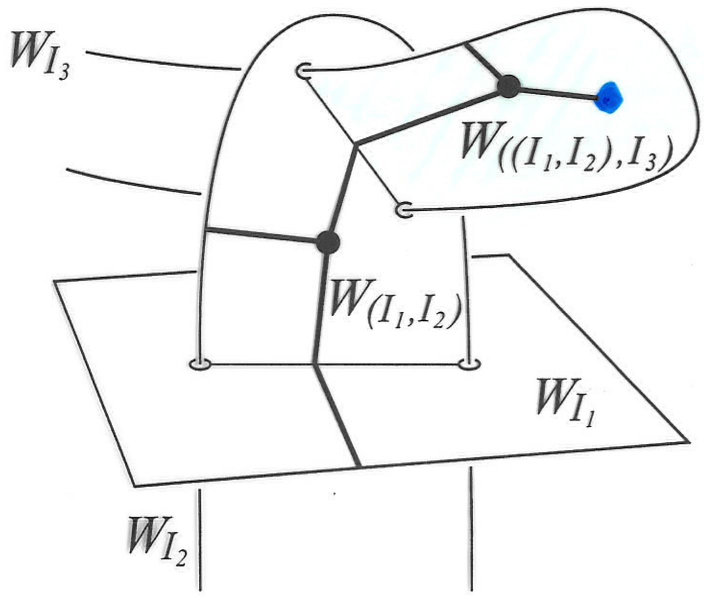
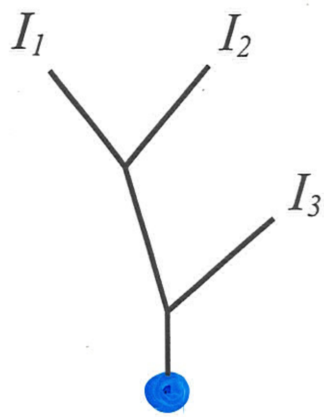
Whitney Towers



$\rightsquigarrow$

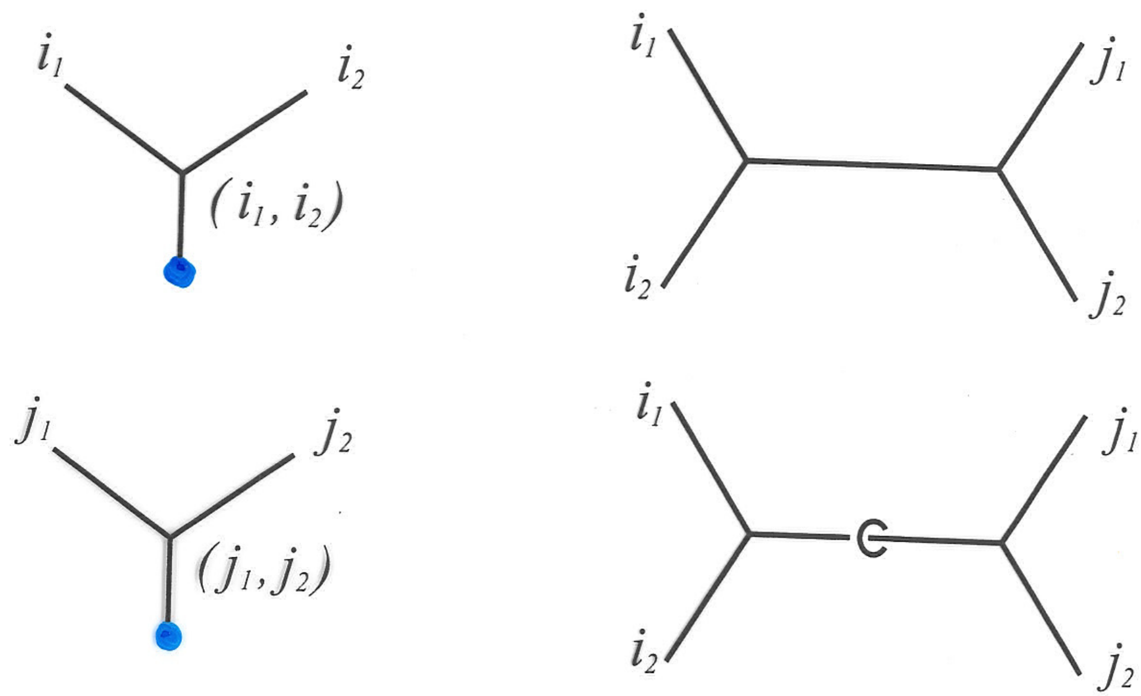


- original sheets A_i $\rightsquigarrow$ univalent vertices
- Whitney disks $\rightsquigarrow$ trivalent vert.
- top order W. disk $\rightsquigarrow$ <u>root</u>
- orientations on W. disks $\rightsquigarrow$ vertex orient.

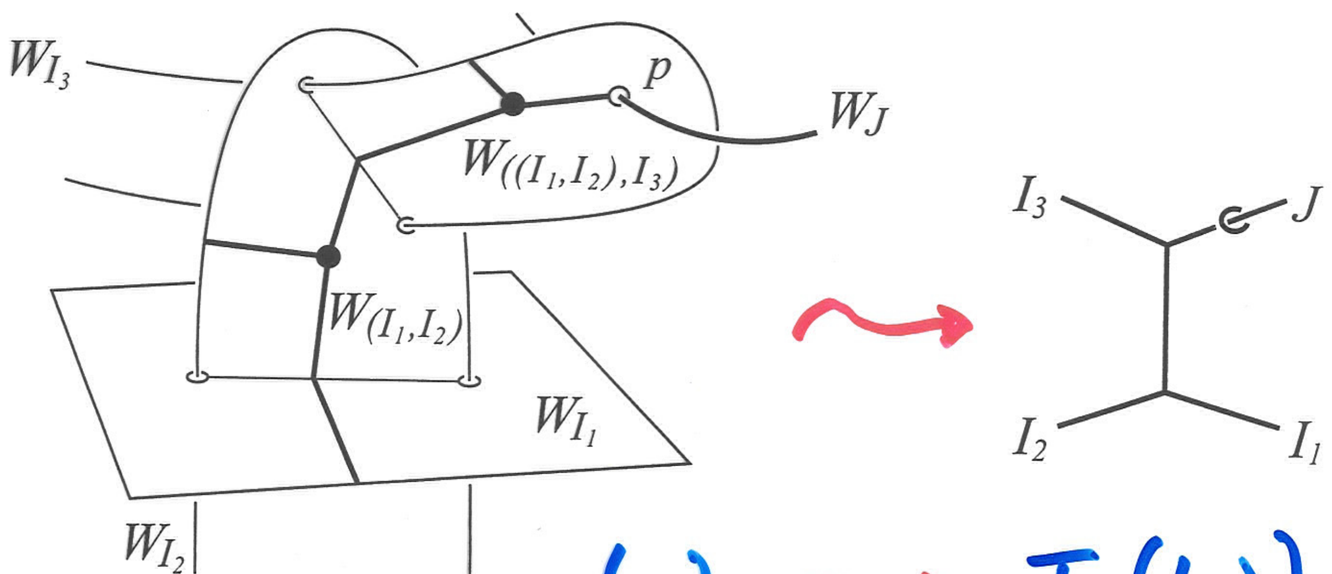
$$\text{tree}(W) \subseteq W$$

order = class $-$ 1

Inner product of rooted trees



corresponds to unpaired intersection points: $p \in W$ gives an intersection tree :



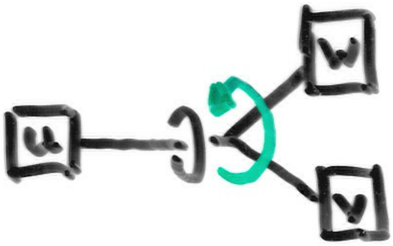
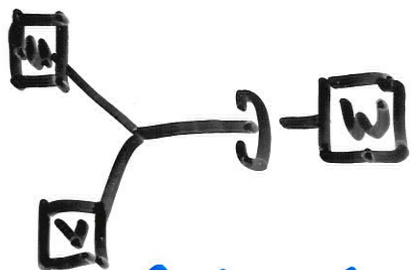
$W \rightsquigarrow \tau_n(W)_p$

