Journal of Nonlinear Mathematical Physics (2025) 32:9
https://doi.org/10.1007/s44198-024-00257-2

RESEARCH ARTICLE

On the Orbital Stability of Gross-Pitaevskii Solitons

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Received: 9 September 2024 / Accepted: 16 December 2024

Abstract
The one-dimensional Gross-Pitaevskii equation, under non-vanishing boundary condition, has a set of solitary solutions. The orbital stability of these solitons has been well established. However, the existing proof methods usually treat the cases of dark solitons and black solitons separately. Here we provide an alternative proof of this orbital stability result, which treats the two cases in a unified framework.

Keywords Gross-Pitaevskii equation · Soliton · Constrained minimization · Ginzburg-Landau energy.

Mathematics Subject Classification 35Q51 · 35Q55 · 35Q40 · 35J20 · 37K40

1 Introduction

We study the Gross-Pitaevskii equation,

$$i\Psi_t + \Psi_{xx} + \Psi\left(1 - |\Psi|^2\right) = 0 \qquad x \in \mathbb{R}, \tag{1}$$

where $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ satisfies the boundary condition

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$$|\Psi| \to 1 \qquad \text{as } |x| \to \infty.$$

Equation (1) appears in various fields in physics, including superfluidity and Bose-Einstein condensation ([1, 3, 4, 20–22]), and it describes the dark soliton in nonlinear optics ([23, 24]). Under the nonzero boundary condition, (1) has a nontrivial dynamics, in contrast with the zero boundary condition case, where the dynamics is essentially dispersion and scattering.

The energy functional

$$E(\Psi) = \int_{\mathbb{R}} |\Psi_x|^2 + \frac{1}{2}\left(|\Psi|^2 - 1\right)^2 dx$$

is a conserved quantity of (1), where $V(|\Psi|^2) = \frac{1}{2}(|\Psi|^2 - 1)^2$ is the potential.

The momentum $P(\Psi)$ is also conserved. Section 2 will give $P(\Psi)$ a rigorous definition.

We consider the traveling wave solution of (1): $\Psi(x,t) = \phi(x+vt)$, where v is velocity. It satisfies

$$iv\phi_x + \phi_{xx} + \phi\left(1 - |\phi|^2\right) = 0 \qquad \text{in } \mathbb{R}. \tag{2}$$

We only focus on the case $v \geq 0$, because a function ϕ solves (2) for some v is equivalent to $\phi(-x)$ solves it with velocity $-v$.

Equation (2) is integrable, by the ordinary differential equation technique (see [6]). If $v \geq \sqrt{2}$, $\phi = 1$ (modulo complex number of magnitude 1). We set $v_s = \sqrt{2}$ (called the sound speed). For $0 \leq v < \sqrt{2}$, the solution is either 1, or

$$b_v = \sqrt{\frac{2 - v^2}{2}} \tanh\left(\frac{\sqrt{2 - v^2}}{2} x\right) - i\frac{v}{\sqrt{2}}, \tag{3}$$

modulo unit length complex number and translation. For $v \neq 0$,b_v are called dark solitons and they do not vanish on $\mathbb{R}$. In the case $v = 0$, $b_0 = 0$ at $x = 0$. b_0 is called the black soliton.

We consider orbital stability of the solution (3). Two ways are used to tackle the orbital stability problem: the first one is concentration-compactness argument in [11], and the other one is Grillakis-Shatah-Strauss theory ([18, 19]). Our goal is to establish orbital stability using [11] for all speed $|v| < \sqrt{2}$, under a general class of perturbations in the energy space.

The overall strategy is to implement (3) as minimizers of E at fixed P, where v serves as the Lagrange multiplier. Then using [11], we get the orbital stability result.

We introduce some function spaces. Let $\phi \in H^1_{\text{loc}}(\mathbb{R})$ and $\Omega \subset \mathbb{R}$ be an open set, we define

$$E_\Omega(\phi) = \int_\Omega |\phi'|^2 + V(|\phi|^2) dx$$

to be the Ginzburg-Landau energy of ϕ in Ω. When $\Omega = \mathbb{R}$, we use $E(\phi)$ rather than $E_{\mathbb{R}}(\phi)$.
We use the notation $\dot{H}^1(\mathbb{R}) = \left\{\phi \in L^1_{\text{loc}}(\mathbb{R}) \;\middle|\; \phi' \in L^2(\mathbb{R})\right\}$. Define the energy space

$$\begin{aligned} \mathcal{E} &= \left\{\phi \in \dot{H}^1(\mathbb{R}) \;\middle|\; |\phi|^2 - 1 \in L^2(\mathbb{R})\right\} \\ &= \left\{\phi \in \dot{H}^1(\mathbb{R}) \;\middle|\; E(\phi) < \infty\right\}. \end{aligned}$$

Denote the distance $(\mathcal{E}, d_{\mathcal{E}})$ as

$$d_{\mathcal{E}}(\phi_1, \phi_2) = \| |\phi_1| - |\phi_2| \|_{L^2(\mathbb{R})} + \|\phi_1' - \phi_2'\|_{L^2(\mathbb{R})} + \|\phi_1 - \phi_2\|_{L^2 + L^\infty(\mathbb{R})} . \quad (4)$$

$(\mathcal{E}, d_{\mathcal{E}})$ is a complete metric space (this can be proved following section 1 in [15], pp. 132–133).
Denote the semi-distance d_0 on $\mathcal{E}$ as

$$d_0(\phi_1, \phi_2) = \| |\phi_1| - |\phi_2| \|_{L^2(\mathbb{R})} + \|\phi_1' - \phi_2'\|_{L^2(\mathbb{R})} . \quad (5)$$

The following theorem is established in ([6, 7]). Our main aim of this article is to provide an alternative proof of this well-known theorem.

Theorem 1.1 ([6, 7]) For $0 < q \leq \pi$, let

$$E_{\min}(q) = \inf_{\phi \in \mathcal{E}} \{E(\phi) \mid P(\phi) = q\}.$$

Then any minimizing sequence $(\phi_n)_{n \geq 1} \subset \mathcal{E}$ verifying $E(\phi_n) \to E_{\min}(q)$ under the constraint $P(\phi) \to q$ has a convergent subsequence, under the semi-distance d_0 (up to translations).
$U_q = \{\phi \in \mathcal{E} \mid E(\phi) = E_{\min}(q),\ P(\phi) = q\}$ has a unique element $b_{v(q)}$ (up to translations and rotations), where *v*(*q*) is the unique speed *v* such that $P(b_v) = q$. The set U_q is orbitally stable, with respect to the semi-distance d_0.

Theorem 1.1 is a summary of Theorem 4.1, Proposition 4.6 and Theorem 5.5.

The orbital stability of dark solitons $v = 0$, under the distance (see Lemma 10 in [12], p. 1338, and [25])

$$d(\phi_1, \phi_2) = |\phi_1(0) - \phi_2(0)| + \| |\phi_1| - |\phi_2| \|_{L^2(\mathbb{R})} + \|\phi_1' - \phi_2'\|_{L^2(\mathbb{R})} , \quad (6)$$

was proved in [25]. The proof exploits the hydrodynamical form of (1), which is a Hamiltonian system and Grillakis-Shatah-Strauss theorem is applied.
This method is not valid for the case $v = 0$, since b_0 vanishes at $x = 0$. Orbital stability for black soliton $(v = 0)$ for distance

$$d_A(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|_{L^\infty[-A,A]} + \||\phi_1| - |\phi_2|\|_{L^2(\mathbb{R})} + \|\phi_1' - \phi_2'\|_{L^2(\mathbb{R})} \quad (7)$$

was established in [7] relying on variational arguments, given any $A > 0$. The orbital stability of b_v ($|v| < \sqrt{2}$) with the distance (7) has been proved in ([6, 7]).
Using Lemma 2.2, it can be shown that the semi-distance d_0, the distance defined in (6) and (7) are equivalent, so we state Theorem 1.1 using the semi-distance d_0.

A motivation of this work is that, previous work, e.g. ([6, 7, 12]), treated the cases $0 < v < \sqrt{2}$ and $v = 0$ separately, while our proof strategy deals with the two cases in a unified framework.

[16] proved orbital stability of black soliton, under a very restricted class of perturbations. See [12] for a detailed study of the stability problem of the traveling waves for the nonlinear Schrödinger equation, under the distance (6). Generalizations of the orbital stability to variations of the 1-dimensional Gross-Pitaevskii equation (with non local terms and general nonlinearities) was shown in ([5, 14]). The asymptotic stability was shown in [8].

In space dimension $N \geq 2$, the constraint minimization procedure is used in [13] to obtain a class of orbitally stable traveling waves, for general nonlinearity (including the Gross-Pitaevskii equation).

We then comment on the proof methods. We rely on the ideas in [13]. An important quantity called modified Ginzburg-Landau energy is indispensable in analyzing the traveling waves in space dimension ≥ 2 ([13, 27]). However, for the one dimensional equation (1), we don't need this modified Ginzburg-Landau energy because the Ginzburg-Landau energy E itself can be used to control $\||\phi| - 1\|_{L^\infty(\mathbb{R})}$, see Lemma 2.2.

We use the concentration-compactness principle (similar to [13]) to prove that $b_{v(q)}$ is minimizer (modulo translations and rotations) for $E_{\min}(q)$. If "vanishing" holds, we have that $\||\phi_n| - 1\|_{L^\infty} \to 0$, provided $(\phi_n)_{n\geq 1}$ is a vanishing minimizing sequence. Then from Lemma 3.2 (ii) we get $E(\phi_n) \geq v\,|P(\phi_n)|$ for all $v \in (0, v_s)$, Taking limit $v \uparrow v_s$ we obtain $E_{\min}(q) \geq v_s q$, which contradicts the upper bound $E_{\min}(q) < v_s q$ (see Lemma 3.3).

If we have "dichotomy", then we show that $E_{\min}(q) = E_{\min}(q_1) + E_{\min}(q - q_1)$,$q_1 \in (0, q)$, which contradicts with $E_{\min}$ is strictly subadditive (see Lemma 3.5).

Hence, we have concentration since vanishing and dichotomy are excluded.

1.1 Outline

Section 2 gives the rigorous definition of momentum. Section 3 contains some properties of $E_{\min}$. Section 4 shows the precompactness of the minimizing sequence. Section 5 presents the orbital stability result. Finally, in the Appendix A, we give a technical result: a splitting lemma, which is used to ruling out dichotomy in the proof of Theorem 4.1.

2 The Definition of Momentum in 1D

To solve (2) via a variational approach, we need a reasonable definition of momentum. In dimension $N \geq 3$, a definition of the momentum for all functions in the energy space has been given in [27]. The definition of momentum in dimension 2 is given in [13]. In dimension 1, a definition called untwisted momentum for any function in $\mathcal{E}$ has been provided by [7]. We propose an alternative definition in 1D, generalizing a strategy given in [27] for dimension ≥ 3, and show that this definition is equivalent to the one given in [7]. We will use this alternative definition in the following sections.

We now give some observations of why we need to give a definition of momentum. The momentum should be defined as

$$P(\phi) = \int_{\mathbb{R}} \langle i\phi', \phi - 1 \rangle \, dx.$$

provided $\phi - 1 \in H^1(\mathbb{R})$. But there are functions $\phi - 1 \in \mathcal{E} \setminus H^1(\mathbb{R})$ satisfying $\langle i\phi', \phi - 1 \rangle \notin L^1(\mathbb{R})$.
If $\phi \in \mathcal{E}$ has a lifting $\phi = \rho e^{i\theta}$, and $\lim_{x\to\infty} \phi$, $\lim_{x\to-\infty} \phi$ exist, a computation gives

$$\int_{\mathbb{R}} \langle i\phi', \phi - 1 \rangle \, dx = -\int_{\mathbb{R}} \left(\rho^2 - 1\right) \theta' dx + [\mathrm{Im}(\phi) - \theta]|_{-\infty}^{\infty}.$$

But there exists $\phi \in \mathcal{E}$ such that ϕ can not be liftted. Also, $\lim_{x\to\infty} \phi(x)$, $\lim_{x\to-\infty} \phi(x)$ may not exist.

