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Bilinear character correlators in superintegrable theory

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Abstract We continue investigating the superintegrability property of matrix models, i.e. factorization of the matrix model averages of characters. This paper focuses on the Gaussian Hermitian example, where the role of characters is played by the Schur functions. We find a new intriguing corollary of superintegrability: factorization of an infinite set of correlators *bilinear* in the Schur functions. More exactly, these are correlators of products of the Schur functions and polynomials K_Δ that form a complete basis in the space of invariant matrix polynomials. Factorization of these correlators with a small subset of these K_Δ follow from the fact that the Schur functions are eigenfunctions of the generalized cut-an-join operators, but the full set of K_Δ is generated by another infinite commutative set of operators, which we manifestly describe.

1 Introduction

In celestial mechanics, superintegrability (SI) implies existence of an additional operator (Laplace-Runge-Lenz vector) which commutes with the Hamiltonian and is somehow different (superficial) as compared to the "obvious" commuting set of operators (rotations), which are responsible for the ordinary integrability.

In the case of matrix models, even this language is still to be developed: our original definition of superintegrability in [1,2] (based on the phenomenon earlier observed in [3–12], see also some preliminary results in [27–32] and later progress in [13–26], [24,33]) implies the mapping between a big space $\mathcal{X}$ (functions of matrix eigenvalues or time-variables p_k) to a small one, $\mathcal{Z}$ (functions of the matrix *size* N) with the Schur functions S_R being a kind of eigenfunc-

tions of this contraction map. Despite the setting looks very poor, the phenomenon clearly exists: a minor deformation of the Gaussian measure (which preserves integrability) is not compensated by a small deformation of the Schur functions so that the *super*integrability property $\left\langle S_R\{p\} \right\rangle \sim S_R\{N\}$ is preserved. Obviously, the setting should be lifted to the case when the both spaces, $\mathcal{X}$ and $\mathcal{Z}$ are "equal". This could mean, for instance, that one can consider the Schur functions not just as a subjects of averaging over matrices, but rather as common eigenfunctions of a commuting set of operators. In fact, integrability and superintegrability are both related to existence of different mutually commuting operators. But how to distinguish between different operators, and separate them into sets which are responsible respectively for integrability and for the *super*integrability?

A related question can be what is the reason for an additional enhancement in the case of Dotsenko-Fateev (double logarithmic) measure, where one gets a whole set of factorized Kadell integrals with not only averages of Schur functions, but also of their peculiar multilinear combinations being described by nice factorization formulas (giving rise to Nekrasov functions) [24,33,34]?

In this letter, we argue that these two questions are related to each other, and demonstrate that, in the Gaussian Hermitian model case, factorization of bilinear averages of Schur functions, which encodes the superintegrability is also due to existence of an infinite set of commuting operators. Note that the very idea is close to what was discussed in [35,36], where the author used an infinite set of commuting operators in order to study correlators in Gaussian matrix models with different gauge groups (exploiting a specific different structure: the embedding structure of gauge groups depending on N).

More exactly, we demonstrate that there is a complete set of symmetric polynomials K_Δ such that the averages $\left\langle K_\Delta S_R \right\rangle$ are proportional to the averages $\left\langle S_R \right\rangle$:

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$$\left\langle K_\Delta S_R \right\rangle = \mu_{\Delta,R} \left\langle S_R \right\rangle \tag{1}$$

where the eigenvalues $\mu_{\Delta,R}$ do not depend on N. This set of polynomials K_Δ is generated by an infinite set of commuting operators W_Δ^-.

The letter is organized as follows. In Sects. 2 and 3, we discuss the generic property of superintegrability. In Sect. 4, we consider the set of cut-and-join operators W_Δ [37,38] as a natural candidate for the infinite set of commuting operators generating K_Δ, and realize that it gives rise to only part of K_Δ. Hence, in Sect. 5, we construct the set W_Δ^- that generates all K_Δ. In Sect. 6, we discuss examples of eigenvalues $\mu_{\Delta,R}$, and, in Sect. 7, we find an explicit formula for $\mu_{\Delta,R}$. Section 8 contains concluding remarks, and, in the Appendix, we explicitly list polynomials K_Δ for all Δ up to level 6.

Notation The Schur functions are symmetric polynomials of variables x_i, $i = 1, \ldots, N$. In particular, x_i can be eigenvalues of a matrix H. We denote through $S_R\{p_k\}$ the Schur functions as functions of power sums $p_k = \sum_i x_i^k$. When we emphasize that x_i are the eigenvalues of H, we use the notation $P_k := \mathrm{tr}\, H^k$.

The Schur function depends on the partition (Young diagram) R, which is a set of lines with lengths $R_1 \geq R_2 \geq \cdots \geq R_{l_R}$. We also denote through $S_{R/T}$ the skew Schur functions, and sometimes use the notation

$$d_R := S_R\{\delta_{k,1}\}. \tag{2}$$

2 SI in matrix models

According to [2], see also [24] and references therein, SI means that there exits a linear basis in the space of observables such that all the elements of the basis have "very simple" averages. In practice, this "very simple" means fully factorized. Moreover, this distinguished basis is usually formed by characters of an underlying symmetry algebra (to which the matrices belong), and the average of each character is again just the same character, only at a different (diminished) space of variables. The typical example is the Gaussian Hermitian model, where averages over Hermitian matrices are defined

$$\langle F(H) \rangle := \int F(H) e^{-\frac{1}{2}\mathrm{tr}\, H^2} dH \tag{3}$$

dH being the Haar measure on Hermitian matrices normalized in such a way that $\left\langle 1 \right\rangle$.