$u =$  $\in \mathcal{L}_{n_1}$

$v =$  $\in \mathcal{L}_{n_2}$

$\langle u, v \rangle :=$  $\overset{180^\circ}{=} \langle v, u \rangle$

$$\langle u, [v, w] \rangle = \langle [u, v], w \rangle$$

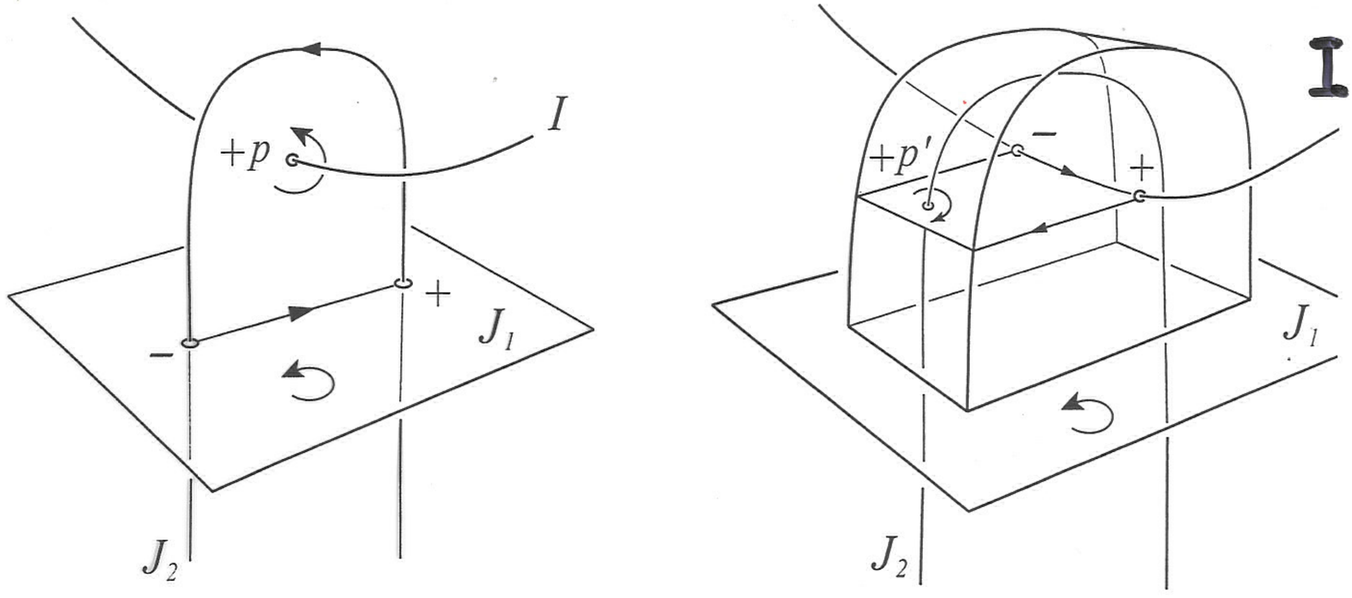
 $\overset{120^\circ}{=}$ 

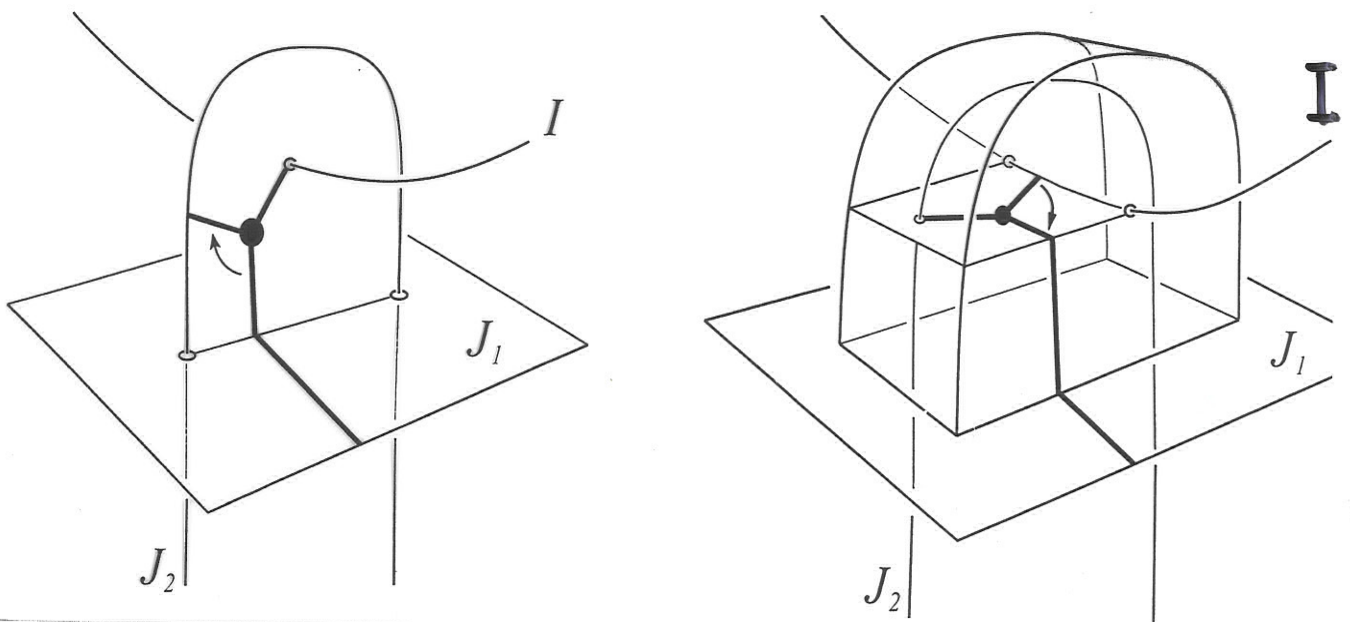
$$\langle u, v \rangle \in \mathcal{T}_n^{(m)} = \frac{\text{m-labeled unrooted trees of length } n}{AS,\ IHX}$$

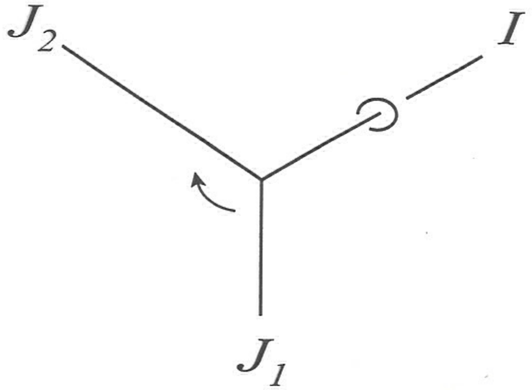
$n = n_1 + n_2$

is the universal symmetric, invariant form on $\mathcal{L}$!

