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RESEARCH ARTICLE

Existence of Solutions for *p*(*x*)-Triharmonic Problem with Navier Boundary Conditions

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Abstract

In this paper, we use the Mountain pass theorem and the Fountain theorem to study the existence of solutions for the following $p(x)$-triharmonic equations:

$$\begin{cases} -\Delta^3_{p(x)} u = \lambda f(x,u), & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega. \end{cases}$$

Keywords Variable exponential space · $p(x)$-triharmonic operator · Navier boundary condition

1 Introduction

In this paper, we study the following nonlinear problem of $p(x)$-triharmonic type with Navier boundary conditions:

$$\begin{cases} -\Delta^3_{p(x)} u = \lambda f(x,u), & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $N \geq 3\Delta^3_{p(x)}$ is defined as

$$\Delta^3_{p(x)} u := div\left(\Delta\left(|\nabla \Delta u|^{p(x)-2}\nabla \Delta u\right)\right),$$

and is called a $p(x)$-triharmonic operator. Here, $p(x) \in C\left(\overline{\Omega}\right)$ satisfies

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$$3 < p^- := \inf_{x \in \overline{\Omega}} p(x) \leq p^+ := \sup_{x \in \overline{\Omega}} p(x).$$

λ is a real number, f satisfies $Carath\acute{e}odory$ condition, and

$$F(x,t) = \int_0^t f(x,s)\,ds. \tag{2}$$

The analysis of problem (1) involves the $p(x)$-triharmonic operator is a natural extension of the p-triharmonic operator and has garnered considerable interest in recent years due to its more complex nonlinear characteristics compared to the p-triharmonic operator. The triharmonic problem has a wide range of applications in science and engineering, particularly requiring precise numerical solutions. The triharmonic equation is typically used to describe the slow rotational flow of highly viscous fluids within small cavities [1] and is also applied to geometric modeling in phase-field crystal models [2], such as surface optimization [3], mixing, and pore filling [4, 5]. Additionally, these equations can be viewed as a linearization of the Euler-Lagrange equations, used to minimize curvature variations [6]. In surface generation and forming, controlling boundary conditions to ensure curvature smoothness is crucial for designing high-quality surfaces. The variable exponent triharmonic equation further extends these applications by accurately describing high-order nonlinear effects and spatial variations in complex physical systems. It is widely used in fields such as material mechanics, fluid mechanics, nonlinear wave and vibration analysis, heat conduction and diffusion, biophysics, and biomedical engineering. This equation effectively models stress distribution in heterogeneous materials, the flow behavior of non-uniform fluids, the vibration modes of complex structures, and heat transfer in functionally graded materials, providing powerful tools for studying and predicting the behavior of these complex systems.

The investigation of Lebesgue and Sobolev spaces with variable exponent has been the subject of intensive research. $Kov\acute{a}\check{c}ik$ and $R\acute{a}kosn\acute{i}k$ [7] introduced the generalized Lebesgue space $L^{p(x)}$ and generalized Sobolev space $W^{k,p(x)}$, which serve as counterparts to the traditional Lebesgue space L^p and Sobolev space $W^{k,p}$, respectively. *Fan* and *Zhao* [8] presented fundamental results regarding the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega)$. *MarekGalewski* [9] provided a generalized sufficient condition for the continuity of the Nemytskij operator between $L^{p_1(x)}$ and $L^{p_2(x)}$, among other results. The study of these spaces has laid a foundation for exploring partial differential equations and variational problems in the context of variable exponent $p(x)$ growth, which has found extensive applications in various areas such as the modeling of electrorheological fluids (see [10–13]), mathematical representation of stationary thermo-rheological viscous flows in non-Newtonian fluids (see [14]), phenomena related to image processing (see [15–17]), elasticity (see [18]), and flow in porous media (see [19]).

A representative elliptic Dirichlet boundary value problem featuring variable exponent growth conditions

$$\begin{cases} -\Delta_{p(x)}u = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

has been studied extensively. For instance, numerous researchers have investigated the following nonhomogeneous eigenvalue problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

The scenario where $p(x) = q(x)$ has been explored by *Fan*, *Zhang*, and *Zhao* in [20]. Utilizing an argument based on the Ljusternik-Schnirelmann critical point theory, the findings in [20] demonstrate that problem (4) possesses infinitely many eigenvalues. By defining Λ as the set of all nonnegative eigenvalues, the authors establish that Λ is a nonempty infinite set and $\sup\Lambda = +\infty$. They further note that, in general, $\inf\Lambda = 0$ can occur, but $\inf\Lambda > 0$ holds under certain specific conditions, which are somewhat related to a type of monotonicity in the function $p(x)$. It is also noted that for the p-Laplace operator (where $p(x) \equiv p$), $\inf\Lambda > 0$ always applies. $M.Mih\breve{a}ilescu$ and $V.R\breve{a}dulescu$ [21] investigate problem (4) under a fundamental assumption:

$$1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x).$$

The authors establish the existence of a continuous family of eigenvalues for problem (4) in a neighborhood of the origin. Specifically, they demonstrate the existence of $\lambda^\star$ such that every $\lambda \in (0, \lambda^\star)$ is an eigenvalue of problem (4). The proof relies on simple variational arguments based on Ekeland's variational principle. This finding was later extended by *X.Fan* in [22]. The authors further investigate the eigenvalues of problem (4) through a constrained variational approach. They prove that for each $t > 0$, problem (4) admits at least one sequence of solutions satisfying $\int_\Omega \frac{1}{p(x)}|\nabla u_{n,t}|^{p(x)}dx = t$ and $\lambda_{n,t} \to \infty$ as $n \to \infty$. They also examine the principal eigenvalues of problem (4) in the general case, as well as in the cases where $\Omega = \Omega_- = \{x \in \Omega : q(x) < p(x)\}$ and $\Omega = \Omega_+ = \{x \in \Omega : q(x) > p(x)\}$, and reveal the similarities and differences between the variable exponent case and the constant exponent case in Problem (4). Additionally, it is noted that the established results for problem (4) can be generalized to the more comprehensive form (3) under suitable conditions on $f(x, u)$.
Similarly, the following eigenvalue problem involving $p(x)$-biharmonic:

$$\begin{cases} -\Delta^2_{p(x)}u = \lambda f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

has also been studied in depth. For instance, the problem

$$\begin{cases} -\Delta^2_{p(x)}u = \lambda|u|^{q(x)-2}u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

has been studied by *A.Ayoujil* and *A.R.ElAmrouss* in [23] for the case $p(x) = q(x)$ and in [24] for the case $p(x) \neq q(x)$. Notably, in [23], using Ljusternik-Schnirelmann theory on C^1-manifolds, the authors demonstrated the existence of a sequence of eigenvalues and showed that $\sup \Lambda = \infty$, where Λ represents the set of all nonnegative eigenvalues. For the variable exponent $p(x)$, unlike the constant exponent scenario, they provided sufficient conditions under which $\inf \Lambda = 0$. In [24], employing the mountain pass lemma and Ekeland's variational principle, the authors further established various existence criteria for eigenvalues.
As research progresses, numerous scholars are now focusing on elliptic boundary value problems involving the $p(x)$-triharmonic operator. For further details, refer to [25–29]. Notably, *BelgacemRahal* [25] was the first to extend the triharmonic problem from a constant exponent framework to a variable exponent setting, marking a significant starting point for the study of variable exponent triharmonic problems. The eigenvalue equation:

$$\begin{cases} -\Delta^3_{p(\cdot)}u = \lambda|u|^{q(\cdot)-2}u, & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

similar to the above has also been studied by *İsmailAydin* in [28]. The author demonstrated the existence of multiple weak solutions for problem (7) using the Fountain theorem under suitable assumptions.
Inspired by the above research process, this paper will supplement and extend the results of [28], and explore more general forms (1) under appropriate assumptions.