If the function F is invariant, i.e. depends on the eigenvalues h_i of H, one can integrate over angular variables, and

$$\langle F(H) \rangle = \int_{-\infty}^{\infty} \prod_{i<j}^{N} F(h_i)(h_i - h_j)^2 \prod_{i=1}^{N} e^{-\frac{1}{2}h_i^2} dh_i \tag{4}$$

SI in this case states that averages of the Schur functions $S_R\{P_k = \mathrm{tr}\, H^k\}$ are

$$\left\langle S_R \right\rangle = \frac{S_R\{N\} \cdot S_R\{\delta_{k,2}\}}{S_R\{\delta_{k,1}\}} \tag{5}$$

At the r.h.s. are the same Schur functions but at very special points: the main one is $S_R\{N\} := S_R\{p_k = N\}$.

There are very similar statements $< character > \sim character$ for a variety of other eigenvalue models, see [24] for an extensive list.

3 Does SI really exist in matrix models?

A natural question is if there is any true sense in the above observation? Perhaps, one can always find such a distinguished basis? It is therefore instructive to look at what happens in the same Hermitian model when one changes the background potential from the Gaussian one to anything else.

The Gaussian partition function

$$Z\{p\} = \left\langle e^{\sum_k \frac{p_k P_k}{k}} \right\rangle = \sum_R S_R\{p\}\left\langle S_R\{P\} \right\rangle \tag{6}$$

allows one to define an average in arbitrary non-Gaussian potential $\sum_k T_k X^k$:

$$Z^{(T)}\{p\} = \left\langle e^{\sum_k \frac{(p_k+T_k)P_k}{k}} \right\rangle = \sum_Q S_Q\{p+T\}\left\langle S_Q\{P\} \right\rangle$$

$$= \sum_R S_R\{p\} \sum_Q S_{Q/R}\{T\}\left\langle S_Q\{P\} \right\rangle \tag{7}$$

On the other hand, one can rewrite it as a sum over averages in the T-background,

$$\left\langle e^{\sum_k \frac{(p_k+T_k)P_k}{k}} \right\rangle = \sum_R S_R\{p\}\left\langle S_R\{P\} \right\rangle^{(T)} \tag{8}$$

i.e. the T-deformed averages (note that the normalization is still Gaussian, i.e. the definition of average is not changed) are

$$\left\langle S_R\{P\} \right\rangle^{(T)} = \sum_Q S_{Q/R}\{T\}\left\langle S_Q\{P\} \right\rangle \tag{9}$$

These averages are non-factorized infinite series that, actually, can not to be simplified even for finitely many non-vanishing T_k. They much more complicated as compared to $\left\langle S_R\{P\} \right\rangle = \frac{S_R\{N\}\cdot S_R\{\delta_{k,2}\}}{S_R\{\delta_{k,1}\}}$, and there is no any (obvious) way to modify the Schur functions in order to produce factorized averages, not to say that the preferred basis, even if existed, would not be formed by the characters of sl_N. The only exceptions are the deformation by T_1 and T_2 only, which preserve Gaussianity.

In this sense, what we call *super*integrability is an obviously non-trivial feature, which, in this concrete example, distinguishes the Gaussian potential among the arbitrary ones. Note that it is clearly a further restriction as compared to the ordinary integrability, the latter one is preserved by arbitrary T-deformations and does *not* require Gaussianity: all $Z^{(T)}\{p\}$ are KP τ-functions, just they are associated with T-dependent points of the universal Grassmannian. Thus, superintegrability exists and is a strong refinement of ordinary integrability.

4 Constructing K_Δ: W-operators

In the next sections, we assume that Schur functions are restricted to the Miwa locus $S_R\{p_k = \operatorname{tr} H^k\}$ with $N \times N$ matrix H. It is a little less general than arbitrary time variables, but still far away from restricting the Schur functions to their Gaussian averages $\langle S_R \rangle$. We will assume that $N \geq |R|$ though this restriction is not necessary, and formulas are basically correct at any N: one just has to be careful with normalizations. For instance, $\mu_{\Delta,R}$ is a ratio of two zeroes unless $N \geq |R|$. In a proper normalization, both sides of formulas typically vanish unless $N \geq |R|$.