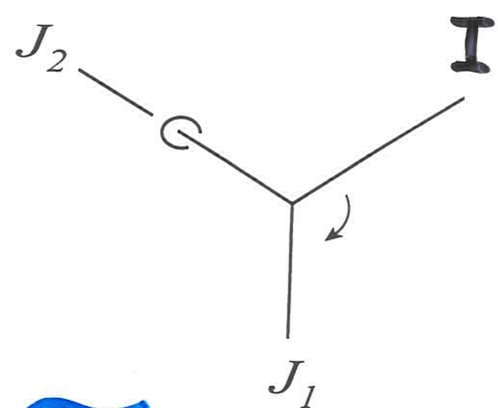
length = # tips = order + 2







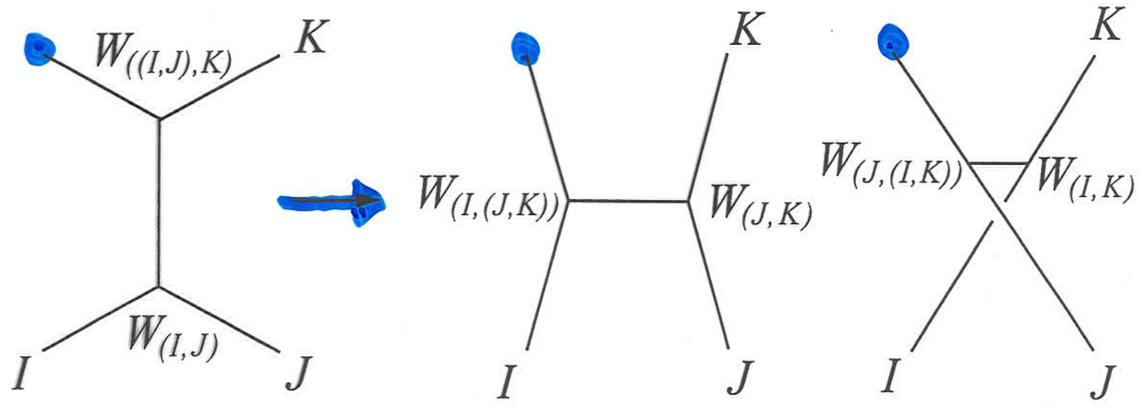
$\overset{!}{=}$



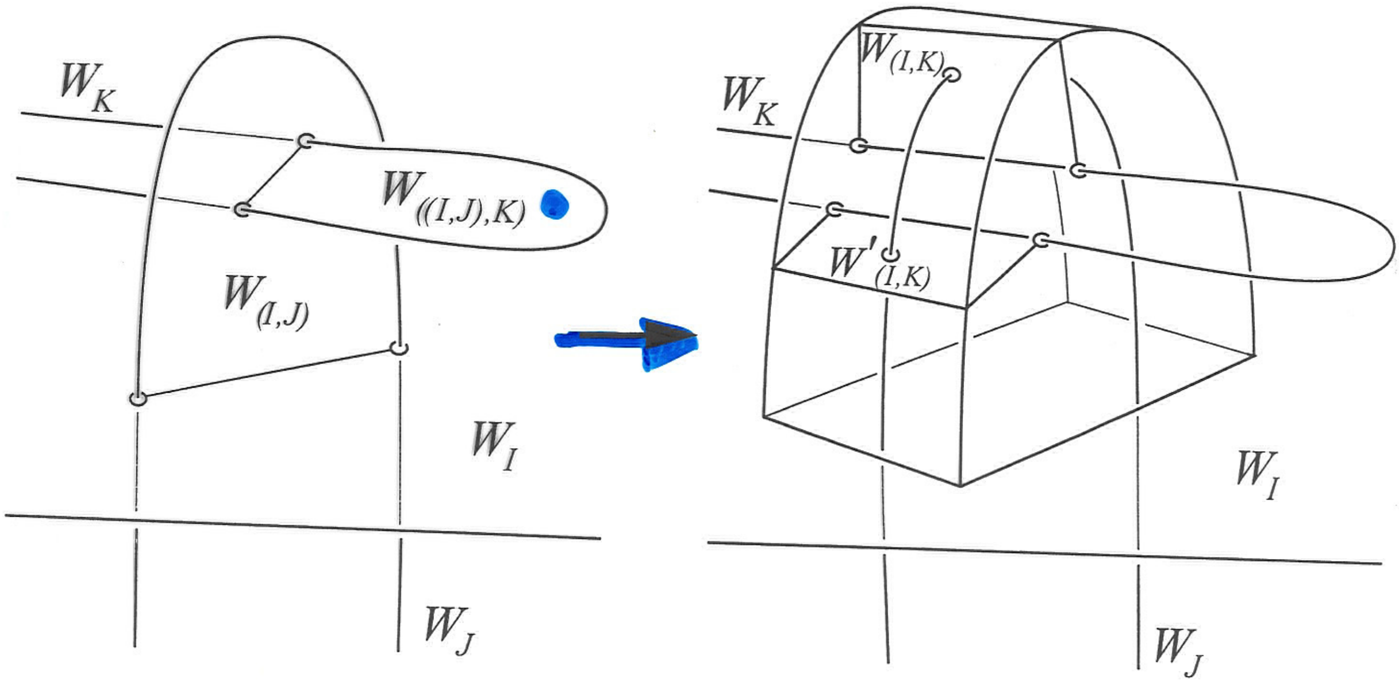
$$T_n(W) := \sum_p \varepsilon_p \cdot T_n(W)_p \in \mathcal{T}_n(M)$$

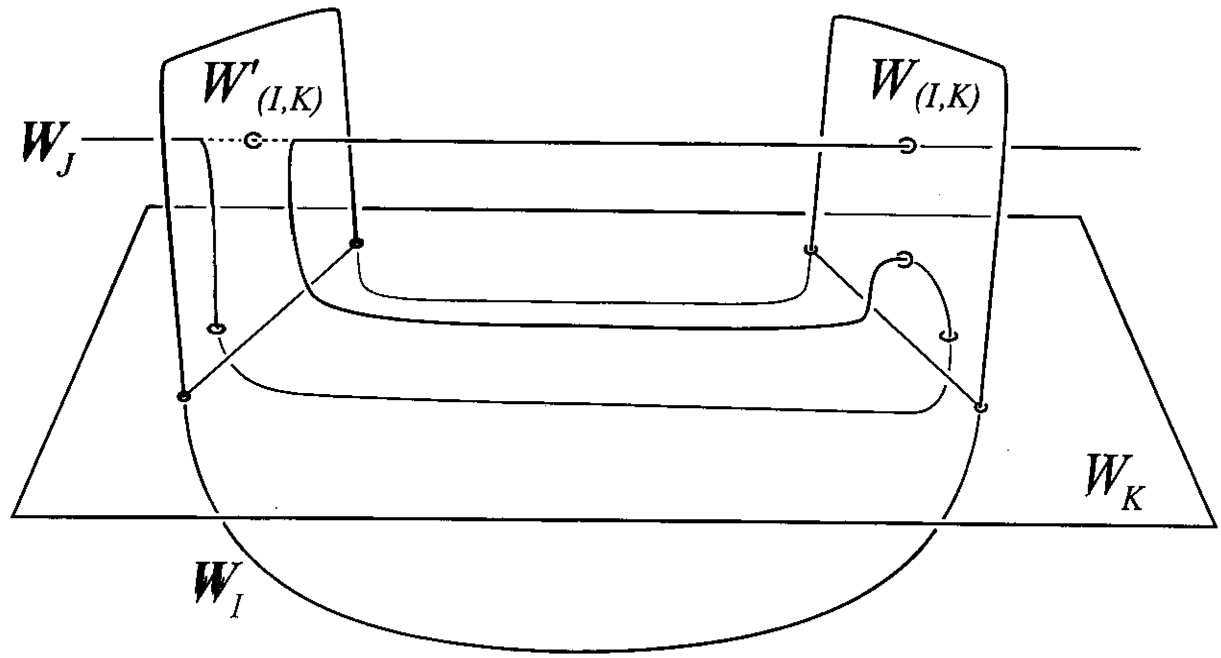
5

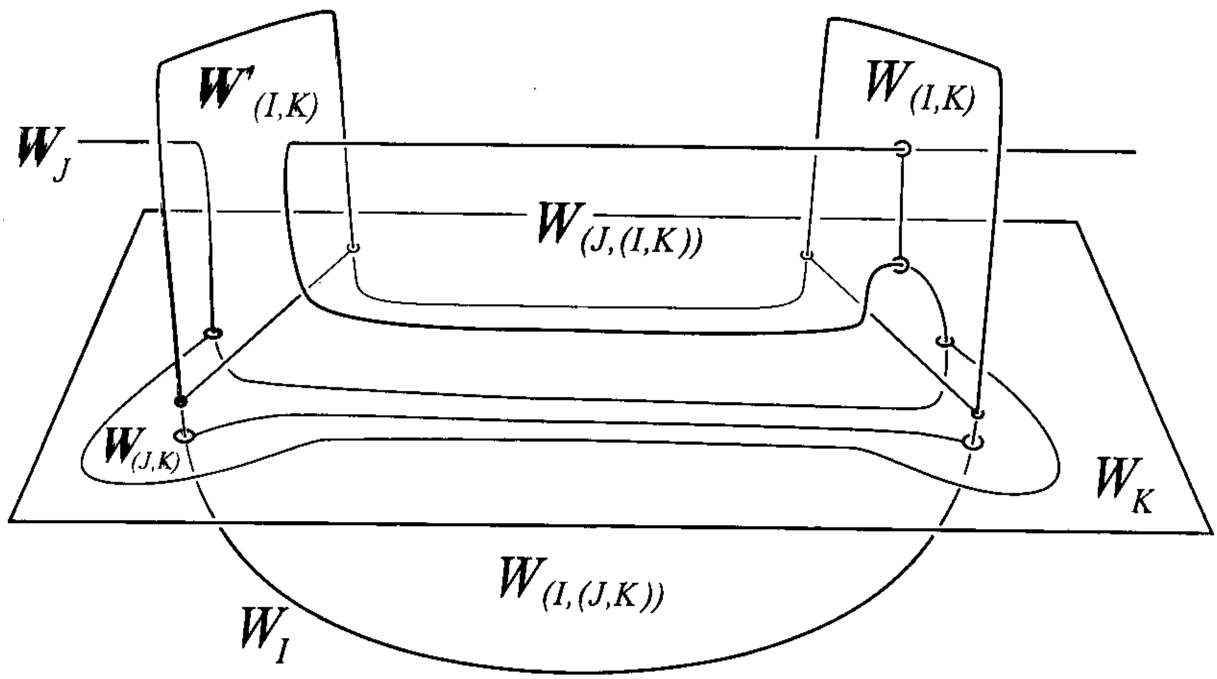
IHX - for trees:

