Appendix

In this appendix, we summarize, usually without proofs, some of the basic machinery that is needed in the book. The first section, on inverse limits, is used in Chapters 12 and 13. Infinite Galois theory and ramification theory are used primarily in Chapter 13. The main points of the section are that the usual Galois correspondence holds if we work with closed subgroups and that we may talk about ramification for infinite extensions, even though the rings involved are not necessarily Dedekind domains (much of this section comes from a course of Iwasawa in 1971). The last section summarizes those topics from class field theory that we use in the book. The reader willing to believe that the Galois group of the maximal unramified abelian extension is isomorphic to the ideal class group (and variants of this statement) will have enough background to read all but certain parts of Chapter 13.

§1 Inverse Limits

Let I be a directed set. This means that there is a partial ordering on I, and for every $i, j \in I$ there exists $k \in I$ with $i \leq k, j \leq k$. For each $i \in I$, let A_i be a set (or group, ring, etc.). We assume that whenever $i \leq j$ there is a map $\phi_{ji}\colon A_j \to A_i$ such that $\phi_{ii} = id$ and $\phi_{ji}\phi_{kj} = \phi_{ki}$ whenever $i \leq j \leq k$. This situation is called an inverse system.

Let $A = \prod A_i$ and define the *inverse limit* by

$$\varprojlim A_i = \{(\ldots, a_i, \ldots) \in A \mid \phi_{kj}(a_k) = a_j \text{ whenever } j \leq k\}.$$

For each i, there is a map $\phi_i\colon \varprojlim A_i \to A_i$ induced by the projection $A \to A_i$. Clearly $\phi_{ji}\phi_j = \phi_i$.

Assume now that each A_i is a Hausdorff topological space. Then A is given the product topology and $\varprojlim A_i$ receives the topology it inherits from A.

We assume the maps ϕ_{ji} are continuous. The maps ϕ_i are always continuous: If U_i is open in A_i then $\phi_i^{-1}(U_i)$ is the intersection in A of an open set of A (definition of product topology) and $\varprojlim A_i$, hence open. The topology of $\varprojlim A_i$ is generated by unions and finite intersections of such sets $\phi_i^{-1}(U_i)$. In fact, every open set contains $\phi_k^{-1}(U_k)$ for some k and some U_k (proof: it suffices to show that $\phi_i^{-1}(U_i) \cap \phi_j^{-1}(U_j) = \phi_k^{-1}(U_k)$ for some k. Choose $k \geq i, j$ and let $U_k = \phi_{kj}^{-1}(U_j) \cap \phi_{ki}^{-1}(U_i)$).

We claim that $\varprojlim A_i$ is closed in A. Suppose $a = (\ldots, a_i, \ldots) \notin \varprojlim A_i$. Then $\phi_{ji}(a_j) \neq a_i$ for some i, j. Let U_1 and U_2 be neighborhoods of $\phi_{ji}(a_j)$ and a_i, respectively, such that $U_1 \cap U_2 = \varnothing$. Let $U_3 = \phi_{ji}^{-1}(U_1)$ and let

$$U = U_2 \times U_3 \times \prod_{k \neq i, j} A_k \subseteq A.$$

Then $a \in U$ but $U \cap \varprojlim A_i = \varnothing$. Since U is open, it follows that $\varprojlim A_i$ is closed.

Suppose now that each A_i is finite, with the discrete topology. Then A is compact, hence $\varprojlim A_i$ is compact. Also, $\varprojlim A_i$ can be shown to be nonempty and totally disconnected (the only connected sets are points). An inverse limit of finite sets is called *profinite*. If each A_i is a finite group and the maps ϕ_{ji} are homomorphisms, then $\varprojlim A_i$ is a compact group in the natural manner. It can be shown that all compact totally disconnected groups are profinite. Also, if G is profinite then $G = \varprojlim G/U$, where U runs through the open normal subgroups (necessarily of finite index, by compactness) of G, ordered by inclusion.

Examples. (1) Let I be the positive integers, $A_i = \mathbb{Z}/p^i\mathbb{Z}$, ϕ_{ji}: $a \bmod p^j \mapsto a \bmod p^i$. Then $\varprojlim \mathbb{Z}/p^i\mathbb{Z} = \mathbb{Z}_p$, the p-adic integers. The maps ϕ_i are the natural maps $\mathbb{Z}_p \to \mathbb{Z}/p^i\mathbb{Z}$. In essence, the ith component represents the ith partial sum of the p-adic expansion.

(2) Let I be the positive integers ordered by $m \leq n$ if $m \mid n$. If $m \mid n$, there is a natural map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. Let $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$. It can be shown, via the Chinese Remainder Theorem, that $\hat{\mathbb{Z}} \simeq \prod_{\text{all } p} \mathbb{Z}_p$.

For more on inverse limits, see Shatz [1] or any book on homological algebra.

§2 Infinite Galois Theory and Ramification Theory

Let K/k be an algebraic extension of fields and assume it is also Galois (normal, and generated by roots of separable polynomials). As usual, $G = \mathrm{Gal}(K/k)$ is the group of automorphisms of K which fix k pointwise. Suppose $k \subseteq F \subseteq K$ with F/k finite. Then $G_F = \mathrm{Gal}(K/F)$ is of finite index in G. The topology on G is defined by letting such G_F form a basis for the neighborhoods of the identity in G. Then G is profinite, and

$$G \simeq \varprojlim G/G_F \simeq \varprojlim \mathrm{Gal}(F/k),$$

where F runs through the normal finite subextensions F/k, or through any subsequence of such F such that $\bigcup F = K$. The ordering on the indices F is via inclusion ($F_1 \subseteq F_2$) and the maps used to obtain the inverse limit are the natural maps $\mathrm{Gal}(F_2/k) \to \mathrm{Gal}(F_1/k)$. The fundamental theorem of Galois theory now reads as follows:

There is a one-one correspondence between closed subgroups H of G and fields L with $k \subseteq L \subseteq K$:

$$H \leftrightarrow \textit{fixed field of } H,$$

$$\mathrm{Gal}(K/L) \leftrightarrow L.$$

Open subgroups correspond to finite extensions, normal subgroups correspond to normal extensions, etc.

EXAMPLES. (1) Consider $\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}$. An element $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ is determined by its action on ζ_{p^n} for all $n \geq 1$. For each n we have $\sigma\zeta_{p^n} = \zeta_{p^n}^{a_n}$ for some $a_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, and clearly $a_n \equiv a_{n-1} \bmod p^{n-1}$. So we obtain an element of

$$\mathbb{Z}_p^\times = \varprojlim(\mathbb{Z}/p^n\mathbb{Z})^\times = \varprojlim \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}).$$

Conversely, if $a \in \mathbb{Z}_p^\times$ then $\sigma\zeta_{p^n} = \zeta_{p^n}^a$ defines an automorphism. The closed (and open) subgroup $1 + p^n\mathbb{Z}_p$ corresponds to its fixed field $\mathbb{Q}(\zeta_{p^n})$.

(2) Let $\mathbb{F}$ be a finite field and let $\bar{\mathbb{F}}$ be its algebraic closure. For each n, there is a unique extension of $\mathbb{F}$ of degree n, and the Galois group is cyclic, generated by the Frobenius. Therefore

$$\mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F}) \simeq \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}.$$

Now suppose that k is an algebraic extension of $\mathbb{Q}$, not necessarily of finite degree. Let $\mathcal{O}_k$ be the ring of all algebraic integers in k and let $\not{p}$ be a nonzero prime ideal of $\mathcal{O}_k$. Then $\not{p} \cap \mathbb{Z}$ is nonzero (if $\alpha \in \not{p}$, $\mathrm{Norm}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) \in \not{p} \cap \mathbb{Z}$) and prime, hence $\not{p} \cap \mathbb{Z} = p\mathbb{Z}$ for some prime number p. Therefore

$$\mathbb{Z}/p\mathbb{Z} \simeq (\mathbb{Z} + \not{p})/\not{p} \subseteq \mathcal{O}_k/\not{p}.$$

It is easy to see that $\mathcal{O}_k/\not{p}$ is a field and is an algebraic extension of $\mathbb{Z}/p\mathbb{Z}$ (since $\mathcal{O}_k$ is integral over $\mathbb{Z}$). In fact, $\mathrm{Gal}((\mathcal{O}_k/\not{p})/(\mathbb{Z}/p\mathbb{Z}))$ is abelian since any finite extension of a finite field is cyclic, and an inverse limit of abelian groups is clearly abelian.

Let K/k be an algebraic extension, again not necessarily finite. Let $\mathcal{P}$ be a nonzero prime ideal of $\mathcal{O}_K$ and let $\not{p} = \mathcal{P} \cap \mathcal{O}_k$, which is a prime ideal of $\mathcal{O}_k$.

Then $\mathcal{O}_K/\mathcal{P}$ is an extension of $\mathcal{O}_k/\mathfrak{p}$; in fact, it is an abelian extension since $\mathcal{O}_K/\mathcal{P}$ is abelian over $\mathbb{Z}/p\mathbb{Z}$. Conversely, suppose we are given a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$. Then there exists $\mathcal{P}$ in $\mathcal{O}_K$ lying above $\mathfrak{p}$; that is, $\mathfrak{p} = \mathcal{P} \cap \mathcal{O}_k$ (see Lang [6], Chapter 9, Proposition 9; or Lang [1], Chapter 1, Proposition 9).

Lemma. *Suppose K/k is a Galois extension. Let $\mathcal{P}$ and $\mathcal{P}'$ be primes of K lying above $\mathfrak{p}$. Then there exists $\sigma \in \mathrm{Gal}(K/k)$ such that $\sigma\mathcal{P} = \mathcal{P}'$.*

Proof. We know the lemma is true for finite extensions (see Lang [6], Chapter 9, Proposition 11, or Lang [1], Chapter 1, Proposition 11). Choose a sequence of fields

$$k = F_0 \subseteq \cdots \subseteq F_n \subseteq \cdots \subseteq K$$

such that $K = \bigcup F_n$ and such that each F_n/k is a finite Galois extension. Such a sequence exists since the algebraic closure of $\mathbb{Q}$ is countable. Let

$$\mathfrak{p}_n = \mathcal{P} \cap \mathcal{O}_{F_n}, \qquad \mathfrak{p}_n' = \mathcal{P}' \cap \mathcal{O}_{F_n}.$$

Since F_n/k is finite, there exists $\tau_n \in \mathrm{Gal}(F_n/k)$ such that $\tau_n(\mathfrak{p}_n) = \mathfrak{p}_n'$. Let $\sigma_n \in \mathrm{Gal}(K/k)$ restrict to τ_n. Since $\mathrm{Gal}(K/k)$ is compact, the sequence $\{\sigma_n\}$ has a cluster point σ. There is a subsequence $\{\sigma_{n_i}\}$ which converges to σ (*a priori*, we would have to use a subnet. But subsequences suffice since $\mathrm{Gal}(K/k)$ satisfies the first countability axiom. This follows from the fact that the set of finite subextensions of K/k is countable). For simplicity, assume $\lim \sigma_n = \sigma$. Let m be arbitrary. Since $\mathrm{Gal}(K/F_m)$ is an open neighborhood of 1, $\sigma^{-1}\sigma_n \in \mathrm{Gal}(K/F_m)$ for $n \geq m$ sufficiently large. Hence, $\sigma^{-1}\sigma_n\mathfrak{p}_m = \mathfrak{p}_m$, so $\sigma\mathfrak{p}_m = \sigma_n\mathfrak{p}_m = \sigma_n(\mathfrak{p}_n \cap \mathcal{O}_{F_m}) = \mathfrak{p}_n' \cap \mathcal{O}_{F_m} = \mathfrak{p}_m'$. Since $\mathcal{P} = \bigcup \mathfrak{p}_m$ and $\mathcal{P}' = \bigcup \mathfrak{p}_m'$, we have $\sigma\mathcal{P} = \mathcal{P}'$. This completes the proof. □

We now want to discuss ramification. However, $\mathcal{O}_k$ and $\mathcal{O}_K$ are not necessarily Dedekind domains. For example, if $k = \mathbb{Q}(\zeta_{p^\infty})$ and $\mathfrak{p} = (\zeta_p - 1, \zeta_{p^2} - 1, \ldots)$ then $\mathfrak{p}^p = \mathfrak{p}$, since $(\zeta_{p^{n+1}} - 1)^p = (\zeta_{p^n} - 1)$. This means that we cannot define ramification via factorization of primes. Instead we use inertia groups. Let K/k be a Galois extension, as above, and let $\mathcal{P}$ lie above $\mathfrak{p}$. Define the *decomposition group* by

$$Z = Z(\mathcal{P}/\mathfrak{p}) = \{\sigma \in \mathrm{Gal}(K/k) \mid \sigma\mathcal{P} = \mathcal{P}\}.$$

We claim Z is closed, hence there is a corresponding fixed field. Let the notations be as in the proof of the lemma and let $Z_n = \{\sigma \mid \sigma(\mathfrak{p}_n) = \mathfrak{p}_n\}$. Then $Z \subseteq Z_n$ for all n, and since $\mathcal{P} = \bigcup \mathfrak{p}_n$ we have $Z = \cap Z_n$. Since $\mathrm{Gal}(K/F_n) \subseteq Z_n$, we have Z_n open, hence closed (it is the complement of its open cosets). Therefore Z is closed, as claimed.

Now define the *inertia group* by

$$T = T(\mathcal{P}/\mathfrak{p}) = \{\sigma \mid \sigma \in Z, \sigma(\alpha) \equiv \alpha \bmod \mathcal{P} \text{ for all } \alpha \in \mathcal{O}_K\}.$$

It is easy to show that T is a closed subgroup. As with the case of finite extensions, we have an exact sequence

$$1 \to T \to Z \to \mathrm{Gal}((\mathcal{O}_K/\mathscr{P})/(\mathcal{O}_k/\not{p})) \to 1.$$

The surjectivity may be proved by using the fact that we have surjectivity for finite extensions (Lang [1] or [6], Proposition 14).