Since the Schur functions are common eigenfunctions [37, 38] of the operators

$$\hat{W}_\Delta := \; : \prod_{a=1}^{l_\Delta} \operatorname{tr}\left(H \frac{\partial}{\partial H_{tr}}\right)^{\Delta_a} : \tag{10}$$

$$\hat{W}_\Delta S_R = \lambda_{\Delta,R} S_R \tag{11}$$

where the eigenvalues are appropriately normalized symmetric-group characters, $\lambda_{\Delta,R} = \varphi_R(\Delta)$ [37,38], and the normal ordering $: \ldots :$ implies all the derivatives put to the right. One can use integration by parts to get

$$\left\langle S_R \cdot \left(e^{\frac{1}{2}\operatorname{tr} H^2} \hat{W}_\Delta^\dagger e^{-\frac{1}{2}\operatorname{tr} H^2}\right)\right\rangle = \lambda_{\Delta,R} \cdot \left\langle S_R \right\rangle \tag{12}$$

Since the expression in brackets at the l.h.s. is a polynomial in H, it can be expanded into a linear combination of the Schur functions,

$$\left\langle \left(\sum_{|Q| \leq 2|\Delta|} C_{\Delta Q}(N) S_Q\right) \cdot S_R \right\rangle = \lambda_{\Delta,R} \cdot \left\langle S_R \right\rangle \tag{13}$$

with

$$\sum_Q C_{\Delta Q}(N) \cdot S_Q\{p_k = \operatorname{tr} H^k\}$$

$$= e^{\frac{1}{2}\operatorname{tr} H^2} \hat{W}_\Delta^\dagger e^{-\frac{1}{2}\operatorname{tr} H^2}$$

$$= e^{\frac{1}{2}\operatorname{tr} H^2} \ddagger \prod_{a=1}^{l_\Delta} \operatorname{tr}\left(-\frac{\partial}{\partial H_{tr}} H\right)^{\Delta_a} \ddagger e^{-\frac{1}{2}\operatorname{tr} H^2} \tag{14}$$

where the normal ordering $\ddagger \cdots \ddagger$ this time implies that all the derivatives are put to the left.

In particular, for $\hat{W}_{[1]}$, which is just a dilatation operator with $\lambda_{[1],R} = |R|$, this means:

$$\left\langle (P_2 - N^2) \cdot S_R\{P\}\right\rangle = |R| \cdot \left\langle S_R\{P\}\right\rangle \tag{15}$$

which is indeed true (for example, one can use that $p_2 S_{[2r]} = S_{[2r+2]} + S_{[2r,2]} - S_{[2r,1,1]}$ and similar relations for non-symmetric representations R). For example,

$$\hat{W}_{[2]} : \left\langle (P_4 - 4N P_2 - P_1^2 + 2N^3) \cdot S_R \right\rangle = \lambda_{[2],R} \cdot \left\langle S_R \right\rangle$$

$$\hat{W}_{[1,1]} : \left\langle \left(P_2^2 - (2N^2 + 3)P_2 + N^2(N^2 + 1)\right) \cdot S_R \right\rangle$$

$$= \lambda_{[1,1],R} \cdot \left\langle S_R \right\rangle$$

$$\hat{W}_{[3]} : \left\langle \left(-P_6 + 6N P_4 + 3P_3 P_1 + 3P_2^2 \right.\right.$$

$$\left.\left. -(15N^2 + 6)P_2 - 6N P_1^2 + 5N^4 + N^2\right) \cdot S_R \right\rangle$$

$$= \lambda_{[3],R} \cdot \left\langle S_R \right\rangle$$

$$\ldots \tag{16}$$

Equation (13) are rather poor – they are *not* sufficient to express all Gaussian pair correlators, they are just sum rules, which impose certain constraints on them. This is because the number of Young diagrams $\#_{2n} > \#_n$, the former number is what we need for complete set of pair correlators, the latter number is what we can actually deduce from (11).

5 Constructing K_Δ: W^--operators

In this section, we discuss that, in order to construct the full set of operators, one has to consider another set of commuting operators, which are a kind of "lowering" operators in the space of Schur functions.

Let us note that, in addition to relations (16), there more bilinear Schur averages of the (1) type: for instance, there is the relation

$$\left\langle (P_1^2 - N) \cdot S_R \right\rangle = \left\langle (S_{[2]} + S_{[1,1]} - N) \cdot S_R \right\rangle$$

$$= \mu_{[1,1],R} \cdot \left\langle S_R \right\rangle \tag{17}$$

The l.h.s. of this formula would appear if we act on $e^{-\frac{1}{2}\operatorname{tr} H^2}$ with the operator $(\operatorname{tr} \frac{\partial}{\partial H})^2$. This operator with $\operatorname{tr} \frac{\partial}{\partial H} = N \frac{\partial}{\partial p_1} + \sum_{k=2}^{\infty} p_{k-1} \frac{\partial}{\partial p_k}$ does not have S_R as an eigenfunction,

$$\left(\operatorname{tr} \frac{\partial}{\partial H}\right)^2 S_R \neq \mu_{[1,1],R} \cdot S_R \tag{18}$$

What happens is that its action is conspired with the SI formula: despite (18) forbids S_R to be an eigenfunction, i.e. equation does not hold at the "operator level", it does hold

for the Gaussian averages:

$$\left\langle \left(\operatorname{tr} \frac{\partial}{\partial H} \right)^2 S_R \right\rangle = \mu_{[1,1],R} \cdot \left\langle S_R \right\rangle \tag{19}$$

Only for restricted set (16) they are promoted to the operator level (11).