We assume the following hypotheses:

($H1$) $\alpha(x),\ \beta(x),\ \omega(x),\ \eta(x),\ p(x),\ q(x) \in C_+\left(\overline{\Omega}\right)$ and $p(x) < \frac{N}{3}$ such that

$$\begin{aligned} &1 < \alpha^- \leq \alpha^+ < \beta^- \leq \beta^+ < p^-, \\ &1 < \omega^- \leq \omega^+ < p^- < p^+ < q^- \leq q^+, \\ &1 < \eta^- \leq \eta^+ < p^- < p^+ < l^- \leq l^+ < q^-, \end{aligned}$$

and

$$3 < p^- \leq p^+ < q^- \leq q^+ < p_3^*(x).$$

($H2$) Denote

$$\gamma_1(x) = \frac{r(x)}{r(x)-\omega(x)}, \quad \gamma_2(x) = \frac{r(x)}{r(x)-q(x)},$$
$$\gamma_3(x) = \frac{r(x)}{r(x)-\alpha(x)}, \quad \gamma_4(x) = \frac{r(x)}{r(x)-\beta(x)},$$
$$\gamma_5(x) = \frac{r(x)}{r(x)-\eta(x)}, \quad \gamma_6(x) = \frac{r(x)}{r(x)-l(x)}.$$

Then $a_i(x) \in L^{\gamma_{2i-1}(x)}(\Omega) \bigcap L^\infty(\Omega)$ and $b_i(x) \in L^{\gamma_{2i}(x)}(\Omega) \bigcap L^\infty(\Omega)$,

for $i = 1, 2, 3,$ where $r(x) \in C_+(\overline{\Omega})$ with $q(x) < r(x) < p_3^*(x)$, $meas\{x \in \Omega : a_i(x) > 0\} > 0$ for $i = 1, 2, 3$, and $meas\{x \in \Omega : b_i(x) > 0\} > 0$ for $i = 1, 2, 3$.

($F1$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the *Carathéodory* condition, which states that $f(\cdot, t)$ is measurable for all $t \in \mathbb{R}$ and $f(x, \cdot)$ is continuous for almost all $x \in \Omega$.

($F2$) *f* satisfies the following condition:

$$|f(x,t)| \le |a_1(x)|\,|t|^{\omega(x)-1} + |b_1(x)|\,|t|^{q(x)-1}, \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

where $\omega(x)$, $q(x)$, $a_1(x)$ and $b_1(x)$ are given in ($H1$) and ($H2$).

($F3$) There are positive constants M_1 and θ_1 such that $\theta_1 > p^+$ and

$$0 < \theta_1 F(x,t) \le f(x,t)\,t, \; |t| \ge M_1.$$

($F4$) $f(x,t) = o\left(|t|^{p^+-1}\right)$, as $t \to 0$ for $x \in \Omega$ uniformly.

($F5$) *f* satisfies the following condition:

$$|f(x,t)| \ge |a_3(x)|\,|t|^{\eta(x)-1} + |b_3(x)|\,|t|^{l(x)-1}, \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

where $\eta(x)$, $l(x)$, $a_3(x)$ and $b_3(x)$ are given in ($H1$) and ($H2$).

($F6$) *f* satisfies the following condition:

$$|f(x,t)| \le |a_2(x)|\,|t|^{\alpha(x)-1} + |b_2(x)|\,|t|^{\beta(x)-1}, \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

where $\alpha(x)$, $\beta(x)$, $a_2(x)$ and $b_2(x)$ are given in ($H1$) and ($H2$). It follows from ($F2$) that:

($F2'$) F satisfies the following growth condition:

$$|F(x,t)| \le \frac{|a_1(x)|}{\omega(x)}|t|^{\omega(x)} + \frac{|b_1(x)|}{q(x)}|t|^{q(x)}, \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

Our main result is given by the following theorem:

Theorem 1.1 Assume ($H1$), ($H2$), ($F1$) and ($F6$) hold. Then problem (1) has a weak solution for any $\lambda > 0$.

Theorem 1.2 Assume ($H1$),($H2$),($F1$)-($F4$) hold, then the problem (1) has a non-trivial weak solution for all $\lambda > 0$.

Theorem 1.3 Assume ($H1$), ($H2$), ($F1$)-($F3$) hold. If

$$f(x,-t) = -f(x,t), \text{ for all } (x,t) \in \Omega \times \mathbb{R}, \tag{8}$$

then for all $\lambda > 0$, I_λ has a list of critical points $\{\pm u_n\}$ in X such that, as $n \to \infty$.

Theorem 1.4 Assume ($H1$), ($H2$), ($F1$)-($F3$) and ($F5$) hold. If

$$f(x,-t) = -f(x,t), \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

then for all $\lambda > 0$, I_λ has a list of critical points $\{\pm u_n\}$ in X such that $I_\lambda(\pm u_n) < 0$ and $I_\lambda(\pm u_n) \to 0$, as $n \to \infty$.
The remainder of this paper is organized as follows: In Section 2, we review some fundamental theories and results related to variable exponent Lebesgue-Sobolev spaces. Then, in Section 3, we establish the existence of weak solutions to problem (1) by employing different methods based on various assumptions on $f(x, u)$.

2 Preliminaries

In order to discuss the problem (1), we need some theories about the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. Below, we first give some basic definitions and properties relevant to this paper.

For any $p(x) \in C_+(\overline{\Omega})$, we introduce the Lebesgue space with variable exponents:

$$L^{p(x)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function in } \Omega, \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

with the corresponding Luxemburg norm:

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

It is worth noting that if $p(x) \equiv p,\ p \ge 1$ is a constant, the norm of $L^{p(x)}(\Omega)$ is equivalent to the standard norm of $L^p(\Omega)$, namely,

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Proposition 2.1 [7, 8] The space $\left(L^{p(x)}(\Omega), \|\bullet\|_{L^{p(x)}(\Omega)}\right)$ is a separable, uniformly convex, reflexive Banach space. Its conjugate space is $L^{p'(x)}(\Omega)$, where $p'(x)$ is conjugate function of $p(x)$, namely,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have

$$\left|\int_\Omega uvdx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)} \leq 2\|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}.$$

Proposition 2.2 [8] Let $\ell_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx$, for all $u \in L^{p(x)}(\mathit{\Omega})$ .We have

$$\|u\|^{\check{p}}_{L^{p(x)}(\Omega)} \leq \ell_{p(x)}(u) \leq \|u\|^{\hat{p}}_{L^{p(x)}(\Omega)}.$$

Next we introduce the variable exponent Sobolev space:

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\right\},$$

with the norm:

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^{p(x)}(\Omega)},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} \cdots \partial {x_N}^{\alpha_N}} u,$$

$\alpha = (\alpha_1, \cdots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$.
In addition, $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Proposition 2.3 [7, 8] Let $p(x) \in C_+(\overline{\Omega})$. Then the Space $\left(W^{k,p(x)}(\Omega), \|\bullet\|_{k,p(x)}\right)$ is a reflexive and separable Banach space.