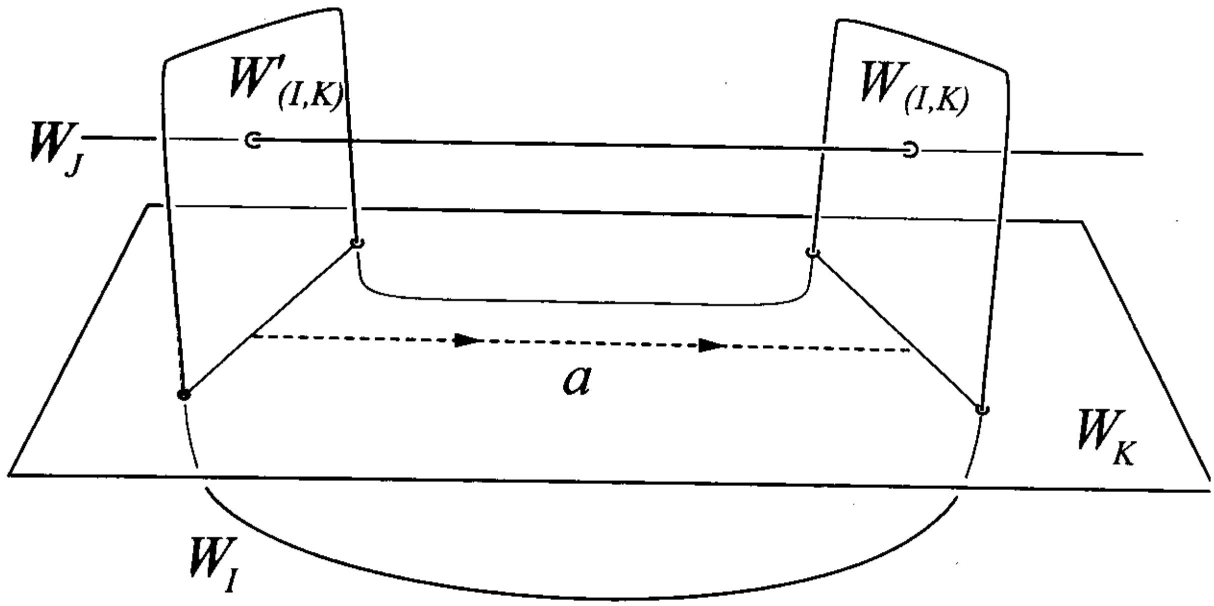


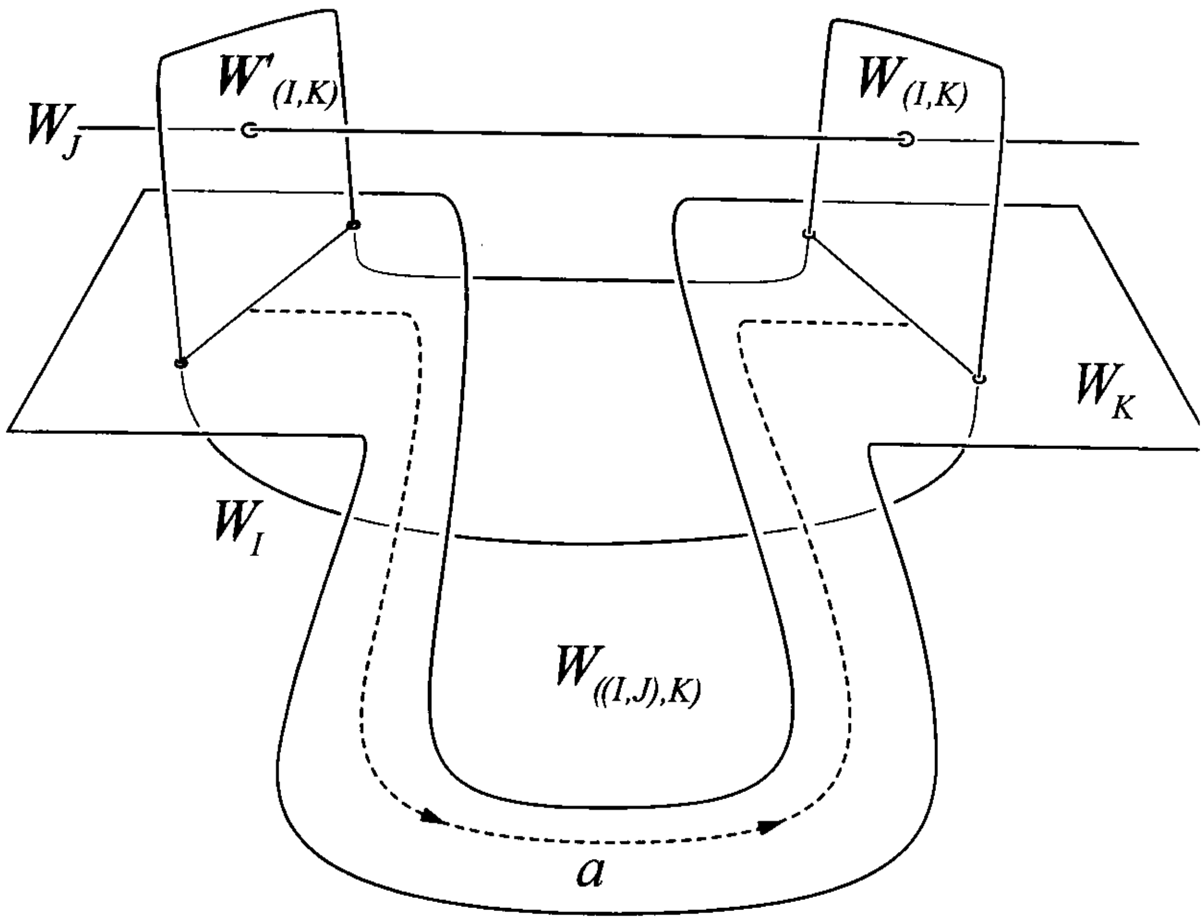
Geometric Incarnation

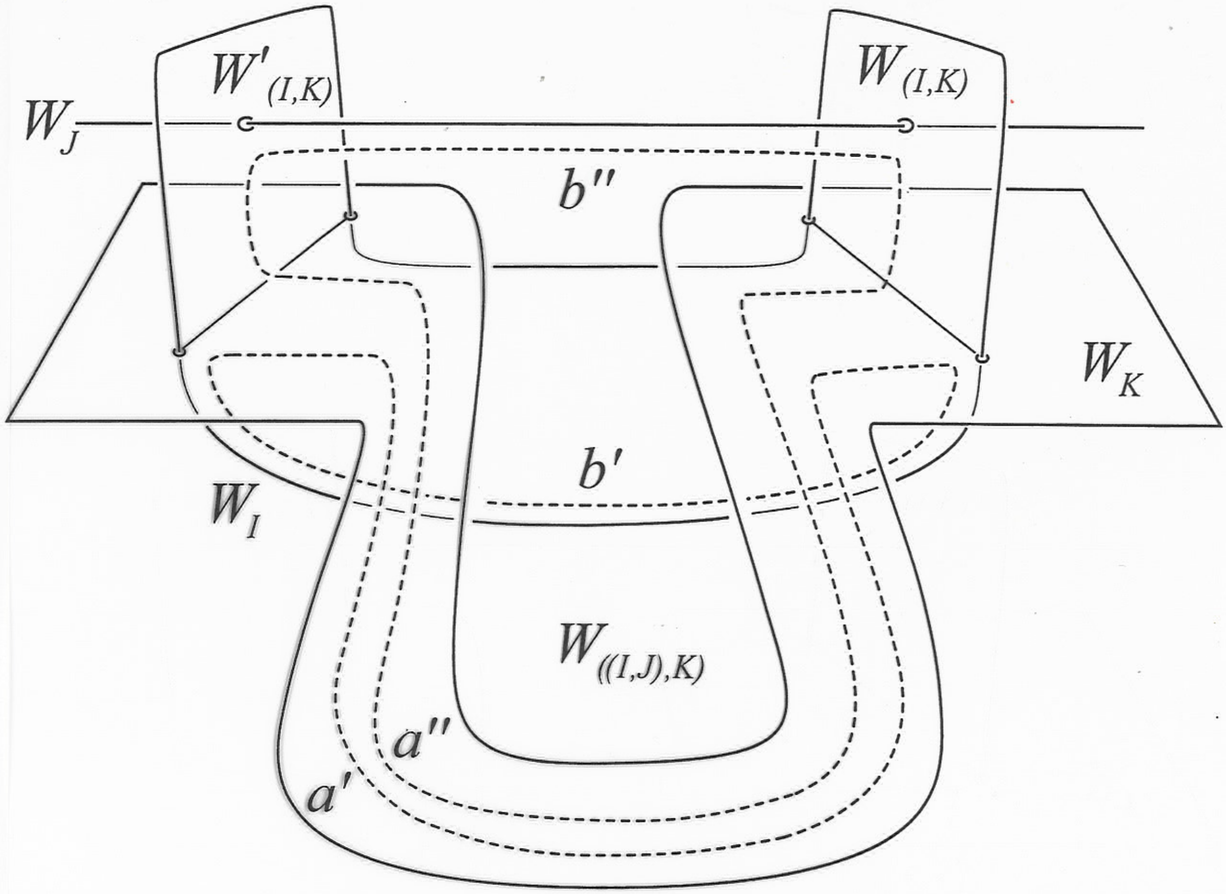


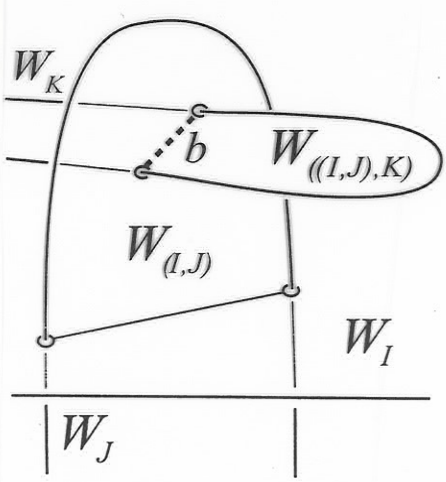

$W'_{(I,K)}$
$W_{(I,K)}$
W_J
W_K
W_I

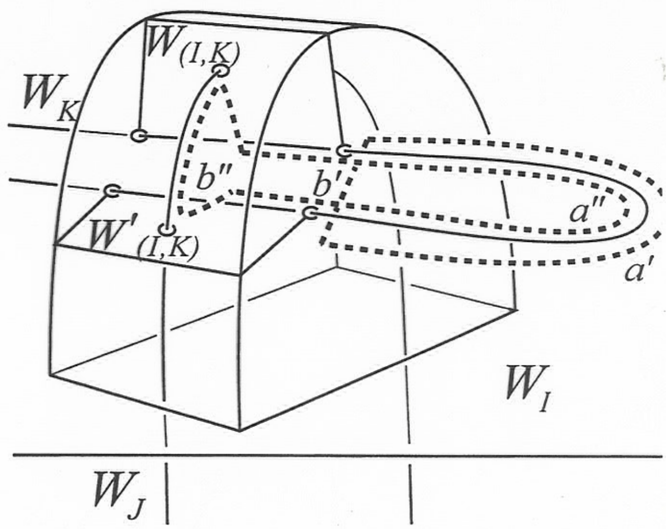

$W'_{(I,K)}$
$W_{(I,K)}$
W_J
$W_{(J,(I,K))}$
$W_{(J,K)}$
W_K
$W_{(I,(J,K))}$
W_I


$W'_{(I,K)}$
$W_{(I,K)}$
W_J
a
W_K
W_I


$W'_{(I,K)}$
$W_{(I,K)}$
W_J
W_K
W_I
$W_{((I,J),K)}$
a

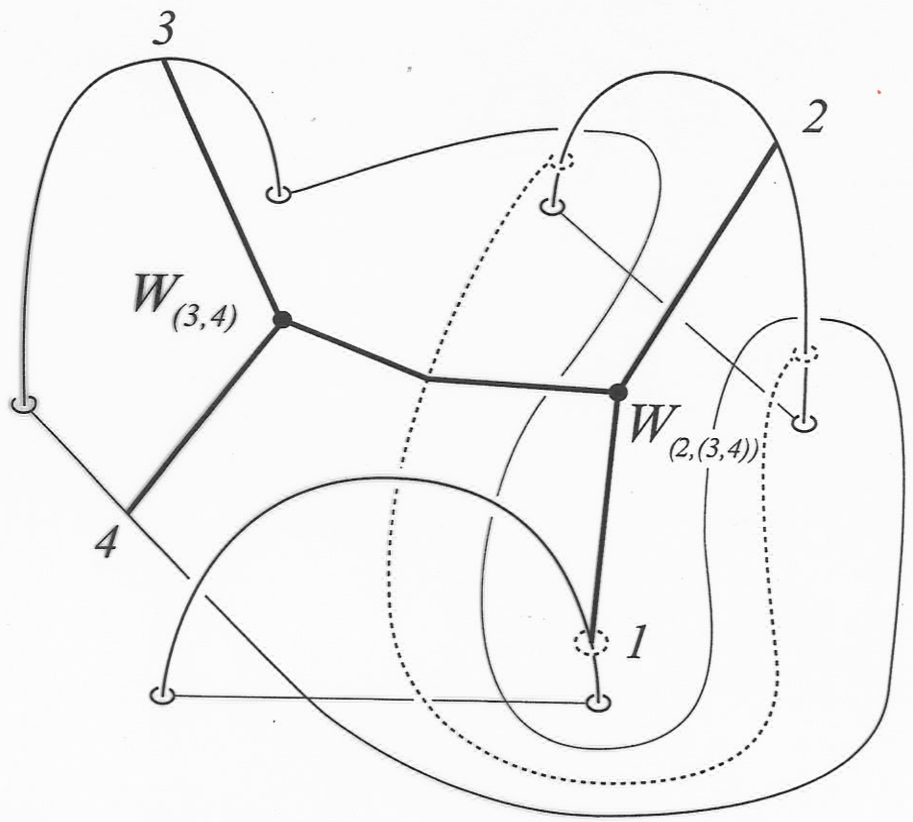






All 3 top order W. disks are parallel copies of given one