Now our main claim is that one can construct in a similar way the full set of polynomials K_Δ for (1). That is, define

$$\hat{W}_k^- := \operatorname{tr} \left(\frac{\partial}{\partial H} \right)^k$$

$$\hat{W}_\Delta^- := \prod_a^{l_\Delta} W_{\Delta_a}^- \tag{20}$$

Then,

$$\boxed{\left\langle \hat{W}_\Delta^- S_R \right\rangle = \left\langle K_\Delta \cdot S_R \right\rangle = \mu_{\Delta,R} \cdot \left\langle S_R \right\rangle} \tag{21}$$

where

$$\boxed{K_\Delta = e^{\frac{1}{2} \operatorname{tr} H^2} \, \hat{W}_\Delta^- \, e^{-\frac{1}{2} \operatorname{tr} H^2}} \tag{22}$$

Note that

$$K_\Delta = P_\Delta + \text{lower degrees} \tag{23}$$

so that they form a complete set polynomials at any given level.

These formulas can be considered as one more reformulation (avatar) of superintegrability. As we already discussed in the Introduction, it is related to an infinite set of commuting operators W_Δ^-, which are manifestly given by (20).

In Sect. 7, we prove these relations, and find explicit expressions for $\mu_{\Delta,R}$. Examples at level 2 are given by (15) and (17), examples at level 4 are (examples up to level 6 can be found in the Appendix)

$$\left\langle K_{[3,1]} \cdot S_R \right\rangle = \left\langle \left(P_3 P_1 - 3 P_2 \underline{-3N P_1^2 + 3N^2} \right) \cdot S_R \right\rangle$$
$$= \mu_{[3,1],R} \cdot \left\langle S_R \right\rangle$$

$$\left\langle K_{[2,1,1]} \cdot S_R \right\rangle = \left\langle \left(P_2 P_1^2 - N P_2 - (N^2 + 4) P_1^2 + N^3 + 2N \right) \cdot S_R \right\rangle$$
$$= \mu_{[2,1,1],R} \cdot \left\langle S_R \right\rangle$$

$$\left\langle K_{[1,1,1,1]} \cdot S_R \right\rangle = \left\langle \left(P_1^4 - 6N P_1^2 + 3N^2 \right) \cdot S_R \right\rangle$$
$$= \mu_{[1,1,1,1],R} \cdot \left\langle S_R \right\rangle$$

$$\left\langle K_{[4]} \cdot S_R \right\rangle = \left\langle \left(P_4 - 4N P_2 - 2 P_1^2 + 2N^3 + N \right) \cdot S_R \right\rangle$$
$$= \mu_{[4],R} \cdot \left\langle S_R \right\rangle$$

$$\left\langle K_{[2,2]} \cdot S_R \right\rangle = \left\langle \left(P_2^2 - 2(N^2 + 2) P_2 + N^2(N^2 + 2) \right) \cdot S_R \right\rangle$$
$$= \mu_{[2,2],R} \cdot \left\langle S_R \right\rangle \tag{24}$$

The underlined term could be eliminated with the help of (17), but this causes an N-dependent shift of the eigenvalue

$\mu_{[3,1]} \longrightarrow \mu_{[3,1]} - 3N\mu_{[1,1]}$. In (24) *per se* all μ_R are independent of N. However, one can use (15) instead in order to remove the second and forth terms in this formula: this would give rise to the N-independent shift $\mu_{[3,1]} \longrightarrow \mu_{[3,1]} + 3\mu_{[2]}$.

Note that two equations of (16) can be compared with the corresponding equations from this list, they differ by adding lower N-independent averages so that $\lambda_{[2]} = \mu_{[4]} + \mu_{[1,1]}$ and $\lambda_{[1,1]} = \mu_{[2,2]} + \mu_{[2]}$. Generally, the identification for the part of relations that can be generated by the W-operators is $\lambda_{\Delta,R} = \mu_{2\Delta,R} + lower\ terms$, where 2Δ denotes a Young diagram with all line lengths doubled, $2\Delta := \{2\Delta_1 \geq 2\Delta_2 \geq \ldots \ldots 2\Delta_{l_\Delta}\}$.

6 Values of $\mu_{\Delta,R}$

Finding the "eigenvalues" $\mu_{\Delta,R}$ is a separate challenge. As we explain in the next section, there is a general formula for them. However, the formula is not that simple, and it is instructive to look at examples. The list of the first few is in the Table.

Clearly, transposition of R preserves the absolute value of μ:

$$\mu_{\Delta,R^\vee} = (-1)^{l_\Delta + |\Delta|/2} \mu_{\Delta,R} \tag{25}$$

We will prove it in the next section.