Suppose now that K/k is an algebraic extension but not necessarily Galois. Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$. Then $\bar{\mathbb{Q}}/K$ and $\bar{\mathbb{Q}}/k$ are Galois extensions. Let $\mathscr{P}$ be a prime of K lying over the prime $\not{p}$ of k. Choose a prime ideal $\mathscr{D}$ of $\mathcal{O}_{\bar{\mathbb{Q}}}$ lying above $\mathscr{P}$. We have

$$T(\mathscr{D}/\not{p}) \subseteq \mathrm{Gal}(\bar{\mathbb{Q}}/k),$$
$$T(\mathscr{D}/\mathscr{P}) \subseteq \mathrm{Gal}(\bar{\mathbb{Q}}/K) \subseteq \mathrm{Gal}(\bar{\mathbb{Q}}/k),$$
$$T(\mathscr{D}/\mathscr{P}) = T(\mathscr{D}/\not{p}) \cap \mathrm{Gal}(\bar{\mathbb{Q}}/k).$$

Define the *ramification index* by

$$e(\mathscr{P}/\not{p}) = [T(\mathscr{D}/\not{p}) : T(\mathscr{D}/\mathscr{P})],$$

which is possibly infinite. If $\mathscr{D}'$ is another prime lying above $\mathscr{P}$ then $\mathscr{D}' = \sigma\mathscr{D}$ for some $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/K)$, and

$$T(\mathscr{D}'/\not{p}) = \sigma T(\mathscr{D}/\not{p})\sigma^{-1},$$
$$T(\mathscr{D}'/\mathscr{P}) = \sigma T(\mathscr{D}/\mathscr{P})\sigma^{-1}.$$

Therefore the index $e(\mathscr{P}/\not{p})$ does not depend on the choice of $\mathscr{D}$. If K/k is Galois then there is the natural restriction map

$$\mathrm{Gal}(\bar{\mathbb{Q}}/k) \to \mathrm{Gal}(K/k)$$

with kernel $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$. It is easy to see that the induced map $T(\mathscr{D}/\not{p}) \to T(\mathscr{P}/\not{p})$ is surjective, with kernel equal to $T(\mathscr{D}/\mathscr{P})$. Therefore

$$T(\mathscr{D}/\not{p})/T(\mathscr{D}/\mathscr{P}) \simeq T(\mathscr{P}/\not{p})$$

and

$$e(\mathscr{P}/\not{p}) = |T(\mathscr{P}/\not{p})|.$$

So the ramification index equals the order of the inertia group, for Galois extensions. It follows that the definition agrees with the usual one for finite extensions.

To consider archimedean primes, we proceed slightly differently. An archimedean place of k is either an embedding $\phi: k \to \mathbb{R}$ or a pair of complex-conjugate embeddings $(\psi, \bar{\psi})$, with $\bar{\psi} \neq \psi$ and $\psi: k \to \mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, any embedding ϕ or ψ may be extended to an embedding $\bar{\mathbb{Q}} \to \mathbb{C}$ (use Zorn's lemma). In particular, we can extend to K. If K/k is Galois and ϕ_1 and ϕ_2 are two extensions of ϕ, then $\phi_2^{-1}\phi_1 \in \mathrm{Gal}(K/k)$. Hence $\phi_1 = \phi_2\sigma$ for some σ. If $(\psi_1, \bar{\psi}_1)$ and $(\psi_2, \bar{\psi}_2)$ extend ϕ, we have $\psi_1 = \psi_2\sigma$, hence $(\psi_1, \bar{\psi}_1) = (\psi_2, \bar{\psi}_2)\sigma$, for some σ. A similar result holds for extensions of complex places, so the Galois group acts transitively on the extensions of a given place.

If K/k is Galois, w is an archimedean place of K, and v is the place of k below w, then we define

$$T(w/v) = Z(w/v) = \{\sigma \in \mathrm{Gal}(K/k) \mid w\sigma = w\}.$$

It is easy to see that T is nontrivial only when v is real, $w = (\psi, \bar{\psi})$ is complex, and $\sigma \neq 1$ is the "complex conjugation" $\psi^{-1}\bar{\psi}$ $(=\bar{\psi}^{-1}\psi)$, which permutes ψ and $\bar{\psi}$ and has order 2. Therefore

$$|T(w/v)| = 1 \text{ or } 2.$$

We may now define the ramification indices for archimedean primes just as we did for finite primes.

For more on the above, see Iwasawa [6], §6.

§3 Class Field Theory

This section consists of three subsections. The first treats global class field theory from the classical viewpoint of ideal groups. The second discusses local class field theory. In the third, we return to the global case, this time using the language of idèles.

We only consider some of the highlights of the theory and give no indications of the proofs. The interested reader can consult, for example, Lang [1], Neukirch [1], Hasse [2], or the articles by Serre and Tate in Cassels and Fröhlich [1].

Global Class Field Theory (first form)

Let k be a number field of finite degree over $\mathbb{Q}$. Let $\mathfrak{M}_0 = \prod \mathfrak{p}_i^{e_i}$ denote an integral ideal of k and let $\mathfrak{M}_\infty$ denote a formal squarefree product (possibly empty) of real archimedean places of k. Then $\mathfrak{M} = \mathfrak{M}_0 \mathfrak{M}_\infty$ is called a *divisor* of k. For example, $\mathfrak{M} = 1$, $\mathfrak{M} = \infty$, $\mathfrak{M} = 5^3 \cdot 17^2 \cdot \infty$, and $\mathfrak{M} = 3 \cdot 37 \cdot 103$ are divisors of $\mathbb{Q}$. If $\alpha \in k^\times$, then we write $\alpha \equiv 1 \bmod^* \mathfrak{M}$ if (i) $v_{\mathfrak{p}_i}(\alpha - 1) \geq e_i$ for all primes $\mathfrak{p}_i$ (with $e_i > 0$) in the factorization of $\mathfrak{M}_0$, and (ii) $\alpha > 0$ at the real embeddings corresponding to the archimedean places in $\mathfrak{M}_\infty$. Let $P_\mathfrak{M}$ denote the group of principal fractional ideals of k which have a generator $\alpha \equiv 1 \bmod^* \mathfrak{M}$. Let $I_\mathfrak{M}$ be the group of fractional ideals relatively prime to $\mathfrak{M}$ (note that $I_\mathfrak{M} = I_{\mathfrak{M}_0}$). The quotient $I_\mathfrak{M}/P_\mathfrak{M}$ is a finite group, called the generalized ideal class group mod $\mathfrak{M}$.

For example, let $k = \mathbb{Q}$, let n be a positive integer, and let $\mathfrak{M} = n$. The group I_n consists of ideals generated by rational numbers relatively prime to

n. Let (r) be such an ideal. Then (r) is generated by $+r$ and by $-r$. If $(r) \in P_n$ then we must have $\pm r \equiv 1 \bmod n$, hence $r \equiv \pm 1 \bmod n$. It follows that

$$I_n/P_n \simeq (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\}.$$

Now suppose $\mathfrak{M} = n\infty$. The group $I_{n\infty}$ is the same as I_n, but if $(r) \in P_{n\infty}$ then we must be able to take a *positive* generator congruent to 1 mod n, so we need $|r| \equiv 1 \bmod n$. If $|r| \equiv -1 \bmod n$ then $(r) \notin P_{n\infty}$ (unless $n = 2$), so the archimedean factor makes $P_{\mathfrak{M}}$ smaller. It follows easily that

$$I_{n\infty}/P_{n\infty} \simeq (\mathbb{Z}/n\mathbb{Z})^\times.$$

The effect of the archimedean primes is apparent in the case of a real quadratic field k. Let $\mathfrak{M}_0 = 1$ and let $\mathfrak{M}_\infty = \infty_1\infty_2$ be the product of the two (real) archimedean places. Suppose the fundamental unit ε has norm -1, so ε is positive at one place and negative at the other. Let $(\alpha) = (-\alpha) = (\varepsilon\alpha) = (-\varepsilon\alpha)$ be a principal ideal of k. One of the generators for (α) is positive at both ∞_1 and ∞_2, so every principal ideal has a totally positive generator, and $P = P_1 = P_{\infty_1\infty_2}$. Of course,

$$I_1/P_1 = \text{ideal class group}.$$

By definition,

$$I_{\infty_1\infty_2}/P_{\infty_1\infty_2} = \text{narrow ideal class group}.$$

So we find that the narrow and ordinary class groups are the same. It will follow from subsequent theorems that the narrow ideal class group corresponds to the maximal abelian extension of k which is unramified at all finite places.

Now suppose ε has norm $+1$. Choose $\alpha \in k$ such that $\alpha > 0$ at ∞_1 and $\alpha < 0$ at ∞_2 (for example, $\alpha = 1 + \sqrt{d}$). Then (α) has no totally positive generator, hence $P_{\infty_1\infty_2} \neq P_1$ (the index is easily seen to be 2). Therefore the narrow ideal class group is twice as large as the ordinary class group in this case.

We return to the general situation, so k is a number field of finite degree over $\mathbb{Q}$. Let $\mathcal{O}_k$ denote the ring of integers of k. Consider a finite Galois extension K/k. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_k$ and $\mathscr{P}$ a prime of $\mathcal{O}_K$ above $\mathfrak{p}$. Let $N\mathfrak{p} = |\mathcal{O}_k/\mathfrak{p}| =$ norm to $\mathbb{Q}$ of $\mathfrak{p}$. The finite field $\mathcal{O}_K/\mathscr{P}$ is a finite extension of $\mathcal{O}_k/\mathfrak{p}$ with Galois group generated by the Frobenius $(x \mapsto x^{N\mathfrak{p}})$. Let $Z(\mathscr{P}/\mathfrak{p})$ be the decomposition group and $T(\mathscr{P}/\mathfrak{p})$ the inertia group. There is an exact sequence

$$1 \to T(\mathscr{P}/\mathfrak{p}) \to Z(\mathscr{P}/\mathfrak{p}) \to \operatorname{Gal}((\mathcal{O}_K/\mathscr{P})/(\mathcal{O}_k/\mathfrak{p})) \to 1.$$

Suppose $\mathscr{P}$ is unramified over $\mathfrak{p}$. Then $T = 1$, so Z is cyclic, generated by the (global) Frobenius $\sigma_{\mathscr{P}}$, which is uniquely determined by the relation

$$\sigma_{\mathscr{P}}\, x \equiv x^{N\mathfrak{p}} \bmod \mathscr{P} \quad \text{for all } x \in \mathcal{O}_K.$$

Suppose τ is an automorphism of K such that $\tau(k) = k$. Then $\tau\mathscr{P}$ is unramified over $\tau\not{p}$. Since $\sigma_{\mathscr{P}}\tau^{-1}x \equiv (\tau^{-1}x)^{N\not{p}} \bmod \mathscr{P}$, we have $\tau\sigma_{\mathscr{P}}\tau^{-1}x \equiv x^{N\not{p}} \bmod \tau\mathscr{P}$. Since $N\not{p} = N\tau\not{p}$, we obtain

$$\sigma_{\tau\mathscr{P}} = \tau\sigma_{\mathscr{P}}\tau^{-1}.$$

If K/k is abelian then $\sigma_{\tau\mathscr{P}} = \sigma_{\mathscr{P}}$ for all $\tau \in \mathrm{Gal}(K/k)$. Hence $\sigma_{\mathscr{P}}$ depends only on the prime $\not{p}$ of k, so we let

$$\sigma_{\not{p}} = \sigma_{\mathscr{P}}.$$

We may extend by multiplicativity to obtain a map, called the *Artin map*,

$$I_{\mathfrak{d}} \to \mathrm{Gal}(K/k),$$

where $\mathfrak{d}$ is the relative discriminant of K/k. What are the kernel and image?

Theorem 1. *Let K/k be a finite abelian extension. Then there exists a divisor $\mathfrak{f}$ of k (the minimal such divisor is called the conductor of K/k) such that the following hold:*

(i) *a prime $\not{p}$ (finite or infinite) ramifies in $K/k \Leftrightarrow \not{p} \mid \mathfrak{f}$.*

(ii) *If $\mathfrak{M}$ is a divisor with $\mathfrak{f} \mid \mathfrak{M}$ then there is a subgroup H with $P_{\mathfrak{M}} \subseteq H \subseteq I_{\mathfrak{M}}$ such that*

$$I_{\mathfrak{M}}/H \simeq \mathrm{Gal}(K/k),$$

the isomorphism being induced by the Artin map. In fact, $H = P_{\mathfrak{M}} N_{K/k}(I_{\mathfrak{M}}(K))$, where $I_{\mathfrak{M}}(K)$ is the group of ideals of K relatively prime to $\mathfrak{M}$.

Theorem 2. *Let $\mathfrak{M}$ be a divisor for k and let H be a subgroup of $I_{\mathfrak{M}}$ with $P_{\mathfrak{M}} \subseteq H \subseteq I_{\mathfrak{M}}$. Then there exists a unique abelian extension K/k, ramified only at primes dividing $\mathfrak{M}$ (however, some primes dividing $\mathfrak{M}$ could be unramified), such that $H = P_{\mathfrak{M}} N_{K/k}(I_{\mathfrak{M}}(K))$ and*

$$I_{\mathfrak{M}}/H \simeq \mathrm{Gal}(K/k)$$

under the Artin map.

Theorem 3. *Let K_1/k and K_2/k be abelian extensions of conductors $\mathfrak{f}_1$ and $\mathfrak{f}_2$, let $\mathfrak{M}$ be a multiple of $\mathfrak{f}_1$ and $\mathfrak{f}_2$, and let $H_1, H_2 \subseteq I_{\mathfrak{M}}$ be the corresponding subgroups. Then*

$$H_1 \subseteq H_2 \Leftrightarrow K_1 \supseteq K_2.$$

The above theorems summarize the most basic facts. We now derive some consequences.

In Theorem 2, let $\mathfrak{M} = 1$ and let $H = P_{\mathfrak{M}} = P$. We obtain an abelian extension K/k with

$$\mathrm{Gal}(K/k) \simeq I/P \simeq \text{ideal class group of } k.$$

By Theorem 1(i), K/k is unramified, and any unramified abelian extension has $\mathfrak{f} = 1$ and corresponds to a subgroup containing $P_1 = P$. By Theorem 3, K is maximal, so we have proved the following important result.

Theorem 4. *Let k be a number field and let K be the maximal unramified (including ∞) abelian extension of k. Then*

$$\mathrm{Gal}(K/k) \simeq \textit{ideal class group of } k,$$

the isomorphism being induced by the Artin map. (The field K is called the Hilbert class field of k).

We note an interesting consequence. Let $\mathfrak{p}$ be a prime ideal of k. Then $\mathfrak{p}$ splits completely in the Hilbert class field $\Leftrightarrow$ the decomposition group for $\mathfrak{p}$ is trivial $\Leftrightarrow \sigma_{\mathfrak{p}} = 1 \Leftrightarrow \mathfrak{p} \in P \Leftrightarrow \mathfrak{p}$ is principal.