Proposition 2.4 [8] Let $p(x), q(x) \in C_+(\overline{\Omega})$ such that: $q(x) \leq p_k^*(x)$. Then there is continuous embedding:

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

If $\leq$ is replaced by $<$, then the embedding is compact.
The problem (1) studied in this paper is discussed in the following Sobolev space:

$$X := W^{3,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega),$$

with the corresponding norm:

$$\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)} + \|u\|_{3,p(x)}.$$

$\|u\|_X$ and $\|\nabla\Delta u\|_{L^{p(x)}(\Omega)}$ are two equivalent norms in X [25]. For any $u \in X$, define

$$\|u\| = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{\nabla\Delta u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

Thus, $\|u\|$ in X is equivalent to $\|u\|_X$.

Proposition 2.5 [25]If $1 < p^- \le p^+ < \infty$, the space $(X, \|\bullet\|_X)$ is a reflexive and separable Banach space.

According to Proposition 2.2, there is the following proposition

Proposition 2.6 Let $\zeta(u) = \int_\Omega |\nabla\Delta u|^{p(x)} dx$, for all $u \in X$. We have

$$\|u\|^{\check{p}} \le \zeta(u) \le \|u\|^{\hat{p}}.$$

Proposition 2.7 [25] Assume $q(x) \in C_+(\overline{\Omega})$ and $q(x) < p_3^*(x)$. Then there is a continuous and compact embedding:

$$X \hookrightarrow L^{q(x)}(\Omega).$$

3 Existence of Solution

Definition 3.1 If

$$\int_\Omega |\nabla\Delta u|^{p(x)-2}\nabla\Delta u \nabla\Delta v dx = \lambda \int_\Omega f(x,u) v dx$$

for all $v \in X$, then $u \in X$ is said to be a weak solution of the problem (1).
Define the functionals φ, ψ and $I_\lambda : X \to \mathbb{R}$ by:

$$\varphi(u) = \int_\Omega \frac{1}{p(x)} |\nabla\Delta u|^{p(x)} dx, \tag{9}$$

$$\psi(u) = \int_\Omega F(x,u)dx, \tag{10}$$

$$I_\lambda(u) = \varphi(u) - \lambda\psi(u). \tag{11}$$

3.1 The proof of Theorem 1.1

Lemma 3.1 $\varphi \in C^1(X,\mathbb{R})$ and its $Fréchet$ derivative is given by:

$$\langle \varphi'(u), v\rangle = \int_\Omega |\nabla\Delta u(x)|^{p(x)-2} \nabla\Delta u(x)\nabla\Delta v(x)dx.$$

Proof Let $u(x),\ v(x) \in X,\ x \in \Omega$ and $0 < |t| < 1$. By the mean value theorem, there exists $s \in [0,1]$ such that

$$\begin{aligned}
&\left|\frac{|\nabla\Delta(u(x)+tv(x))|^{p(x)} - |\nabla\Delta u(x)|^{p(x)}}{p(x)t}\right| \\
&= |\nabla\Delta(u(x)+tsv(x))|^{p(x)-1}|\nabla\Delta v(x)| \\
&\le (|\nabla\Delta u(x)| + |\nabla\Delta v(x)|)^{p(x)-1}|\nabla\Delta v(x)|.
\end{aligned}$$

Using the inequality [8]

$$|u(x)+v(x)|^{p(x)} \le 2^{p^+-1}\left(|u(x)|^{p(x)} + |v(x)|^{p(x)}\right),$$

Proposition 2.1 and Proposition 2.6, We can obtain

$$\begin{aligned}
&\int_\Omega (|\nabla\Delta u(x)| + |\nabla\Delta v(x)|)^{p(x)-1}|\nabla\Delta v(x)|dx \\
&\le 2^{p^+-2}\int_\Omega \left(|\nabla\Delta u(x)|^{p(x)-1}|\nabla\Delta v(x)| + |\nabla\Delta v(x)|^{p(x)}\right)dx \\
&\le 2^{p^+-1}\left\||\nabla\Delta u(x)|^{p(x)-1}\right\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}\|\nabla\Delta v(x)\|_{L^{p(x)}(\Omega)} + 2^{p^+-2}\|v(x)\|^{\hat{p}} \\
&\le 2^{p^+-1}\|u(x)\|^{\hat{p}-1}\|v(x)\| + 2^{p^+-2}\|v(x)\|^{\hat{p}}.
\end{aligned}$$

Thus,

$$(|\nabla\Delta u(x)| + |\nabla\Delta v(x)|)^{p(x)-1}|\nabla\Delta v(x)| \in L^1(\Omega).$$

By the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\langle \varphi'(u(x)), v(x)\rangle &= \lim_{t\to 0}\frac{\varphi(u(x)+tv(x))-\varphi(u(x))}{t}\\ &= \lim_{t\to 0}\int_\Omega \frac{|\nabla\Delta(u(x)+tv(x))|^{p(x)}-|\nabla\Delta u(x)|^{p(x)}}{tp(x)}dx\\ &= \int_\Omega \lim_{t\to 0}|\nabla\Delta(u(x)+tsv(x))|^{p(x)-2}\nabla\Delta(u(x)+tsv(x))\nabla\Delta v(x)dx\\ &= \int_\Omega |\nabla\Delta u(x)|^{p(x)-2}\nabla\Delta u(x)\nabla\Delta v(x)dx.\end{aligned}$$

Let $u_n(x)\to u(x)$ in X, i.e., $\nabla\Delta u_n(x)\to\nabla\Delta u(x)$ in $L^{p(x)}(\Omega)$. Then,

$$\begin{aligned}|\langle\varphi'(u_n)-\varphi'(u),v\rangle| &= \left|\int_\Omega \left(|\nabla\Delta u_n|^{p(x)-2}\nabla\Delta u_n-|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\right)\nabla\Delta v dx\right|\\ &\le \int_\Omega \left||\nabla\Delta u_n|^{p(x)-2}\nabla\Delta u_n-|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\right||\nabla\Delta v|\,dx\\ &\le 2\left\||\nabla\Delta u_n|^{p(x)-2}\nabla\Delta u_n-|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\right\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}\|\nabla\Delta v\|_{L^{p(x)}(\Omega)}\\ &= 2\left\||\nabla\Delta u_n|^{p(x)-2}\nabla\Delta u_n-|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\right\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}\|v\|.\end{aligned}$$

Let $P\big(x,\nabla\Delta u\big)=\big|\nabla\Delta u\big|^{p(x)-2}\nabla\Delta u$. We deduce from theorem 1.16 of [8] or theorem 1.1 of [9] that $P(x,\bullet):L^{p(x)}(\Omega)\to L^{\frac{p(x)}{p(x)-1}}(\Omega)$ is continuous,which shows that

$$P(x,\nabla\Delta u_n)\to P(x,\nabla\Delta u)\ \text{ in } L^{\frac{p(x)}{p(x)-1}}(\Omega).$$

Therefore,

$$\begin{aligned}\|\varphi'(u_n)-\varphi'(u)\| &= \sup_{0\ne v\in X}\frac{|\langle\varphi'(u_n)-\varphi'(u),v\rangle|}{\|v\|}\\ &\le \sup_{0\ne v\in X}\frac{2\left\||\nabla\Delta u_n|^{p(x)-2}\nabla\Delta u_n-|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\right\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}\|v\|}{\|v\|}\\ &= 2\|P(x,\nabla\Delta u_n)-P(x,\nabla\Delta u)\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}\to 0,\ n\to\infty.\end{aligned}$$

To sum up, we can conclude that $\varphi\in C^1(X,\mathbb{R})$. □

Lemma 3.2 φ' is a mapping of type (S_+), i.e., if $u_n\rightharpoonup u$ in X and

$$\limsup_{n\to\infty}\langle\varphi'(u_n)-\varphi'(u),u_n-u\rangle\le 0,$$

then $u_n\to u$ in X, as $n\to\infty$.