In fact, the quantities in the Table are given by product formulas, in particular:

$$\mu_{[2m],[2r]} = \frac{\left\langle K_{[2m]} S_{[2r]} \right\rangle}{\left\langle S_{[2r]} \right\rangle} = \frac{S_{[2r-2m]}}{S_{[2r]}} \{\delta_{k,2}\}$$
$$= \frac{(2r)!!}{(2r - 2m)!!}$$

$$\mu_{[2m],[2r-1,1]} = \frac{\left\langle K_{[2m]} S_{[2r-1,1]} \right\rangle}{\left\langle S_{[2r-1,1]} \right\rangle} = \frac{S_{[2r-2m-1,1]}}{S_{[2r-1,1]}} \{\delta_{k,2}\}$$
$$= \frac{(2r)!!}{(2r - 2m)!!} \quad \text{for } r > 2$$

$$\mu_{[2m],[2r-2,2]} = \frac{\left\langle K_{[2m]} S_{[2r-2,2]} \right\rangle}{\left\langle S_{[2r-2,2]} \right\rangle} = \frac{S_{[2r-2m-2,2]}}{S_{[2r-2,2]}} \{\delta_{k,2}\}$$
$$= \frac{(2r - 2)!!}{(2r - 2m - 2)!!} \quad \text{for } m \geq 2$$
$$\ldots \tag{26}$$

These $\mu_{\Delta,R}$ are shown boldfaced in the table. Expression through the values of Schur functions at $p_k = \delta_{k,2}$ are explained in (39) below.

All averages $\left\langle K_Q \cdot S_{[3,2,1]} \right\rangle = 0$, because $\left\langle S_{[3,2,1]} \right\rangle = 0$, this is, in turn, because the factor $S_{[3,2,1]}\{\delta_{k,2}\} = 0$ in (5). Therefore, the corresponding $\mu_{Q,[3,2,1]}$ are not defined. The

R\Δ	[2]	[1²]	[4]	[3,1]	[2²]	[2,1²]	[1⁴]	[6]	[5,1]	[4,2]	[4,1²]	[3²]	[3,2,1]	[2³]	[3,1³]	[2²,1²]	[2,1⁴]	[1⁶]
[2]	2	2																
[1²]	2	−2																
[4]	4	4	8	8	8	8	8											
[3,1]	4	−4	8	0	8	−8	−24											
[2²]	4	0	0	−4	8	0	8											
[2,1²]	4	4	−8	0	8	8	−24											
[1⁴]	4	−4	−8	8	8	−8	8											
[6]	6	6	24	24	24	24	24	48	48	48	48	48	48	48	48	48	48	48
[5,1]	6	−6	24	0	24	−24	−72	48	0	48	−48	48	0	48	−96	−48	−144	−240
[4,2]	6	2	8	0	24	8	24	0	−16	16	−16	0	0	48	0	16	48	144
[4,1²]	6	6	0	12	24	24	−24	−24	0	0	0	−24	24	48	−24	48	−48	−240
[3²]	6	−2	8	−8	24	−8	−8	0	0	16	16	−32	−16	48	16	−16	−16	−80
[3,2,1]	6	0	0	0	24	0	0	48	0	0	24	−16	0	48	24	0	0	48
[2³]	6	2	−8	−8	24	8	−8	0	0	−16	16	32	−16	48	−16	16	−16	80
[3,1³]	6	−6	0	12	24	−24	−24	−24	0	0	0	24	24	48	24	−48	−48	240
[2²,1²]	6	−2	−8	0	24	−8	24	0	16	−16	−16	0	0	48	0	−16	48	−144
[2,1⁴]	6	6	−24	0	24	24	−72	48	0	−48	−48	−48	0	48	96	48	−144	240
[1⁶]	6	−6	−24	24	24	−24	24	48	−48	−48	48	−48	48	48	−48	−48	48	−48

same is true for all $S_{[\ldots 4321]}$, which are independent of even time-variables. In fact, it is sufficient for vanishing of the average that the Schur polynomials does not contain the item $p_2^{|R|/2}$, this happens for $S_{[5,2,1]}$, $S_{[1,1,1,2,3]}$ and a number of other examples of bigger sizes.

The table has clearly a triangle structure, since, if $|\Delta| > |R|$, the corresponding W_Δ^- contains more derivatives than the degree of H in S_R.

The first example is provided by $\mu_{[1,1],R}$ at level 2. While

$$\mu_{[2],R} = \lambda_{[1],R} = |R|$$

is very simple, expression for $\mu_{[1,1],R}$ is quite involved: it depends on the number l_R of columns in the diagram $R = [r1 \geq r_2 \geq r_3 \geq \cdots \geq r_{l_R} > 0]$:

$$\mu_{[1,1],[r_1]} = r_1 \cdot P_e,$$
$$\mu_{[1,1],[r_1,r_2]} = (r_1 - r_2) \cdot P_{ee}$$
$$+(-r_1 + r_2 - 2) \cdot P_{oo}$$
$$\mu_{[1,1],[r_1,r_2,r_3]} = (r_1 - r_2 + r_3) \cdot P_{eee}$$
$$+(-r_1 + r_2 + r_3 - 2) \cdot P_{ooe}$$
$$+(r_1 + r_2 - r_3 + 2) \cdot P_{eoo}$$
$$\mu_{[1,1],[r_1,r_2,r_3,r_4]} = (r_1 - r_2 + r_3 - r_4) \cdot P_{eeee}$$
$$+(-r_1 + r_2 + r_3 - r_4 - 2) \cdot P_{ooee}$$
$$+(r_1 + r_2 - r_3 - r_4 + 2) \cdot P_{eooe}$$
$$+(r_1 - r_2 - r_3 + r_4 - 2) \cdot P_{eeoo}$$
$$+(-r_1 - r_2 + r_3 + r_4 - 6) \cdot P_{oeeo}$$
$$+(-r_1 + r_2 - r_3 + r_4 - 4) \cdot P_{oooo} \quad (27)$$

Projector P_{eoo} here, for instance, means that r_1 is even, r_2 and r_3 are odd so that all the values of $|R|$ are even. Averages of the type oeo, $eoeo$ and $oeoe$ are all vanishing.