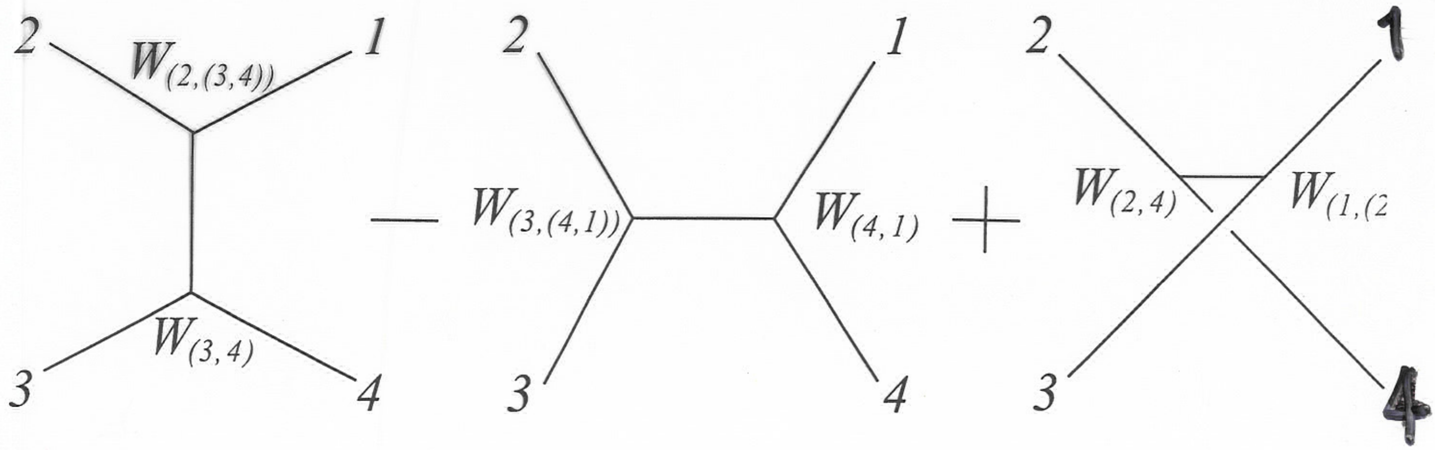
QED



On intersection trees,

the IHX-relation

or Jacobi-Identity reads as:



Milnor's μ-invariants

π a group, set $\pi/_n := \pi/\pi_n$, quotient by n-fold commutators.

$L = (\ell_1, \dots, \ell_m)$ a link in S^3

$\pi(L) := \pi_1(S^3 \setminus L) \xleftarrow{\varphi} F$,

F free group on meridians $x_1, \dots, x_m$.

Stallings' Theorem: $\forall n$, the relations $[x_i, \ell_i]$, $i = 1, \dots, m$ generate $\mathrm{Ker}\left(F/_n \xrightarrow{\varphi_n} \pi(L)/_n\right)$

Corollary: φ_n is an isom.