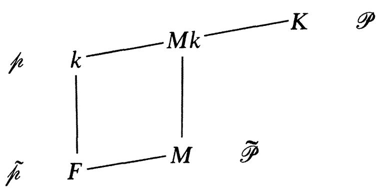
Similarly, for a prime number p, we may choose $H \supseteq P$ such that $H/P =$ non-p-part of I/P. Then $I/H \simeq p$-Sylow subgroup of I/P. The field (= Hilbert p-class field) corresponding to H is the maximal unramified abelian p-extension of k.

We now justify a statement made in Section 10.2. Let K be the Hilbert class field (or p-class field) of k, let $F \subseteq k$, and suppose k/F is Galois. Then K/F is also Galois, by the maximality of K. As in Chapter 10, $G = \mathrm{Gal}(k/F)$ acts on $\mathrm{Gal}(K/k)$ (let $\tau \in G$; extend to $\tilde{\tau} \in \mathrm{Gal}(K/F)$; then $\sigma^{\tau} = \tilde{\tau}\sigma\tilde{\tau}^{-1}$). Also, G acts on the ideal class group of k. Let $\mathfrak{p}$ be a prime ideal of k. Then $\mathfrak{p} \mapsto \sigma_{\mathfrak{p}}$ under the Artin map, and $\tau\mathfrak{p} \mapsto \sigma_{\tau\mathfrak{p}} = \tilde{\tau}\sigma_{\mathfrak{p}}\tilde{\tau}^{-1} = (\sigma_{\mathfrak{p}})^{\tau}$, by a formula preceding Theorem 1. Therefore

$$\mathrm{Gal}(K/k) \simeq \text{ideal class group of } k$$

as $\mathrm{Gal}(k/F)$-modules, as was claimed in Chapter 10.

We now need another property of the Artin map. Suppose we have fields F, k, M, and K, as in the diagram, with K/k and M/F abelian.



(we do not assume $M \cap k = F$). Let $\mathfrak{p}$ be a prime ideal of k, unramified in K/k, and let $\mathfrak{P}$ lie above $\mathfrak{p}$. Similarly, let $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{P}}$ be the primes of F and M lying below $\mathfrak{p}$ and $\mathfrak{P}$, respectively. We also assume that $\tilde{\mathfrak{p}}$ is unramified in M/F. Let $f = [\mathcal{O}_k/\mathfrak{p} : \mathcal{O}_F/\tilde{\mathfrak{p}}]$ be the residue class degree. Then $\mathrm{Norm}_{k/F}\,\mathfrak{p} = \tilde{\mathfrak{p}}^f$ and $N\mathfrak{p} = (N\tilde{\mathfrak{p}})^f$. Since $\mathcal{O}_M \subseteq \mathcal{O}_K$, we have

$$\sigma_{\mathfrak{p}}^{K/k}|_M x \equiv x^{N\mathfrak{p}} \bmod \tilde{\mathfrak{P}}, \quad \text{for } x \in \mathcal{O}_M.$$

We have used the notation $\sigma_{\mathfrak{p}}^{K/k}|_M$ to mean "$\sigma_{\mathfrak{p}}$ for the extension K/k, restricted to M." But

$$\sigma_{\mathrm{Norm}\,\mathfrak{p}}^{M/F}x = (\sigma_{\tilde{\mathfrak{p}}}^{M/F})^f x \equiv x^{N\tilde{\mathfrak{p}}^f} = x^{N\mathfrak{p}} \bmod \tilde{\mathscr{P}}.$$

Therefore

$$\sigma_{\mathfrak{p}}^{K/k}|_M = \sigma_{\mathrm{Norm}\,\mathfrak{p}}^{M/F}.$$

We give an application. Suppose M is the Hilbert class field of F and K is the Hilbert class field of k. Furthermore, assume $M \cap k = F$. Then $\mathrm{Gal}(Mk/k) \simeq \mathrm{Gal}(M/F)$, via restriction; hence $\mathrm{Gal}(K/k) \to \mathrm{Gal}(M/F)$ surjectively via restriction. We have the following diagram ($I_k/P_k =$ ideal class group of k, etc.):

$$\begin{array}{ccc} I_k/P_k & \xrightarrow{\sim} & \mathrm{Gal}(K/k) \\ \Big\downarrow \text{Norm} & & \Big\downarrow \text{restr.} \\ I_F/P_F & \xrightarrow{\sim} & \mathrm{Gal}(M/F). \end{array}$$

The horizontal maps are the Artin maps. The diagram commutes by what we just proved. Since our assumptions imply that the arrow on the right is surjective, Norm is also surjective. So we have proved the following.

Theorem 5 (= Theorem 10.1). *Suppose the extension of number fields k/F contains no unramified abelian subextensions L/K with $L \neq K$. Then the norm map from the ideal class group of k to the ideal class group of F is surjective and the class number h_F divides h_k.*

We now relate the above theorems to abelian extensions of $\mathbb{Q}$. Let n be a positive integer and consider $\mathbb{Q}(\zeta_n)$. Let $p \nmid n$. As we showed in Chapter 2, the Frobenius σ_p is given by $\sigma_p(\zeta_n) = \zeta_n^p$. Thus we have a map

$$I_n \to \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}.).$$

If $(a, n) = 1$ and $a > 0$, then $(a) \mapsto \sigma_a$, so the map is surjective (in fact, by Dirichlet's theorem, it is surjective when restricted to prime ideals). We now determine the kernel. Let $r \in \mathbb{Q}$ with $(r) \in I_n$. Write $|r| = \prod p_i^{b_i}$. Then, as ideals, $(r) = \prod (p_i)^{b_i}$, so

$$\sigma_{(r)} = \prod \sigma_{p_i}^{b_i} = \sigma_{|r|},$$

where $\sigma_{|r|}(\zeta_n) = \zeta_n^{|r|}$ ($|r| \bmod n$ is a well-defined element of $(\mathbb{Z}/n\mathbb{Z})^\times$). Therefore

$$\begin{aligned} \sigma_{(r)} = 1 &\Leftrightarrow |r| \equiv 1 \bmod n \\ &\Leftrightarrow (r) \in P_{n\infty}. \end{aligned}$$

Since $I_n = I_{n\infty}$, we obtain

$$I_{n\infty}/P_{n\infty} \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$$

under the Artin map. This of course agrees with the fact that $I_{n\infty}/P_{n\infty} \simeq (\mathbb{Z}/n\mathbb{Z})^\times$.

What happens if we leave off ∞ and consider I_n/P_n? By Theorem 1(i), we cannot have ramification at ∞ and it is not hard to show that the corresponding field is $\mathbb{Q}(\zeta_n)^+$. This agrees with our previous calculation that $I_n/P_n \simeq (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\}$.

Suppose now that K is a number field and $K/\mathbb{Q}$ is abelian. By Theorem 1, there exists a divisor $\mathfrak{M}$ and a subgroup H with $P_{\mathfrak{M}} \subseteq H \subseteq I_{\mathfrak{M}}$. We may assume $\mathfrak{M} = n\infty$, with $n \in \mathbb{Z}$. By Theorem 3, K is contained in the field corresponding to $P_{n\infty}$, namely $\mathbb{Q}(\zeta_n)$. We obtain the following.

Theorem 6 (Kronecker–Weber). *Let K be an abelian extension of $\mathbb{Q}$. Then K is contained in a cyclotomic field.*

Let $K/\mathbb{Q}$ be abelian and let $H \supseteq P_{n\infty}$ be the corresponding subgroup. Since

$$I_{n\infty}/P_{n\infty} \simeq (\mathbb{Z}/n\mathbb{Z})^\times,$$

the group $H/P_{n\infty}$ corresponds to a subgroup of congruence classes mod n. Since

$$(p) \text{ splits completely} \Leftrightarrow \sigma_p = 1 \Leftrightarrow (p) \in H,$$

we find that the primes that split completely are determined by congruence conditions mod n. In fact, this property characterizes abelian extensions.

Let $p \equiv 1 \pmod 4$ and let $q \neq p$ be an odd prime. Then q splits in $\mathbb{Q}(\sqrt{p}) \Leftrightarrow (p/q) = 1 \Leftrightarrow$ (by Quadratic Reciprocity) $(q/p) = 1 \Leftrightarrow q$ is a square mod p, which is equivalent to q lying in certain congruence classes mod p. Let $\{1, \tau\} = \mathrm{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q})$. Since q splits $\Leftrightarrow \sigma_q = 1$, we have shown that $\sigma_q = 1$ if q is a square mod p, $\sigma_q = \tau$ if not. Now let $r \in \mathbb{Q}$ with $(r) \in I_p$ (i.e., $(r, p) = 1$). Write $|r| = \prod q^b$ and $\sigma_{(r)} = \prod \sigma_q^b$. It is easy to see that

$$\sigma_{(r)} = 1 \Leftrightarrow |r| \text{ is a square mod } p$$
$$\Leftrightarrow r \text{ is a square mod } p$$

(since $p \equiv 1 \bmod 4$). Let H denote the group of ideals in I_p generated by squares mod p. We have shown (the main step was Quadratic Reciprocity) that H is the kernel of the Artin map. In particular,

$$P_p \subseteq H.$$

Conversely, the fact that $P_p \subseteq H$ implies Quadratic Reciprocity for p: Since $H \subset I_p$ has index 2, it must consist of the squares mod p, because

$$I_p/P_p \simeq (\mathbb{Z}/p\mathbb{Z})^\times/\{\pm 1\}$$

is cyclic. Therefore

$$\left(\frac{p}{q}\right) = 1 \Leftrightarrow q \text{ splits} \Leftrightarrow \sigma_q = 1 \Leftrightarrow q \text{ is a square mod } p$$

$$\Leftrightarrow \left(\frac{q}{p}\right) = 1.$$

In general, the fact that the kernel of the Artin map contains $P_{\mathfrak{M}}$ (Theorem 1(ii)) is one of the most important parts of the theory. For example, it was the major step in the above proof of the Kronecker–Weber theorem.

Local Class Field Theory

Let k be a finite extension of $\mathbb{Q}_p$. We may write

$$k^\times = \pi^{\mathbb{Z}} \times U = \pi^{\mathbb{Z}} \times W' \times U_1,$$

where π = a uniformizing parameter for k,
$\pi^{\mathbb{Z}} = \{\pi^n \mid n \in \mathbb{Z}\}$,
U = local units,
W' = the roots of unity in k of order prime to p,
$U_1 = \{x \in U \mid x \equiv 1 \bmod \pi\}$.

Theorem 7. *Let K/k be a finite abelian extension. There is a map (called the Artin map)*

$$k^\times \to \mathrm{Gal}(K/k)$$

$$a \mapsto (a, K/k)$$

which induces an isomorphism

$$k^\times / N_{K/k} K^\times \simeq \mathrm{Gal}(K/k),$$

where $N_{K/k}$ denotes the norm mapping. Let T denote the inertia subgroup of Gal(K/k). *Then*

$$U_k / N_{K/k} U_K \simeq T.$$

If K/k is unramified then Gal(K/k) *is cyclic, generated by the Frobenius F, and*

$$(a, K/k) = F^{v_\pi(a)},$$

Theorem 8. *Let $H \subseteq k^\times$ be an open subgroup of finite index. Then there exists a unique abelian extension K/k such that $H = N_{K/k} K^\times$.*

Theorem 9. *Let K_1 and K_2 be finite abelian extensions of k. Then $K_1 \subseteq K_2 \Leftrightarrow N_{K_1/k} K_1^\times \supseteq N_{K_2/k} K_2^\times$.*

The Artin map satisfies the expected properties. For example, if σ is an automorphism of the algebraic closure of k then

$$(\sigma a, \sigma K/\sigma k) = \sigma(a, K/k)\sigma^{-1}.$$

Also, if K/k and M/F are abelian, with $F \subseteq k$ and $M \subseteq K$ (see the diagram in the previous subsection), then, for $a \in k^\times$,

$$(a, K/k)|_M = (N_{k/F}a, M/F).$$

The above theorems may be modified to include infinite abelian extensions K/k. Let $\hat{k}^\times$ be the profinite completion of $k^\times$. This means

$$\hat{k}^\times \stackrel{\text{def}}{=} \varprojlim k^\times/H$$

where H runs through (a cofinal subsequence of) open subgroups of finite index. Write $k^\times \simeq \pi^{\mathbb{Z}} \times W' \times U_1$, as above, and let H be of finite index. By taking a smaller H if necessary, we may assume

$$k^\times/H \simeq (\mathbb{Z}/m\mathbb{Z}) \times W' \times U_1/U_1^{p^n}$$

for some m and n. It is easy to see that

$$U_1 = \varprojlim U_1/U_1^{p^n}, \qquad W' = \varprojlim W'.$$

But

$$\varprojlim \mathbb{Z}/m\mathbb{Z} = \hat{\mathbb{Z}} \simeq \prod_p \mathbb{Z}_p$$

(see the section on inverse limits). Therefore, we may formally write

$$\hat{k}^\times \simeq \pi^{\hat{\mathbb{Z}}} \times W' \times U_1 \simeq \pi^{\hat{\mathbb{Z}}} \times U.$$

Theorem 10. *Let k be a finite extension of $\mathbb{Q}_p$ and let k^{ab} denote the maximal abelian extension of k. There is a continuous isomorphism*

$$\hat{k}^\times \simeq \mathrm{Gal}(k^{ab}/k).$$

This induces a one–one correspondence between abelian extensions K/k and closed subgroups $H \subseteq \hat{k}^\times$. If H corresponds to K,

$$\hat{k}^\times/H \simeq \mathrm{Gal}(K/k).$$

Let $\tilde{N}_{K/k}(U_K) = \bigcap_L N_{L/k}(U_L)$, where L runs through all finite subextensions of K/k. Then

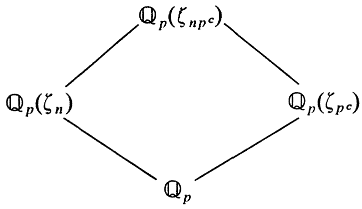
$$U_k/\tilde{N}_{K/k}(U_K) \simeq T(K/k),$$

the inertia subgroup of Gal(K/k).