Lemma 2.1 Let $\phi \in \mathcal{E}$ satisfies $0 < c_1 \leq |\phi| < \infty$ on $\mathbb{R}$ for a constant c_1. Then we can write $\phi = \rho e^{i\theta}$ with $\rho - 1 \in H^1(\mathbb{R}),\ \theta \in \dot{H}^1(\mathbb{R})$,

$$\langle i\phi', \phi - 1 \rangle = \frac{d}{dx}(\mathrm{Im}(\phi) - \theta) - \left(\rho^2 - 1\right) \theta' \text{ a.e. on } \mathbb{R}. \tag{8}$$

In addition, $\int_{\mathbb{R}} |\ (\rho^2 - 1)\theta'\ |\ dx \leq \frac{1}{\sqrt{2}c_1} E(\phi)$.

Proof Since $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$, the existence of $\rho,\ \theta \in H^1_{\mathrm{loc}}(\mathbb{R})$ such that $\phi = \rho e^{i\theta}$ a.e. can be obtained using Theorem 1 in ([10], p. 37). Direct calculation shows

$$|\phi'|^2 = |\rho'|^2 + \rho^2 |\theta'|^2. \tag{9}$$

Since $\rho = |\phi| \geq c_1$ and $\phi' \in L^2(\mathbb{R})$, it follows that $\rho', \theta' \in L^2(\mathbb{R})$. We have $\rho^2 - 1 \in L^2(\mathbb{R})$ because $\phi \in \mathcal{E}$. Since $|\rho - 1| = \frac{|\rho^2 - 1|}{\rho + 1} \leq |\rho^2 - 1|$, then $\rho - 1 \in L^2(\mathbb{R})$.
A short computation yields

$$\langle i\phi', \phi - 1\rangle = \langle i\phi', -1\rangle - \rho^2\theta' = \frac{d}{dx}(\mathrm{Im}(\phi) - \theta) - \left(\rho^2 - 1\right)\theta'.$$

Using 9, we have $|\theta'| \leq \frac{1}{\rho}|\phi'| \leq \frac{1}{c_1}|\phi'|$, and

$$\int_{\mathbb{R}} |\,(\rho^2-1)\theta'|dx \leq \|(\rho^2-1)\|_{L^2}\|\theta'\|_{L^2} \leq \frac{1}{c_1}\|(\rho^2-1)\|_{L^2}\|\phi'\|_{L^2}$$
$$\leq \frac{1}{\sqrt{2}c_1}(\frac{1}{2}\|(\rho^2-1)\|_{L^2}^2 + \|\phi'\|_{L^2}^2) = \frac{1}{\sqrt{2}c_1}E(\phi).$$

□

We use the notation

$$X^1(\mathbb{R}) = \left\{\phi \in L^\infty(\mathbb{R}) \;\middle|\; \phi' \in L^2(\mathbb{R})\right\}.$$

Lemma 2.2 We have $\mathcal{E} \subset L^\infty(\mathbb{R})$. There exists a universal constant C such that

$$\|\phi\|_{L^\infty(\mathbb{R})} \leq C(1 + \sqrt{E(\phi)}).$$

Moreover,

$$|\phi|^2 - 1 \in H^1(\mathbb{R}), \quad \forall \phi \in \mathcal{E}. \tag{10}$$

Proof Let $\chi_1 \in C_0^\infty(\mathbb{C})$ with $0 \leq \chi_1 \leq 1$, $\chi_1(x) = 1$ for $|x| \leq 2$, and $\chi_1(x) = 0$ for $|x| \geq 3$. Let us decompose

$$\phi = \phi_1 + \phi_2, \quad \phi_1 = \chi_1(\phi)\phi, \quad \phi_2 = (1 - \chi_1(\phi))\,\phi.$$

Using Lemma 1.5 in ([15], p. 132), we have

$$\|\phi_1\|_{X^1(\mathbb{R})} + \|\phi_2\|_{H^1(\mathbb{R})} \leq C_1 + C_2\sqrt{E(\phi)}.$$

By Sobolev inequality in 1D ([9], pp. 212–213),

$$\|\phi\|_{L^\infty(\mathbb{R})} \leq \|\phi_1\|_{L^\infty(\mathbb{R})} + \|\phi_2\|_{L^\infty(\mathbb{R})} \leq \|\phi_1\|_{X^1(\mathbb{R})} + C\,\|\phi_2\|_{H^1(\mathbb{R})}$$
$$\leq C(1 + \sqrt{E(\phi)}).$$

Since $\left(|\phi|^2 - 1\right)' = 2\,\langle\phi, \phi'\rangle$, we have

$$\|(|\phi|^2-1)'\|_{L^2(\mathbb{R})} = 2\,\|\langle\phi,\phi'\rangle\|_{L^2(\mathbb{R})} \leq 2\left(\int_{\mathbb{R}} |\phi|^2|\phi'|^2 dx\right)^{\frac{1}{2}}$$
$$\leq 2\|\phi\|_{L^\infty(\mathbb{R})}\,\|\phi'\|_{L^2(\mathbb{R})} \leq C(1 + \sqrt{E(\phi)})\sqrt{E(\phi)} < \infty,$$

thus, $\left(|\phi|^2-1\right)' \in L^2(\mathbb{R})$. Combining with the fact that $|\phi|^2-1 \in L^2(\mathbb{R})$, we have $|\phi|^2-1 \in H^1(\mathbb{R})$. □

Remark 2.3 [7] uses the energy space

$$\chi^1 = \left\{\gamma \in L^\infty(\mathbb{R}) \;\middle|\; 1-|\gamma|^2 \in L^2(\mathbb{R}) \text{ and } \gamma' \in L^2(\mathbb{R})\right\}.$$

Using Lemma 2.2, we see that $\mathcal{E} = \chi^1$.

Lemma 2.4 Let $\chi \in C_c^\infty(\mathbb{C},\mathbb{R})$ satisfies $\chi = 1$ on $\{x \mid ||x|-1| < \frac{1}{4}\}$, $0 \le \chi \le 1$ and $\operatorname{supp}(\chi) \subset \{x \mid ||x|-1| < \frac{1}{2}\}$. For any $\phi \in \mathcal{E}$, denote $\phi_1 - 1 = \chi(\phi)(\phi-1)$ and $\phi_2 - 1 = (1-\chi(\phi))(\phi-1)$. Then $\phi_1 \in \mathcal{E}$, $\phi_2 - 1 \in H^1(\mathbb{R})$ and we have the following:

$$|\phi_i'| \le C|\phi'| \qquad i=1,2, \text{ with C depends only on } \chi; \tag{11}$$

$$\begin{gathered} \|\phi_2-1\|_{L^2(\mathbb{R})} \le C_1 \left\||\phi|^2-1\right\|_{L^2(\mathbb{R})} \text{ and} \\ \left\|\left(1-\chi^2(\phi)\right)(\phi-1)\right\|_{L^2(\mathbb{R})} \le C_2 \left\||\phi|^2-1\right\|_{L^2(\mathbb{R})}; \end{gathered} \tag{12}$$

$$\int_{\mathbb{R}} (|\phi_1|^2-1)^2 dx \le C_3 \int_{\mathbb{R}} \left(|\phi|^2-1\right)^2 dx; \tag{13}$$

$$\int_{\mathbb{R}} (|\phi_2|^2-1)^2 dx \le C_3 \int_{\mathbb{R}} \left(|\phi|^2-1\right)^2 dx. \tag{14}$$

Let $\phi_1 = \rho e^{i\theta}$ be the lifting of ϕ_1, provided by Lemma 2.1. Then

$$\langle i\phi', \phi-1\rangle = \left(1-\chi^2(\phi)\right)\langle i\phi', \phi-1\rangle - \left(\rho^2-1\right)\theta' + \frac{d}{dx}\left(\operatorname{Im}(\phi_1)-\theta\right). \tag{15}$$

Proof Since $|\phi_i| \le |\phi-1|+1$ we have $\phi_i \in L^\infty(\mathbb{R})$ for $i=1,2$ by Lemma 2.2. It can be shown that $\phi_i \in H^1_{\text{loc}}(\mathbb{R})$ (see Lemma C1 in [10], p. 66) and we have

$$\phi_1' = \left(\partial_1\chi(\phi)\frac{d(\operatorname{Re}(\phi))}{dx} + \partial_2\chi(\phi)\frac{d\left(\operatorname{Im}(\phi)\right)}{dx}\right)(\phi-1) + \chi(\phi)\phi'. \tag{16}$$

For ϕ_2 we have a similar formula. Since $\partial_i\chi(\phi)(\phi-1)$ are bounded, $i=1,2$, we have (11).
Since $||\phi|-1| \ge \frac{1}{4}$ on the support of $(1-\chi(\phi))\phi$, there exists $C_1 > 0$ such that

$$\|\phi_2-1\|_{L^2(\mathbb{R})} = \|(1-\chi(\phi))(\phi-1)\|_{L^2(\mathbb{R})} \le \||\phi|+1\|_{L^2(\mathbb{R})} \le C_1 \left\||\phi|^2-1\right\|_{L^2(\mathbb{R})}.$$

Thus we get the first part in (12). Similarly we have the second part.

Since $\phi_1(x)=\phi(x)$ when $||\phi|-1|\leq \frac{1}{4}$, so

$$\int_{\{||\phi|-1|\leq \frac{1}{4}\}} (|\phi_1|^2-1)^2 dx = \int_{\{||\phi|-1|\leq \frac{1}{4}\}} \left(|\phi|^2-1\right)^2 dx.$$

There exists $C_3>0$ such that

$$(|\phi_1|^2-1)^2 \leq C_3 \left(|\phi|^2-1\right)^2$$

if $||\phi|-1|\geq \frac{1}{4}$. Thus

$$\int_{\{||\phi|-1|>\frac{1}{4}\}} (|\phi_1|^2-1)^2 dx \leq C_3 \int_{\left\{||\phi|-1|>\frac{1}{4}\right\}} \left(|\phi|^2-1\right)^2 dx.$$

This implies (13). (14) is similar.
Since $\partial_1\chi(\phi)\frac{d(\mathrm{Re}(\phi))}{dx}+\partial_2\chi(\phi)\frac{d(\mathrm{Im}(\phi))}{dx}\in\mathbb{R}$, using (16) to get

$$\langle i\phi_1', \phi_1-1\rangle = \chi^2(\phi)\,\langle i\phi', \phi-1\rangle\,.$$

From Lemma 2.1,

$$\langle i\phi_1', \phi_1-1\rangle = \chi^2(\phi)\,\langle i\phi', \phi-1\rangle = \frac{d}{dx}\left(\mathrm{Im}(\phi_1)-\theta\right) - \left(\rho^2-1\right)\theta', \quad (17)$$

hence,

$$\langle i\phi', \phi-1\rangle = \left(1-\chi^2(\phi)\right)\langle i\phi', \phi-1\rangle - \left(\rho^2-1\right)\theta' + \frac{d}{dx}\left(\mathrm{Im}(\phi_1)-\theta\right)$$

and this gives (15). □
Consider the Banach space $\mathcal{Y}=\{u' \mid u\in \dot{H}^1(\mathbb{R})\}$ (see [27], p. 122). Defining the norm as $\|u'\|_{\mathcal{Y}}=\|u\|_{\dot{H}^1(\mathbb{R})}=\|u'\|_{L^2(\mathbb{R})}$.

For any $\phi\in\mathcal{E}$, from (15), Lemma 2.1 and Lemma 2.4, we see that $\langle i\phi', \phi-1\rangle\in L^1(\mathbb{R})+\mathcal{Y}$. It motivates us to give:

Definition 2.5 For any $\phi\in\mathcal{E}$, let $\chi,\ \phi_1,\ \phi_2,\ \rho,\ \theta$ are as in Lemma 2.4, the momentum of ϕ is

$$P(\phi)=\int_{\mathbb{R}} \left(1-\chi^2(\phi)\right)\langle i\phi', \phi-1\rangle - \left(\rho^2-1\right)\theta' dx. \quad (18)$$

The above formula is independent of the choice of the χ.