$\Longleftrightarrow \ell_i \in \pi(L)_{n-1}$

Examples:

$n = 2 \quad : \quad \pi(L)/_2 \cong H_1(S^3 \setminus L) \cong \mathbb{Z}^m$

$n = 3 \quad : \quad \ell_i \in \pi(L)_2 \iff$ all linking numbers vanish.

Milnor's boot strap:

① $\pi(L)/_2 \cong F/_2 \cong \mathbb{Z}^m$ always

$[\ell_i] \longmapsto (\ell k(\ell_i, \ell_j))_{j=1,\dots,m}$

$\mu_2(L) \in \mathbb{Z}^m \otimes \mathcal{L}_1 \qquad \overset{!!}{\mu_2}(i,j)(L)$

② assume $\mu_2(L) = 0 \iff \ell_i \in \pi(L)_2$

$$\pi(L)_2 / \pi(L)_3 \cong F_2 / F_3 \cong \mathcal{L}_2$$

$$\mu_3(L) := \sum_{i=1}^{m} x_i \otimes \ell_i \in \mathbb{Z}^m \otimes \mathcal{L}_2$$

(n) assume $\mu_n(L) = 0 \iff \ell_i \in \pi(L)_n$

$$\pi(L)_n / \pi(L)_{n+1} \cong F_n / F_{n+1} \cong \mathcal{L}_n$$

$$\mu_{n+1}(L) := \sum_{i=1}^{m} x_i \otimes \ell_i \in \mathbb{Z}^m \otimes \mathcal{L}_n$$

$$\mathbb{Z}^m \otimes \mathcal{L}_n \cong \frac{\text{rooted trees of class } n \text{ with } m \text{ labels also on root}}{\text{AS, IHX}}$$

$x_1 \otimes [x_2, x_1] \longleftarrow$

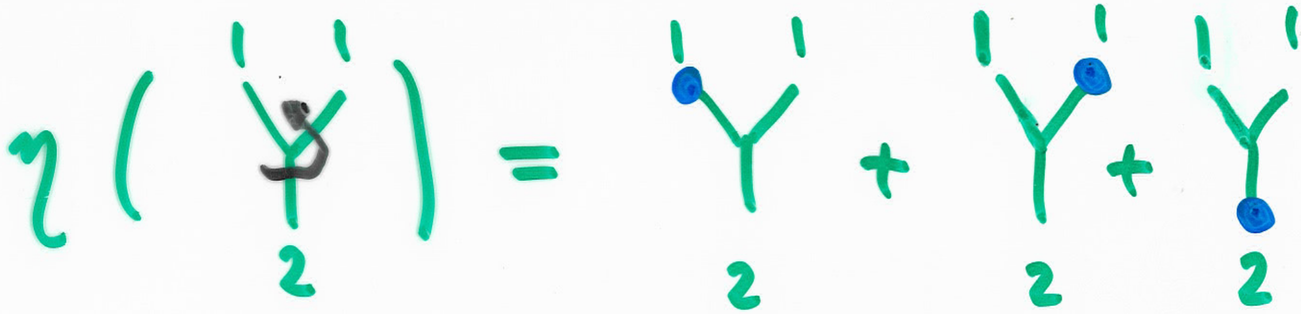
Cyclic symmetry: $\mu_{n+1}(L)$ lies in

$$\mathcal{D}_{n+1} := \operatorname{Ker}\left(\mathbb{Z}^m \otimes \mathcal{L}_n \xrightarrow{[\,,\,]} \mathcal{L}_{n+1}\right)$$

$S^3 \setminus L$ is zero bordism: $H_2(S^3 \setminus L) \to H_2(F/F_{n+1})$

Levine's map $\eta : \mathcal{T}_n \to \mathcal{D}_n$

takes a tree of length n and sends it to the sum of making each tip the root:



$$= x_1 \otimes \left([x_2, x_1] + [x_1, x_2] \right) + x_2 \otimes [x_1, x_1] = 0$$

Theorem 1:

If $L = \partial W$, W a W.tower of class $n-1$ $(n \geq 2)$

(a) $\mu_{<n}(L) = 0$

(b) $\mu_n(L) = \eta(\tau_n(W))$

Proof:

$$\begin{tikzcd} \pi(W) := \pi_1(D^4 \setminus W) & \\ \pi(L) \arrow[u, "j"] & F \arrow[l, "\varphi"'] \arrow[ul, "\psi = j \circ \varphi"'] \end{tikzcd}$$

Step 1: W of class $n-1$ $\Rightarrow$

$$F/n \xrightarrow[\cong]{\psi_n} \pi(W)/n \qquad \text{grope duality}$$

hence φ_n is isom. $\Rightarrow$ $\mu_{<n}(L) = 0$

Step 2: $\mu_n(L)$ determined by

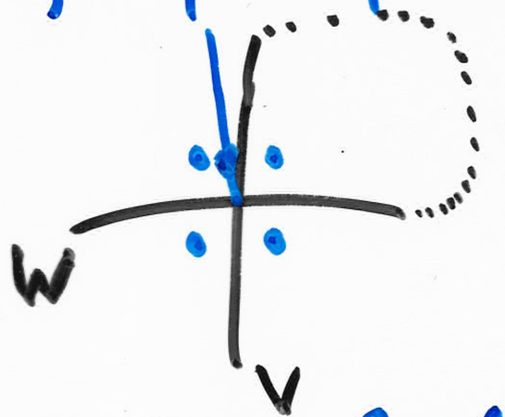
$$j : \pi(L)_w/n \xleftarrow{\cong} F/n \xrightarrow{\cong} \pi(W)/n$$

$[\ell_i]$ so may read off the word $j(\ell_i)$ in $\pi(W)/n$.

Step 1 follows again from Stallings' Thm. because

$H_2(D^4 \setminus \omega)$ contains all linking tori T_p, p unpaired;

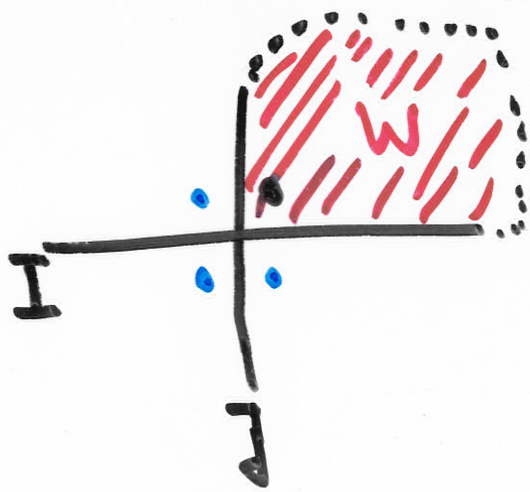
Alexander duality



$\forall k$, get relations

in $\pi(\omega)/k$: $[m_V, m_W] = 1$ $\left(\begin{matrix}2\text{ cell}\\ \text{of } T_p\end{matrix}\right)$

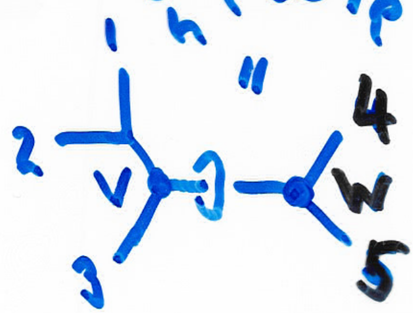
$m_W = [m_I, m_J]$



For $T_n(\omega)_p$

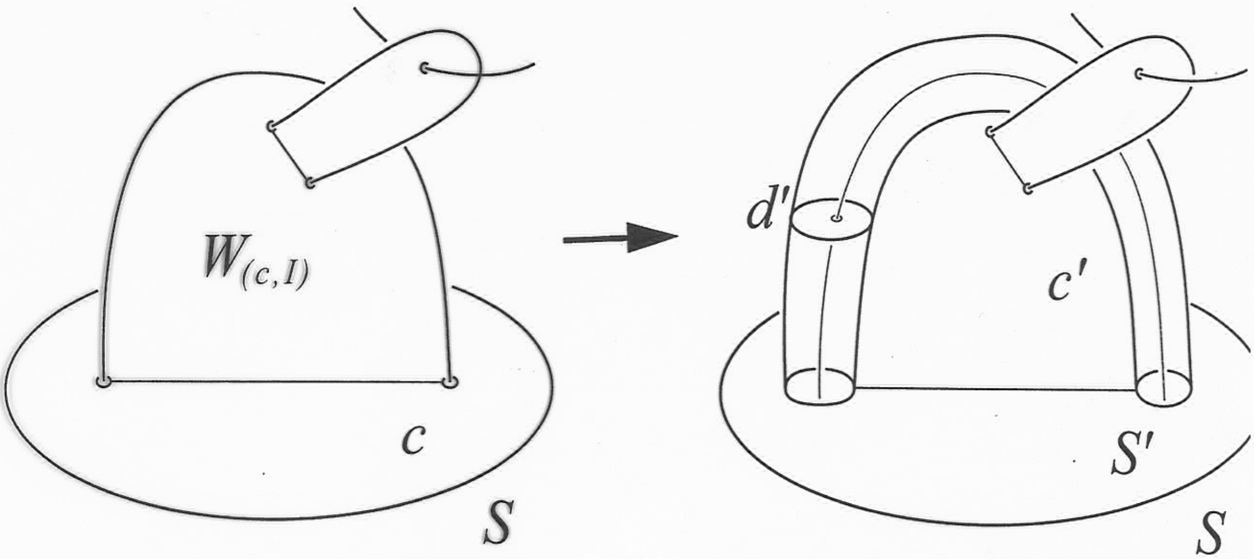
"