We give an example. Let $k = \mathbb{Q}_p$. Then

$$\mathbb{Q}_p^\times \simeq p^{\mathbb{Z}} \times W_{p-1} \times (1 + p\mathbb{Z}_p) \simeq p^{\mathbb{Z}} \times \mathbb{Z}_p^\times.$$

Let $(n, p) = 1$ and let $c \geq 0$. We have the following diagram:



Let $a = p^b u \in \mathbb{Q}_p^\times$. Then

$$\begin{aligned}(a, \mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p) &= (p^b, \mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p)\\ &= F^b\colon \zeta_n \mapsto \zeta_n^{p^b}\end{aligned}$$

(F = Frobenius). The group U maps to the inertia subgroup, which is isomorphic to $\text{Gal}(\mathbb{Q}_p(\zeta_{p^c})/\mathbb{Q}_p)$. It can be shown that $(u, \mathbb{Q}(\zeta_{np^c})/\mathbb{Q}_p)$ yields the map $\zeta_{p^c} \mapsto \zeta_{p^c}^{u^{-1}}$, where $\zeta_{p^c}^{u^{-1}}$ is defined in the usual manner. It is now easy to see that W_{p-1} corresponds to the (tamely ramified) extension $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$ and that $1 + p\mathbb{Z}_p$ corresponds to the (wildly ramified) extension $\mathbb{Q}_p(\zeta_{p^c})/\mathbb{Q}_p(\zeta_p)$.

Now consider the infinite extension $\mathbb{Q}_p^{ab}/\mathbb{Q}_p$. We have

$$\text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \simeq \hat{\mathbb{Q}}_p^\times \simeq p^{\hat{\mathbb{Z}}} \times \mathbb{Z}_p^\times$$

We know (Chapter 14) that

$$\begin{aligned}\mathbb{Q}_p^{ab} &= \mathbb{Q}_p(\zeta_3, \zeta_4, \ldots)\\ &= \mathbb{Q}_p(\zeta_{p^\infty})\mathbb{Q}_p(\{\zeta_n \mid (p, n) = 1\}).\end{aligned}$$

We have

$$\text{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \simeq \mathbb{Z}_p^\times.$$

Since Galois groups of unramified extensions are isomorphic to Galois groups of extensions of finite fields, it follows that

$$\text{Gal}(\mathbb{Q}_p(\{\zeta_n \mid (p, n) = 1\})/\mathbb{Q}_p) \simeq \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \simeq \hat{\mathbb{Z}} \simeq p^{\hat{\mathbb{Z}}}.$$

Global Class Field Theory (second form)

Let k be a number field and let $\mathfrak{p}$ be a prime (finite or infinite) of k. Let $k_\mathfrak{p}$ and $U_\mathfrak{p}$ denote the completion of k at $\mathfrak{p}$ and the local units of $k_\mathfrak{p}$, respectively. If $\mathfrak{p}$ is archimedean, let $U_\mathfrak{p} = k_\mathfrak{p}^\times$. Define the *idèle group* of k by

$$J_k = \{(\ldots, x_\mathfrak{p}, \ldots) \in \prod_p k_\mathfrak{p}^\times \mid x_\mathfrak{p} \in U_\mathfrak{p} \text{ for almost all } \mathfrak{p}\}$$

("almost all" means "for all but finitely many"). Topologize J_k by giving

$$U = \prod U_\mathfrak{p}$$

the product topology and letting U be an open set of J_k. Then J_k becomes a locally compact group.

It is easy to see that there is an embedding

$$k^\times \hookrightarrow J_k$$

(diagonally) and it can be shown that the image is discrete. The image is called the subgroup of principal idèles. Let

$$C_k = J_k/k^\times$$

be the group of idèle classes.

Let K/k be a finite extension. If $\mathscr{P}$ is a prime of K above the prime $\not{p}$ of k, then we have a norm map on the completions $N_{\mathscr{P}/\not{p}}\colon K_{\mathscr{P}} \to k_{\not{p}}$. Let $x = (\ldots, x_{\mathscr{P}}, \ldots) \in J_K$. Define

$$N_{K/k}(x) = (\ldots, y_{\not{p}}, \ldots) \in J_k,$$

where

$$y_{\not{p}} = \prod_{\mathscr{P} | \not{p}} N_{\mathscr{P}/\not{p}} x_{\mathscr{P}}.$$

It is not hard to show that if $x = (\ldots, x, \ldots)$ is principal, then $N_{K/k}x = (\ldots, N_{K/k}x, \ldots)$, which is also principal. Therefore we have a map

$$N_{K/k}\colon C_K \to C_k.$$

Theorem 11. *Let K/k be a finite abelian extension. There is an isomorphism*

$$J_k/k^\times N_{K/k}J_K = C_k/N_{K/k}C_K \simeq \mathrm{Gal}(K/k).$$

The prime $\not{p}$ (finite or infinite) is unramified in $K/k \Leftrightarrow U_{\not{p}} \subseteq k^\times N_{K/k}J_K$. ($U_{\not{p}}$ embeds in J_k via $u_{\not{p}} \mapsto (1, \ldots, u_{\not{p}}, \ldots, 1)$).

Theorem 12. *If H is an open subgroup of C_k of finite index then there is a unique abelian extension K/k such that $N_{K/k}C_K = H$. Equivalently, if H is open of finite index in J_k, and $k^\times \subseteq H$, then there exists a unique abelian extension K/k such that $k^\times N_{K/k}J_K = H$.*

Theorem 13. *Let K_1 and K_2 be finite abelian extensions of k. Then*

$$K_1 \subseteq K_2 \Leftrightarrow k^\times N_{K_1/k}J_{K_1} \supseteq k^\times N_{K_2/k}J_{K_2}.$$

The above theorems may also be stated for infinite extensions. Let D_k denote the connected component of the identity in C_k.

Theorem 14. (a) *If K/k is abelian, then there is a closed subgroup H with $D_k \subseteq H \subseteq C_k$, such that*

$$C_k/H \Leftrightarrow \mathrm{Gal}(K/k).$$

The prime $\not{p}$ is unramified $\Leftrightarrow k^\times U_{\not{p}}/k^\times \subseteq H$.

(b) *Given a closed subgroup* H *with* $D_k \subseteq H \subseteq C_k$ *(equivalently,* C_k/H *is totally disconnected), there is a unique abelian extension corresponding to* H*, as in* (a).

As a simple example, let K be the Hilbert class field of k. Since K/k is unramified everywhere, $U = \prod U_{\mathfrak{p}} \subseteq k^\times N_{K/k} J_K$. Since K is maximal, $k^\times U$ is the subgroup corresponding to K, hence

$$J_k/k^\times U \simeq \mathrm{Gal}(K/k).$$

There is a natural map

$$J_k \to \text{ideals of } k$$
$$(\ldots, x_{\mathfrak{p}}, \ldots) \mapsto \prod_{\text{finite } \mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}.$$

The kernel is U. If we consider the induced map to the ideal class group, we obtain

$$J_k/k^\times U \simeq \text{ideal class group of } k.$$

Therefore $\mathrm{Gal}(K/k)$ is isomorphic to the ideal class group, as we showed previously.

Tables

§1 Bernoulli Numbers

This table from H. Davis [1], pp. 230–231, gives the value of $(-1)^{n+1}B_{2n}$ for $1 \leq n \leq 62$. In this book we have numbered the Bernoulli numbers so that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_{2n+1} = 0$ for $n \geq 1$. Some authors use different numbering systems and a different choice of signs. For more Bernoulli numbers, see H. Davis [1] and Knuth–Buckholtz [1]. For prime factorizations, see Wagstaff [1].

n	Numerator	Denominator	n
1	1	6	1
2	1	30	2
3	1	42	3
4	1	30	4
5	5	66	5
6	691	2730	6
7	7	6	7
8	3617	510	8
9	43867	798	9
10	1 74611	330	10
11	8 54513	138	11
12	2363 64091	2730	12
13	85 53103	6	13
14	2 37494 61029	870	14
15	861 58412 76005	14322	15
16	770 93210 41217	510	16
17	257 76878 58367	6	17
18	26315 27155 30534 77373	1919190	18
19	2 92999 39138 41559	6	19
20	2 61082 71849 64491 22051	13530	20

n	Numerator	Denominator	n
21	15 20097 64391 80708 02691	1806	21
22	278 33269 57930 10242 35023	690	22
23	5964 51111 59391 21632 77961	282	23
24	560 94033 68997 81768 62491 27547	46410	24
25	49 50572 05241 07964 82124 77525	66	25
26	80116 57181 35489 95734 79249 91853	1590	26
27	29 14996 36348 84862 42141 81238 12691	798	27
28	2479 39292 93132 26753 68541 57396 63229	870	28
29	84483 61334 88800 41862 04677 59940 36021	354	29
30	121 52331 40483 75557 20403 04994 07982 02460 41491	56786730	30
31	123 00585 43408 68585 41953 03985 74033 86151	6	31
32	10 67838 30147 86652 98863 85444 97914 26479 42017	510	32
33	1 47260 00221 26335 65405 16194 28551 93234 22418	64722	33
	99101 … … … … … … … …		
34	7877 31308 58718 72814 19091 49208 47460 62443 47001	30	34
35	1505 38134 73333 67003 80307 65673 77857 20851 14381	4686	35
	60235 … … … … … … … …		
36	58279 54961 66994 41104 38277 24464 10673 65282 48830	140100870	36
	18442 60429 … … … … … … …		
37	34152 41728 92211 68014 33007 37314 72635 18668 83077	6	37
	83087 … … … … … … … …		
38	246 55088 82593 53727 07687 19604 05851 99904 36526	30	38
	78288 65801 … … … … …		
39	41 48463 65575 40082 82951 79035 54954 20734 92199	3318	39
	37537 24004 83487 … … … … … …		
40	4 60378 42994 79457 64693 55749 69019 04684 97942	230010	40
	57872 75128 89196 56867 … … … … …		
41	1 67701 41491 85145 83682 31545 09786 26990 02077	498	41
	36027 57025 34148 81613 … … … … …		
42	20 24576 19593 52903 60231 13116 01117 31009 98991	3404310	42
	73911 98090 87728 10839 32477 … … … …		
43	660 71461 94176 78653 57384 78474 26261 49627 78306	6	43
	86653 38893 17619 96983 … … … … …		
44	13114 26488 67401 75079 95511 42401 93118 43345 75027	61410	44
	55720 28644 29691 98905 74047 … … … …		
45	117 90572 79021 08279 98841 23351 24921 50837 75254	272118	45
	94966 96471 16231 54521 57279 22535 … … …		
46	129 55859 48207 53752 79894 27828 53857 67496 59341	1410	46
	48371 94351 43023 31632 68299 46247 … … …		
47	122 08138 06579 74446 96073 01679 41320 12039 58508	6	47
	41520 26966 21436 21510 52846 49447 … … …		
48	2 11600 44959 72665 13097 59772 81098 24233 67304	4501770	48
	39543 89060 23415 06387 33420 05066 83499 87259 …		
49	67 90826 06729 05495 62405 11175 46403 60560 73421	6	49
	95728 50448 75090 73961 24999 29470 58239 … …		
50	945 98037 81912 21252 95227 43306 94937 21872 70284	33330	50
	15330 66936 13338 56962 04311 39541 51972 47711 …		
51	32040 19410 86090 70782 43020 78211 62417 75491 81719	4326	51
	71527 17450 67900 25010 86861 53083 66781 58791 …		

n	Numerator	Denominator	n
52	31 95336 31363 83001 12871 03352 79617 42746 71189	1590	52
	60607 82727 38327 10347 01628 49568 36554 97212 24053		
53	3637 39031 72617 41440 81518 20151 59342 71692 31298	642	53
	64058 16900 38930 81637 82818 79873 38620 23465 72901		
54	34 69342 24784 78287 89552 08865 93238 52541 39976	209191710	54
	67857 60491 14687 00058 91371 50126 63197 24897 59230		
	65973 38057		
55	7645 99294 04847 42892 24813 42467 24347 50052 87524	1518	55
	13412 30790 66835 93870 75979 76062 69585 77997 79302		
	17515		
56	26508 79602 15509 97133 52597 21468 51620 14443 15149	1671270	56
	91925 09896 45178 84276 80966 75651 48755 15366 78120		
	35526 00109		
57	217 37832 31936 91633 33310 76108 66529 91475 72115	42	57
	66790 90831 36080 61101 14933 60548 42345 93650 90418		
	86185 62649		
58	30 95539 16571 84297 69125 13458 03384 14168 69004	1770	58
	12806 43298 44245 50404 57210 08957 52457 19682 71388		
	19959 57547 52259		
59	36 69631 19969 71311 15349 47151 58558 50066 84606	6	59
	36108 06992 04301 05944 06764 14485 04580 64618 89371		
	77635 45170 95799		
60	515 07486 53507 91090 61843 99685 78499 83274 09517	2328255930	60
	03532 62675 21309 28691 67199 29747 49229 85358 81132		
	93670 77682 67780 32820 70131		
61	49 63366 60792 62581 91253 26374 75990 75743 87227	6	61
	90311 06013 97703 09311 79315 06832 14100 43132 90331		
	13678 09803 79685 64431		
62	95876 77533 42471 28750 77490 31075 42444 62057 88300	30	62
	13297 33681 95535 12729 35859 33544 35944 41363 19436		
	10268 47268 90946 09001		

§2 Irregular Primes

This table lists the irregular primes $p \leq 4001$ along with the even indices $2a$, $0 \leq 2a \leq p - 3$, such that $p | B_{2a}$. It is essentially the table of Lehmer–Lehmer–Vandiver–Selfridge–Nicol which is printed in Borevich–Shafarevich [1], but there are four additional entries (for $p = 1381, 1597, 1663, 1877$), which were originally missed because of machine error and which were later found by W. Johnson (see Johnson [1]; this paper gives a list of irregular primes for $p < 8000$).