Proof By Lemma 3.1, it follows that

$$\langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle$$
$$= \int_\Omega \left(|\nabla \Delta u_n|^{p(x)-2} \nabla \Delta u_n - |\nabla \Delta u|^{p(x)-2} \nabla \Delta u \right) \nabla \Delta (u_n - u) dx.$$

Using the following elementary inequality [30]:

$$|x-y|^\gamma \leq 2^\gamma \left(|x|^{\gamma-2} x - |y|^{\gamma-2} y \right) \cdot (x-y), \ \gamma \geq 2, \tag{12}$$

we can conclude that for all $u, \ v \in X$ with $u \neq v$,

$$\langle \varphi'(u) - \varphi'(v), u - v \rangle > 0.$$

This implies φ' is strictly monotone. Let $\{u_n\} \subset X$ be a sequence such that $u_n \rightharpoonup u$, and

$$\limsup_{n\to\infty} \langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle \leq 0.$$

By Proposition 2.6, it suffices to show that $\int_\Omega |\nabla \Delta u_n - \nabla \Delta u|^{p(x)} dx \to 0$. By the monotonicity of φ', we have

$$\langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle \geq 0.$$

Thus,

$$\langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle \to 0, \ n \to \infty. \tag{13}$$

From (12) and (13), we deduce that

$$\langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle \geq \int_\Omega \frac{1}{2^{p(x)}} |\nabla \Delta (u_n - u)|^{p(x)}$$
$$\geq \frac{1}{2^{p^+}} \int_\Omega |\nabla \Delta (u_n - u)|^{p(x)} \to 0, \ n \to \infty.$$

Therefore, $u_n \to u$ in X, which implies that φ' is a mapping of type (S_+). □

Lemma 3.3 Assume that (H1) holds. Then φ is weakly lower semi-continuous, meaning that if $u_n \rightharpoonup u$ as $n \to \infty$ in X, then $\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n)$.

Proof Let $u_n \rightharpoonup u$ in X. From Lemma 3.1 and Lemma 3.2, we know that φ is convex functional [31] and

$$\varphi(u_n) \geq \varphi(u) + \left\langle \varphi'(u), u_n - u \right\rangle, \text{ for any } n.$$

According to the definition of weak convergence we can obtain

$$\langle \varphi' (u), u_n - u \rangle \to 0.$$

Thus,

$$\liminf_{n\to\infty} \varphi (u_n) \geq \varphi (u) + \liminf_{n\to\infty} \left\langle \varphi^{'} (u), u_n - u \right\rangle = \varphi (u).$$

□

Lemma 3.4 Assume

$$|f(x,t)| \leq |a_2(x)| \, |t|^{\alpha(x)-1} + |b_2(x)| \, |t|^{\beta(x)-1}, \text{ for all } (x,t) \in \Omega \times \mathbb{R}, \quad (14)$$

and that conditions $(H1)$, $(H2)$ and $(F1)$ hold. Then for almost all $x \in \Omega$ and all $t \in \mathbb{R}$, the following estimate can be obtained:

$$F(x,t) \leq \frac{1}{\alpha^- {\gamma_3}^-} |a_2(x)|^{\gamma_3(x)} + \frac{1}{\beta^- {\gamma_4}^-} |b_2(x)|^{\gamma_4(x)} + 2L|t|^{r(x)},$$

where $L = max \left\{ \frac{1}{\alpha^- ({\gamma_3}^+)'}, \ \frac{1}{\beta^- ({\gamma_4}^+)'} \right\}$. Also, the Nemytskij operator

$$u \mapsto F(x,u)$$

is continuous from $L^{r(x)}(\Omega)$ to $L^1(\Omega)$.

Proof According to (14) and the Young inequality, we can deduce that

$$\begin{aligned} F(x,t) &\leq \frac{|a_2(x)|}{\alpha(x)} |t|^{\alpha(x)} + \frac{|b_2(x)|}{\beta(x)} |t|^{\beta(x)} \\ &\leq \frac{1}{\alpha^-} |a_2(x)| \, |t|^{\alpha(x)} + \frac{1}{\beta^-} |b_2(x)| \, |t|^{\beta(x)} \\ &\leq \frac{1}{\alpha^-} \left(\frac{|a_2(x)|^{\gamma_3(x)}}{{\gamma_3}^-} + \frac{|t|^{\alpha(x)(\gamma_3(x))'}}{({\gamma_3}^+)'} \right) + \frac{1}{\beta^-} \left(\frac{|b_2(x)|^{\gamma_4(x)}}{{\gamma_4}^-} + \frac{|t|^{\beta(x)(\gamma_4(x))'}}{({\gamma_4}^+)'} \right) \\ &= \frac{1}{\alpha^-} \left(\frac{|a_2(x)|^{\gamma_3(x)}}{{\gamma_3}^-} + \frac{|t|^{r(x)}}{({\gamma_3}^+)'} \right) + \frac{1}{\beta^-} \left(\frac{|b_2(x)|^{\gamma_4(x)}}{{\gamma_4}^-} + \frac{|t|^{r(x)}}{({\gamma_4}^+)'} \right) \\ &\leq \frac{1}{\alpha^- {\gamma_3}^-} |a_2(x)|^{\gamma_3(x)} + \frac{1}{\beta^- {\gamma_4}^-} |b_2(x)|^{\gamma_4(x)} + 2L|t|^{r(x)}. \end{aligned}$$

Let $u \in L^{r(x)}$. By Proposition 2.2, we have

$$\begin{aligned}\int_{\Omega}F\left(x,u\right)dx &\leq \frac{1}{\alpha^{-}\gamma_{3}^{-}}\int_{\Omega}\left|a_{2}\left(x\right)\right|^{\gamma_{3}(x)}dx+\frac{1}{\beta^{-}\gamma_{4}^{-}}\int_{\Omega}\left|b_{2}\left(x\right)\right|^{\gamma_{4}(x)}dx\\&\quad+2L\int_{\Omega}\left|t\right|^{r(x)}dx\\&\leq \frac{1}{\alpha^{-}\gamma_{3}^{-}}\left\|a_{2}\left(x\right)\right\|_{L^{\gamma_{3}(x)}(\Omega)}^{\hat{\gamma_{3}}}+\frac{1}{\beta^{-}\gamma_{4}^{-}}\left\|b_{2}\left(x\right)\right\|_{L^{\gamma_{4}(x)}(\Omega)}^{\hat{\gamma_{4}}}\\&\quad+2L\left\|u\right\|_{L^{r(x)}(\Omega)}^{\hat{r}}\\&<\infty.\end{aligned}$$

So $F\left(x,u\right)\in L^{1}\left(\Omega\right)$. Let $u_{n}\to u$ in $L^{r(x)}\left(\Omega\right)$. By Lemma 2.2 of [9], in the subsequence sense, there is $g\in L^{r(x)}$ such that

$$\begin{aligned}\left|F\left(x,u_{n}\right)-F\left(x,u\right)\right| &\leq \frac{1}{\alpha^{-}\gamma_{3}^{-}}|a_{2}\left(x\right)|^{\gamma_{3}(x)}+\frac{1}{\beta^{-}\gamma_{4}^{-}}|b_{2}\left(x\right)|^{\gamma_{4}(x)}+2L|u_{n}|^{r(x)}\\&\quad+\frac{1}{\alpha^{-}\gamma_{3}^{-}}|a_{2}\left(x\right)|^{\gamma_{3}(x)}+\frac{1}{\beta^{-}\gamma_{4}^{-}}|b_{2}\left(x\right)|^{\gamma_{4}(x)}+2L|u|^{r(x)}\\&\leq \frac{2}{\alpha^{-}\gamma_{3}^{-}}|a_{2}\left(x\right)|^{\gamma_{3}(x)}+\frac{2}{\beta^{-}\gamma_{4}^{-}}|b_{2}\left(x\right)|^{\gamma_{4}(x)}\\&\quad+2L|g|^{r(x)}+2L|u|^{r(x)}.\end{aligned}$$