If $\phi\in\mathcal{E}$ can be lifted, that is, $\phi=\rho e^{i\theta}$ with $\rho-1\in H^1(\mathbb{R})$ and $\theta\in\dot{H}^1(\mathbb{R})$, then from lemma 2.1 and Definition 2.5 we have

$$P(\phi) = -\int_{\mathbb{R}} \left(\rho^2 - 1\right) \theta' dx. \tag{19}$$

Remark 2.6 We have $|\phi|^2 - 1 \in H^1(\mathbb{R})$ by Lemma 2.2, then necessarily $\lim_{x\to\infty} |\phi(x)| = \lim_{x\to-\infty} |\phi(x)| = 1$, and then $\lim_{x\to\pm\infty} (\phi_1 - \phi) = \lim_{x\to\pm\infty}(\chi(\phi)(\phi - 1) + 1 - \phi) = 0$. From (15) and (18), we have

$$\begin{aligned}
P(\phi) &= \int_{\mathbb{R}} \left(1 - \chi^2(\phi)\right) \langle i\phi', \phi - 1\rangle - \left(\rho^2 - 1\right) \theta' dx \\
&= \int_{\mathbb{R}} \langle i\phi', \phi - 1\rangle - \frac{d}{dx} \left(\mathrm{Im}(\phi_1) - \theta\right) dx \\
&= \int_{\mathbb{R}} \langle i\phi', -1\rangle - \frac{d}{dx}\mathrm{Im}(\phi_1) + \langle i\phi', \phi\rangle + \theta' dx \\
&= \lim_{x_0\to\infty} \left[\left(\mathrm{Im}(\phi) - \mathrm{Im}(\phi_1)\right)\Big|_{-x_0}^{x_0} + \int_{-x_0}^{x_0} \langle i\phi', \phi\rangle\, dx + \theta\Big|_{-x_0}^{x_0} \right] \\
&= \lim_{x_0\to\infty} \left[\int_{-x_0}^{x_0} \langle i\phi', \phi\rangle + \arg\phi\Big|_{-x_0}^{x_0} \right].
\end{aligned}$$

The last formula above is an alternative definition for momentum of ϕ in $\mathcal{E}$ and is precisely the untwisted momentum defined in ([7], Lemma 1.8), when mod 2π.

Remark 2.7 We have

$$\begin{aligned}
&P\left(b_v\right) = -v\sqrt{2 - v^2} - 2\arctan\frac{v}{\sqrt{2 - v^2}} + \pi. \\
&\frac{d}{dv}P\left(b_v\right) = -2\sqrt{2 - v^2}.
\end{aligned}$$

$P(b_v)$ is a diffeomorphism from $(0, \sqrt{2})$ to $(0, \pi)$. It follows from Proposition 2.6 in ([6], p. 63) that $E\left(b_v\right) = \frac{2\left(2-v^2\right)^{\frac{3}{2}}}{3}$. It can be easily shown as in ([6], p. 64) that the map $P \mapsto E(P)$ satisfies $E(P) < v_s P$ on $(0, \pi]$.

Corollary 2.8 For any constant $c_1 \in \mathbb{C}$ and $\phi \in \mathcal{E}$ such that $\phi + c_1 \in \mathcal{E}$, we have $P\left(\phi + c_1\right) = P(\phi)$.

Proof For any $\phi \in \mathcal{E}$, let $\phi_1,\ \rho,\ \theta$ are given by Lemma 2.4. Then (17) gives

$$\begin{aligned}
\langle i\phi', \phi + c_1 - 1\rangle &= \left(1 - \chi^2(\phi)\right) \langle i\phi', \phi - 1\rangle + \chi^2(\phi) \langle i\phi', \phi - 1\rangle + \langle i\phi', c_1\rangle \\
&= \left(1 - \chi^2(\phi)\right) \langle i\phi', \phi - 1\rangle + \langle i\phi', c_1\rangle + \frac{d}{dx}\left(\mathrm{Im}(\phi_1) - \theta\right) - \left(\rho^2 - 1\right) \theta'.
\end{aligned}$$

Then using a calculation similar to Remark 2.6, we have

$$
\begin{aligned}
P(\phi + c_1) &= \lim_{R\to\infty}\int_{-R}^{R}\langle i\phi', \phi + c_1 - 1\rangle - \frac{d}{dx}\left(\mathrm{Im}(\phi_1)\right) - \langle i\phi', c_1\rangle + \theta' dx \\
&= \lim_{R\to\infty}\int_{-R}^{R}\langle i\phi', \phi\rangle + \theta' dx \\
&= P(\phi).
\end{aligned}
$$

□

Lemma 2.9 Let $\phi \in \mathcal{E}$ and $w \in H^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \langle i\phi', w\rangle + \langle i\phi, w'\rangle\, dx = 0. \tag{20}$$

Proof Since $w, \phi' \in L^2(\mathbb{R})$, then $\langle i\phi', w\rangle \in L^1(\mathbb{R})$. Let $\chi,\ \phi_1,\ \phi_2$ be given by Lemma 2.4. Set $w_1 = \chi(w)w,\ w_2 = (1-\chi(w))w$. We have $\phi = \phi_1 + \phi_2 - 1$, $w = w_1 + w_2$, $\phi_1 - 1 \in \dot{H}^1 \cap L^\infty(\mathbb{R})$ and $\phi_2 - 1,\ w_1,\ w_2 \in H^1(\mathbb{R})$.

We see that $\langle i\phi_2', w\rangle,\ \langle i(\phi_2 - 1), w'\rangle \in L^1(\mathbb{R})$ by Cauchy-Schwarz inequality. We have

$$\int_{\mathbb{R}} \langle i\phi_2', w\rangle + \langle i(\phi_2 - 1), w'\rangle\, dx = 0. \tag{21}$$

Since $\phi_1 - 1 \in \dot{H}^1 \cap L^\infty(\mathbb{R})$ and $w_1 \in H^1 \cap L^\infty(\mathbb{R})$, we have $\langle i(\phi_1 - 1), w_1\rangle \in \dot{H}^1 \cap L^\infty(\mathbb{R})$ and $\frac{d}{dx}\langle i(\phi_1 - 1), w_1\rangle = \langle i\phi_1', w_1\rangle + \langle i(\phi_1 - 1), w_1'\rangle$. Since $w_1 \in H^1(\mathbb{R})$, then necessarily $\lim_{|x|\to\infty} w_1(x) = 0$ on $\mathbb{R}$, and together with $\phi_1 - 1 \in L^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \frac{d}{dx}\langle i(\phi_1 - 1), w_1\rangle\, dx = \left[\langle i(\phi_1 - 1), w_1\rangle\right]\Big|_{-\infty}^{\infty} = 0.$$

Then

$$\int_{\mathbb{R}} \langle i\phi_1', w_1\rangle + \langle i(\phi_1 - 1), w_1'\rangle\, dx = 0. \tag{22}$$

Let $B = \{x \in \mathbb{R} \mid ||w| - 1| \ge \frac{1}{4}\}$. We have $\frac{1}{16}|B| \le \int_B |w|^2 dx \le \|w\|_{L^2}^2$ and B has finite measure. It can be seen that $w_2 = 0$ and $w_2' = 0$ a.e. on $\mathbb{R}\setminus B$. By Sobolev inequality in $1D$ ([9], pp. 212–213), we have $w_2 \in L^\infty(\mathbb{R})$. Combined with $w_2' \in L^2(\mathbb{R})$, we deduce that $w_2 \in L^1 \cap L^\infty(\mathbb{R})$ and $w_2' \in L^1 \cap L^2(\mathbb{R})$. Using $\phi_1 - 1 \in L^\infty(\mathbb{R})$ and $\phi_1' \in L^2(\mathbb{R})$, this gives $\langle i(\phi_1 - 1), w_2\rangle \in L^1 \cap L^\infty(\mathbb{R})$, $\langle i\phi_1', w_2\rangle \in L^1(\mathbb{R})$ and $\langle i(\phi_1 - 1), w_2'\rangle \in L^1 \cap L^2(\mathbb{R})$. We have

$$\frac{d}{dx}\left\langle i\left(\phi_1-1\right),w_2\right\rangle=\left\langle i\phi_1',w_2\right\rangle+\left\langle i\left(\phi_1-1\right),w_2'\right\rangle.$$

The above information implies $\langle i(\phi_1-1),w_2\rangle\in W^{1,1}(\mathbb{R})$, thus

$$\int_{\mathbb{R}}\left\langle i\phi_1',w_2\right\rangle+\left\langle i\left(\phi_1-1\right),w_2'\right\rangle dx=\int_{\mathbb{R}}\frac{d}{dx}\left\langle i\left(\phi_1-1\right),w_2\right\rangle dx=0. \tag{23}$$

Now from (21), (22) and (23) we have

$$\int_{\mathbb{R}}\langle i\phi',w\rangle+\langle i(\phi-1),w'\rangle\,dx=0.$$

Since $\int_{\mathbb{R}}\langle -i,w'\rangle dx=0$, we have (20). □

Corollary 2.10 Let $\phi_1,\ \phi_2\in\mathcal{E}$ be such that $\phi_1-\phi_2\in L^2(\mathbb{R})$. Then

$$|P(\phi_1)-P(\phi_2)|\leq\|\phi_1-\phi_2\|_{L^2(\mathbb{R})}\left(\|\phi_1'\|_{L^2(\mathbb{R})}+\|\phi_2'\|_{L^2(\mathbb{R})}\right) \tag{24}$$

Proof The proof uses formula (20) and is the same as ([27], Corollary 2.6). □

3 Some Preliminary Results

Let $\Omega\subset\mathbb{R}$ be an open set, and it may not be bounded or connected.

Lemma 3.1 Let $\phi\in\mathcal{E}$. For any $0<\delta_0<1$ and $R>0$, there exists a constant $M=M(\delta_0,R)>0$, such that if $E_\Omega(\phi)<M$, then

$$-\delta_0<|\phi(x)|-1<\delta_0,$$

for $x\in\Omega$ satisfies $\operatorname{dist}(x,\partial\Omega)>2R$.

Proof Using the 1D Morrey inequality,

$$\begin{aligned}|\phi(x)-\phi(y)|&\leq\|w'\|_{L^2(\Omega)}|x-y|^{\frac12}\\&\leq\left(E_\Omega(\phi)\right)^{\frac12}|x-y|^{\frac12}\quad\forall x,y\in B\left(x_0,R\right).\end{aligned} \tag{25}$$

Fix $\delta_0>0$. Suppose $\operatorname{dist}(x_0,\partial\Omega)>2R$ and $||\phi(x_0)|-1|\geq\delta_0$. Let $r_{\delta_0}=\min\{R,\frac{\delta_0^2}{4E_\Omega(\phi)}\}$. Since $\big|||\phi(x)|-1|-||\phi(x_0)|-1|\big|\leq|\phi(x)-\phi(x_0)|$, using (25) we get

$$||\phi(x)|-1| \geq \frac{\delta_0}{2} \quad \forall x \in B\left(x_0, r_{\delta_0}\right).$$

We have

$$\begin{aligned} E_\Omega(\phi) &\geq \frac{1}{2}\int_{B\left(x_0,r_{\delta_0}\right)} \left(|\phi|^2-1\right)^2 dx \\ &\geq \frac{1}{2}\int_{B\left(x_0,r_{\delta_0}\right)} (|\phi|-1)^2 dx \\ &\geq \frac{\delta_0^2}{4} r_{\delta_0} = \frac{\delta_0^2}{4}\min\left\{R, \frac{\delta_0^2}{4E_\Omega(\phi)}\right\}. \end{aligned} \tag{26}$$

Solving(26),wehave$E_\Omega(\phi) \geq \frac{\delta_0^2}{4}\min\{R,1\}$.Let$M = M(R,\delta_0) := \frac{\delta_0^2}{4}\min\{R,1\}$, then the lemma holds. □

Lemma 3.2 (i) If $\phi \in \mathcal{E}$ satisfies $||\phi|-1| \leq \delta$ with $\delta \in (0,1)$, then

$$E(\phi) \geq \sqrt{2}(1-\delta)|P(\phi)|.$$

(ii) Let $\phi \in \mathcal{E}$, $0 \leq v < \sqrt{2}$ and $\varepsilon \in (0, 1-\frac{v}{\sqrt{2}})$. There exists a constant $M = M(v,\varepsilon) > 0$, such that if $E(\phi) < M$, then

$$E(\phi) - v|P(\phi)| \geq \varepsilon E(\phi).$$

Proof (i) Writing $\phi = \rho e^{i\theta}$, where $\rho,\ \theta$ are provided by Lemma 2.1. Using (19),

$$P(\phi) = -\int_{\mathbb{R}} \left(\rho^2-1\right)\theta' dx.$$

We have the following:

$$\begin{aligned} \sqrt{2}(1-\delta)|P(\phi)| &\leq \sqrt{2}(1-\delta)\left\|\rho^2-1\right\|_{L^2(\mathbb{R})}\|\theta'\|_{L^2(\mathbb{R})} \\ &\leq (1-\delta)^2\|\theta'\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\rho^2-1\right\|^2_{L^2(\mathbb{R})} \\ &\leq E(\phi). \end{aligned}$$