Different lines in these formulas are connected smoothly, one should just pick up the terms with E at the very right position and put the highest $r_{l_R} = 0$.

7 Derivation of (21) and explicit formula for $\mu_{\Delta,R}$

Alternative representation of $\mu_{[1,1],R}$ can be deduced from (19), which we are going to derive now. The first examples of this relation are

$$\left(\text{tr}\,\frac{\partial}{\partial H}\right)^2 S_{[2]} = \left(\text{tr}\,\frac{\partial}{\partial H}\right)^2 \left(\frac{\text{tr}\,H^2 + (\text{tr}\,H)^2}{2}\right) = N(N+1)$$
$$\left(\text{tr}\,\frac{\partial}{\partial H}\right)^2 S_{[1,1]} = \left(\text{tr}\,\frac{\partial}{\partial H}\right)^2 \left(\frac{-\text{tr}\,H^2 + (\text{tr}\,H)^2}{2}\right) = N(N-1)$$
$$\left(\text{tr}\,\frac{\partial}{\partial H}\right)^2 S_{[k]} = (N+k-2)(N+k-1)S_{[k-2]}$$
$$\left(\text{tr}\,\frac{\partial}{\partial H}\right)^2 S_{[k_1,k_2]} = (N+k_1-1)(N+k_1-2)S_{[k_1-2,k_2]}$$
$$+2(N+k_1-1)(N+k_2-2)S_{[k_1-1,k_2-1]}$$
$$+(N+k_2-1)(N+k_2-2)S_{[k_1,k_2-2]}$$
$$\cdots \quad (28)$$

Clearly, these formulas are obtained by successive application of the operator

$$\left(\operatorname{tr}\frac{\partial}{\partial H}\right)S_R = \sum_\square (N+j-i)S_{R-\square_{i,j}} \tag{29}$$

where the box $\square_{i,j}$ with coordinates (i,j) is removed from the Young diagram R so that $R-\square_{i,j}$ is still a Young diagram. This formula is a kind of inverse of the Pieri rule.

Despite this time the operator changes S_R, which is no longer its eigenfunction, like it was in (11), it does not change the average! This is because of the very special coefficient at the r.h.s. of (29). Indeed, note that

$$\frac{S_R\{N\}}{d_R} = \prod_{\square_{i,j}\in R}(N+j-i) \tag{30}$$

and hence

$$(N+j-i) = \frac{S_R\{N\}}{S_{R-\square_{i,j}}\{N\}} \cdot \frac{d_{R-\square_{i,j}}}{d_R} \tag{31}$$

This means that

$$\left\langle (P_1^2 - N)\cdot S_R\right\rangle \overset{(24)}{=} \left\langle \left(\operatorname{tr}\frac{\partial}{\partial H}\right)^2 S_R\right\rangle =$$

$$\overset{(29)}{=} \sum_{\square_1,\square_2\in R}\frac{S_R(N)}{S_{R-\square_1-\square_2}(N)}\cdot\frac{d_{R-\square_1-\square_2}}{d_R}\cdot\left\langle S_{R-\square_1-\square_2}\right\rangle$$

$$\overset{(5)}{=} \mu_{[1,1],R}\cdot\left\langle S_R\right\rangle \tag{32}$$

with

$$\mu_{[1,1],R} = \frac{1}{S_R\{\delta_{k,2}\}}\sum_{\square_1,\square_2\in R}S_{R-\square_1-\square_2}\{\delta_{k,2}\} \tag{33}$$

The sum over the boxes of the Young diagram is such that the diagram obtained after removing any of these two boxes and both of them still remains a Young diagram. When there are two different ways to achieve the final state, a combinatorial coefficient 2 appears. For example,

$$\mu_{[1,1],[7,1]} = \frac{S_{[5,1]}+2S_{[6]}}{S_{[7,1]}}\{\delta_{k,2}\}$$

$$\mu_{[1,1],[8,4]} = \frac{S_{[6,4]}+2S_{[7,3]}+S_{[8,2]}}{S_{[8,4]}}\{\delta_{k,2}\}$$

$$\dots \tag{34}$$

When $S_R\{\delta_{k,2}\} = 0$, the corresponding Gaussian average vanishes and $\mu_{[1,1],R}$ is not defined. One can check that in these cases the numerator vanishes as well.

One can similarly consider the action higher (even) degrees of operator $\left(\operatorname{tr}\frac{\partial}{\partial H}\right)^{2n}$, in this case with the same line of reasoning, one obtains

$$\mu_{[1^{2n}],R} = \frac{1}{S_R\{\delta_{k,2}\}}\sum_{\{\square_i\}\in R}S_{R-\sum_i^{2n}\square_i}\{\delta_{k,2}\} \tag{35}$$

where one has to remove $2n$ boxes from the Young diagram preserving the property of being the Young diagram.