Also using Proposition 2.2, we get

$$\begin{aligned}\int_{\Omega}|F(x,u_{n})-F(x,u)|dx &\leq \frac{2}{\alpha^{-}\gamma_{3}^{-}}\int_{\Omega}|a_{2}\left(x\right)|^{\gamma_{3}(x)}dx+\frac{2}{\beta^{-}\gamma_{4}^{-}}\int_{\Omega}|b_{2}\left(x\right)|^{\gamma_{4}(x)}dx\\&\quad+2L\int_{\Omega}|g|^{r(x)}dx+2L\int_{\Omega}|u|^{r(x)}dx\\&\leq \frac{2}{\alpha^{-}\gamma_{3}^{-}}\left\|a_{2}\left(x\right)\right\|_{L^{\gamma_{3}(x)}(\Omega)}^{\hat{\gamma_{3}}}+\frac{2}{\beta^{-}\gamma_{4}^{-}}\left\|b_{2}\left(x\right)\right\|_{L^{\gamma_{4}(x)}(\Omega)}^{\hat{\gamma_{4}}}\\&\quad+2L\left\|g\right\|_{L^{r(x)}(\Omega)}^{\hat{r}}+2L\left\|u\right\|_{L^{r(x)}(\Omega)}^{\hat{r}}\\&<\infty.\end{aligned}$$

Thus, $\left|F\left(x,u_{n}\right)-F\left(x,u\right)\right|\in L^{1}\left(\Omega\right)$. It is clear that F satisfies the Carathedory condition, so by the Lebesgue dominated convergence theorem, we have $F\left(x,u_{n}\right)\to F\left(x,u\right)$ in $L^{1}\left(\Omega\right)$. This concludes the proof of the theorem. □

Lemma 3.5 Assuming that (14), ($H1$), ($H2$) and ($F1$) hold, then ψ is weak-strong continuous, which means that if $u_{n}\rightharpoonup u$ in X, then $\psi\left(u_{n}\right)\to\psi\left(u\right)$.

Proof Let $\{u_{n}\}\subset X$ such that $u_{n}\rightharpoonup u$. By proposition 2.7, we can deduce that $u_{n}\to u$ in $L^{r(x)}\left(\Omega\right)$. Then, $\psi\left(u_{n}\right)\to\psi\left(u\right)$ follows from Lemma 3.4. □

The Proof of Theorem 1.1 Let $u\in X$ with $\|u\|>1$. From Proposition 2.1, Proposition 2.6 and Proposition 2.7, we can deduce that

$$\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\nabla \Delta u|^{p(x)}\, dx - \lambda \int_\Omega F(x,u)\, dx \\
&\geq \frac{1}{p^+} \int_\Omega |\nabla \Delta u|^{p(x)} dx - \lambda \int_\Omega \left(\frac{|a_2(x)|}{\alpha(x)} |u|^{\alpha(x)} + \frac{|b_2(x)|}{\beta(x)} |u|^{\beta(x)} \right) dx \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{2\lambda}{\alpha^-} \|a_2(x)\|_{L^{\gamma_3}(\Omega)} \left\| |u|^{\alpha(x)} \right\|_{L^{(\gamma_3)'}(\Omega)} \\
&\quad - \frac{2\lambda}{\beta^-} \|b_2(x)\|_{L^{\gamma_4}(\Omega)} \left\| |u|^{\beta(x)} \right\|_{L^{(\gamma_4)'}(\Omega)} \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{2\lambda}{\alpha^-} \|a_2(x)\|_{L^{\gamma_3}(\Omega)} \|u\|^{\hat{\alpha}}_{L^{r(x)}(\Omega)} - \frac{2\lambda}{\beta^-} \|b_2(x)\|_{L^{\gamma_4}(\Omega)} \|u\|^{\hat{\beta}}_{L^{r(x)}(\Omega)} \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{2\lambda C}{\alpha^-} \|a_2(x)\|_{L^{\gamma_3}(\Omega)} \|u\|^{\hat{\alpha}} - \frac{2\lambda C}{\beta^-} \|b_2(x)\|_{L^{\gamma_4}(\Omega)} \|u\|^{\hat{\beta}} \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - 2L_1 \|u\|^{\hat{\beta}},
\end{aligned}$$

where $L_1 = max\left\{ \frac{2\lambda C}{\alpha^-} \|a_2(x)\|_{L^{\gamma_3}(\Omega)}, \frac{2\lambda C}{\beta^-} \|b_2(x)\|_{L^{\gamma_4}(\Omega)} \right\}$. Since $1 \leq \alpha^- \leq \alpha^+ < \beta^- \leq \beta^+ < p^-$ implies that $I_\lambda(u) \to \infty$ as $\|u\| \to \infty$ for $\lambda > 0$, namely, I_λ is coercive. By Lemma 3.3 and Lemma 3.5, the functional I_λ is weakly lower semi-continuous, so there exists a global minimizer, denoted as *u*, of I_λ in the function space X. Moreover, *u* is a weak solution to problem (1). This completes the proof. □

Example 3.1 Assume that

$$|f(x,t)| \leq C_1(1 + |t|^{\beta(x)-1}), \text{ for all } (x,t) \in \Omega \times \mathbb{R}, \tag{15}$$

where C_1 is a positive constant. If conditions ($H1$) and ($F1$) hold, then problem (1) has a weak solution for any $\lambda > 0$.

Proof According to (15), we get

$$\begin{aligned}
F(x,t) &\leq C_1 \left(|t| + \frac{1}{\beta(x)} |t|^{\beta(x)} \right) \\
&\leq C_1 \left(|t| + \frac{1}{\beta^-} |t|^{\beta(x)} \right).
\end{aligned}$$

Let $u \in L^{\beta(x)}$. By Theorem 2.8 of [7] and Proposition 2.2, we have

$$\begin{aligned}\int_{\Omega} F(x,u)dx &\le C_1 \int_{\Omega}\left(|u| + \frac{1}{\beta^-}|u|^{\beta(x)}\right)dx\\ &\le C_1\|u\|_{L^1(\Omega)} + \frac{C_1}{\beta^-}\|u\|_{L^{\beta(x)}(\Omega)}^{\hat{\beta}}\\ &\le C\|u\|_{L^{\beta(x)}(\Omega)} + \frac{C_1}{\beta^-}\|u\|_{L^{\beta(x)}(\Omega)}^{\hat{\beta}}\\ &< \infty.\end{aligned}$$

So, $F(x,u) \in L^1(\Omega)$. Let $u_n \to u$ in $L^{\beta(x)}(\Omega)$. By Lemma 2.2 of [9], in the subsequence sense, there is $g \in L^{\beta(x)}$ such that

$$\begin{aligned}|F(x,u_n) - F(x,u)| &\le C_1\left(|u_n| + \frac{1}{\beta^-}|u_n|^{\beta(x)} + |u| + \frac{1}{\beta^-}|u|^{\beta(x)}\right)\\ &\le C_1\left(|g| + |u| + \frac{1}{\beta^-}|g|^{\beta(x)} + \frac{1}{\beta^-}|u|^{\beta(x)}\right).\end{aligned}$$