(ii) Set $\varepsilon < 1-\frac{v}{\sqrt{2}}$. Let $\delta > 0$ satisfies $\varepsilon \leq 1-\frac{v}{\sqrt{2}(1-\delta)}$. Let $M = M(\delta,1)$ be given by Lemma 3.1. Let $\phi \in \mathcal{E}$ satisfies $E(\phi) < M$. Using Lemma 3.1, $-\delta < |\phi|-1 < \delta$. Using Lemma 3.2 (i), we have

$$v|P(\phi)| \leq \frac{v}{\sqrt{2}(1-\delta)} E(\phi) \leq (1-\varepsilon)\, E(\phi). \tag{27}$$

Using (27) we obtain

$$E(\phi) - v|P(\phi)| \geq \varepsilon E(\phi),$$

then (ii) follows. □

For $0 < q \leq \pi$, we define

$$E_{\min}(q) = \inf_{\phi \in \mathcal{E}} \{E(\phi) \mid P(\phi) = q\}.$$

For any $\phi \in \mathcal{E}$, the function $\phi_1(x) = \phi(-x) \in \mathcal{E}$ and $E(\phi_1) = E(\phi)$, $P(\phi_1) = -P(\phi)$, then $E_{\min}(-q) = E_{\min}(q)$. That is, $E_{\min}$ is an even function. This is the reason why we only need to consider $E_{\min}(q)$ at the interval $q \in (0, \pi]$.

Lemma 3.3 Let $0 < q \leq \pi$, we have $E_{\min}(q) < \sqrt{2}q$.

Proof From Remark 2.7, we have $E_{\min}(q) \leq E(b_{v(q)}) < v_s q$, where $v(q)$ is the unique velocity v such that $P(b_v) = q$. □

Lemma 3.4 For any $\varepsilon > 0$, there exists $q_1(\varepsilon) > 0$ with

$$E_{\min}(q) \geq \left(\sqrt{2} - \varepsilon\right) q \quad \forall q \in (0, q_1(\varepsilon)).$$

Proof Lemma 3.2 (ii) implies

$$E(\phi) \geq (\sqrt{2} - \varepsilon)|P(\phi)|$$

for all $\phi \in \mathcal{E}$ verifying $E(\phi) < M(\varepsilon)$. Set $q_1(\varepsilon) = \frac{M(\varepsilon)}{\sqrt{2}+c_1} < \pi$, where c_1 is a positive constant. Fix $q \in (0, q_1(\varepsilon))$. There exists $\phi \in \mathcal{E}$ satisfying $P(\phi) = q$, $E(\phi) < E_{\min}(q) + c_1 q$. Using Lemma 3.3, we have

$$E(\phi) < \left(\sqrt{2} + c_1\right) q < \left(\sqrt{2} + c_1\right) q_1(\varepsilon) = M(\varepsilon),$$

thus $E(\phi) \geq (\sqrt{2} - \varepsilon)|P(\phi)| = (\sqrt{2} - \varepsilon)q$. This yields $E_{\min}(q) \geq (\sqrt{2} - \varepsilon)q$. □

Lemma3.5 (i)Forany$0 \leq q_1 \leq q \leq \pi$,wehave$E_{\min}(q) \leq E_{\min}(q_1) + E_{\min}(q - q_1)$.

(ii) $E_{\min}$ is nondecreasing. It is continuous with best Lipchitz constant $\sqrt{2}$. It is concave.

(iii) The conclusion of (i) can be upgraded to strictly subadditive, i.e., for any $0 < q_1 < q < \pi$, $E_{\min}(q) < E_{\min}(q_1) + E_{\min}(q - q_1)$.

Proof (i) Corollary A.2 in Appendix A provides $\phi_1,\ \phi_2 \in \mathcal{E}$ with $P(\phi_1) = q_1,\ P(\phi_2) = q - q_1,\ E(\phi_1) < E_{\min}(q_1) + \frac{\varepsilon}{2},\ E(\phi_2) < E_{\min}(q - q_1) + \frac{\varepsilon}{2}$, where $\varepsilon > 0 \phi_1 = 1$ on $[R_1, \infty),\ \phi_2 = 1$ on $(-\infty, R_2]$. Define $\phi(x) = \begin{cases} \phi_1(x), & \text{if } x \le R_1 \\ \phi_2(x - 2(R_1 + R_2)) & \text{otherwise.} \end{cases}$ Then $\phi \in \mathcal{E},\ P(\phi) = P(\phi_1) + P(\phi_2) = q$ and $E(\phi) = E(\phi_1) + E(\phi_2)$. Thus $E_{\min}(q) \le E(\phi) < E_{\min}(q_1) + E_{\min}(q - q_1) + \varepsilon$. This gives (i).

(ii) Let $0 < q_1 < q_2 < \pi$ and $\sigma = \frac{q_1}{q_2} < 1$. Assume that $\phi \in \mathcal{E}$ satisfies $\inf_{x\in\mathbb{R}} |\phi(x)| > 0$ and $P(\phi) = q_2$ (such ϕ exists according to Remark 2.7). We write $\phi = \rho e^{i\theta}$, by Theorem 1 in ([10], p. 37). Then for $\phi_\sigma = \rho e^{i\sigma\theta}$ we have $P(\phi_\sigma) = P(\rho e^{i\sigma\theta}) = \sigma P(\phi) = q_1$. Using (9) we have $E_{\min}(q_1) \le E(\phi_\sigma) \le E(\phi)$. Taking the infimum over all ϕ satisfying $P(\phi) = q_2$, we see that $E_{\min}(q_1) \le E_{\min}(q_2)$. We thus have that $E_{\min}$ is nondecreasing.

The conclusion of (i) and Lemma 3.3 implies

$$E_{\min}(q_2) - E_{\min}(q_1) \le \sqrt{2}(q_2 - q_1).$$

Combining with Lemma 3.4, we see that $E_{\min}$ is Lipchitz continuous with best Lipchitz constant $\sqrt{2}$.
For $f : \mathbb{R} \to \mathbb{C}$ and $c \in \mathbb{R}$, denote

$$Q_c^+ f(x) = \begin{cases} f(x) & \text{if } x \ge c \\ e^{i\theta}\overline{f(2c - x)} & \text{if } x < c, \end{cases}$$
$$Q_c^- f(x) = \begin{cases} e^{i\theta}\overline{f(2c - x)} & \text{if } x \ge c \\ f(x) & \text{if } x < c, \end{cases}$$

where $\theta \in \mathbb{R}$ is a constant satisfying $f(c) = e^{i\theta}\overline{f(c)}$, which ensures that $Q_c^+ f(x), Q_c^- f(x)$ is continuous at $x = c$. For any $\phi \in \mathcal{E}$ we have $Q_c^+\phi, Q_c^-\phi \in \mathcal{E},\ E(Q_c^+\phi) + E(Q_c^-\phi) = 2E(\phi)$ and $P(Q_c^+\phi) + P(Q_c^-\phi) = 2P(\phi)$. The map $c \mapsto P(Q_c^+\phi)$ is continuous on $\mathbb{R}$, goes to 0 as $c \to \infty$ and to $2P(\phi)$ as $c \to -\infty$. Then proceeding similarly as in ([13], p. 176), we can show the concavity of $E_{\min}$.
(iii) Let $0 < q_1 < q \le \pi$. The result of (ii) implies that $E_{\min}(q_1) \ge \frac{q_1}{q} E_{\min}(q)$, with equality holds if and only if $E_{\min}(q_1) = a_1 q_1$ for a constant $a_1 \in \mathbb{R}$. Using Lemma 3.3, we see that $a_1 < \sqrt{2}$. However, using Lemma 3.4 we see that

$a_1 \geq \sqrt{2} - \varepsilon$. Hence a_1 doesn't exist. This means that we have the strict inequality $E_{\min}(q_1) > \frac{q_1}{q} E_{\min}(q)$. □

4 Minimizing E at Fixed P

We will implement b_v as solution of the constrained minimization problem using concentration-compactness principle. We will show the precompactness of minimizing sequences.

Theorem 4.1 Set $0 < q \leq \pi$. Let $(\phi_n)_{n\geq 1} \subset \mathcal{E}$ be a minimizing sequence, that is, suppose that

$$P(\phi_n) \to q \quad \text{and} \quad E(\phi_n) \to E_{\min}(q).$$

Then, up to a subsequence and translations, we have the following:
(i) there exist $\phi \in \mathcal{E}$ such that $\phi_n \to \phi$ a.e. on $\mathbb{R}$ and $d_0(\phi_n, \phi) \to 0$, i.e.,

$$\begin{aligned} &\|\phi_n' - \phi'\|_{L^2(\mathbb{R})} \to 0, \\ &\||\phi_n| - |\phi|\|_{L^2(\mathbb{R})} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

(ii) $P(\phi) = q,\ E(\phi) = E_{\min}(q)$.

Proof Let $\beta_0 = E_{\min}(q)$. We have that $E(\phi_n) \to \beta_0 > 0$ as $n \to \infty$.

The concentration-compactness principle [26] will be used. Let $\xi_n(t)$ be the concentration function of $E(\phi_n)$:

$$\xi_n(t) = \sup_{y\in\mathbb{R}} E_{B(y,t)}(\phi_n).$$

Following [26], up to a subsequence, there exists $\xi : [0, \infty) \to \mathbb{R}$ and $\beta \in [0, \beta_0]$ satisfying

$$\xi_n(t) \to \xi(t) \text{ when } n \to \infty \quad \text{and} \quad \xi(t) \to \beta \text{ when } t \to \infty.$$

Using similar arguments as Theorem 5.3 in [27], there exists a nondecreasing sequence $r_n \to \infty$ satisfying

$$\lim_{n\to\infty} \xi_n(r_n) = \lim_{n\to\infty} \xi_n\left(\frac{r_n}{2}\right) = \beta. \tag{28}$$

Step 1 (Ruling out vanishing) We will prove that vanishing will not hold, i.e., there exists a constant $c_1 > 0$ such that $\sup_{y\in\mathbb{R}} E_{B(y,1)}(\phi_n) \geq c_1$ as $n \to \infty$. Suppose in contradiction that up to a subsequence

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} E_{B(y,1)}(\phi_n) = 0, \tag{29}$$

then we show that $\||\phi_n| - 1\|_{L^\infty(\mathbb{R})} \to 0$ as $n \to \infty$.
Since $(E(\phi_n))_{n\geq 1}$ is bounded, then $\|\phi_n'\|_{L^2(\mathbb{R})}$ is bounded for any *n*. Using Morrey inequality, there exists $C_1 > 0$ such that

$$|\phi_n(x) - \phi_n(y)| \leq C_1|x-y|^{\frac{1}{2}} \quad \forall \text{ x, y } \in \mathbb{R}. \tag{30}$$

Since $\phi_n \in \mathcal{E}$, using Lemma 2.2, $\phi_n \in L^\infty(\mathbb{R})$. Let $\delta_n = \||\phi_n| - 1\|_{L^\infty(\mathbb{R})}$. Choose $x_n \in \mathbb{R}$ such that $||\phi_n(x_n)| - 1| \geq \frac{\delta_n}{2}$. From (30) we have $||\phi_n(x)| - 1| \geq \frac{\delta_n}{4}$ for any $x \in B(x_n, r_n)$, with $r_n = (\frac{\delta_n}{4C_1})^2$. We have

$$\int_{B(x_n,r_n)} \left(|\phi_n|^2 - 1\right)^2 dx \geq \frac{\delta_n^2}{8} r_n. \tag{31}$$