$n=5$

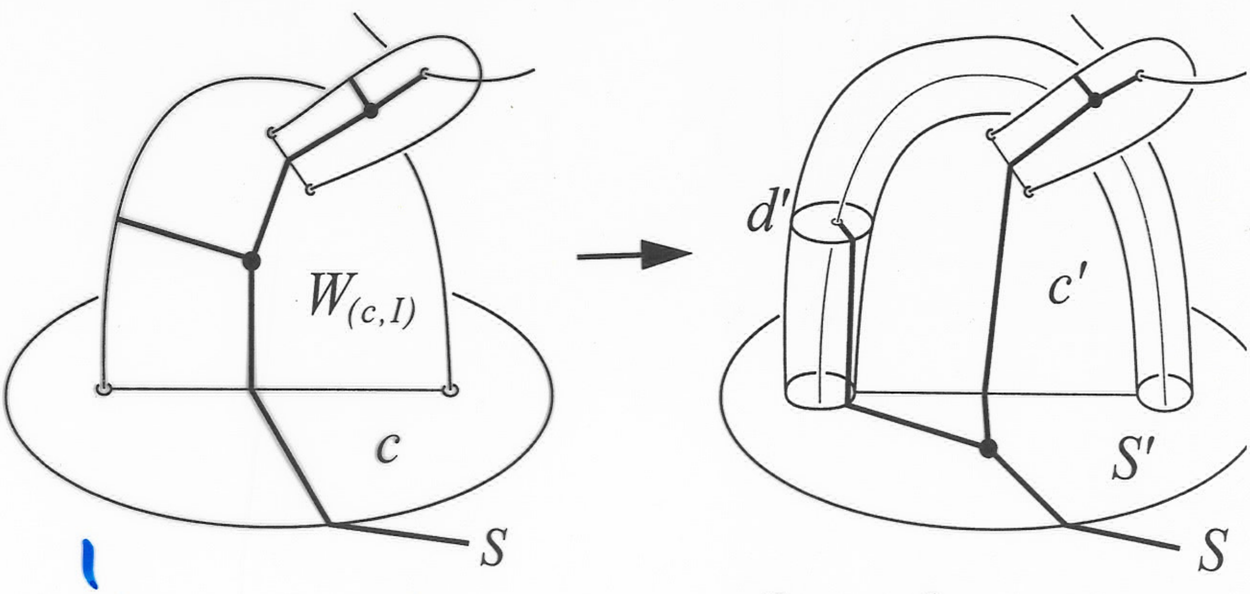


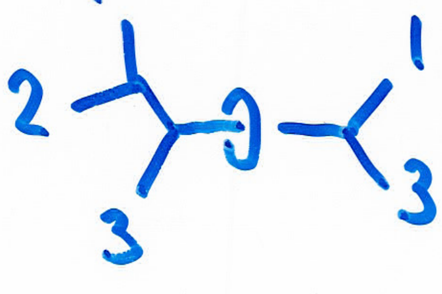
$$\left[\left[[x_1, x_2], x_3\right], [x_4, x_5]\right] \in \pi(\omega)_n$$

7 **Step 2** builds a grope G_i of class $n-1$, in $D^4 \setminus \omega$, with $\partial G_i = \ell_i$



inductive step: trees are preserved

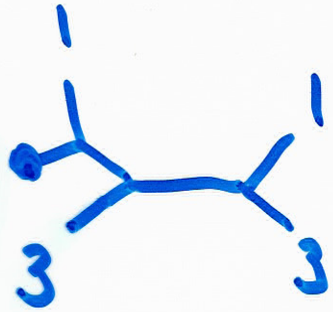




$\rightsquigarrow$ G_2 has genus 1
G_1, G_3 -"- 2

tree of G_i = $T(\omega_p)$ roots at i

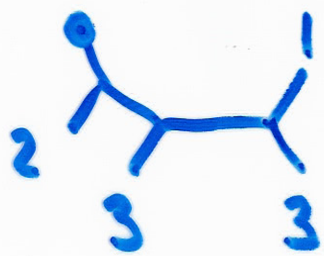
Example:

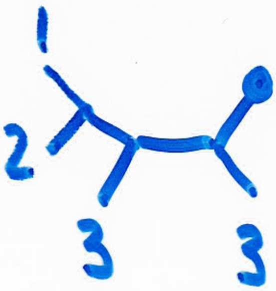
$t(G_2) =$ 

$\in \mathcal{L}_4 \cong F_4/F_5$

$t(G_2) = \ell_2$

and

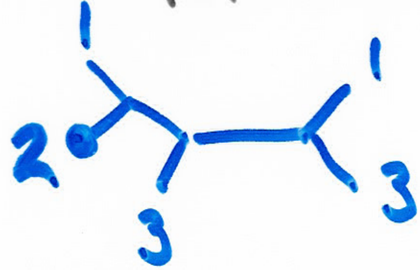
$\ell_1 =$ 

$+$ 

$\in \mathcal{L}_4$

Together $\sum_{i=1}^{m} x_i \otimes \ell_i = \eta(\tau_n(W))$

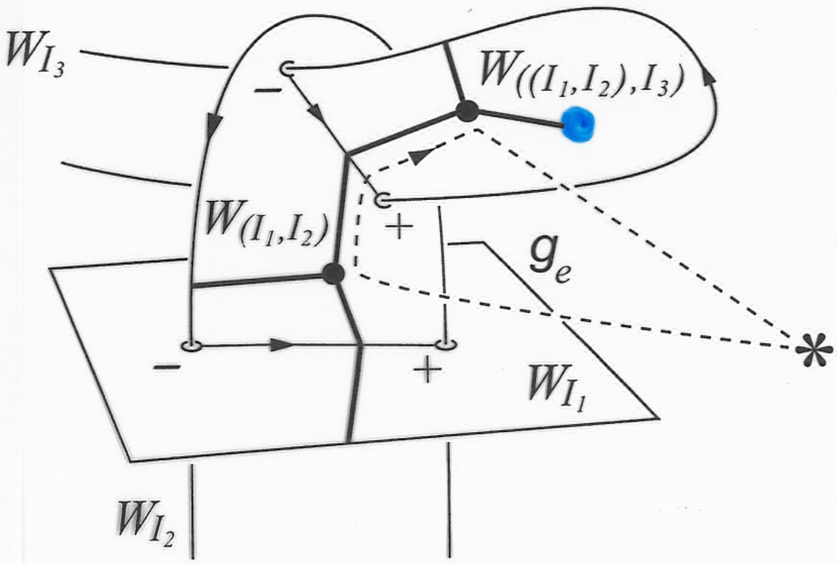
in $\mathbb{Z}^m \otimes \mathcal{L}_{n-1} \cong \bigoplus_{i=1}^{m} \mathcal{L}_{n-1}$ ∎