Using Theorem 2.8 from [7] and Proposition 2.2, we can derive

$$\begin{aligned}\int_{\Omega}|F(x,u_n) - F(x,u)|dx &\le C_1\int_{\Omega}\left(|g| + |u| + \frac{1}{\beta^-}|g|^{\beta(x)} + + \frac{1}{\beta^-}|u|^{\beta(x)}\right)dx\\ &\le C_1\|g\|_{L^1(\Omega)} + C_1\|u\|_{L^1(\Omega)} + \frac{C_1}{\beta^-}\|g\|_{L^{\beta(x)}(\Omega)}^{\hat{\beta}}\\ &\quad + \frac{C_1}{\beta^-}\|u\|_{L^{\beta(x)}(\Omega)}^{\hat{\beta}}\\ &\le C\|g\|_{L^{\beta(x)}(\Omega)} + C\|u\|_{L^{\beta(x)}(\Omega)} + \frac{C_1}{\beta^-}\|g\|_{L^{\beta(x)}(\Omega)}^{\hat{\beta}}\\ &\quad + \frac{C_1}{\beta^-}\|u\|_{L^{\beta(x)}(\Omega)}^{\hat{\beta}}\\ &< \infty.\end{aligned}$$

Thus, $|F(x,u_n) - F(x,u)| \in L^1(\Omega)$. It is clear that F satisfies the Caratheodory condition since condition $(F1)$, so by the Lebesgue dominated convergence theorem, we have $F(x,u_n) \to F(x,u)$ in $L^1(\Omega)$. Let $u_n \rightharpoonup u$ in X. By Proposition 2.7, we deduce that $u_n \to u$ in $L^{\beta(x)}(\Omega)$, which implies $\psi(u_n) \to \psi(u)$. Hence, ψ is weak-strong continuous. Combining this with Lemma 3.3, we know that the functional I_λ is weakly lower semi-continuous. Let $u \in X$ with $\|u\| > 1$. From Proposition 2.6 and Proposition 2.7, we can conclude

$$
\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\nabla \Delta u|^{p(x)} dx - \lambda \int_\Omega F(x,u)\, dx \\
&\geq \frac{1}{p^+} \int_\Omega |\nabla \Delta u|^{p(x)} dx - C_1 \lambda \int_\Omega \left(|u| + \frac{1}{\beta(x)} |u|^{\beta(x)} \right) dx \\
&\geq \frac{1}{p^+} \int_\Omega |\nabla \Delta u|^{p(x)} dx - C_1 \lambda \int_\Omega |u|\, dx - \frac{C_1 \lambda}{\beta^-} \int_\Omega |u|^{\beta(x)} dx \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - C_1 \lambda \|u\|_{L^1(\Omega)} - \frac{C_1 \lambda}{\beta^-} \|u\|^{\hat{\beta}}_{L^{\beta(x)}(\Omega)} \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - C\lambda \|u\| - \frac{C\lambda}{\beta^-} \|u\|^{\hat{\beta}} \\
&\geq \frac{1}{p^+} \|u\|^{p^-} - 2L_2 \|u\|^{\hat{\beta}},
\end{aligned}
$$

where $L_2 = max\{C\lambda, \frac{C\lambda}{\beta^-}\}$. Since $1 \leq \beta^- \leq \beta^+ < p^-$, this implies $I_\lambda(u) \to \infty$ as $\|u\| \to \infty$ for $\lambda > 0$. Thus, I_λ is coercive. Therefore, there exists a global minimizer *u* of I_λ in the function space X. Moreover, *u* is a weak solution to the problem (1). This completes the proof. □

Example 3.2 Assume that conditions (H1), (H2), (F1) hold. Then problem

$$
\begin{cases}
-\Delta^3_{p(x)} u = \lambda |b_2(x)| |t|^{\beta(x)-2} t, & \text{in } \Omega, \\
u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega,
\end{cases} \tag{16}
$$

has a nontrivial weak solution for any $\lambda > 0$.

Proof The same proof technique used in Theorem 1.1 and Example 3.1 can be applied to derive the following result: there exists a global minimizer, denoted as u_0, of I_λ in the function space X. Moreover, u_0 is a weak solution to problem (16). Let $v \in X$ such that $\|v\| \neq 0$ and $0 < t < 1$. Then Since $\beta^+ < p^-$, we can conclude that $I_\lambda(tv) < 0$ for sufficiently small *t*. Thus, $I_\lambda(u_0) < 0$. Hence, the weak solution u_0 is non-trivial. □

3.2 The Proof of Theorem 1.2

Lemma 3.6 Assume that (F1) and (F3) hold. Then I_λ satisfies (PS)-condition for all $\lambda > 0$.

Proof Since ψ' is compact, it's clear that ψ' is a mapping of type (S_+).

Let $\{u_n\} \subset X$ be a (PS)-sequence, i.e., there exists a constant *M* such that

$$
\sup_n |I_\lambda(u_n)| \leq M \quad \text{and} \quad I'_\lambda(u_n) \to 0, \text{ as } n \to \infty.
$$

Because I'_λ is a mapping of type (S_+) and X is reflexive, it suffices to prove that $\{u_n\}$ is bounded in X.
Suppose $\|u_n\| \to \infty$, in the subsequence sense. For n large enough, assume $\|u_n\| > 1$. From ($F3$), Proposition 2.6 and $\langle I'_\lambda(u_n), u_n\rangle \geq -\left\|I'_\lambda(u_n)\right\|_{X^*} \|u_n\|_X$, we have

$$\begin{aligned} M &\geq I_\lambda(u_n) \\ &= \int_\Omega \frac{1}{p(x)} |\nabla \Delta u_n|^{p(x)}\, dx - \lambda \int_\Omega F(x, u_n)\, dx \\ &\geq \frac{1}{p^+} \int_\Omega |\nabla \Delta u_n|^{p(x)} dx - \lambda \int_\Omega \frac{1}{\theta_1} f(x, u_n) u_n dx \\ &= \left(\frac{1}{p^+} - \frac{1}{\theta_1}\right) \int_\Omega |\nabla \Delta u_n|^{p(x)} dx + \frac{1}{\theta_1} \int_\Omega (|\nabla \Delta u_n|^{p(x)} - \lambda f(x, u_n) u_n) dx \\ &= \left(\frac{1}{p^+} - \frac{1}{\theta_1}\right) \int_\Omega |\nabla \Delta u_n|^{p(x)}\, dx + \frac{1}{\theta_1} \langle I'_\lambda(u_n), u_n\rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta_1}\right) \int_\Omega |\nabla \Delta u_n|^{p(x)} dx - \frac{1}{\theta_1} \|I'_\lambda(u_n)\|_{X^*} \|u_n\|_X \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta_1}\right) \|u_n\|^{p^-} - \frac{1}{\theta_1} \|I'_\lambda(u_n)\|_{X^*} \|u_n\|_X . \end{aligned}$$

Due to $p^+ < \theta_1$ and $p^- > 3$, we have $I_\lambda(u_n) \to \infty$ as $\|u_n\| \to \infty$. This leads to a contradiction. □

The Proof of Theorem 1.2 It's clear from (2) and (11) that $I_\lambda(0) = 0$. It follows from Lemma 3.6 that I_λ satisfies (PS)-condition, so it suffices to prove the geometric conditions in the Mountain pass theorem.