Combining (29) with (31), $\lim_{n\to\infty} \delta_n^2 r_n = 0$. Clearly this implies $\lim_{n\to\infty} \delta_n = 0$. Then Lemma 3.2 (i) implies

$$E(\phi_n) \geq \sqrt{2}(1-\delta_n)|P(\phi_n)|.$$

Letting $n \to \infty$, we have

$$\liminf_{n\to\infty} E(\phi_n) \geq \sqrt{2}q. \tag{32}$$

However, using Lemma 3.3,

$$\limsup_{n\to\infty} E(\phi_n) < \sqrt{2}q. \tag{33}$$

We see that (32) contradicts with (33).
Step 2 (Ruling out dichotomy) We will prove that $\beta \notin (0, \beta_0)$. Suppose that $0 < \beta < \beta_0$. Let r_n be as in (28) and set $R_n = \frac{r_n}{2}$. After translation, we have $E_{B(0,R_n)}(\phi_n) \geq \xi_n(R_n) - \frac{1}{n}$. Using (28) we obtain

$$\varepsilon_n := E_{B(0,r_n)\setminus B(0,R_n)}(\phi_n) \leq \xi_n(r_n) - \left(\xi_n(R_n) - \frac{1}{n}\right) \to 0.$$

Applying Lemma A.1 (in Appendix A), set $R = R_n,\ A = 2,\ \varepsilon = \varepsilon_n$ in that Lemma, then there exist two functions $\phi_{n,1},\ \phi_{n,2}$ such that $E(\phi_{n,1}) \geq E_{B(0,R_n)}(\phi_n)$

$\geq \xi_n(R_n) - \frac{1}{n},\ E(\phi_{n,2}) \geq E_{\mathbb{R}\setminus B(0,2R_n)}(\phi_n) \geq E(\phi_n) - \xi(2R_n)$ and

$$|E(\phi_n) - E(\phi_{n,1}) - E(\phi_{n,2})| \to 0 \quad \text{as } n \to \infty.$$

From (28), we deduce that necessarily

$$E(\phi_{n,1}) \to \beta \quad \text{and} \quad E(\phi_{n,2}) \to \beta_0 - \beta \quad \text{as } n \to \infty.$$

From Lemma A.1 (v) (in Appendix A) we have

$$|P(\phi_n) - P(\phi_{n,1}) - P(\phi_{n,2})| \to 0 \quad \text{as } n \to \infty. \tag{34}$$

Proceeding as ([13], p. 181), we infer that up to a subsequence, there exists $q_1,\ q_2 \in (0, q)$, such that $P(\phi_{n,1}) \to q_1$ and $P(\phi_{n,2}) \to q_2$ and $q_1 + q_2 = q$.
Since $E(\phi_{n,1}) \geq E_{\min}(P(\phi_{n,1}))$ and $E(\phi_{n,2}) \geq E_{\min}(P(\phi_{n,2}))$, taking limit we obtain $\beta \geq E_{\min}(q_1),\ \beta_0 - \beta \geq E_{\min}(q_2)$. We then have

$$E_{\min}(q) = \beta + (\beta_0 - \beta) \geq E_{\min}(q_1) + E_{\min}(q_2),$$

which is in contradiction with strictly subadditivity of $E_{\min}$ (Lemma 3.5 (iii)). Thus we have $\beta \notin (0, \beta_0)$.
Step 3 (Concentration-compactness) After finishing step 1 and step 2, we thus have concentration, i.e., $\beta = \beta_0$. Then after translation, for any $\varepsilon > 0$, there exists positive A_ε and $n_\varepsilon \in \mathbb{N}$ satisfying

$$E_{\mathbb{R}\setminus B(0,A_\varepsilon)}(\phi_n) < \varepsilon \quad \forall n \geq n_\varepsilon. \tag{35}$$

Let χ be provided by Lemma 2.4 and set $\phi_{n,1} = \chi(\phi_n)(\phi_n - 1) + 1$, $\phi_{n,2} = (1 - \chi(\phi_n))(\phi_n - 1) + 1$. From Lemma 2.4 we see that $(\phi_{n,1})_{n\geq 1} \subset \mathcal{E}$, $(\phi_{n,2} - 1)_{n\geq 1} \subset H^1(\mathbb{R})$ and $(E(\phi_{n,1}))_{n\geq 1}$, $(E(\phi_{n,2}))_{n\geq 1}$ are bounded.

Using Lemma 2.1, write $\phi_{n,1} = \rho_n e^{i\theta_n}$ with $\frac{1}{2} \leq \rho_n \leq \frac{3}{2}$ and $\theta_n \in \dot{H}^1(\mathbb{R})$,

$(\rho_n - 1)_{n\geq 1} \subset H^1(\mathbb{R})$. $(\phi_n)'_{n\geq 1} \subset L^2(\mathbb{R})$ and $(\phi_n)_{n\geq 1} \subset L^2(B(0, A))$ for any $A > 0$ (using Lemma 2.2). We see that up to a subsequence $(n_k)_{k\geq 1}$, there exist $\phi \in H^1_{\text{loc}}(\mathbb{R})$ with $\phi' \in L^2(\mathbb{R})$, $\phi_1 \in H^1_{\text{loc}}(\mathbb{R})$ with $\phi_1' \in L^2(\mathbb{R})$, $\phi_2 - 1 \in H^1(\mathbb{R}),\ \theta \in \dot{H}^1(\mathbb{R}),\ \rho - 1 \in H^1(\mathbb{R})$ such that

$$
\begin{aligned}
&(\phi_{n_k})' \rightharpoonup \phi', \ (\phi_{n_k,1})' \rightharpoonup \phi_1', \quad \text{and} \quad (\theta_{n_k})' \rightharpoonup \theta' \text{ weakly in } L^2(\mathbb{R}),\\
&\phi_{n_k,2}-1 \rightharpoonup \phi_2 - 1 \quad \text{and} \quad \rho_{n_k}-1 \rightharpoonup \rho - 1 \text{ weakly in } H^1(\mathbb{R}),\\
&\phi_{n_k} \rightharpoonup \phi \text{ weakly in } H^1(B(0,A)) \quad \forall A>0,\\
&\phi_{n_k,1} \to \phi_1, \ \phi_{n_k,2} \to \phi_2, \ \theta_{n_k} \to \theta, \ \rho_{n_k}-1 \to \rho - 1, \ \phi_{n_k} \to \phi\\
&\ \text{strongly in } L^p(B(0,A)) \text{ and a.e. on } \mathbb{R}, \quad \forall A>0, \ p \in [1,\infty].
\end{aligned} \tag{36}
$$

Weak convergence implies

$$
\int_{\mathbb{R}} |\phi'|^2 dx \le \liminf_{k\to\infty} \int_{\mathbb{R}} |(\phi_{n_k})'|^2 dx. \tag{37}
$$

Fatou's Lemma implies

$$
V(|\phi|^2) \le \liminf_{k\to\infty} V(|\phi_{n_k}|^2). \tag{38}
$$

From (37) and (38),

$$
E(\phi) \le \liminf_{k\to\infty} E(\phi_{n_k}) = E_{\min}(q). \tag{39}
$$

Step 4: Lemmas 4.2 and 4.3 will be used.

Lemma 4.2 Suppose the following hold for $(\omega_n)_{n\ge1} \subset \mathcal{E}$:

(i) $(E(\omega_n))_{n\ge1}$ is bounded, and (35) holds for ω_n;

(ii) There exists $\omega \in \mathcal{E}$ such that $\|\omega_n - \omega\|_{L^2(B(0,A))} \to 0$ for $A>0$ and $\omega_n \to \omega$ a.e. on $\mathbb{R}$.

Then we have $|\omega_n| \to |\omega|$ in $L^2(\mathbb{R})$, $(1-|\omega_n|^2)^2 \to (1-|\omega|^2)^2$ in $L^1(\mathbb{R})$ as $n \to \infty$.

Proof Fix $\varepsilon > 0$, assumption (i) implies that

$$
\||\omega_n|^2 - 1\|^2_{L^2(\mathbb{R}\setminus B(0,A_\varepsilon))} \le 2\varepsilon \quad \text{for } n \ge n_\varepsilon. \tag{40}
$$

ω has a similar estimate. From 40 we have

$$
\begin{aligned}
&\||\omega_n| - |\omega|\|_{L^2(\mathbb{R}\setminus B(0,A_\varepsilon))}\\
&\le \||\omega_n|^2 - 1\|_{L^2(\mathbb{R}\setminus B(0,A_\varepsilon))} + \left\||\omega|^2 - 1\right\|_{L^2(\mathbb{R}\setminus B(0,A_\varepsilon))} \le 2\sqrt{2}\sqrt{\varepsilon}.
\end{aligned} \tag{41}
$$

Using assumption (ii) and the fact that $|\omega_n| \in L^p(B(0,A))$ for $1 \le p \le \infty$ (using Lemma 2.2), we obtain $\omega_n \to \omega$ in $L^p(B(0,A))$ for $1 \le p \le \infty$. Therefore for large

n, we have $\||\omega_n| - |\omega|\|_{L^2(B(0,A_\varepsilon))} \leq \varepsilon$, $\|V(|\omega_n|^2) - V(|\omega|^2)\|_{L^1(B(0,A_\varepsilon))} \leq \varepsilon$,

Combining with (40) and (41), we have $\||\omega_n| - |\omega|\|_{L^2(\mathbb{R})} \leq 2\sqrt{2}\sqrt{\varepsilon} + \varepsilon$,

$\|V(|\omega_n|^2) - V(|\omega|^2)\|_{L^1(\mathbb{R})} \leq 3\varepsilon$ for large n. Lemma 4.2 follows when letting ε goes to 0. □

The following lemma is a 1D counterpart of Lemma 4.12 in [13], where the space dimension is assumed to be $N \geq 2$. The conformal transform method is used in the prove of Lemma 4.12 in [13], however, this method is not valid for the 1D case. We use a method which is inspired by ([27], pp. 163–164).

Lemma 4.3 Suppose the following hold for $(\omega_n)_{n\geq 1} \subset \mathcal{E}$:

(i) $(E(\omega_n))_{n\geq 1}$ is bounded, and (35) holds for ω_n;

(ii) There is $\omega \in \mathcal{E}$ with $\omega_n' \rightharpoonup \omega'$ weakly in $L^2(\mathbb{R})$, and $\|\omega_n - \omega\|_{L^2(B(0,A))} \to 0$ for any $A > 0$

Then $P(\omega_n) \to P(\omega)$ as $n \to \infty$.

Proof Consider a subsequence of $(\omega_n)_{n\geq 1}$. For simplicity, we still denote it by $(\omega_n)_{n\geq 1}$. Let $\varepsilon,\ A_\varepsilon,\ n_\varepsilon$ be as in (35). From 12 we get

$$\left\|\left(1-\chi^2(\omega_n)\right)(\omega_n - 1)\right\|_{L^2(\mathbb{R})} \leq C\left\||\omega_n|^2 - 1\right\|_{L^2(\mathbb{R})} \leq C(E(\omega_n))^{\frac{1}{2}}.$$

The Cauchy-Schwartz inequality implies

$$\begin{aligned}
&\int_{\mathbb{R}\setminus B(0,A_\varepsilon)} \left|\left(1-\chi^2(\omega_n)\right)\langle i\omega_n', \omega_n - 1\rangle\right| dx \\
\leq &\left\|\left(1-\chi^2(\omega_n)\right)(\omega_n - 1)\right\|_{L^2(\mathbb{R})} \|\omega_n'\|_{L^2(\mathbb{R}\setminus B(0,A_\varepsilon))} \\
\leq &C\sqrt{M}\sqrt{\varepsilon}
\end{aligned} \tag{42}$$

for any $n \geq n_\varepsilon$, and $M > 0$ is such that $E(\omega_n) \leq M$ for any n.

Let χ be provided by Lemma 2.4 and set $\omega_{n,1} = \chi(\omega_n)(\omega_n - 1) + 1$, $\omega_{n,2} = (1-\chi(\omega_n))(\omega_n - 1) + 1$. From Lemma 2.4, we see that $(\omega_{n,1})_{n\geq 1} \subset \mathcal{E}$, $(\omega_{n,2} - 1)_{n\geq 1} \subset H^1(\mathbb{R})$. Using Lemma 2.1, we write $\omega_{n,1} = \rho_n e^{i\theta_n}$ with $\frac{1}{2} \leq \rho_n \leq \frac{3}{2}$, $\theta_n \in \dot{H}^1(\mathbb{R})$. Using assumption (i) and (ii), we deduce that up to a subsequence, there exist $\{\rho_n\}$, $\{\theta_n\}$, $\{\omega_n\}$, $\{\omega_{n,1}\}$, $\{\omega_{n,2}\}$, ρ, θ, ω that satisfy (36).