Furthermore,

$$W_2^- S_R = \operatorname{tr}\left(\frac{\partial^2}{\partial H^2}\right)S_R = \sum_{\square\square}c_{\square\square}S_{R-\square\square} - \sum_{\substack{\square\\\square}}c_{\substack{\square\\\square}}S_{R-\substack{\square\\\square}} \tag{36}$$

where the two boxes are removed from rightmost part of the R in such a way that obtained is still a Young diagram. The general formula is

$$W_k^- S_R = \operatorname{tr}\left(\frac{\partial^k}{\partial H^k}\right)S_R = \sum_{h_k}(-1)^{\operatorname{ht}(h_k)}c_{R,h_k}S_{R-h_k} \tag{37}$$

where h_k is a border strip of length k (i.e. containing k boxes) [39], $\operatorname{ht}(h_k)$ is its height (defined to be one less than the number of rows it occupies), the sum runs over all such border strips, and all the coefficients c_{R,h_k} are products of $N-i+j$ over subtracted boxes. Note that the border strips are called rim hooks in [12]. In order to get W_Δ^- with a few rows, one has to apply (37) sequentially.

Formula (37) is a kind of inverse to the Pieri rule

$$p_k S_R = \sum_{Q:\,Q-R=h_k}(-1)^{\operatorname{ht}(h_k)}S_Q \tag{38}$$

Now one immediately obtains the explicit formula for $\mu_{\Delta,R}$:

$$\boxed{\mu_{\Delta,R} = \frac{1}{S_R\{\delta_{k,2}\}}\sum_{\{h_{\Delta_a}\}\in R}(-1)^{\operatorname{ht}(h_{\Delta_a})}S_{R-\sum_a h_{\Delta_a}}\{\delta_{k,2}\}} \tag{39}$$

where h_{Δ_a} are the border strips removed from the Young diagram R in accordance with (37), with the corresponding sign taken into account.

Note that, since $S_{R^\vee}\{p_k\} = (-1)^{|R|}S_R\{-p_k\}$ and $S_R\{p_k = \delta_{k,2}\} \sim p_2^{|R|/2}$, one immediately obtains from this formula relation (25). Indeed, let us look, for instance, at the case of $K_{[2]}$. Then,

$$\mu_{[2],R} = \frac{1}{S_R\{\delta_{k,2}\}}\left(\sum_{\square\square\in R}S_{R-\square\square}\{\delta_{k,2}\} - \sum_{\substack{\square\\\square}\in R}S_{R-\substack{\square\\\square}}\{\delta_{k,2}\}\right) \tag{40}$$

and every term in the sum gets the sign $(-1)^{|\Delta|/2}$ under conjugation of the Young diagram. Besides, in the course of this conjugation, one has to permute the two terms in (40). This gives total sign $(-1)^{|\Delta|/2+1}$. This case is certainly very trivial, because the difference of two sums (40) is equal to $|R|$.

For the generic K_Δ, there is a sign factor $(-1)^{|\Delta|}$ from the ratios of $S_{R-\Delta}/S_R$ and an additional factor of $(-1)^{l_\Delta}$ that follows from the fact that the conjugation of R changes additionally the sign of $\mu_{\Delta,R}$ for every even part of partition

Δ because of formula (37) (the height of the border strip changes the parity under conjugation in this case). This gives $(-1)^{|\Delta|+\#_e}$, where $\#_e$ is the number of even parts of partition Δ. However, as soon as the number of odd parts (which do not give rise to this additional sign changing) is even, this formula can be changed for $(-1)^{|\Delta|+l_\Delta}$.

8 Conclusion

To conclude, we have found a new indirect implication of superintegrability: factorization of peculiar pair correlators, where one component of the pair is just the character, while the other one is its N-dependent deformation. This factorization is due to existence of an infinite set of commuting operators W_Δ^-, (20) that provide a mapping from the space $\mathcal{X}$ of invariant matrix polynomials to the space $\mathcal{Z}$ of polynomials of matrix size, $F_{W_\Delta^-} : \mathcal{X} \longrightarrow \mathcal{Z}$. This mapping is manifestly given by the matrix averaging

$$\forall f \in \mathcal{X}: \ F_{W_\Delta^-}(f) = \left\langle W_\Delta^- \cdot f \right\rangle \in \mathcal{Z} \quad (41)$$

The space $\mathcal{Z}$ can be spanned (ambiguously) by matrix averages of the Schur functions $\left\langle S_R \right\rangle$. As we demonstrated in this paper, the Schur functions turn out to be eigenvectors of the mapping:

$$F_{W_\Delta^-}(S_R) = \left\langle W_\Delta^- \cdot S_R \right\rangle = \mu_{\Delta,R} \left\langle S_R \right\rangle \quad (42)$$

while, on the space $\mathcal{X}$, i.e. before the averaging, this is not the case, since the action of W_Δ^- on S_R decrease its grading to $|R| - |\Delta|$.

It is an open question, if this construction persists in other superintegrable models, which could help to understand if it is in fact a *direct* corollary of SI or not. Note also that, in other models, the space $\mathcal{Z}$ may need an extension to the space of rational functions of the matrix size, see, for instance, the case of two-logarithm (Selberg) models, [24].

