quotient $\mathbb{Z}$ and b corresponds to $\frac{1}{k}$):

$$\Gamma_k \cong \mathbb{Z}\left[\tfrac{1}{k}\right] \rtimes \mathbb{Z}.$$

Here $n \in \mathbb{Z}$ acts on $\mathbb{Z}[\frac{1}{k}]$ by multiplication by $\left(\frac{1}{k}\right)^n$.

Lemma 6.1. The 2-complex corresponding to the above presentation of Γ_k is aspherical. Let N_n be the index n normal subgroup of Γ_k generated by a^n and b. If $k \neq 1$, and n is odd in the case $k = -1$, then for all other k, n we have

$$H_i(N_n) = 0 \quad \forall\, i > 1. \qquad \square$$

Proof. The presentation of Γ has a unique relation which is not a proper power. Then the corresponding 2-complex is aspherical [12], just like for surface groups. It is easy to see that $N_n \cong \mathbb{Z}[\frac{1}{k}] \rtimes n \cdot \mathbb{Z}$. Then the Wang sequence shows the homology result. ■

Before we prove Theorem 1.6, we collect some useful information that applies to closed oriented 4-manifolds M with arbitrary fundamental group Γ. Denote by Λ the group ring $\mathbb{Z}[\Gamma]$ and let A be a Λ-algebra.

Lemma 6.2. Consider the equivariant intersection form with coefficients in A

$$\lambda_A : H_2(M; A) \longrightarrow \mathrm{Hom}_A(H_2(M; A), A).$$

given by Poincaré duality composed with the Kronecker evaluation. If $A = \Lambda$ then the radical $\mathrm{Ker}(\lambda_\Lambda)$ is isomorphic to $H^2(\Gamma; \Lambda)$. If $A = \mathbb{Z}[G]$ where $G = \Gamma/N$ is a quotient group, then the image of $H^2(\Gamma; A)$ in $H^2(M; A) \cong H_2(M; A)$ is contained in the radical $\mathrm{Ker}(\lambda_A)$ if $H_2(N)$ is finite. $\square$

Proof. The universal coefficient spectral sequence gives an exact sequence

$$0 \longrightarrow H^2(\Gamma, \Lambda) \longrightarrow H^2(M; \Lambda) \longrightarrow \operatorname{Hom}_\Lambda(H_2(M; \Lambda), \Lambda)$$

Applying Poincaré duality $H^2(M; \Lambda) \cong H_2(M; \Lambda)$ and identifying the composition with λ (respectively $\lambda_{\Gamma/N}$) yields the first result. For the second statement we note that by naturality of the evaluation map, it suffices to prove the vanishing of

$$\operatorname{Hom}_A(H_2(\Gamma; A), A) = \operatorname{Hom}_A(H_2(N; \mathbb{Z}), A).$$

This follows from our assumption that $H_2(N; \mathbb{Z})$ is finite. ■

Proof of Theorem 1.6. The construction of the 4-manifold M is very similar to that of Section 4. Start with a 4-dimensional thickening K of the finite aspherical 2-complex, i.e. K is a handlebody with handles of index ≤ 2 corresponding to the cells of the 2-complex. We may assume that there is a 1-handle h_g that corresponds to the given element $1 \neq g \in \Gamma$.

Next we attach four more 2-handles in a neighborhood of the meridian to the 1-handle h_g. This is done just like in Section 4 so that the intersections between the null homotopies for the handles represent the matrix L. In particular, these 4-handles are attached homotopically trivially, so that the resulting 4-manifold X has the homotopy type

$$X \simeq K \vee 4 \cdot S^2.$$

If we had attached these four 2-handles to $S^1 \times D^3$, we showed in Section 4 that the resulting 4-manifold has boundary equal to 0-surgery on a knot with trivial Alexander polynomial. By Freedman's theorem, this knot is $\mathbb{Z}$-slice, i.e. the 0-surgery also bounds a topological 4-manifold C which is a homotopy circle. Since we have attached the four 2-handles in X only to a meridian of h_g, it follows that the boundary ∂X also bounds a 4-manifold Y that is obtained from C by adding 1- and 2-handles (corresponding to the handles in K). Since C is a homotopy circle, it follows that Y is another $K(\Gamma, 1)$. We define

$$M := X \cup_\partial Y$$

and we claim that it has the two properties stated in Theorem 1.6. The inclusion $Y \hookrightarrow M$ induces an exact sequence (with coefficients in any Λ-module A)

$$H_2(Y; A) \longrightarrow H_2(M; A) \xrightarrow{j} H_2(M, Y; A) \xrightarrow{0} H_1(Y; A) \xrightarrow{\cong} H_1(M; A). \qquad (*)$$

The rightmost map is an isomorphism since the inclusion $Y \hookrightarrow M$ induces an isomorphism on fundamental groups.