From (13) we have

$$\|\rho_n^2 - 1\|_{L^2(\mathbb{R})} \leq C(E(\omega_n))^{\frac{1}{2}} \leq CM^{\frac{1}{2}}.$$

Using (9) and (11) we get

$$\left|\theta_n'\right| \leq 2\left|\frac{d\left(\chi\left(\omega_n\right)\left(\omega_n-1\right)\right)}{dx}\right| \leq C\left|\omega_n'\right|.$$

Then assumption (i) implies that

$$\|\theta_n'\|_{L^2(\mathbb{R}\setminus B(0,A_\varepsilon))} \leq C\sqrt{\varepsilon} \quad \forall n \geq n_\varepsilon.$$

We have

$$\int_{\mathbb{R}\setminus B(0,A_\varepsilon)} \left|\left(\rho_n^2-1\right)\theta_n'\right| dx \leq \left\|\rho_n^2-1\right\|_{L^2(\mathbb{R})} \|\theta_n'\|_{L^2(\mathbb{R}\setminus B(0,A_\varepsilon))} \leq C\sqrt{M}\sqrt{\varepsilon} \tag{43}$$

$\forall n \geq n_\varepsilon$. We see that (42) and (43) also hold with $\omega,\ \rho$, and θ replacing $\omega_n,\ \rho_n$ and θ_n.

Since $\omega_n \to \omega$ and $\rho_n - 1 \to \rho - 1$ in $L^2(B(0,A_\varepsilon))$ and a.e., then

$$\left(1-\chi^2\left(\omega_n\right)\right)\left(\omega_n-1\right) \to \left(1-\chi^2(\omega)\right)(\omega-1) \quad \text{and} \quad \rho_n^2-1 \to \rho^2-1$$

in $L^2(B(0,A_\varepsilon))$. Combining with the fact that $\omega_n' \rightharpoonup \omega'$ and $\theta_n' \rightharpoonup \theta'$ weakly, we have

$$\int_{B(0,A_\varepsilon)} \left\langle i\omega_n', \left(1-\chi^2\left(\omega_n\right)\right)\left(\omega_n-1\right)\right\rangle dx \to \int_{B(0,A_\varepsilon)} \left\langle i\omega', \left(1-\chi^2(\omega)\right)(\omega-1\right\rangle dx \tag{44}$$

and

$$\int_{B(0,A_\varepsilon)} \left(\rho_n^2-1\right)\theta_n' dx \to \int_{B(0,A_\varepsilon)} \left(\rho^2-1\right)\theta' dx. \tag{45}$$

Using (42)-(45) and (18), we deduce that there exist $n_1(\varepsilon) \geq n_\varepsilon$ such that for any $n \geq n_1(\varepsilon)$,

$$\left|P\left(\omega_n\right) - P(\omega)\right| \leq C\sqrt{\varepsilon}.$$

Since every subsequence of $(\omega_n)_{n\geq 1}$ includes a further subsequence satisfying $P\left(\omega_n\right) \to P(\omega)$ as $n \to \infty$, thus Lemma 4.3 follows. □

We will finish the proof of Theorem 4.1. From (35), (36) and Lemma 4.3 we see that $q = \lim_{k\to\infty} P(\phi_{n_k}) = P(\phi)$. Necessarily we have $\lim_{k\to\infty} E(\phi_{n_k}) = E_{\min}(q) \leq E(\phi)$. Together with (39), we see that $E(\phi) = E_{\min}(q)$. From (35), (36) and Lemma 4.2, we see that $|\phi_{n_k}| \to |\phi|$ in $L^2(\mathbb{R})$, $V(|\phi_{n_k}|^2) \to V(|\phi|^2)$ in $L^1(\mathbb{R})$. Combining (37), (38) and $E(\phi_{n_k}) \to E(\phi)$ leads to $\int_{\mathbb{R}} |(\phi_{n_k})'|^2 dx \to \int_{\mathbb{R}} |\phi'|^2 dx$. Combining with the weak convergence $(\phi_{n_k})' \rightharpoonup \phi'$ in $L^2(\mathbb{R})$, we have the strong convergence $\|(\phi_{n_k})' - \phi'\|_{L^2(\mathbb{R})} \to 0$ as $k \to \infty$. □

Corollary 4.4 The momentum P and energy E defined on $\mathcal{E}$ are continuous functionals, under the semi-distance d_0.

Proof For momentum P, the proof uses Lemma 4.3 and Corollary 2.8. For energy E, the proof uses Lemma 4.2. The details are similar to ([13], Corollary 4.13) and we omit it. □

Remark 4.5 (1) It is proved in Lemma 2.7 of [7] that the momentum P is locally Lipschitz continuous on $\mathcal{E}$ for the distance d_A defined as Eq. (7). It is proved in ([6], pp. 75–76) that P is continuous on $\mathcal{E}$ for the distance d_A. Hence, Corollary 4.4 is an improvement of these results.

(2) For $0 < q < \pi$, assume $\phi \in \mathcal{E}$ satisfies $P(\phi) = q$, $E(\phi) = E_{\min}(q)$. For $(\phi_n)_{n\geq 1} \subset \mathcal{E}$ such that $d_0(\phi_n, \phi) \to 0$, by Corollary 4.4, we have $P(\phi_n) \to q$ and $E(\phi_n) \to E_{\min}(q)$, modulo translation. Therefore, Theorem 4.1 offers an optimal convergence result. The corresponding optimality in dimension $N \geq 2$ is pointed out by ([13], p. 187).
Now we will show that the minimizers are traveling waves b_v.

Proposition 4.6 Let $0 < q \leq \pi$. Suppose $\phi \in \mathcal{E}$ minimizes E subject to $P(\phi) = q$. Then

(i) There exists v such that

$$iv\phi' + \phi'' + \phi\left(1 - |\phi|^2\right) = 0 \quad \text{in } D'(\mathbb{R}). \tag{46}$$

(ii) There exist constants $\theta_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$ and $\phi = e^{i\theta_0} b_v(\cdot + x_0) \in \mathcal{E}$ such that $P(\phi) = q$, $E(\phi) = E_{\min}(q)$ and ϕ satisfies (46) with speeds $v = E'_{\min}(q)$ for $0 < q < \pi$ and $v = d^- E_{\min}(\pi) = 0$ for $q = \pi$ ($d^- E_{\min}(\pi)$ is the left derivative of $E_{\min}$ at π), where b_v is given by (3).
More precisely, for $0 < q \leq \pi$,

$$U_q = \left\{\phi \in \mathcal{E} \;\middle|\; P(\phi) = q, \;\text{ and } E(\phi) = E_{\min}(q)\right\}$$

has a unique element $b_{v(q)}$ (up to translations and rotations), where $v(q)$ denote the unique speed v such that $P(b_v) = q$.

Proof (i) Proceeding exactly as Proposition 4.14 in ([13], pp. 187–188), for any $\psi \in C_c^\infty(\mathbb{R})$, there exists v such that

$$\int_{\mathbb{R}} \left\langle iv\phi' + \phi'' + \phi\left(1 - |\phi|^2\right), \psi\right\rangle dx = 0,$$

and this implies (46).
(ii) Consider a sequence $q_n \to q$ (when $q = \pi$, this sequence should be $q_n \uparrow q$). Assume $q_n > 0$. Let $\phi_n \in \mathcal{E}$ be such that $P(\phi_n) = q_n \to q$ and

$E(\phi_n) = E_{\min}(q_n) \to E_{\min}(q)$ (using continuity of $E_{\min}$). Using Theorem 4.1, we see that up to translation and subsequence, there exist $\phi_1 \in \mathcal{E}$ verifying $P(\phi_1) = q,\ E(\phi_1) = E_{\min}(q)$ and $\phi_n \to \phi_1$ a.e. on $\mathbb{R}$ and

$$d_0\left(\phi_n, \phi_1\right) \to 0 \quad \text{when } n \to \infty.$$

Using (i), ϕ_n satisfies (46). Taking limit $n \to \infty$, we see that ϕ_1 satisfies (46). Using the fact that (46) is integrable, we infer that $\phi = \phi_1$, and there exist constants $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$, such that

$$\phi = e^{i\theta_0} b_v(\cdot + x_0)$$

and the statement in Proposition 4.6 (ii) holds. □

5 Orbital Stability

The Cauchy problem of (1) was solved in [28], see Theorem 2.3 in ([15], p. 142) for a summary of the case in space dimension $N = 1$.

Theorem 5.1 ([15]). For any $\phi_0 \in \mathcal{E}$, there exists a unique solution $\phi(t) \in C([0,\infty), \mathcal{E})$ of (1) with $\phi(0) = \phi_0$. The following properties of solution hold:

(1) For any $T > 0$, if $d_{\mathcal{E}}(\phi_0^n, \phi_0) \to 0$, then $d_{\mathcal{E}}(\phi_n(t), \phi(t)) \to 0$ uniformly on [0, T] as $n \to \infty$, where $\phi_n(t)$ is solution with initial data ϕ_0^n.
(2) For any $t \in [0,\infty)$, $E(\phi(t)) = E(\phi_0)$.
(3) $\phi - \phi_0 \in C([0,\infty), H^1(\mathbb{R}))$.
(4) If $\Delta\phi_0 \in L^2(\mathbb{R})$, then $\Delta\phi \in C([0,\infty), L^2(\mathbb{R}))$.

The following two Lemmas 5.2 and 5.3 are a regularization of functions in $\mathcal{E}$. The regularization technique was exploited in [13, 27, 2].

For $\phi \in \mathcal{E}$ and $s > 0$, consider

$$G^{\phi}_{s,\Omega}(\gamma) = E_{\Omega}(\gamma) + \frac{1}{s^2}\int_{\Omega} |\gamma - \phi|^2 dx.$$

We see that $G^{\phi}_{s,\Omega}(\gamma) < \infty$ when $\gamma \in \mathcal{E}$ and $\gamma - \phi \in L^2(\Omega)$, We define $H_0^1(\Omega) := \left\{w \in H^1(\mathbb{R}) \mid w = 0 \ in\ \mathbb{R} \setminus \Omega\right\}$, and $H_\phi^1(\Omega) := \left\{\gamma \in \mathcal{E} \mid \gamma - \phi \in H_0^1(\Omega)\right\}.$

Lemma 5.2 (i) There exists a minimizer of $G^{\phi}_{s,\Omega}$ in $H^1_\phi(\Omega)$.

(ii) Denote the minimizer provided by (i) by γ_s. Then

$$E_\Omega(\gamma_s) \le E_\Omega(\phi); \tag{47}$$

$$\|\gamma_s - \phi\|^2_{L^2(\Omega)} \le s^2 E_\Omega(\phi). \tag{48}$$

(iii) Denote $F(z) = z(|z|^2 - 1)$ for $z \in \mathbb{C}$. Then

$$-\gamma_s'' + F(\gamma_s) + \frac{1}{s^2}(\gamma_s - \phi) = 0 \qquad \text{in } D'(\Omega). \tag{49}$$

For set $\Omega_1 \subset\subset \Omega$, $\gamma_s \in W^{2,p}(\Omega_1)$, $\forall p \in (1,\infty)$. Hence, $\gamma_s \in C^{1,\alpha}(\Omega_1)$ for $\alpha \in (0,1)$.

Proof (i) We see that $\phi \in H^1_\phi(\Omega)$. Let $(\gamma_n)_{n\ge 1}$ be a minimizing sequence for $G^\phi_{s,\Omega}$ in $H^1_\phi(\Omega)$. Suppose $G^\phi_{s,\Omega}(\gamma_n) \le G^\phi_{s,\Omega}(\phi) = E_\Omega(\phi)$. This implies $\int_\Omega |\gamma_n'|^2 dx \le E_\Omega(\phi)$. We have

$$\int_\Omega |\gamma_n - \phi|^2 \, dx \le s^2 E_\Omega(\phi).$$

It follows that $\gamma_n - \phi \in H^1_0(\Omega)$. Then, up to a subsequence, there exists $w \in H^1_0(\Omega)$ such that $\gamma_n - \phi \rightharpoonup w$ weakly in $H^1_0(\Omega)$, $\gamma_n - \phi \to w$ a.e. and $\gamma_n - \phi \to w$ in $L^p_{\text{loc}}(\Omega)$ with $p \in [1,\infty]$. Let $\gamma = \phi + w$, we have $\gamma_n' \rightharpoonup \gamma'$ weakly in $L^2(\mathbb{R})$, together with an application of Fatou's Lemma, we have $G^\phi_{s,\Omega}(\gamma) \le \liminf_{n\to\infty} G^\phi_{s,\Omega}(\gamma_n)$. Hence, γ is a minimizer.