By ($F2$) and ($F4$), for any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that for all $(x, t) \in \Omega \times \mathbb{R}$,

$$|F(x, t)| \leq \varepsilon |t|^{p^+} + C(\varepsilon) |b_1(x)| |t|^{q(x)}. \tag{17}$$

Assume $\|u\| < 1$. Then, by Proposition 2.1, Proposition 2.6 and Proposition 2.7, we have

$$\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx - \lambda\int_\Omega F(x,u)\,dx \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \lambda\int_\Omega\left(\varepsilon|u|^{p^+} + C(\varepsilon)|b_1(x)||u|^{q(x)}\right)dx \\
&= \frac{1}{p^+}\|u\|^{p^+} - \lambda\varepsilon\int_\Omega |u|^{p^+}dx - \lambda C(\varepsilon)\int_\Omega |b_1(x)||u|^{q(x)}dx \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \lambda\varepsilon\|u\|^{p^+}_{L^{p^+}(\Omega)} - 2\lambda C(\varepsilon)\|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)}\|\,|u|^{q(x)}\,\|_{L^{(\gamma_2(x))'}(\Omega)} \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \lambda\varepsilon C\|u\|^{p^+} - 2\lambda C(\varepsilon)\,C_2\|u\|^{\hat{q}}_{L^{r(x)}(\Omega)} \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \lambda\varepsilon C\|u\|^{p^+} - 2\lambda C(\varepsilon)CC_2\|u\|^{q^-},
\end{aligned}$$

where $C_2 = \|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)}$. Choosing $\varepsilon > 0$ small enough such that $0 < \lambda\varepsilon C_1 < \frac{1}{2p^+}$. Then

$$I_\lambda(u) \geq \frac{1}{2p^+}\|u\|^{p^+} - C(\lambda,\varepsilon)CC_2\|u\|^{q^-}.$$

Since $q^- > p^+$, there exist $R > 0$ small enough and $\delta > 0$ such that

$$\inf_{\|u\|=R} I_\lambda(u) \geq \delta > I_\lambda(0) = 0.$$

It follows from ($F3$) that for all $(x,t) \in \Omega\times\mathbb{R}$ and some constants $C_3,\ C_4 > 0$,

$$F(x,t) \geq C_3|t|^{\theta_1} - C_4. \tag{18}$$

Thus, by (18), for $v \in X\setminus\{0\}$, we have

$$\begin{aligned}
I_\lambda(tv) &= \int_\Omega \frac{1}{p(x)}|\nabla\Delta tv|^{p(x)}dx - \lambda\int_\Omega F(x,tv)\,dx \\
&\leq \frac{t^{p^+}}{p^-}\int_\Omega |\nabla\Delta v|^{p(x)}dx - \lambda C_3 t^{\theta_1}\int_\Omega |v|^{\theta_1}dx + \lambda C_4|\Omega|,
\end{aligned}$$

where $t \geq 1$ and $|\Omega|$ denotes the Lebesgue measure of Ω. Since $\theta_1 > p^+$, we have $I_\lambda(tv) \to -\infty$, as $t\to\infty$. That is, there exists $e := tv$ such that $\|e\| > R$ and $I_\lambda(e) \leq 0$. Therefore, I_λ satisfies the geometric structure of the Mountain pass theorem, and the theorem is proved. □

Example 3.3 Assume $|f(x,t)| \leq |b_1(x)|\,|t|^{q(x)-1}$, for all $(x,t) \in \Omega\times\mathbb{R}$, and that conditions ($H1$), ($H2$), ($F1$), ($F3$), ($F4$) hold. Then the problem

$$\begin{cases} -\Delta^3_{p(x)}u = f(x,u), & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega, \end{cases} \tag{19}$$

has a non-trivial weak solution.

3.3 The Proof of Theorem 1.3

Lemma 3.7 [32] Let X be a reflexive and separable Banach space. Then there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span\{e_j | j = 1, 2, \cdots\}}, \; X^* = \overline{span\{e_j^* | j = 1, 2, \cdots\}},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we write $X_j = \text{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$.

Lemma 3.8 Denote

$$\alpha_k := \sup \left\{ \|u\|_{L^{r(x)}(\Omega)} \mid \|u\| = 1, u \in Z_k \right\}.$$

Then $\lim\limits_{k\to\infty} \alpha_k \to 0$, for all $x \in \overline{\Omega}$.

Proof It's clear that $0 < \alpha_{k+1} \leq \alpha_k$, so $\alpha_k \to \alpha \geq 0$.

Let $u_k \in Z_k$ be such that $\|u_k\| = 1$ and $0 \leq \alpha_k - \|u_k\|_{L^{r(x)}(\Omega)} < \frac{1}{k}$. Then there exist subsequences $\{u_k\}$, which we also denote by $\{u_k\}$, such that $u_k \rightharpoonup u$ and

$$\langle e_j^*, u \rangle = \lim_{k\to+\infty} \langle e_j^*, u_k \rangle = 0, j = 1, 2, \cdots.$$

This implies that $u = 0$, so $u_k \rightharpoonup 0$, as $k \to \infty$. It is obtainable from Proposition 2.7 that $u_k \to 0$ in $L^{r(x)}(\Omega)$. Therefore we get $\alpha_k \to 0$, as $k \to \infty$. □

Lemma 3.9 Assume $(F1)$ and $(F3)$ hold. Then I_λ satisfies $(PS)_c^*$-condition for all $\lambda > 0$.

Proof Let $\{u_n\} \subset X$ be a $(PS)_c^*$-sequence, meaning $u_n \in Y_n$, $I_\lambda(u_n) \to c$, $(I_\lambda|_{Y_n})'(u_n) \to 0$, $n \to \infty$. May as well set $\|u_n\| > 1$, for sufficiently large n. Using $(F3)$ and Proposition 2.6 we have

$$
\begin{aligned}
\mathrm{c}+1 \geq I_\lambda\left(u_n\right) &= \int_\Omega \frac{1}{p(x)}\left|\nabla\Delta u_n\right|^{p(x)} dx - \lambda\int_\Omega F\left(x, u_n\right) dx \\
&\geq \frac{1}{p^+}\int_\Omega \left|\nabla\Delta u_n\right|^{p(x)} dx - \lambda\int_\Omega \frac{1}{\theta_1} f\left(x, u_n\right) u_n dx \\
&= \left(\frac{1}{p^+}-\frac{1}{\theta_1}\right)\int_\Omega \left|\nabla\Delta u_n\right|^{p(x)} dx + \frac{1}{\theta_1}\int_\Omega (\left|\nabla\Delta u_n\right|^{p(x)} - \lambda f\left(x, u_n\right) u_n) dx \\
&\geq \left(\frac{1}{p^+}-\frac{1}{\theta_1}\right)\|u_n\|^{p^-} + \frac{1}{\theta_1}\left\langle I_\lambda'(u_n), u_n\right\rangle \\
&= \left(\frac{1}{p^+}-\frac{1}{\theta_1}\right)\|u_n\|^{p^-} + \frac{1}{\theta_1}\left\langle \left(I_\lambda\,|_{Y_n}\right)'(u_n), u_n\right\rangle \\
&\geq \left(\frac{1}{p^+}-\frac{1}{\theta_1}\right)\|u_n\|^{p^-} - \frac{1}{\theta_1}\left\|\left(I_\lambda\,|_{Y_n}\right)'(u_n)\right\|_{X^*}\|u_n\|_X .
\end{aligned}
$$