In order to obtain information about generalized Bernoulli numbers and about class groups, see Corollary 5.15 and Theorems 6.17 and 6.18. For a report on the irregular primes $p < 125000$, see Wagstaff [1].

p	$2a$	p	$2a$	p	$2a$
37	32	577	52	1061	474
59	44	587	90, 92	1091	888
67	58	593	22	1117	794
101	68	607	592	1129	348
103	24	613	522	1151	534, 784, 968
131	22	617	20, 174, 338	1153	802
149	130	619	428	1193	262
157	62, 110	631	80, 226	1201	676
233	84	647	236, 242, 554	1217	784, 866, 1118
257	164	653	48	1229	784
263	100	659	224	1237	874
271	84	673	408, 502	1279	518
283	20	677	628	1283	510
293	156	683	32	1291	206, 824
307	88	691	12, 200	1297	202, 220
311	292	727	378	1301	176
347	280	751	290	1307	382, 852
353	186, 300	757	514	1319	304
379	100, 174	761	260	1327	466
389	200	773	732	1367	234
401	382	797	220	1381	266
409	126	809	330, 628	1409	358
421	240	811	544	1429	996
433	366	821	744	1439	574
461	196	827	102	1483	224
463	130	839	66	1499	94
467	94, 194	877	868	1523	1310
491	292, 336, 338	881	162	1559	862
523	400	887	418	1597	842
541	86	929	520, 820	1609	1356
547	270, 486	953	156	1613	172
557	222	971	166	1619	560

p	$2a$	p	$2a$	p	$2a$
1621	980	2357	2204	3181	3142
1637	718	2371	242, 2274	3203	2368
1663	270, 1508	2377	1226	3221	98
1669	388, 1086	2381	2060	3229	1634
1721	30	2383	842, 2278	3257	922
1733	810, 942	2389	776	3313	2222
1753	712	2411	2126	3323	3292
1759	1520	2423	290, 884	3329	1378
1777	1192	2441	366, 1750	3391	2232, 2534
1787	1606	2503	1044	3407	2076, 2558
1789	848, 1442	2543	2374	3433	1300
1811	550, 698, 1520	2557	1464	3469	1174
1831	1274	2579	1730	3491	2544
1847	954, 1016, 1558	2591	854, 2574	3511	1416, 1724
1871	1794	2621	1772	3517	1836, 2586
1877	1026	2633	1416	3529	3490
1879	1260	2647	1172	3533	2314, 3136
1889	242	2657	710	3539	2082, 2130
1901	1722	2663	1244	3559	344, 1592
1933	1058, 1320	2671	404, 2394	3581	1466
1951	1656	2689	926	3583	1922
1979	148	2753	482	3593	360, 642
1987	510	2767	2528	3607	1976
1993	912	2777	1600	3613	2082
1997	772, 1888	2789	1984, 2154	3617	16, 2856
2003	60, 600	2791	2554	3631	1104
2017	1204	2833	1832	3637	2526, 3202
2039	1300	2857	98	3671	1580
2053	1932	2861	352	3677	2238
2087	376, 1298	2909	400, 950	3697	1884
2099	1230	2927	242	3779	2362
2111	1038	2939	332, 1102, 2748	3797	1256
2137	1624	2957	138, 788	3821	3296
2143	1916	2999	776	3833	1840, 1998, 3286
2153	1832	3011	1496	3851	216, 404
2213	154	3023	2020	3853	748
2239	1826	3049	700	3881	1686, 2138
2267	2234	3061	2522	3917	1490
2273	876, 2166	3083	1450	3967	106
2293	2040	3089	1706	3989	1936
2309	1660, 1772	3119	1704	4001	534

§3 Class Numbers

The following table gives the value and prime factorization of the relative class number h_n^- of $\mathbb{Q}(\zeta_n)$ for $1 \le \phi(n) \le 256$, $n \not\equiv 2 \pmod 4$. It is extracted from Schrutka von Rechtenstamm [1], which also lists the contributions from the various odd characters in the analytic class number formula. Some of the large factors were only checked for primality by a pseudo-primality test, so there is a small chance that some of the "prime" factorizations include composites. For values of h_p^- for $257 < p < 521$, see Lehmer–Masley [1]. A few of the factorizations below have been obtained from this paper.

Since the size of h_n^- depends more on the size of $\phi(n)$ than of n, we have arranged the table according to degree.

For h^+ there are the following results (see van der Linden [1]):

(a) If n is a prime power with $\phi(n) \le 66$ then $h_n^+ = 1$.
(b) If n is not a prime power and $n \le 200$, $\phi(n) \le 72$, then $h_n^+ = 1$, except for $h_{136}^+ = 2$ and the possible exceptions $n = 148$ and $n = 152$. Also, we have $h_{165}^+ = 1$.

If we assume the generalized Riemann hypothesis, then the following hold:

(c) If n is a prime power with $\phi(n) < 162$ then $h_n^+ = 1$. We have $h_{163}^+ = 4$.
(d) If n is not a prime power and $n \le 200$, then $h_n^+ = 1$, with the following exceptions: $h_{136}^+ = 2$, $h_{145}^+ = 2$, $h_{183}^+ = 4$.

It is possible to obtain examples of $h_p^+ > 1$ using quadratic subfields (Ankeny-Chowla–Hasse [1], S.-D. Lang [1]), or using cubic subfields (see the tables in M.-N. Gras [3] and Shanks [1]), or using both (Cornell–Washington [1]). See also Takeuchi [1].

Kummer determined the structure of the minus part of the class group of $\mathbb{Q}(\zeta_p)$ for $p < 100$. By (a) above, this is the whole class group for $p \le 67$; by (c), it is the whole class group for $p < 100$ if we assume the generalized Riemann hypothesis. All the groups have square-free order, hence are cyclic, with the following possible exceptions: 29, 31, 41, and 71. In these cases, 29 yields $(2) \times (2) \times (2)$, 31 yields (9), 41 yields $(11) \times (11)$, and 71 yields $(7^2 \cdot 79241)$, Here (m) denotes the cyclic group $\mathbb{Z}/m\mathbb{Z}$. See Kummer [5, pp. 544, 907–918], Iwasawa [16], and Section 10.1. For more techniques, see Cornell–Rosen [1] and Gerth [5].

n	$\phi(n)$	h^-	n	$\phi(n)$	h^-	n	$\phi(n)$	h^-	n	$\phi(n)$	h^-
1	1	1	36	12	1	56	24	2	41	40	$121 = 11^2$
3	2	1	17	16	1	72	24	3	55	40	$10 = 2 \cdot 5$
4	2	1	32	16	1	84	24	1	75	40	11
5	4	1	40	16	1	29	28	$8 = 2^3$	88	40	$55 = 5 \cdot 11$
8	4	1	48	16	1	31	30	$9 = 3^2$	100	40	$55 = 5 \cdot 11$
12	4	1	60	16	1	51	32	5	132	40	11
7	6	1	19	18	1	64	32	17	43	42	211
9	6	1	27	18	1	68	32	$8 = 2^3$	49	42	43
15	8	1	25	20	1	80	32	5	69	44	$69 = 3 \cdot 23$
16	8	1	33	20	1	96	32	$9 = 3^2$	92	44	$201 = 3 \cdot 67$
20	8	1	44	20	1	120	32	$4 = 2^2$	47	46	$695 = 5 \cdot 139$
24	8	1	23	22	3	37	36	37	65	48	$64 = 2^6$
11	10	1	35	24	1	57	36	$9 = 3^2$	104	48	$351 = 3^3 \cdot 13$
13	12	1	39	24	2	63	36	7	105	48	13
21	12	1	45	24	1	76	36	19	112	48	$468 = 2^2 \cdot 3^2 \cdot 13$
28	12	1	52	24	3	108	36	19			

n	$\phi(n)$	h^-	n	$\phi(n)$	h^-
140	48	$39 = 3 \cdot 13$	135	72	$75961 = 37 \cdot 2053$
144	48	$507 = 3 \cdot 13^2$	148	72	$4\,827501 = 3^2 \cdot 7 \cdot 19 \cdot 37 \cdot 109$
156	48	$156 = 2^2 \cdot 3 \cdot 13$	152	72	$1\,666737 = 3^5 \cdot 19^3$
168	48	$84 = 2^2 \cdot 3 \cdot 7$	216	72	$1\,714617 = 3^2 \cdot 19 \cdot 37 \cdot 271$
180	48	$75 = 3 \cdot 5^2$	228	72	$238203 = 3^2 \cdot 7 \cdot 19 \cdot 199$
53	52	4889	252	72	$71344 = 2^4 \cdot 7^3 \cdot 13$
81	54	2593	79	78	$100\,146415 = 5 \cdot 53 \cdot 377911$
87	56	$1536 = 2^9 \cdot 3$	123	80	$8\,425472 = 2^{12} \cdot 11^2 \cdot 17$
116	56	$10752 = 2^9 \cdot 3 \cdot 7$	164	80	$82\,817240 = 2^3 \cdot 5 \cdot 11^2 \cdot 71 \cdot 241$
59	58	$41421 = 3 \cdot 59 \cdot 233$	165	80	$92620 = 2^2 \cdot 5 \cdot 11 \cdot 421$
61	60	$76301 = 41 \cdot 1861$	176	80	$29\,371375 = 5^3 \cdot 11 \cdot 41 \cdot 521$
77	60	$1280 = 2^8 \cdot 5$	200	80	$14\,907805 = 5 \cdot 11^2 \cdot 41 \cdot 601$
93	60	$6795 = 3^2 \cdot 5 \cdot 151$	220	80	$856220 = 2^2 \cdot 5 \cdot 31 \cdot 1381$
99	60	$2883 = 3 \cdot 31^2$	264	80	$1\,875500 = 2^2 \cdot 5^3 \cdot 11^2 \cdot 31$
124	60	$45756 = 2^2 \cdot 3^2 \cdot 31 \cdot 41$	300	80	$1\,307405 = 5 \cdot 11^2 \cdot 2161$
85	64	$6205 = 5 \cdot 17 \cdot 73$	83	82	$838\,216959 = 3 \cdot 279405653$
128	64	$359057 = 17 \cdot 21121$	129	84	$37\,821539 = 7 \cdot 29 \cdot 211 \cdot 883$
136	64	$111744 = 2^7 \cdot 3^2 \cdot 97$	147	84	$5\,874617 = 7 \cdot 29 \cdot 43 \cdot 673$
160	64	$31365 = 3^2 \cdot 5 \cdot 17 \cdot 41$	172	84	$792\,653572 = 2^2 \cdot 43 \cdot 211 \cdot 21841$
192	64	$61353 = 3^2 \cdot 17 \cdot 401$	196	84	$82\,708823 = 43 \cdot 71 \cdot 27091$
204	64	$15440 = 2^4 \cdot 5 \cdot 193$	89	88	$13379\,363737 = 113 \cdot 118401449$
240	64	$6400 = 2^8 \cdot 5^2$	115	88	$44\,697909 = 3 \cdot 331 \cdot 45013$
67	66	$853513 = 67 \cdot 12739$	184	88	$1486\,137318 = 2 \cdot 3 \cdot 23 \cdot 67^2 \cdot 2399$
71	70	$3\,882809 = 7^2 \cdot 79241$	276	88	$131\,209986 = 2 \cdot 3 \cdot 23^2 \cdot 67 \cdot 617$
73	72	$11\,957417 = 89 \cdot 134353$	141	92	$1257\,700495 = 5 \cdot 47 \cdot 139^2 \cdot 277$
91	72	$53872 = 2^4 \cdot 7 \cdot 13 \cdot 37$	188	92	$24260\,850805 = 5 \cdot 47 \cdot 139 \cdot 742717$
95	72	$107692 = 2^2 \cdot 13 \cdot 19 \cdot 109$	97	96	$411322\,842001 = 577 \cdot 3457 \cdot 206209$
111	72	$480852 = 2^2 \cdot 3^2 \cdot 19^2 \cdot 37$	119	96	$1238\,459625 = 3^4 \cdot 5^3 \cdot 13 \cdot 97^2$
117	72	$132678 = 2 \cdot 3^6 \cdot 7 \cdot 13$	153	96	$2416\,282880 = 2^8 \cdot 5 \cdot 11^2 \cdot 15601$