We now turn to the proof of part (1) of Theorem 1.6. Consider the exact sequence ($*$) with $A = \Lambda = \mathbb{Z}[\Gamma]$. Note that $H_2(Y;\Lambda) = 0$ since Y is a $K(\Gamma, 1)$. Therefore, j is an isomorphism and we get by excision and Poincaré duality

$$H_2(M;A) \cong H_2(M, Y;A) \cong H_2(X, \partial X;A) \cong H^2(X;A) \cong H^2(K;A) \oplus H^2(S^2;A)^4$$

where $H^2(K;\Lambda) \cong H^2(\Gamma;\Lambda)$ is the radical of the Λ-valued intersection form on $\pi_2 M$ by Lemma 6.2. The intersection form on $H^2(S^2;\Lambda)^4$ (and hence the intersection form λ_Λ on $\pi_2 M$ modulo its radical) is by construction given by extending the form L from $\mathbb{Z}[\mathbb{Z}]$ to $\mathbb{Z}[\Gamma]$ via i_g. Since g has infinite order, the proof of [11, Lemma 4] can easily be seen to carry through to show that λ_Λ is not extended from the integers. Alternatively the fact, proven in (2), that λ_Λ gives rise to a non-standard form over $\mathbb{Z}$ also shows that λ_Λ is not extended.

For part (2) of Theorem 1.6, let $A = \mathbb{Z}[G] = \mathbb{Z}[\Gamma/N]$ and recall that by assumption

$$H_2(Y;A) \cong H_2(\Gamma;A) \cong H_2(N;\mathbb{Z}) = 0.$$

By Lemma 6.2, the group $H^2(K;A) \cong H^2(\Gamma;A)$ lies in the radical of λ_A and so the above exact sequence ($*$) and the exact sequence slightly below it show that the A-valued intersection pairing λ_A (modulo its radical) is given by extending the form L from $\mathbb{Z}[\mathbb{Z}]$ to $A = \mathbb{Z}[G]$ via the map $\mathbb{Z} \to \Gamma \to G$. Denote by n the order of $\varphi(g)$ and write $k = \frac{1}{n}|G|$. Pick representatives $h_1, \ldots, h_k$ for $G/\langle g\rangle$. Then any $h \in G$ is of the form $h = h_i g^j$ for unique $i \in \{1, \ldots, k\}$ and $j \in \{0, \ldots, n-1\}$.

Denoting the standard basis of $\mathbb{Z}^4$ again by $e_1, \ldots, e_4$, the linearity of λ_G gives

$$\lambda_G(h_j g^k e_i, h_{j'} g^{k'} e_{i'}) = h_{j'} \lambda_G(g^k e_i, g^{k'} e_{i'}) h_j^{-1}.$$

Note that by the definition of λ_G we have $\lambda_G(g^k e_i, g^{k'} e_{i'}) \in \mathbb{Z}[\langle g\rangle] \in \mathbb{Z}[\Gamma]$. Furthermore, $h_{j'} g^i h_j^{-1} = e$ if and only if $j' = j$ and $i = 0$. Therefore

$$\lambda_G(h_j g^k e_i, h_{j'} g^{k'} e_{i'})_1 = (h_{j'} \lambda_G(g^k e_i, g^{k'} e_{i'}) h_j^{-1})_1 = \delta_{jj'}\, \lambda(e_i g^k, e_{i'} g^{k'})_1.$$

This shows that the ordinary intersection form of M_φ is the direct sum of k copies of L_n. This is clearly positive definite. Taking k copies of the vector w' from the proof of

Theorem 1.2 we get a characteristic vector of norm

$$(4n-8)k < 4|G| = \mathrm{rank}(H_2(M_\varphi)). \qquad \blacksquare$$

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