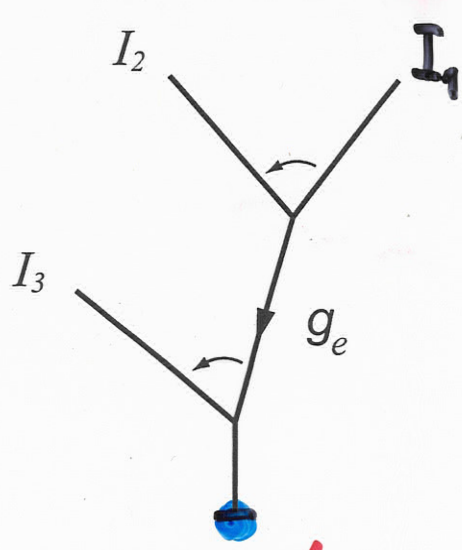
e.g. $x_2 \otimes \ell_2 =$ 

∎

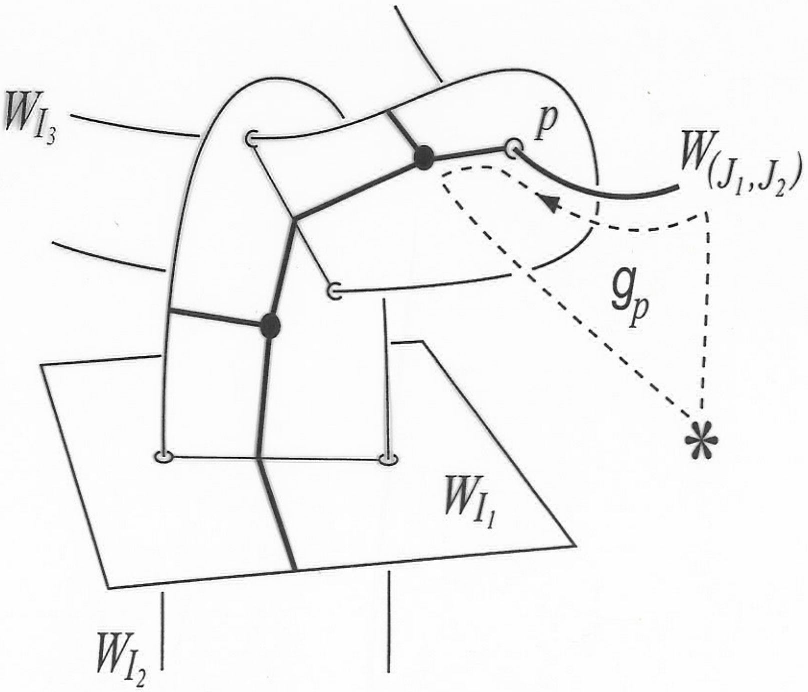
Let's briefly return to W-towers in a 4-manifold M:

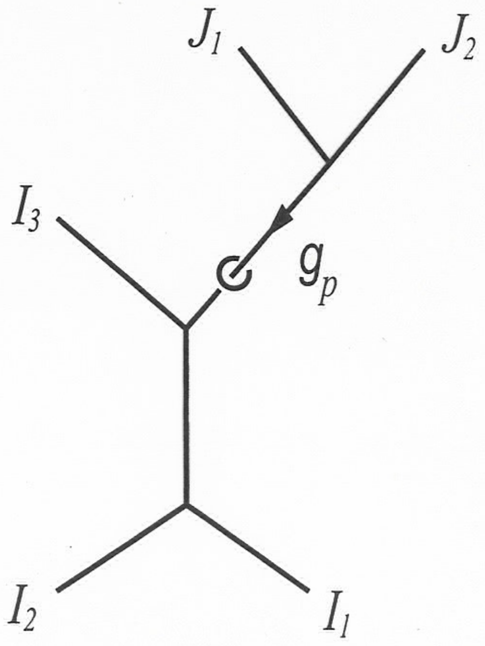
$\mathcal{W}$. tower $\mathcal{W}$ it's tree $t(T)$





e any edge $\rightsquigarrow$ $g_e \in \pi := \pi_1(M^4)$.



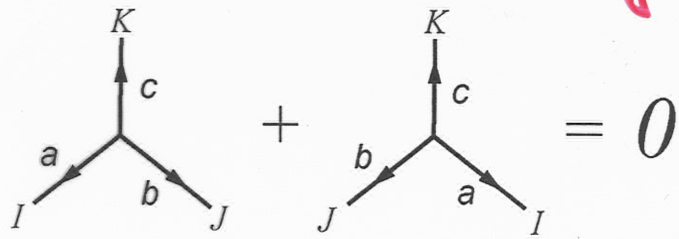


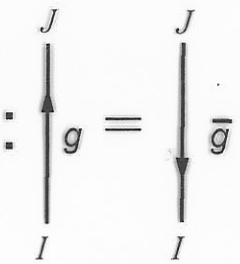
$p \in \mathcal{W}$ unpaired intersection $I \cap J$ $\rightsquigarrow$ unrooted tree $t_p := t(T_I\, g_p\, T_J) = \tau(\mathcal{W})_p$ with π-labels

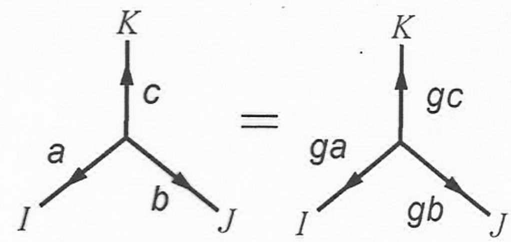
10

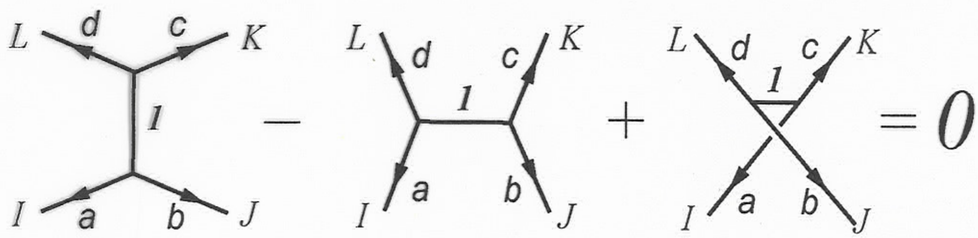
$\mathcal{T}(\pi, m) :=$ free abelian group on decorated, oriented trees, modulo isom. & the following relations

graded by # tips

AS:


OR:


HOL:


IHX:


orientation:
- arrows on edges
- cyclic order on vertices

decoration:
- group elements on edges
- indices from $\{1, 2, \dots, m\}$ on univalent vertices

Def.: If W is a W. tower of class $(n-1)$, define its n-th <u>intersection tree</u>

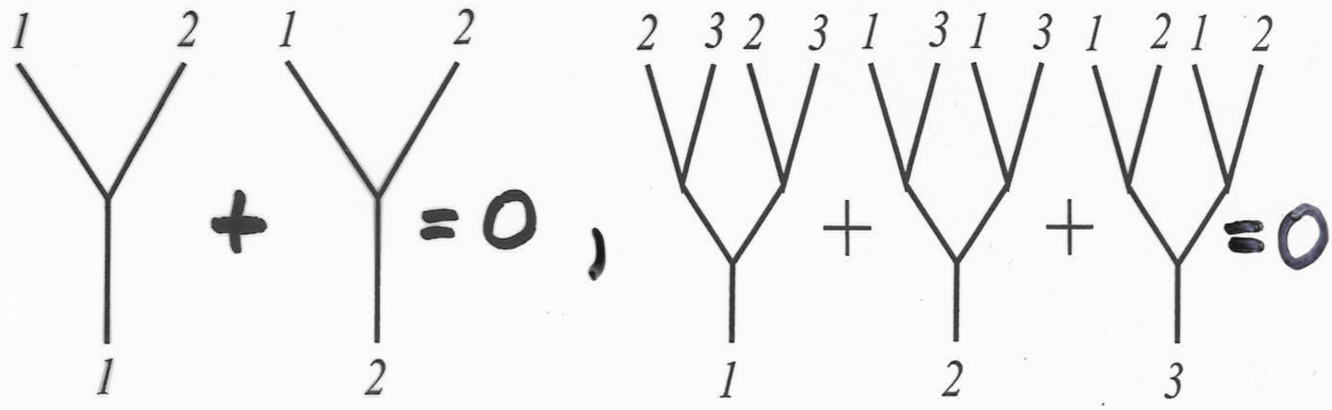
$$\tau_n(W) := \sum_p \varepsilon_p \cdot \tau_n(W)_p \in \mathcal{T}_n(\pi, m)$$

sum over all intersection points p of W which are <u>unpaired</u>.

Thm 2: $\tau_n(W) = 0 \Longrightarrow$ $\exists\, T'$ of order n on A'_k s.t. $[A'_k] = [A_k] \in \pi_2(M)$.

Q: $\tau_n(W) = \tau_n([A_1], \dots, [A_m])$?
In some quotient of $\mathcal{T}_n(\pi, m)$?

Framing relations: FR



etc.

Intersection relations: INT

Tube a sphere into a W. disk.

<u>Conjecture</u>: $T_n(\pi, m) / \text{FR, INT}$

is the right quotient.

True for $n = 1$ [Wall]

$n = 2$ [S.T. 99]

and in special cases.