n	$\phi(n)$	h^-
195	96	$22\,151168 = 2^{17} \cdot 13^2$
208	96	$29904\,190875 = 3^3 \cdot 5^3 \cdot 13^3 \cdot 37 \cdot 109$
224	96	$14989\,501800 = 2^3 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17 \cdot 769$
260	96	$531\,628032 = 2^{20} \cdot 3 \cdot 13^2$
280	96	$265\,454280 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 37 \cdot 73$
288	96	$32899\,636107 = 3^5 \cdot 13^2 \cdot 457 \cdot 1753$
312	96	$1621\,069632 = 2^6 \cdot 3^3 \cdot 7 \cdot 13^3 \cdot 61$
336	96	$930\,436416 = 2^6 \cdot 3^3 \cdot 7 \cdot 13 \cdot 61 \cdot 97$
360	96	$523\,952100 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 109^2$
420	96	$10\,229232 = 2^4 \cdot 3 \cdot 13^3 \cdot 97$
101	100	$3\,547404\,378125 = 5^5 \cdot 101 \cdot 601 \cdot 18701$
125	100	$57708\,445601 = 2801 \cdot 20\,602801$
103	102	$9\,069094\,643165 = 5 \cdot 103 \cdot 1021 \cdot 17\,247691$
159	104	$223233\,182255 = 5 \cdot 53^2 \cdot 3251 \cdot 4889$
212	104	$6\,789574\,466337 = 3 \cdot 13 \cdot 1093 \cdot 4889 \cdot 32579$
107	106	$63\,434933\,542623 = 3 \cdot 743 \cdot 9859 \cdot 2\,886593$
109	108	$161\,784800\,122409 = 17 \cdot 1009 \cdot 9431\,866153$
133	108	$157577\,452812 = 2^2 \cdot 3^{10} \cdot 13 \cdot 19 \cdot 37 \cdot 73$
171	108	$503009\,425548 = 2^2 \cdot 3^6 \cdot 7 \cdot 19 \cdot 73 \cdot 109 \cdot 163$
189	108	$105778\,197511 = 7 \cdot 37 \cdot 109 \cdot 127 \cdot 163 \cdot 181$
324	108	$5\,770749\,978919 = 19 \cdot 2593 \cdot 117\,132157$
121	110	$12\,188792\,628211 = 67 \cdot 353 \cdot 20021 \cdot 25741$
113	112	$1612\,072001\,362952 = 2^3 \cdot 17 \cdot 11\,853470\,598257$
145	112	$1\,467250\,393088 = 2^{14} \cdot 281 \cdot 421 \cdot 757$
232	112	$248\,372639\,563776 = 2^{18} \cdot 3 \cdot 7 \cdot 13 \cdot 43^2 \cdot 1877$
348	112	$5\,889026\,949120 = 2^{18} \cdot 3^2 \cdot 5 \cdot 7 \cdot 71317$
177	116	$81\,730647\,171051 = 3 \cdot 59 \cdot 233 \cdot 523 \cdot 3\,789257$
236	116	$4509\,195165\,737013 = 3 \cdot 59 \cdot 233 \cdot 109337\,677693$
143	120	$36\,027143\,124175 = 5^2 \cdot 7 \cdot 61^2 \cdot 661 \cdot 83701$
155	120	$84\,473643\,916800 = 2^9 \cdot 3^4 \cdot 5^2 \cdot 631 \cdot 129121$
175	120	$4\,733255\,370496 = 2^8 \cdot 61 \cdot 271 \cdot 601 \cdot 1861$
183	120	$767\,392851\,521600 = 2^6 \cdot 5^2 \cdot 31^3 \cdot 41 \cdot 211 \cdot 1861$
225	120	$15\,175377\,535571 = 11 \cdot 61 \cdot 331 \cdot 2791 \cdot 24481$
231	120	$298807\,787520 = 2^{16} \cdot 3^2 \cdot 5 \cdot 11 \cdot 61 \cdot 151$
244	120	$30953\,273659\,007535 = 3^3 \cdot 5 \cdot 11 \cdot 41 \cdot 61 \cdot 691 \cdot 1861 \cdot 6481$
248	120	$12239\,782830\,975744 = 2^8 \cdot 3^2 \cdot 11^2 \cdot 31^2 \cdot 41 \cdot 211 \cdot 5281$
308	120	$12\,767325\,061120 = 2^{21} \cdot 5 \cdot 7 \cdot 31^2 \cdot 181$
372	120	$307\,999672\,562880 = 2^6 \cdot 3^2 \cdot 5 \cdot 31 \cdot 41^2 \cdot 151 \cdot 13591$
396	120	$44\,485944\,574929 = 3 \cdot 11 \cdot 13 \cdot 31^3 \cdot 181 \cdot 19231$
127	126	$2\,604529\,186263\,992195 = 5 \cdot 13 \cdot 43 \cdot 547 \cdot 883 \cdot 3079 \cdot 626599$
255	128	$16\,881405\,898800 = 2^4 \cdot 3 \cdot 5^2 \cdot 17^2 \cdot 73 \cdot 353 \cdot 1889$
256	128	$10\,449592\,865393\,414737 = 17 \cdot 21121 \cdot 29\,102880\,226241$
272	128	$239445\,927053\,918208 = 2^{15} \cdot 3^2 \cdot 13 \cdot 17 \cdot 41 \cdot 97 \cdot 577 \cdot 1601$
320	128	$39497\,094130\,144005 = 3^2 \cdot 5 \cdot 17^4 \cdot 41 \cdot 97 \cdot 337 \cdot 7841$
340	128	$1212\,125245\,952000 = 2^{12} \cdot 5^3 \cdot 17 \cdot 73 \cdot 593 \cdot 3217$
384	128	$107878\,055185\,500777 = 3^2 \cdot 17 \cdot 401 \cdot 1697 \cdot 21121 \cdot 49057$
408	128	$4710\,612981\,841920 = 2^{16} \cdot 3^2 \cdot 5 \cdot 41 \cdot 97 \cdot 193 \cdot 2081$
480	128	$617\,689081\,497600 = 2^{11} \cdot 3^4 \cdot 5^2 \cdot 7^4 \cdot 17 \cdot 41 \cdot 89$
131	130	$28\,496379\,729272\,136525 = 3^3 \cdot 5^2 \cdot 53 \cdot 131 \cdot 1301 \cdot 4673\,706701$
161	132	$17033\,926767\,658911 = 3^2 \cdot 11 \cdot 67^3 \cdot 22111 \cdot 25873$
201	132	$252655\,290579\,982532 = 2^2 \cdot 11 \cdot 23^2 \cdot 67^2 \cdot 12739 \cdot 189817$

n	$\phi(n)$	h^-
207	132	57569 648362 893621 = $3^2 \cdot 23 \cdot 67 \cdot 727 \cdot 17491 \cdot 326437$
268	132	28 431682 983759 502069 = $7 \cdot 23 \cdot 67^2 \cdot 1607 \cdot 12739 \cdot 1\,921657$
137	136	646 901570 175200 968153 = $17^2 \cdot 47737 \cdot 46\,890540\,621121$
139	138	1753 848916 484925 681747 = $3^2 \cdot 47^2 \cdot 277^2 \cdot 967 \cdot 1188\,961909$
213	140	20 748314 966568 340907 = $7^2 \cdot 41 \cdot 43 \cdot 281 \cdot 421 \cdot 25621 \cdot 79241$
284	140	1858 128446 456993 562103 = $7^2 \cdot 29 \cdot 71 \cdot 113 \cdot 281 \cdot 79241 \cdot 7\,319621$
185	144	13 767756 481797 006325 = $5^2 \cdot 7^2 \cdot 13 \cdot 37^2 \cdot 53^2 \cdot 9433 \cdot 23833$
219	144	219 406633 996698 095616 = $2^{12} \cdot 3^2 \cdot 17^2 \cdot 37 \cdot 89 \cdot 46549 \cdot 134353$
273	144	21198 594942 959616 = $2^{20} \cdot 3^2 \cdot 7 \cdot 13^2 \cdot 19 \cdot 37^2 \cdot 73$
285	144	34397 734347 893592 = $2^3 \cdot 3^4 \cdot 13 \cdot 19 \cdot 37^2 \cdot 73 \cdot 109^2 \cdot 181$
292	144	26883 466789 548427 261560 = $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 89 \cdot 109 \cdot 181^2 \cdot 433 \cdot 577 \cdot 134353$
296	144	8269 489911 111632 618625 = $3^2 \cdot 5^3 \cdot 7^3 \cdot 17^2 \cdot 19 \cdot 37^2 \cdot 109 \cdot 397 \cdot 65881$
304	144	1764 209801 444986 506285 = $3^5 \cdot 5 \cdot 19^3 \cdot 37^3 \cdot 73 \cdot 109 \cdot 525241$
315	144	3990 441973 190400 = $2^8 \cdot 3^4 \cdot 5^2 \cdot 7^3 \cdot 13^2 \cdot 37^2 \cdot 97$
364	144	2 153601 104578 560000 = $2^{14} \cdot 3^7 \cdot 5^4 \cdot 7 \cdot 13^5 \cdot 37$
380	144	3 118301 079203 997232 = $2^4 \cdot 7 \cdot 13 \cdot 19^2 \cdot 53^2 \cdot 73 \cdot 109 \cdot 433 \cdot 613$
432	144	859 095743 251563 370449 = $3^2 \cdot 13^2 \cdot 19 \cdot 37^2 \cdot 109 \cdot 271 \cdot 541 \cdot 1\,358821$
444	144	55 382724 129516 879312 = $2^4 \cdot 3^4 \cdot 7 \cdot 19^3 \cdot 37^2 \cdot 109^2 \cdot 54721$
456	144	17 643537 152468 843364 = $2^2 \cdot 3^7 \cdot 7^2 \cdot 19^4 \cdot 199 \cdot 487 \cdot 3259$
468	144	6 618931 810639 948800 = $2^{10} \cdot 3^{10} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13^4 \cdot 181$
504	144	2 077452 902069 895168 = $2^{16} \cdot 3^{13} \cdot 7^6 \cdot 13^2$
540	144	1 892923 169092 229025 = $3^2 \cdot 5^2 \cdot 19^2 \cdot 37 \cdot 73 \cdot 109 \cdot 2053 \cdot 38557$
149	148	687887 859687 174720 123201 = $3^2 \cdot 149 \cdot 512\,966338\,320040\,805461$
151	150	2 333546 653547 742584 439257 = $7 \cdot 11^2 \cdot 281 \cdot 25951 \cdot 1\,207501 \cdot 312\,885301$
157	156	56 234327 700401 832767 069245 = $5 \cdot 13^2 \cdot 157^2 \cdot 1093 \cdot 1873 \cdot 418861 \cdot 3\,148601$
169	156	546489 564291 684778 075637 = $313 \cdot 1873 \cdot 4733 \cdot 196\,953296\,289361$
237	156	130445 289884 021402 281355 = $5 \cdot 7 \cdot 13 \cdot 53 \cdot 157 \cdot 3433 \cdot 4421 \cdot 6007 \cdot 377911$
316	156	22 036970 003952 429517 953845 = $5 \cdot 13^2 \cdot 53 \cdot 79 \cdot 2393 \cdot 377911 \cdot 6887\,474101$
187	160	38816 037673 830728 480329 = $17^2 \cdot 41 \cdot 241 \cdot 4801 \cdot 299681 \cdot 9\,447601$
205	160	78821 910689 378365 476000 = $2^5 \cdot 3^2 \cdot 5^3 \cdot 11^2 \cdot 41 \cdot 101^2 \cdot 661 \cdot 4261 \cdot 15361$
328	160	82 221729 062003 473169 480000 = $2^6 \cdot 5^4 \cdot 11^2 \cdot 17 \cdot 31 \cdot 71 \cdot 101 \cdot 241 \cdot 521 \cdot 35\,801081$
352	160	5 578700 230786 671358 855375 = $5^3 \cdot 11 \cdot 41^2 \cdot 113 \cdot 281 \cdot 521 \cdot 1801 \cdot 2801 \cdot 28921$
400	160	1 692044 042657 239185 550625 = $5^4 \cdot 11^4 \cdot 41 \cdot 61 \cdot 101 \cdot 601 \cdot 26261 \cdot 46381$
440	160	3690 827552 653792 584000 = $2^6 \cdot 3 \cdot 5^3 \cdot 11 \cdot 31^2 \cdot 61^2 \cdot 181 \cdot 1381 \cdot 15641$
492	160	331431 584848 686177 320960 = $2^{20} \cdot 5 \cdot 11^2 \cdot 17 \cdot 41 \cdot 71 \cdot 241 \cdot 1321 \cdot 33161$
528	160	20215 309155 022994 375000 = $2^3 \cdot 5^7 \cdot 11^2 \cdot 31 \cdot 41 \cdot 61 \cdot 101 \cdot 521 \cdot 65521$
600	160	7166 325608 289022 528100 = $2^2 \cdot 5^2 \cdot 11^3 \cdot 41 \cdot 101 \cdot 131 \cdot 601 \cdot 2161 \cdot 76421$
660	160	20 090237 237998 576000 = $2^7 \cdot 5^3 \cdot 11^2 \cdot 31 \cdot 181 \cdot 421 \cdot 1381 \cdot 3181$
163	162	2708 534744 692077 051875 131636 = $2^2 \cdot 181 \cdot 23167 \cdot 365473 \cdot 441\,845817\,162679$
243	162	14 948557 667133 129512 662807 = $2593 \cdot 5764\,966319\,758245\,087799$ (composite)
249	164	13 898958 132089 743179 099753 = $3 \cdot 279\,405653 \cdot 16581\,575906\,876567$
332	164	2233 138758 192814 382133 816279 = $3 \cdot 80279 \cdot 612377 \cdot 54\,192407 \cdot 279\,405653$
167	166	28121 189830 322933 178315 382891 = $11 \cdot 499 \cdot 5\,123189\,985484\,229035\,947419$
203	168	4 413278 155436 385292 173312 = $2^{14} \cdot 3^2 \cdot 7^2 \cdot 29 \cdot 3907 \cdot 26041 \cdot 207\,015901$
215	168	8 562946 718506 556895 170449 = $7^2 \cdot 19 \cdot 29 \cdot 37 \cdot 211 \cdot 757 \cdot 2017 \cdot 22709 \cdot 1\,171633$
245	168	122845 138181 874350 560487 = $13^2 \cdot 43 \cdot 127 \cdot 631 \cdot 43793 \cdot 4816\,871221$
261	168	18 379288 588511 605529 995776 = $2^9 \cdot 3^2 \cdot 61 \cdot 421 \cdot 883 \cdot 10753 \cdot 38011 \cdot 430333$
344	168	10789 946893 536931 852748 197440 = $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 29 \cdot 43 \cdot 197 \cdot 211 \cdot 21841 \cdot 929419 \cdot 1\,525987$
392	168	112 070797 379361 142494 415714 = $2 \cdot 43^2 \cdot 71 \cdot 617 \cdot 953 \cdot 27091 \cdot 28393 \cdot 943741$