(ii) We see that $G^\phi_{s,\Omega}(\gamma_s) \le G^\phi_{s,\Omega}(\phi) = E_\Omega(\phi)$, then (47) and (48) hold.

(iii) Since $\frac{d}{dh}\big|_{h=0}(G^\phi_{s,\Omega}(\gamma_s + h\zeta)) = 0$, $\forall \zeta \in C^\infty_c(\Omega)$, we then have (49).

Since $\gamma_s \in \mathcal{E}$, we have $|\gamma_s|^2 - 1 \in L^2(\mathbb{R})$. We also have $\gamma_s \in L^\infty$ by Lemma 2.2. Using $\|F(\gamma_s)\|_{L^\infty} \le \|\gamma_s\|_{L^\infty}(\|\gamma_s\|^2_{L^\infty} + 1)$, we have $F(\gamma_s) \in L^\infty(\mathbb{R})$. We then have

$$\|F(\gamma_s)\|_{L^2(\mathbb{R})} \le \|\gamma_s\|_{L^\infty(\mathbb{R})} \, \||\gamma_s|^2 - 1\|_{L^2(\mathbb{R})},$$

this gives $F(\gamma_s) \in L^2(\mathbb{R})$. Then $F(\gamma_s) \in L^2 \cap L^\infty(\mathbb{R})$. We have $\gamma_s,\ \phi \in H^1_{\text{loc}}(\mathbb{R})$. We deduce that $\gamma_s,\ \phi \in L^p_{\text{loc}}(\mathbb{R})$ for $p \in [1,\infty]$ by 1D Sobolev embedding. Using (49) we deduce that $\gamma_s'' \in L^p_{\text{loc}}(\Omega)$ for $p \in [1,\infty]$. Then using the elliptic estimates ([17], Theorem 9.11), we get (iii). □

The following lemma provides a way of using higher regularity functions to approximate the functions in $\mathcal{E}$.

Lemma 5.3 Fix $\phi \in \mathcal{E}$ and $k \in \mathbb{N}$. For any $\varepsilon > 0$, there exists $\gamma \in \mathcal{E}$ satisfying $\gamma' \in H^k(\mathbb{R})$, $E(\gamma) \le E(\phi)$ and $\|\gamma - \phi\|_{H^1(\mathbb{R})} < \varepsilon$.

Proof The proof uses Lemma 5.2 and is similar to Lemma 3.5 in ([13], pp. 170–171). □

Lemma 5.4 (Conservation of the momentum) Let ϕ solves (1) (provided by Theorem 5.1) with initial condition $\phi_0 \in \mathcal{E}$. Then

$$P(\phi(t)) = P(\phi_0) \quad \forall t \in [0, \infty).$$

Proof We first assume that $\Delta\phi_0 \in L^2(\mathbb{R})$. By Theorem 5.1 (4) we have $\phi_x \in C\left([0,\infty), H^1(\mathbb{R})\right)$. For $t,\ t+t_1 > 0$, Theorem 5.1 (3) says $\phi(t+t_1) - \phi(t) \in H^1(\mathbb{R})$, we thus have $\langle i\phi_x(t+t_1) + i\phi_x(t), \phi(t+t_1) - \phi(t)\rangle \in L^1(\mathbb{R})$. Using (20) we get

$$\frac{1}{t_1}(P(\phi(t+t_1)) - P(\phi(t))) = \int_{\mathbb{R}} \langle i\phi_x(t+t_1) + i\phi_x(t), \frac{1}{t_1}(\phi(t+t_1) - \phi(t))\rangle dx.$$

Taking limit $t_1 \to 0$ and using (1),

$$\frac{d}{dt}P(\phi(t)) = 2\int_{\mathbb{R}} \left\langle \frac{\partial\phi(t)}{\partial x}, \phi_{xx}(t) + \phi(t)\left(1 - |\phi|^2\right) \right\rangle dx. \tag{50}$$

Since $\phi_x(t) \in H^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \langle \phi_x(t), \phi_{xx}(t)\rangle\, dx = \frac{1}{2}\int_{\mathbb{R}} \frac{\partial}{\partial x}\left(|\phi_x(t)|^2\right) dx. \tag{51}$$

Since $|\phi_x|^2 \in L^1(\mathbb{R})$ and $\frac{\partial}{\partial x}(|\phi_x|^2) = 2\langle\phi_x, \phi_{xx}\rangle \in L^1(\mathbb{R})$, hence $|\phi_x|^2 \in W^{1,1}(\mathbb{R})$.

Using (51) we get $\int_{\mathbb{R}}\langle\phi_x, \phi_{xx}\rangle dx = 0$.
We have $2\langle\phi_x, \phi(1-|\phi|^2)\rangle = -\frac{1}{2}\frac{\partial}{\partial x}(1-|\phi|^2)^2$. Since $\phi_x \in L^2(\mathbb{R})$, and $\phi(1-|\phi|^2) \in L^2(\mathbb{R})$ by Lemma 2.2, we have $\frac{\partial}{\partial x}(1-|\phi|^2)^2 = -4\langle\phi_x, \phi(1-|\phi|^2)\rangle \in L^1(\mathbb{R})$, hence $(1-|\phi|^2)^2 \in W^{1,1}(\mathbb{R})$. Thus, $\int_{\mathbb{R}} \frac{\partial}{\partial x}(1-|\phi|^2)^2 dx = 0$. Then we obtain $\frac{d}{dt}P(\phi(t)) = 0$ using (50), i.e., $P(\phi(\cdot))$ is constant on $[0,\infty)$.

Then we deal with arbitrary function $\phi_0 \in \mathcal{E}$. By Lemma 5.3, there exists $(\phi_0^n)_{n\geq 1} \subset \mathcal{E}$ with $(\phi_0^n)_x \in H^2(\mathbb{R})$, $\|\phi_0^n - \phi_0\|_{H^1(\mathbb{R})} \to 0$ as $n \to \infty$ (thus, $d_{\mathcal{E}}(\phi_0^n, \phi_0) \to 0$). From Theorem 5.1 (1), for any $T > 0$, $d_{\mathcal{E}}(\phi_n(t), \phi(t)) \to 0$ uniformly on [0, *T*] for large *n*, where ϕ_n solves (1) with initial condition ϕ_0^n. Then we have $d_0(\phi_n(t), \phi(t)) \to 0$ uniformly on [0, *T*]. We deduce that $P(\phi_n(t)) \to P(\phi(t))$ by Corollary 4.4. We get $P(\phi_n(t)) = P(\phi_0^n)$ using the conclusion of the first part of the proof. Since $\|\phi_0^n - \phi_0\|_{H^1(\mathbb{R})} \to 0$, using Corollary 2.10 we get $P(\phi_0^n) \to P(\phi_0)$. Thus, we have $P(\phi(t)) = P(\phi_0)$. □

Using the arguments in [11], we have the following orbital stability result, with respect to the semi-distance d_0.

Theorem 5.5 Let $0 < q \leq \pi$, and let

$$U_q = \{\phi \in \mathcal{E} \mid E(\phi) = E_{\min}(q),\ P(\phi) = q\}$$

be defined as in Proposition 4.6. Then U_q is orbitally stable, under the semi-distance d_0. That is, for any $\varepsilon > 0$ there exists $\delta > 0$, if $d_0(\phi_0, U_q) < \delta$, then $d_0(\phi(t), U_q) < \varepsilon$ for any $t > 0$, where $\phi(t)$ is a solution with initial condition ϕ_0.

Proof If the converse is true, then there exists $\varepsilon_0 > 0$ and $\phi_0^n \in \mathcal{E}$ satisfying $d_0(\phi_0^n, U_q) < \frac{1}{n}$ for any $n \geq 1$, $d_0(\phi_n(t_n), U_q) \geq \varepsilon_0$ for some $t_n > 0$, where ϕ_n is the solution of (1) with $\phi_n(0) = \phi_0^n$.

We claim that $E(\phi_0^n) \to E_{\min}(q)$, $P(\phi_0^n) \to q$. Consider an arbitrary subsequence of $(\phi_0^n)_{n\geq 1}$ (still use $(\phi_0^n)_{n\geq 1}$). Using Theorem 4.1, we see that up to subsequence and translation, there exist $\phi \in U_q$ such that $d_0(\phi_0^n, \phi) \to 0$. Using Corollary 4.4 we get $P(\phi_0^n) \to P(\phi) = q$ and $E(\phi_0^n) \to E(\phi) = E_{\min}(q)$. Because any subsequence of $(\phi_0^n)_{n\geq 1}$ includes a further subsequence satisfying the property, we conclude that the claim holds.

By Theorem 5.1 (2): $E(\phi_n(t_n)) = E(\phi_0^n) \to E_{\min}(q)$. Lemma 5.4 implies $P(\phi_n(t_n)) = P(\phi_0^n) \to q$. Using again Theorem 4.1, we see that up to translation, there exist a subsequence $(\phi_{n_k})_{k\geq 1}$ and $\phi_1 \in U_q$ satisfying $d_0(\phi_{n_k}(t_{n_k}), \phi_1) \to 0$, which contradicts $d_0(\phi_n(t_n), U_q) \geq \varepsilon_0$ for all *n*. □

A Splitting lemma

The following technical lemma is used to ruling out dichotomy of minimizing sequences. The proof is an adaptation of Lemma 3.3 in [27] and Lemma 3.3 in [13] to our 1D setting. For, set $\Omega_{R_1,R_2} = B(0,R_2) \setminus \overline{B}(0,R_1)$.

Lemma A.1 Let $R \geq 1$, $1 < A_1 < A_2 < A$. There are $\varepsilon_0 > 0 C_1, C_2, C_3 > 0$, for $0 < \varepsilon < \varepsilon_0$ and $\phi \in \mathcal{E}$ with $E_{\Omega_{R,AR}}(\phi) \leq \varepsilon$, there exist $\phi_1, \phi_2 \in \mathcal{E}$ and a constant $\theta_0 \in [0, 2\pi)$, such that:

(i) $\phi_1 = \phi$ **on** $(-\infty, A_1 R]$, $\phi_1 = e^{i\theta_0}$ **on** $[A_2 R, \infty)$;

(ii) $\phi_2 = \phi$ **on** $[A_2 R, \infty)$, $\phi_2 = e^{i\theta_0}$ **on** $(-\infty, A_1 R]$;

(iii) $\int_{\mathbb{R}} \left| |\phi'|^2 - |\phi_1'|^2 - |\phi_2'|^2 \right| dx \leq C_1 \varepsilon$;

(iv) $\int_{\mathbb{R}} |(|\phi|^2 - 1)^2 - (|\phi_1|^2 - 1)^2 - (|\phi_2|^2 - 1)^2| dx \leq C_2 \varepsilon$;

(v) $|P(\phi) - P(\phi_1) - P(\phi_2)| \leq C_3 \varepsilon$.

Proof Let $k > 01 + 2k < A_1 < A_2 < A - 2k$. Set $\delta = \frac{1}{2}$. Let $M(\delta, R)$ be provided by Lemma 3.1. Set $\varepsilon_0 = M(\frac{1}{2}, k)$.