Also an important issue is what are interrelations between the set of operators W_Δ^- and a commutative set of generalized cut-and-join operators W_Δ. Indeed, the Schur function are the eigenvectors of the latter already on the space $\mathcal{X}$. However, as a price of it, if one rewrites (42) as a statement about the pair correlator (21),

$$\left\langle K_\Delta \cdot S_R \right\rangle = \mu_{\Delta,R} \cdot \left\langle S_R \right\rangle \quad (43)$$

the set of W_Δ generates too little set of polynomials K_Δ, while the set of W_Δ^- gives rise to the complete basis of K_Δ. The operators W_Δ^- are of a kind of lowering operators in an algebra, and W_Δ are Cartan-like operators in it. Indeed, as one can notice, the operators W_k^- looks as positive harmonics in the w_∞-algebra, while W_k, as zeroth harmonics in it: as $\hat{W}_k^{(k+1)}$ and $\hat{W}_0^{(k+1)}$ correspondingly in terms of [10, secs.8-

9]. Note that these zeroth harmonic operators can generate the Virasoro and W-algebra constraints [40].

Appendix

We list here the first few operators K_Δ. They celebrate the property that their bilinear averages with all S_R, are proportional to the averages of S_R: $\left\langle K_\Delta S_R \right\rangle = \mu_{\Delta,R} \left\langle S_R \right\rangle$. However, this condition does not fix K_Δ unambiguously, therefore we use a precise definition: K_Δ are obtained by the action of operators $\hat{W}^- \Delta$ on the Gaussian weight $e^{-\frac{1}{2} \mathrm{tr}\, H^2}$, see Eq. (22). Then,

$$\begin{aligned}
K_{[2]} &= P_2 - N^2 \\
K_{[1,1]} &= P_1^2 - N \quad (44) \\
K_{[4]} &= P_4 - 4NP_2 - 2P_1^2 + N(2N^2 + 1) \\
K_{[3,1]} &= P_3 P_1 - 3P_2 - 3NP_1^2 + 3N^2 \\
K_{[2,2]} &= P_2^2 - 2(N^2 + 2)P_2 + N^2(N^2 + 2) \\
K_{[2,1,1]} &= P_2 P_1^2 - NP_2 - (N^2 + 4)P_1^2 + N(N^2 + 2) \\
K_{[1,1,1,1]} &= P_1^4 - 6NP_1^2 + 3N^2 \quad (45) \\
K_{[6]} &= P_6 - 6NP_4 - 6P_1 P_3 - 3P_2^2 + 15(N^2 + 1)P_2 \\
&\quad + 15NP_1^2 - 5N^2(N^2 + 2) \\
K_{[5,1]} &= P_5 P_1 - 5P_4 - 5NP_1 P_3 - 3P_2^2 - 5P_1^2 P_2 + 20NP_2 \\
&\quad + 5(2N^2 + 3)P_1^2 - 5N(2N^2 + 1) \\
K_{[4,2]} &= P_4 P_2 - (N^2 + 8)P_4 - 4NP_2^2 - 2P_1^2 P_2 \\
&\quad + (6N^3 + 25N)P_2 \\
&\quad + 2(N^2 + 6)P_1^2 - N(2N^2 + 1)(N^2 + 4) \\
K_{[4,1,1]} &= P_4 P_1^2 - NP_4 - 8P_1 P_3 - 4NP_1^2 P_2 \\
&\quad - 2P_1^4 + 4(N^2 + 3)P_2 \\
&\quad + (2N^3 + 27N)P_1^2 - N^2(2N^2 + 13) \\
K_{[3,3]} &= P_3^2 - 9P_4 - 6NP_1 P_3 + 27NP_2 \\
&\quad + 9(N^2 + 1)P_1^2 - 3N(4N^2 + 1)
\end{aligned}$$

$$K_{[3,2,1]} = P_3 P_2 P_1 - (N^2 + 8) P_1 P_3 - 3 P_2^2 - 3N P_1^2 P_2$$
$$+6(N^2+3)P_2 + 3N(N^2+6)P_1^2 - 3N^2(N^2+4)$$

$$K_{[2,2,2]} = P_2^3 - 3(N^2+4)P_2^2$$
$$+3(N^2+2)(N^2+4)P_2 - N^2(N^2+2)(N^2+4)$$

$$K_{[3,1,1,1]} = P_3 P_1^3 - 3N P_3 P_1 - 9 P_2 P_1^2 - 3N P_1^4 + 9N P_2$$
$$+18(N^2+1)P_1^2 - 3N(3N^2+2)$$

$$K_{[2,2,1,1]} = P_2^2 P_1^2 - N P_2^2 - 2(N^2+6)P_2 P_1^2 + 2N(N^2+4)P_2$$
$$+(N^2+4)(N^2+6)P_1^2 - N(N^2+2)(N^2+4)$$

$$K_{[2,1,1,1,1]} = P_2 P_1^4 - 6N P_2 P_1^2 - (N^2+8)P_1^4 + 3N^2 P_2$$
$$+6N(N^2+6)P_1^2 - 3N^2(N^2+4)$$

$$K_{[1,1,1,1,1,1]} = P_1^6 - 15N P_1^4 + 45N^2 P_1^2 - 15N^3 \qquad (46)$$

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