n	$\phi(n)$	h^-
516	168	38 888604 320171 861798 243568 = $2^{4}\cdot 3^{2}\cdot 7\cdot 29\cdot 43^{2}\cdot 71\cdot 211\cdot 883\cdot 21841\cdot 2\,490307$
588	168	482059 253351 850013 395157 = $7\cdot 29\cdot 43\cdot 71\cdot 673\cdot 2017\cdot 3571\cdot 5923\cdot 27091$
173	172	1 702546 266654 155847 516780 034265 = $5\cdot 20297\cdot 231169\cdot 72\,571729\,362851\,870621$
267	176	12963 312320 905811 283854 380235 = $5\cdot 23\cdot 113\cdot 1123\cdot 5237\cdot 26687\cdot 53681\cdot 118\,401449$
345	176	506186 308788 058155 105915 = $3\cdot 5\cdot 11\cdot 23\cdot 331\cdot 4159\cdot 45013\cdot 2152\,502881$
356	176	4 707593 989354 615385 004311 705592 = $2^{3}\cdot 3\cdot 11\cdot 23\cdot 113\cdot 463\cdot 15269\cdot 19207\cdot 426757\cdot 118\,401449$
368	176	243320 115114 433657 103908 135020 = $2^{2}\cdot 3\cdot 5\cdot 11^{2}\cdot 23^{3}\cdot 67^{2}\cdot 89\cdot 2069\cdot 2399\cdot 8537\cdot 162713$
460	176	197 739166 909616 827795 207545 = $3\cdot 5\cdot 11\cdot 67\cdot 331\cdot 617\cdot 17029\cdot 45013\cdot 114\,259861$
552	176	767 354245 926929 350377 606384 = $2^{4}\cdot 3\cdot 23^{5}\cdot 67^{2}\cdot 617\cdot 2399\cdot 10781\cdot 34673$
179	178	77 281577 212030 298592 756974 721745 = $5\cdot 1069\cdot 14458\,667392\,334948\,286764\,635121$
181	180	211 421757 749987 541697 225501 539625 = $5^{3}\cdot 37\cdot 41\cdot 61\cdot 1321\cdot 2521\cdot 5\,488435\,782589\,277701$
209	180	4551 326160 887085 824176 768000 = $2^{10}\cdot 5^{3}\cdot 11\cdot 61\cdot 271\cdot 264\,250891\cdot 739\,979551$
217	180	3724 911233 451940 358045 813517 = $3^{5}\cdot 7\cdot 11\cdot 37\cdot 241\cdot 541\cdot 571\cdot 691\cdot 2161\cdot 2791\cdot 17341$
279	180	18164 714706 446857 534815 843195 = $3^{6}\cdot 5\cdot 7\cdot 13\cdot 151\cdot 211\cdot 1321\cdot 2551\cdot 4591\cdot 5011\cdot 22171$
297	180	1078 851803 253231 276755 717661 = $3^{2}\cdot 31^{2}\cdot 199\cdot 8191\cdot 1\,674991\cdot 45687\,081331$
235	184	81765 924684 755483 300654 973515 = $5\cdot 139\cdot 1657\cdot 453377\cdot 156604\,975201\,463093$
376	184	237 637802 564280 802840 123241 975060 = $2^{2}\cdot 5\cdot 47\cdot 139\cdot 18493\cdot 742717\cdot 3\,536987\cdot 37437\,658303$
564	184	431950 475833 835326 053345 383630 = $2\cdot 5\cdot 47^{3}\cdot 139^{3}\cdot 277\cdot 599\cdot 742717\cdot 1\,257089$
191	190	165008 365487 223656 458987 611326 929859 = $11\cdot 13\cdot 51263\cdot 612\,771\,091\cdot 36\,733950\,669733\,713761$
193	192	546617 105913 568165 545650 752630 767041 = $6529\cdot 15361\cdot 29761\cdot 91969\cdot 10\,369729\cdot 192026\,280449$
221	192	5 562629 629465 863945 291002 496000 = $2^{10}\cdot 3^{6}\cdot 5^{3}\cdot 17\cdot 31^{2}\cdot 61\cdot 73\cdot 113\cdot 193\cdot 1297\cdot 3529\cdot 8209$
291	192	161 230789 161196 289366 922423 524464 = $2^{4}\cdot 7\cdot 13^{2}\cdot 17^{2}\cdot 577\cdot 1489\cdot 3457\cdot 5641\cdot 206209\cdot 8\,531233$
357	192	1504 490803 465665 772083 088125 = $3^{4}\cdot 5^{4}\cdot 7^{4}\cdot 13^{2}\cdot 37\cdot 97^{3}\cdot 1873\cdot 1\,157953$
388	192	145666 644086 003914 044409 030660 616112 = $2^{4}\cdot 3^{2}\cdot 7^{2}\cdot 13\cdot 19\cdot 37\cdot 577\cdot 3457\cdot 5857\cdot 13441\cdot 206209\cdot 69\,761089$
416	192	1370 350108 087898 680332 276597 421875 = $3^{9}\cdot 5^{7}\cdot 7^{2}\cdot 13^{5}\cdot 37\cdot 73\cdot 97\cdot 109\cdot 241\cdot 409\cdot 17401$
448	192	327 965590 186830 575092 883770 837200 = $2^{4}\cdot 3^{2}\cdot 5^{2}\cdot 7^{2}\cdot 13\cdot 17^{2}\cdot 577^{2}\cdot 769\cdot 13697\cdot 299569\cdot 471073$
476	192	1 099745 163233 204819 353212 762000 = $2^{4}\cdot 3^{6}\cdot 5^{3}\cdot 11^{2}\cdot 13\cdot 47^{2}\cdot 97^{4}\cdot 241\cdot 1489\cdot 6833$
520	192	285052 110419 192727 742709 760000 = $2^{42}\cdot 3^{4}\cdot 5^{4}\cdot 7^{3}\cdot 13^{3}\cdot 17\cdot 37^{2}\cdot 73$

n	$\phi(n)$	h^-
560	192	54738 664378 286829 420235 392000 = $2^{10} \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 13^2 \cdot 17 \cdot 37 \cdot 73 \cdot 97^2 \cdot 181 \cdot 193 \cdot 241 \cdot 409$
576	192	1157 874338 412588 470629 857952 431771 = $3^5 \cdot 13^2 \cdot 17 \cdot 401 \cdot 457 \cdot 1753 \cdot 1873 \cdot 1\,751377 \cdot 1573\,836529$
612	192	4 600831 021854 761317 711337 226240 = $2^{20} \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \cdot 73 \cdot 97 \cdot 193 \cdot 241 \cdot 15601 \cdot 7\,712737$
624	192	2 180486 664807 803314 987752 000000 = $2^9 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 13^5 \cdot 17^3 \cdot 37 \cdot 61 \cdot 97 \cdot 109 \cdot 409$
672	192	438246 323791 968232 985203 468800 = $2^9 \cdot 3^7 \cdot 5^2 \cdot 7^3 \cdot 13 \cdot 17 \cdot 61 \cdot 73 \cdot 97 \cdot 769 \cdot 8761 \cdot 70969$
720	192	222312 165238 308958 816217 760000 = $2^8 \cdot 3^4 \cdot 5^4 \cdot 7^2 \cdot 13^3 \cdot 19^2 \cdot 37^2 \cdot 109^2 \cdot 277 \cdot 313^2$
780	192	409 113496 073931 085358 039040 = $2^{46} \cdot 3 \cdot 5 \cdot 13^5 \cdot 61 \cdot 109 \cdot 157$
840	192	84 878288 737639 882168 320000 = $2^{14} \cdot 3^4 \cdot 5^4 \cdot 7^2 \cdot 13^4 \cdot 19 \cdot 37^2 \cdot 73 \cdot 97 \cdot 397$
197	196	5 532802 218713 600706 095993 713290 631720 = $2^3 \cdot 5 \cdot 1877 \cdot 7841 \cdot 9398\,302684\,870866\,656225\,611549$
199	198	18 844055 286602 530802 019847 012721 555487 = $3^4 \cdot 19 \cdot 727 \cdot 25\,645093 \cdot 207293\,548177 \cdot 3\,168190\,412839$
275	200	18 124664 091430 165276 567871 093750 = $2 \cdot 5^{12} \cdot 11^3 \cdot 41^2 \cdot 61 \cdot 71 \cdot 101 \cdot 241 \cdot 461 \cdot 541 \cdot 631$
303	200	32442 006711 177310 012824 426376 953125 = $5^{10} \cdot 61 \cdot 101 \cdot 601 \cdot 5701 \cdot 6701 \cdot 18701 \cdot 1255\,817401$
375	200	22 533972 115769 639175 905217 196211 = $11 \cdot 2801 \cdot 12101 \cdot 244301 \cdot 20\,602801 \cdot 12007\,682201$
404	200	28 160409 852152 369458 876449 426375 546875 = $5^7 \cdot 7 \cdot 41 \cdot 61 \cdot 101^2 \cdot 601 \cdot 2351 \cdot 18701 \cdot 40351 \cdot 1892\,989601$
500	200	20244 072859 233305 618155 148176 257775 = $5^2 \cdot 11 \cdot 401 \cdot 2801 \cdot 20\,602801 \cdot 94\,315301 \cdot 33728\,676001$
309	204	360807 306655 167078 388646 788532 317360 = $2^4 \cdot 5 \cdot 17 \cdot 103^2 \cdot 239 \cdot 1021 \cdot 3299 \cdot 233683 \cdot 7\,707223 \cdot 17\,247691$
412	204	311 393365 861041 316591 357682 493761 574005 = $5 \cdot 7 \cdot 103 \cdot 1021 \cdot 2347 \cdot 306511 \cdot 17\,247691 \cdot 54\,115489 \cdot 125\,998867$
265	208	169406 792495 647432 946133 820476 066925 = $5^2 \cdot 53 \cdot 1093 \cdot 4889 \cdot 12377 \cdot 19813 \cdot 11\,452741 \cdot 8519\,216837$
424	208	1435 850573 295225 659918 796765 068953 277637 = $3^4 \cdot 13 \cdot 79 \cdot 677 \cdot 1093 \cdot 4889 \cdot 13469 \cdot 32579 \cdot 2\,805713 \cdot 3875\,328913$
636	208	1 127233 629616 849856 487768 072597 188295 = $3 \cdot 5 \cdot 13^3 \cdot 53^2 \cdot 1093^2 \cdot 3251 \cdot 4889 \cdot 32579 \cdot 19684\,564069$
211	210	49238 446584 179914 120276 706365 116286 443831 = $3^2 \cdot 7^2 \cdot 41 \cdot 71 \cdot 181 \cdot 281^2 \cdot 421 \cdot 1051 \cdot 12251 \cdot 113\,981701 \cdot 4343\,510221$
321	212	41 597545 536058 643707 857919 997509 485501 = $3 \cdot 743 \cdot 9859 \cdot 2\,886593 \cdot 10\,109009 \cdot 64868\,018727\,424243$
428	212	70300 542035 941044 246482 693928 842589 712617 = $3 \cdot 743 \cdot 3181 \cdot 9859 \cdot 2\,886593 \cdot 348390\,669416\,638151\,886259$
247	216	13 453389 127871 713260 541632 243338 018775 = $3^9 \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 19^2 \cdot 73^2 \cdot 109^2 \cdot 127 \cdot 157 \cdot 163 \cdot 181 \cdot 397 \cdot 613 \cdot 1009$
259	216	15 168897 693915 178656 178325 215530 382842 = $2 \cdot 3^{20} \cdot 7^6 \cdot 13^2 \cdot 17^2 \cdot 19^3 \cdot 37 \cdot 73^3 \cdot 271 \cdot 14149$
327	216	503 374795 561927 637884 794232 382274 404226 = $2 \cdot 3^7 \cdot 13 \cdot 17 \cdot 37 \cdot 379 \cdot 1009 \cdot 2377 \cdot 47629 \cdot 34\,465933 \cdot 9431\,866153$

n	$\phi(n)$	h^-
333	216	84 239369 799126 310123 807613 556409 560000 = $2^{6} \cdot 3^{6} \cdot 5^{4} \cdot 7^{2} \cdot 13^{2} \cdot 19^{5} \cdot 37^{2} \cdot 43 \cdot 73 \cdot 523 \cdot 111637 \cdot 561529$
351	216	2 881839 794389 013705 029278 932481 257394 = $2 \cdot 3^{12} \cdot 7 \cdot 13 \cdot 19^{6} \cdot 37^{2} \cdot 73 \cdot 631 \cdot 2341 \cdot 31393 \cdot 136657$
399	216	1178 892414 491021 808120 869355 574272 = $2^{10} \cdot 3^{20} \cdot 7 \cdot 13 \cdot 19^{2} \cdot 37 \cdot 61 \cdot 73^{2} \cdot 577 \cdot 829 \cdot 1747$
405	216	289942 114683 805443 433002 828021 894577 = $37 \cdot 487 \cdot 541 \cdot 2053 \cdot 2593 \cdot 1\ 583767 \cdot 3527\ 772707\ 308141$
436	216	893749 713826 042123 652446 227238 954966 290576 = $2^{4} \cdot 3^{7} \cdot 17 \cdot 19^{2} \cdot 163 \cdot 757 \cdot 1009 \cdot 3\ 016927 \cdot 1174\ 772971 \cdot 9431\ 866153$
532	216	1 995278 293629 608216 703343 220411 633664 = $2^{12} \cdot 3^{10} \cdot 7^{3} \cdot 13 \cdot 19^{3} \cdot 31 \cdot 37^{2} \cdot 73^{2} \cdot 109 \cdot 1693 \cdot 2377 \cdot 2719$
648	216	4207 762445 242777 294033 981083 075596 417079 = $3^{3} \cdot 19 \cdot 37 \cdot 271^{2} \cdot 2593 \cdot 117\ 132157 \cdot 157\ 470427 \cdot 63112\ 572037$
684	216	9 549392 972039 711651 917872 649044 836352 = $2^{14} \cdot 3^{6} \cdot 7^{2} \cdot 13 \cdot 19^{2} \cdot 37^{2} \cdot 73 \cdot 109 \cdot 127 \cdot 163 \cdot 199 \cdot 1693 \cdot 3637 \cdot 12583$
756	216	434848 520210 868494 245767 938408 147152 = $2^{4} \cdot 7^{3} \cdot 13 \cdot 19^{3} \cdot 37^{3} \cdot 109 \cdot 127^{2} \cdot 163 \cdot 181^{2} \cdot 271 \cdot 757 \cdot 9109$
253	220	256 271685 260834 247944 985594 908530 991952 = $2^{4} \cdot 3 \cdot 11^{4} \cdot 1409 \cdot 3301 \cdot 26951 \cdot 79861 \cdot 13\ 962631 \cdot 2608\ 886831$
363	220	23 207253 826992 628179 863710 751562 290176 = $2^{10} \cdot 67 \cdot 89 \cdot 353 \cdot 20021 \cdot 25741 \cdot 20\ 891667\ 283264\ 099631$
484	220	29678 406487 322012 695719 894464 039435 383271 = $67 \cdot 353 \cdot 14411 \cdot 20021 \cdot 25741 \cdot 167971 \cdot 1\ 005892\ 255694\ 569981$
223	222	217 076412 323050 485246 172261 728619 107578 141363 = $7 \cdot 43 \cdot 17\ 909933\ 575379 \cdot 11\ 757537\ 731851 \cdot 3424\ 804483\ 726447$
339	224	87309 027165 405351 637092 447907 404827 688960 = $2^{15} \cdot 3 \cdot 5 \cdot 17 \cdot 71 \cdot 113 \cdot 127 \cdot 281 \cdot 2137 \cdot 14449 \cdot 99709 \cdot 11\ 853470\ 598257$
435	224	299190 086533 933244 039620 216234 180608 = $2^{39} \cdot 3 \cdot 13 \cdot 29^{2} \cdot 113^{2} \cdot 281 \cdot 421 \cdot 757 \cdot 1289 \cdot 11257$
452	224	229 865767 233324 575111 010848 122335 548084 846592 = $2^{23} \cdot 3^{2} \cdot 7 \cdot 13^{2} \cdot 17 \cdot 29 \cdot 281 \cdot 24809 \cdot 168617 \cdot 374669 \cdot 11\ 853470\ 598257$
464	224	12 164820 242320 422627 042467 644729 294439 055360 = $2^{30} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17^{2} \cdot 29^{5} \cdot 43^{2} \cdot 1877 \cdot 4621 \cdot 226129 \cdot 386093$
580	224	776 785847 831995 632448 594543 440172 154880 = $2^{39} \cdot 3 \cdot 5 \cdot 7^{2} \cdot 29 \cdot 281 \cdot 421 \cdot 463 \cdot 757 \cdot 1\ 131397 \cdot 1\ 413077$
696	224	6438 349938 668172 599554 162206 096280 780800 = $2^{38} \cdot 3^{3} \cdot 5^{2} \cdot 7^{2} \cdot 13 \cdot 29 \cdot 43^{3} \cdot 113 \cdot 1093 \cdot 1429 \cdot 1877 \cdot 71317$
227	226	2888 747573 690533 630075 559971 022165 906726 932055 = $5 \cdot 2939^{3} \cdot 1692\ 824021\ 974901 \cdot 13\ 444015\ 915122\ 722869$
229	228	10934 752550 628778 589695 733157 034481 831976 032377 = $13 \cdot 17 \cdot 457 \cdot 7753 \cdot 705053 \cdot 47\ 824141 \cdot 414153\ 903321\ 692666\ 991589$
233	232	348185 729880 711782 527290 176798 948867 695747 163449 = $233 \cdot 1433 \cdot 1\ 042818\ 810684\ 723912\ 819200\ 922459\ 107271\ 266041$ (composite)
295	232	670508 644900 926208 004253 553219 885108 451604 = $2^{2} \cdot 3 \cdot 59 \cdot 233 \cdot 349 \cdot 41413 \cdot 9\ 342293 \cdot 3483\ 942493 \cdot 8\ 640296\ 021597$
472	232	19371 983746 349662 149124 469187 254723 339443 284387 = $3^{2} \cdot 29 \cdot 59^{5} \cdot 233 \cdot 42283 \cdot 135257 \cdot 168143 \cdot 4\ 237829 \cdot 109337\ 677693$
708	232	7 622833 744450 532364 757064 890176 317824 613409 = $3 \cdot 59 \cdot 233 \cdot 523 \cdot 2\ 069383 \cdot 3\ 789257 \cdot 109337\ 677693 \cdot 412212\ 149161$