Set $\varepsilon < \varepsilon_0$. Consider $\phi \in \mathcal{E}$ satisfies $E_{\Omega_{R,AR}}(\phi) \leq \varepsilon$. Using Lemma 3.1,

$$\frac{1}{2} \leq |\phi(x)| \leq \frac{3}{2} \quad \text{for } R + 2k \leq |x| \leq AR - 2k.$$

Ω_{A_1R,A_2R} has two connected components $(-A_2R, -A_1R)$ and (A_1R, A_2R). We consider the lifting of ϕ in the open interval (A_1R, A_2R). We can write

$$\phi(x) = \rho(x)e^{i\theta(x)} \quad \text{in } (A_1R, A_2R)$$

with $\rho,\ \theta \in W^{1,p}\left((A_1R, A_2R)\right), 1 < p < \infty$ (see Theorem 1 in [10], p. 37). Using (9) we have

$$\int_{(A_1R,A_2R)} |\rho'|^2 dx \leq \int_{\Omega_{A_1R,A_2R}} |\phi'|^2 dx \leq \varepsilon, \tag{52}$$

$$\frac{1}{2}\int_{(A_1R,A_2R)} \left(\rho^2 - 1\right)^2 dx \leq E_{\Omega_{A_1R,A_2R}}(\phi) \leq \varepsilon, \tag{53}$$

$$\int_{(A_1R,A_2R)} |\theta'|^2 dx \leq 4\int_{\Omega_{A_1R,A_2R}} |\phi'|^2 dx \leq 4\varepsilon. \tag{54}$$

The Poincaré inequality implies that

$$\int_{(A_1R,A_2R)} |f - m(f, (A_1R, A_2R))|^2 dx \leq C\left(A_1, A_2\right) R \int_{(A_1R,A_2R)} |f'|^2 dx \tag{55}$$

for any $f \in H^1((A_1R, A_2R))$ and

$$m\left(f, (A_1R, A_2R)\right) = \frac{1}{(A_2 - A_1)R}\int_{(A_1R,A_2R)} f(x)dx.$$

Using (54) and (55), we get

$$\int_{(A_1R,A_2R)} |\theta - \theta_0|^2 dx \leq C\left(A_1, A_2\right) R \int_{\Omega_{A_1R,A_2R}} |\phi'|^2 dx \leq C\left(A_1, A_2\right) R\varepsilon, \tag{56}$$

where $\theta_0 = m(\theta, (A_1R, A_2R))$.

Consider $\varphi_1 \in C^\infty(\mathbb{R})$ with $\varphi_1 = 1$ in $(-\infty, A_1]$, $\varphi_1 = 0$ in $[A_2, \infty)$, and φ_1 is nonincreasing on $\mathbb{R}$. Consider $\varphi_2 \in C^\infty(\mathbb{R})$ with $\varphi_2 = 0$ on $(-\infty, A_1]$, $\varphi_2 = 1$ on $[A_2, \infty)$, and φ_2 is nondecreasing on $\mathbb{R}$.

We define ϕ_1 and ϕ_2 by the following:

$$\phi_1(x) = \begin{cases} \phi(x) \quad \text{if } x \in (-\infty, A_1R], \\ \left(1 + \varphi_1\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right) e^{i\left(\theta_0 + \varphi_1\left(\frac{|x|}{R}\right)(\theta(x) - \theta_0)\right)} \text{ if } x \in (A_1R, A_2R), \\ e^{i\theta_0} \text{ if } x \in [A_2R, \infty), \end{cases} \tag{57}$$

$$\phi_2(x) = \begin{cases} e^{i\theta_0} \quad \text{if } x \in (-\infty, A_1R], \\ \left(1 + \varphi_2\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right) e^{i\left(\theta_0 + \varphi_2\left(\frac{|x|}{R}\right)(\theta(x) - \theta_0)\right)} \text{ if } x \in (A_1R, A_2R), \\ \phi(x) \quad \text{if } x \in [A_2R, \infty). \end{cases} \tag{58}$$

Then $\phi_1, \ \phi_2 \in \mathcal{E}$. (i) and (ii) hold.

Using $\rho + 1 \geq \frac{3}{2}$ on (A_1R, A_2R) and (53), we get

$$\|\rho - 1\|^2_{L^2((A_1R, A_2R))} \leq \frac{8}{9}\varepsilon. \tag{59}$$

We have

$$\frac{d}{dx}\left(1 + \varphi_i\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right) = \frac{x}{R|x|}\varphi_i'\left(\frac{|x|}{R}\right)(\rho(x) - 1) + \varphi_i\left(\frac{|x|}{R}\right)\rho'.$$

By (52), (59) and $R \geq 1$, we get

$$\begin{aligned} &\left\|\frac{d}{dx}\left(1 + \varphi_i\left(\frac{|x|}{R}\right)(\rho(x) - 1)\right)\right\|_{L^2((A_1R, A_2R))} \\ &\leq \frac{1}{R}\sup|\varphi_i'| \cdot \|\rho - 1\|_{L^2((A_1R, A_2R))} \\ &+ \left\|\varphi_i\left(\frac{|x|}{R}\right)\rho'\right\|_{L^2((A_1R, A_2R))} \leq C\sqrt{\varepsilon}. \end{aligned} \tag{60}$$

Using (54) and (56), we have

$$\begin{aligned}&\frac{d}{dx}\left(\theta_0+\varphi_i\left(\frac{|x|}{R}\right)(\theta(x)-\theta_0)\right)\|_{L^2((A_1R,A_2R))}\\&\leq\frac{1}{R}\sup|\varphi_i'|\cdot\|\theta-\theta_0\|_{L^2((A_1R,A_2R))}\\&+\left\|\varphi_i\left(\frac{|x|}{R}\right)\theta'\right\|_{L^2((A_1R,A_2R))}\leq C\sqrt{\varepsilon}.\end{aligned}\tag{61}$$

From (60), (61) and the definition of ϕ_1,ϕ_2 it follows that

$$\|\phi_1'\|_{L^2((A_1R,A_2R))}\leq C\sqrt{\varepsilon}\quad\text{and}\quad\|\phi_2'\|_{L^2((A_1R,A_2R))}\leq C\sqrt{\varepsilon}.$$

Then

$$\int_{\mathbb{R}}||\phi'|^2-|\phi_1'|^2-|\phi_2'|^2|dx=\int_{(A_1R,A_2R)}|\phi'|^2+|\phi_1'|^2+|\phi_2'|^2dx\leq C_1\varepsilon.$$

So we have proved (iii).

On (A_1R,A_2R), we have $\rho\in\left[\frac{1}{2},\frac{3}{2}\right]$. Then

$$\begin{aligned}\left(\left(1+\varphi_i\left(\frac{|x|}{R}\right)(\rho(x)-1)\right)^2-1\right)^2&=(\rho-1)^2\varphi_i^2\left(\frac{|x|}{R}\right)\left(2+\varphi_i\left(\frac{|x|}{R}\right)(\rho-1)\right)^2\\&\leq C|\rho(x)-1|.\end{aligned}\tag{62}$$

From (57), (58) and (62), we see that $\left\||\phi_i|^2-1\right\|_{L^2((A_1R,A_2R))}\leq C\sqrt{\varepsilon}$. We get

$$\begin{aligned}&\int_{\mathbb{R}}\left|\left(|\phi|^2-1\right)^2-\left(|\phi_1|^2-1\right)^2-\left(|\phi_2|^2-1\right)^2\right|dx\\&\leq\int_{(A_1R,A_2R)}\left(|\phi|^2-1\right)^2+\left(|\phi_1|^2-1\right)^2+\left(|\phi_2|^2-1\right)^2dx\\&\leq C_2\varepsilon.\end{aligned}$$

So (iv) holds.

Using Definition 2.5, (8) and (57), (58), we obtain

$$\begin{aligned} & P(\phi)-P\left(\phi_1\right)-P\left(\phi_2\right) \\ = & \int_{\left(A_1 R, A_2 R\right)} \operatorname{Im}\left(\phi^{\prime}-\phi_1^{\prime}-\phi_2^{\prime}\right) d x \\ & -\int_{\left(A_1 R, A_2 R\right)} \frac{d}{d x}\left(\theta-\sum_{i=1}^2\left(\theta_0+\varphi_i\left(\frac{|x|}{R}\right)\left(\theta(x)-\theta_0\right)\right)\right) d x \\ & -\int_{\left(A_1 R, A_2 R\right)}\left(\rho^2-1\right) \theta^{\prime} d x \\ + & \int_{\left(A_1 R, A_2 R\right)} \sum_{i=1}^2\left(\left(1+\varphi_i\left(\frac{|x|}{R}\right)(\rho-1)\right)^2-1\right) \frac{d}{d x}\left(\theta_0+\varphi_i\left(\frac{|x|}{R}\right)\left(\theta-\theta_0\right)\right) d x . \end{aligned} \tag{63}$$

We have $\phi-\phi_1-\phi_2=-e^{-i \theta_0}=$ constant, $\theta_1:=\theta-\sum_{i=1}^2\left(\theta_0+\varphi_i\left(\frac{|x|}{R}\right)\left(\theta-\theta_0\right)\right)=$ constant on $\mathbb{R} \backslash\left(A_1 R, A_2 R\right)$. Therefore,

$$\int_{\left(A_1 R, A_2 R\right)} \frac{d}{d x}\left(\operatorname{Im}\left(\phi-\phi_1-\phi_2\right)\right) d x=0 \text { and } \int_{\left(A_1 R, A_2 R\right)} \frac{d \theta_1}{d x} d x=0. \tag{64}$$

Using (53), (54) we have

$$\left|\int_{\left(A_1 R, A_2 R\right)}\left(\rho^2-1\right) \theta^{\prime} d x\right| \leq 2 \sqrt{2} \varepsilon. \tag{65}$$

From (59), (61), (62) we get

$$\left|\int_{\left(A_1 R, A_2 R\right)}\left(\left(1+\varphi_i\left(\frac{|x|}{R}\right)(\rho-1)\right)^2-1\right) \frac{d}{d x}\left(\theta_0+\varphi_i\left(\frac{|x|}{R}\right)\left(\theta-\theta_0\right)\right) d x\right| \leq C \varepsilon. \tag{66}$$

From (63)-(66) we get $\left|P(\phi)-P\left(\phi_1\right)-P\left(\phi_2\right)\right| \leq C \varepsilon$. So (v) holds. □

Corollary A.2 For any $\phi \in \mathcal{E}$, there exist $\left(\phi_n\right)_{n \geq 1} \subset \mathcal{E}$ verifying:

(i) $\phi_n=\phi$ **on** $\left(-\infty, 2^n\right]$, $\phi_n=e^{i \theta_n}=$ constant on $\left[2^{n+1}, \infty\right)$;

(ii) $\int_{\mathbb{R}}\left|\left|\phi_n^{\prime}\right|^2-\left|\phi^{\prime}\right|^2\right| d x \rightarrow 0$;

(iii) $\int_{\mathbb{R}}\left|V\left(\left|\phi_n\right|^2\right)-V\left(|\phi|^2\right)\right| d x \rightarrow 0$;

(iv) $P\left(\phi_n\right) \rightarrow P(\phi)$ **as** $n \rightarrow \infty$.

Similarly, there is a sequence $\left(\gamma_n\right)_{n \geq 1} \subset \mathcal{E}$ with $\gamma_n=\phi$ in $\left[2^{n+1}, \infty\right)$, $\gamma_n=e^{i \theta_n}=$ constant in $\left(-\infty, 2^n\right]$. Moreover, results of (ii)-(iv) hold for $\left(\gamma_n\right)_{n \geq 1}$.

Proof Let $\varepsilon_n = E_{\mathbb{R}\setminus B(0,2^n)}(\phi)$, so we have $\varepsilon_n \to 0$ as $n \to \infty$. Using Lemma A.1 with $R = 2^n$ and $A = 2$, we obtain two functions $\phi_1^n,\ \phi_2^n$ fulfill properties (i)-(v) in Lemma A.1. Let $\phi_n = \phi_1^n$, then $(\phi_n)_{n\geq 1}$ satisfies (i)-(iv) above. Similar results hold for $(\gamma_n)_{n\geq 1}$. □

Acknowledgements This work is supported by the National Key Research and Development Program of China (grant No. 2020YFA0309600) and the NSFC (grant No. 12374122). X. Wang acknowledges the support from University Development Fund of the Chinese University of Hong Kong, Shenzhen and Hong Kong RGC Grants (No. 16300522, 16300523 and 16302321).

Author Contributions X.D. and X.W. designed research; X.D. performed research; X.D. and X.W. wrote the paper.

Declarations

Conflict of interest The authors have no conflicts of interest to declare.

Ethics approval and consent to participate Not applicable.

Consent for publication All authors consent for publication.

Data Availability No new data were created or analyzed in this study. Data sharing is not applicable to this article.

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