n	$\phi(n)$	h^-
239	238	19 252683 042543 984486 813299 844961 436592 191498 141760 =
		$2^{6} \cdot 3 \cdot 5 \cdot 511123 \cdot 14\,136487 \cdot 123373\,184789 \cdot 22497\,399987\,891136\,953079$
241	240	74 361351 053524 744837 764467 869162 082791 741351 378657 =
		$47^{2} \cdot 13921 \cdot 15601 \cdot 2\,359873 \cdot 126\,767281 \cdot 518123\,008737\,871423\,891201$
287	240	75 414262 624860 852745 819151 571359 184834 222400 =
		$2^{6} \cdot 5^{2} \cdot 7 \cdot 11^{7} \cdot 13 \cdot 31^{2} \cdot 61 \cdot 521 \cdot 1201 \cdot 1609 \cdot 2521 \cdot 8641 \cdot 20673\,617161$
305	240	135 088091 280028 160307 240417 262034 056281 285000 =
		$2^{3} \cdot 3^{2} \cdot 5^{4} \cdot 13^{2} \cdot 37 \cdot 41^{4} \cdot 61^{3} \cdot 1861 \cdot 2281 \cdot 3061 \cdot 24061 \cdot 37501 \cdot 63841$
325	240	958286 131671 211592 542476 979144 265746 218304 =
		$2^{6} \cdot 61^{3} \cdot 101 \cdot 1201 \cdot 2141 \cdot 7681 \cdot 11701 \cdot 194521 \cdot 849721 \cdot 17\,098621$
369	240	528 852535 797845 727358 844974 839889 196910 080000 =
		$2^{12} \cdot 5^{4} \cdot 11^{6} \cdot 17 \cdot 19 \cdot 31 \cdot 271 \cdot 421 \cdot 4801 \cdot 16921 \cdot 1256\,507775\,765241$
385	240	18 696191 070960 590983 421400 100896 768000 =
		$2^{31} \cdot 3^{2} \cdot 5^{3} \cdot 11 \cdot 19^{2} \cdot 31 \cdot 157 \cdot 1021 \cdot 9661 \cdot 16141 \cdot 2\,514961$
429	240	1880 049931 342806 129486 552279 849583 657000 =
		$2^{3} \cdot 5^{3} \cdot 7 \cdot 11^{3} \cdot 31 \cdot 61^{3} \cdot 181 \cdot 571 \cdot 661 \cdot 39521 \cdot 83701 \cdot 126\,901681$
465	240	6056 875285 186558 003929 869566 624727 040000 =
		$2^{19} \cdot 3^{6} \cdot 5^{4} \cdot 7 \cdot 31 \cdot 61 \cdot 151 \cdot 181 \cdot 631 \cdot 1481 \cdot 1801 \cdot 129121 \cdot 322501$
488	240	3 971856 968532 956975 396384 265567 521800 430781 628875 =
		$3^{3} \cdot 5^{3} \cdot 11^{2} \cdot 31^{2} \cdot 41^{2} \cdot 43 \cdot 61 \cdot 101 \cdot 151 \cdot 421 \cdot 691 \cdot 1861 \cdot 4721 \cdot 6481 \cdot 34171 \cdot 265\,892761$
495	240	151 284295 307196 895954 238278 778191 913580 =
		$2^{2} \cdot 3 \cdot 5 \cdot 7 \cdot 11^{3} \cdot 29^{2} \cdot 31^{3} \cdot 181^{2} \cdot 229 \cdot 241^{2} \cdot 421 \cdot 2131 \cdot 3361 \cdot 8221$
496	240	686038 372620 782033 886901 075737 481803 287781 408768 =
		$2^{15} \cdot 3^{2} \cdot 11^{4} \cdot 31^{4} \cdot 37 \cdot 41 \cdot 61^{2} \cdot 97 \cdot 211 \cdot 241 \cdot 601 \cdot 4621 \cdot 5281 \cdot 14281 \cdot 29501$
525	240	29 585677 490787 726928 862791 955910 586368 =
		$2^{12} \cdot 3^{4} \cdot 11 \cdot 13 \cdot 31^{2} \cdot 61^{4} \cdot 271 \cdot 331 \cdot 601 \cdot 1861 \cdot 467 \cdot 132041$
572	240	5 290237 648692 385160 711880 570308 851548 534375 =
		$3 \cdot 5^{5} \cdot 7 \cdot 19^{2} \cdot 31 \cdot 41 \cdot 61^{2} \cdot 421 \cdot 661 \cdot 27631 \cdot 72271 \cdot 83701 \cdot 1015\,122781$
616	240	894031 197420 910862 005847 489304 819295 846400 =
		$2^{40} \cdot 5^{2} \cdot 7 \cdot 11^{5} \cdot 13 \cdot 31^{4} \cdot 181 \cdot 211 \cdot 2161 \cdot 4621 \cdot 6301$
620	240	19 441064 004704 709948 640099 632484 806819 840000 =
		$2^{26} \cdot 3^{4} \cdot 5^{4} \cdot 11 \cdot 31 \cdot 41 \cdot 61 \cdot 421 \cdot 631 \cdot 5821 \cdot 66931 \cdot 129121 \cdot 502081$
700	240	126016 649965 778239 405605 204267 365457 285120 =
		$2^{12} \cdot 3^{5} \cdot 5 \cdot 11 \cdot 13 \cdot 31 \cdot 59^{2} \cdot 61^{2} \cdot 271 \cdot 601 \cdot 1861 \cdot 9181 \cdot 44641 \cdot 3\,549901$
732	240	1339 692320 604469 611903 838974 531410 116492 800000 =
		$2^{12} \cdot 3^{3} \cdot 5^{5} \cdot 11 \cdot 13 \cdot 19^{2} \cdot 31^{3} \cdot 41 \cdot 61^{2} \cdot 211 \cdot 691 \cdot 1861 \cdot 6481 \cdot 25301 \cdot 371341$
744	240	181 082733 783181 938577 850646 686177 657202 278400 =
		$2^{17} \cdot 3^{5} \cdot 5^{2} \cdot 11^{5} \cdot 31^{2} \cdot 41^{2} \cdot 101 \cdot 131 \cdot 151 \cdot 211 \cdot 541 \cdot 5281 \cdot 13591 \cdot 53401$
792	240	5 042681 390633 567588 773182 959215 349464 474500 =
		$2^{2} \cdot 3^{2} \cdot 5^{3} \cdot 11^{2} \cdot 13^{2} \cdot 19 \cdot 31^{5} \cdot 61^{2} \cdot 181 \cdot 1381 \cdot 5521 \cdot 5791 \cdot 19231 \cdot 176161$
900	240	744248 582096 150452 589487 856013 489542 134375 =
		$3 \cdot 5^{5} \cdot 11^{2} \cdot 61 \cdot 211 \cdot 331 \cdot 811 \cdot 2161 \cdot 2791 \cdot 24481 \cdot 334261 \cdot 3847\,430341$
924	240	228 281655 906261 469381 852055 785911 091200 =
		$2^{39} \cdot 3^{5} \cdot 5^{2} \cdot 7 \cdot 11^{2} \cdot 31^{4} \cdot 61 \cdot 101 \cdot 151 \cdot 181 \cdot 691 \cdot 751$
251	250	95469 181654 584518 651828 574432 658888 070113 445087 403827 =
		$7 \cdot 11 \cdot 348270001 \cdot 9\,631365\,977251 \cdot 369631\,114567\,755437\,243663\,626501$
301	252	205430 142293 947345 943779 193986 871148 546394 604544 =
		$2^{10} \cdot 3^{3} \cdot 7^{7} \cdot 19 \cdot 43^{2} \cdot 211 \cdot 631 \cdot 6301 \cdot 14827 \cdot 16843 \cdot 19531 \cdot 122599 \cdot 511939$
381	252	11 479286 278091 328075 258484 555696 616781 110509 888215 =
		$3^{2} \cdot 5 \cdot 7^{2} \cdot 13 \cdot 37 \cdot 43^{2} \cdot 547 \cdot 631 \cdot 673 \cdot 883 \cdot 3079 \cdot 6007 \cdot 626599 \cdot 2\,185471 \cdot 1126\,755757$

n	$\phi(n)$	h^-
387	252	1 348400 009635 509434 335776 865706 103793 086610 214753 =
		$7^3 \cdot 13^2 \cdot 19^2 \cdot 29 \cdot 43 \cdot 211^2 \cdot 463 \cdot 883 \cdot 967 \cdot 1933 \cdot 3067 \cdot 3319 \cdot 4621 \cdot 125287 \cdot 257713$
441	252	2427 799098 355426 760759 007408 851329 652222 396831 =
		$7^4 \cdot 29 \cdot 43^5 \cdot 127 \cdot 337 \cdot 673 \cdot 2731 \cdot 11173 \cdot 43051 \cdot 1\,271383 \cdot 4\,930381$
508	252	103042 170932 346966 742775 797541 839182 084871 642467 503360 =
		$2^8 \cdot 5 \cdot 7^2 \cdot 13^3 \cdot 19 \cdot 43^3 \cdot 547 \cdot 757 \cdot 883^2 \cdot 2143 \cdot 3079 \cdot 626599 \cdot 2\,664901 \cdot 139\,159441$
257	256	5 452485 023419 230873 223822 625555 964461 476422 854662 168321 =
		257 · 20738 946049 · 1 022997 744563 911961 561298 698183 419037 149697
512	256	6 262503 984490 932358 745721 482528 922841 978219 389975 605329 =
		17 · 21121 · 76 532353 · 29 102880 226241 · 7830 753969 553468 937988 617089
544	256	4584 742688 639592 322280 890443 396756 015190 545059 020800 =
		$2^{30} \cdot 3^8 \cdot 5^2 \cdot 7^4 \cdot 13 \cdot 17^6 \cdot 31^2 \cdot 41^4 \cdot 97 \cdot 353 \cdot 433 \cdot 577 \cdot 929 \cdot 1601$
640	256	112 066740 284710 541318 559132 951039 771578 615246 011365 =
		$3^2 \cdot 5 \cdot 17^4 \cdot 41 \cdot 97^2 \cdot 337 \cdot 7841 \cdot 9473 \cdot 21121 \cdot 376801 \cdot 69\,470881 \cdot 5584\,997633$
680	256	77483 560514 02244 288033 941979 251535 291351 040000 =
		$2^{41} \cdot 3^7 \cdot 5^4 \cdot 13 \cdot 17^3 \cdot 41 \cdot 73 \cdot 97 \cdot 593 \cdot 977 \cdot 3217 \cdot 19489 \cdot 38273$
768	256	1067 969144 915565 716868 049522 568978 331378 093561 484521 =
		$3^2 \cdot 17 \cdot 401 \cdot 1697 \cdot 13313 \cdot 21121 \cdot 49057 \cdot 175361 \cdot 198593 \cdot 733697 \cdot 29\,102880\,226241$
816	256	793553 314770 547109 801192 086472 747224 274042 880000 =
		$2^{38} \cdot 3^8 \cdot 5^4 \cdot 13 \cdot 17^4 \cdot 41^2 \cdot 97 \cdot 113 \cdot 193 \cdot 577 \cdot 1601 \cdot 2081 \cdot 94849$
960	256	20130 907061 992729 156753 037152 064135 304760 934400 =
		$2^{14} \cdot 3^4 \cdot 5^2 \cdot 7^6 \cdot 17^7 \cdot 41 \cdot 89 \cdot 97 \cdot 337 \cdot 401 \cdot 433 \cdot 593 \cdot 7841 \cdot 130513$
1020	256	11 412817 953927 959213 205123 673154 912256 000000 =
		$2^{42} \cdot 3^3 \cdot 5^6 \cdot 17^3 \cdot 73 \cdot 193 \cdot 353 \cdot 593 \cdot 1889 \cdot 3217 \cdot 69857$

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List of Symbols

$J(\chi_1, \chi_2)$	Jacobi sum, 88
θ	Stickelberger element, 93
$\{x\}$	fractional part, 93
ε_χ, ε_i	idempotents, 100
A_i	ith component of class group, 101
A^-	minus component, 101, 192
λ, μ, ν	Iwasawa invariants, 127
K_∞	$\mathbb{Z}_p$-extension, 264
Λ	$\mathbb{Z}_p[[T]]$, 268
$A \sim B$	pseudo-isomorphism, 271
Γ	276
ν_n	278
$\nu_{n,e}$	280
ω_n	291

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