Because $(I_\lambda\,|_{Y_n})'(u_n) \to 0,$ as $n \to \infty$, the sequence $\{u_n\}$ is bounded. In a subsequence sense, we have $u_n \rightharpoonup u$. Denoted $X = \overline{\bigcup_n Y_n}$ and take a sequence $\{v_n\} \subset Y_n$ such that $v_n \to u$. Therefore,

$$
\begin{aligned}
\lim_{n\to\infty}\left\langle I_\lambda'(u_n), u_n - u\right\rangle &= \lim_{n\to\infty}\left\langle I_\lambda'(u_n), u_n - v_n\right\rangle + \lim_{n\to\infty}\left\langle I_\lambda'(u_n), v_n - u\right\rangle \\
&= \lim_{n\to\infty}\left\langle \left(I_\lambda\,|_{Y_n}\right)'(u_n), u_n - v_n\right\rangle \\
&= 0.
\end{aligned}
$$

By Lemma 3.6, I_λ' is a mapping of type (S_+), so $u_n \to u$. From $I_\lambda(u) \in C^1(X, R)$, it follows that $I_\lambda'(u_n) \to I_\lambda'(u)$. For any $\omega_k \in Y_k$, when $n \geq k$, we have

$$
\begin{aligned}
\left\langle I_\lambda'(u), \omega_k\right\rangle &= \left\langle I_\lambda'(u) - I_\lambda'(u_n), \omega_k\right\rangle + \left\langle I_\lambda'(u_n), \omega_k\right\rangle \\
&= \left\langle I_\lambda'(u) - I_\lambda'(u_n), \omega_k\right\rangle + \left\langle \left(I_\lambda\,|_{Y_n}\right)'(u_n), \omega_k\right\rangle .
\end{aligned}
$$

In the limit of *n*, we have $\langle I_\lambda'(u), \omega_k\rangle = 0,$ for any $\omega_k \in Y_k$. Since $X = \overline{\bigcup_k Y_k}$, it follows that $I_\lambda'(u) = 0$. Hence, we have proven that I_λ satisfies $(PS)_c^*$-condition for any $c \in \mathbb{R}$. □

The Proof of Theorem 1.3 It follows from (2) and (8) that I_λ is even functional. From Lemma 3.9 we know that I_λ satisfies (PS)-condition for all $\lambda > 0$. Thus, it suffices to prove that for sufficiently large *k*, there exists $\rho_k > \delta_k > 0$ such that

(1) $b_k := \inf\{I_\lambda(u) | u \in Z_k,\ \|u\| = \delta_k\} \to \infty,$ as $k \to \infty$,
(2) $a_k := \max\{I_\lambda(u) | u \in Y_k,\ \|u\| = \rho_k\} \leq 0.$

Suppose $u \in Z_k$ with $\|u\| \geq 1$. By $(F2')$ and Propositions 2.1, Propositions 2.6 and Propositions 2.7, we have

$$
\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\nabla\Delta u|^{p(x)}\,dx - \lambda \int_\Omega F(x,u)\,dx \\
&\geq \frac{1}{p^+}\int_\Omega |\nabla\Delta u|^{p(x)} dx - \lambda\int_\Omega \left(\frac{|a_1(x)|}{\omega(x)}|u|^{\omega(x)} + \frac{|b_1(x)|}{q(x)}|u|^{q(x)}\right) dx \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{2\lambda}{\omega^-}\|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)} \left\||u|^{\omega(x)}\right\|_{L^{(\gamma_1(x))'}(\Omega)} \\
&\quad - \frac{2\lambda}{q^-}\|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)} \left\||u|^{q(x)}\right\|_{L^{(\gamma_2(x))'}(\Omega)} \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{2\lambda}{\omega^-}\|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)}\|u\|^{\hat{\omega}}_{L^{r(x)}(\Omega)} - \frac{2\lambda}{q^-}\|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)}\|u\|^{\hat{q}}_{L^{r(x)}(\Omega)} \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{2\lambda}{\omega^-}C_4\big(\alpha_k\|u\|\big)^{\hat{\omega}} - \frac{2\lambda}{q^-}C_2\big(\alpha_k\|u\|\big)^{\hat{q}},
\end{aligned}
$$

where $C_4 = \|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)}$, $C_2 = \|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)}$. We can choose $\frac{2}{\omega^-}C_4{\alpha_k}^{\hat{\omega}} < \frac{1}{2\lambda p^+}$ since $\alpha_k \to 0$, as $k \to \infty$. Therefore,

$$
I_\lambda(u) \geq \frac{1}{2p^+}\|u\|^{p^-} - \frac{2\lambda}{q^-}C_2\big(\alpha_k\|u\|\big)^{\hat{q}}.
$$

Choosing $\delta_k = \left(\frac{2\lambda}{q^-}C_2 2q^+\alpha_k^{\hat{q}}\right)^{\frac{1}{p^- - \hat{q}}}$. Then $\delta_k \to \infty$ since $p^- < \hat{q}$ and $\alpha_k \to 0$, as $k \to \infty$. If $u \in Z_k$ with $\|u\| = \delta_k \to \infty$, as $k \to \infty$ and $(H1)$ holds, we can deduce that

$$
I_\lambda(u) \geq \left(\frac{1}{2p^+} - \frac{1}{2q^+}\right){\delta_k}^{p^-} \to \infty, \text{ as } k \to \infty.
$$

This completes the proof of (1).
Let $u \in Y_k$ such that $\rho_k > \delta_k > 1$. According to (18), we have

$$
\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx - \lambda\int_\Omega F(x,u)dx \\
&\leq \frac{1}{p^-}\|u\|^{p^+} - \lambda C_3\int_\Omega |u|^{\theta_1}dx + \lambda C_4|\Omega|.
\end{aligned}
$$

Since $\dim Y_k < \infty$ (this implies that all norms are equivalent) and $\theta_1 > p^+$

$$
I_\lambda(u) \leq \frac{1}{p^-}\|u\|^{p^+} - \lambda C_3\|u\|^{\theta_1} + \lambda C_4|\Omega|.
$$

For $\|u\| = \rho_k$, we have $I_\lambda(u) \leq 0$. This completes the proof of (2). Therefore, we can choose $\rho_k > \delta_k > 0$, which completes the proof of the theorem. □

3.4 The Proof of Theorem 1.4

The proof of Theorem 1.4 It follows from the assumptions of the theorem and Lemma 3.9 that I_λ is even functional and satisfies the $(PS)_c^*$-condition. Therefore, it suffices to show that for any $k \geq k_0$, there exists $\rho_k > \delta_k > 0$ such that

(1) $c_k := \inf\{I_\lambda(u) : \ u \in Z_k, \ \|u\| = \rho_k\} \geq 0,$
(2) $d_k := \max\{I_\lambda(u) : \ u \in Y_k, \ \|u\| = \delta_k\} < 0,$
(3) $f_k := \inf\{I_\lambda(u) : \ u \in Z_k, \ \|u\| \leq \rho_k\} \to 0, \ k \to \infty.$

Let $v \in Z_k$ be such that $\|v\| = 1$. Taking $u = tv, \ 0 < t < 1$, and using $(F2')$ as well as Proposition 2.1, Proposition 2.6 and Proposition 2.7, we have

$$\begin{aligned}
I_\lambda(tv) &= \int_\Omega \frac{1}{p(x)} \left|\nabla\Delta tv\right|^{p(x)} dx - \lambda \int_\Omega F\big(x, tv\big) dx \\
&\geq \frac{t^{p^+}}{p^+} \int_\Omega |\nabla\Delta v|^{p(x)}\, dx - \frac{\lambda t^{\omega^-}}{\omega^-} \int_\Omega |a_1(x)|\, |v|^{\omega(x)} dx \\
&\quad - \frac{\lambda t^{q^-}}{q^-} \int_\Omega |b_1(x)|\, |v|^{q(x)} dx \\
&\geq \frac{t^{p^+}}{p^+} \|v\|^{p^+} - \frac{2\lambda t^{\omega^-}}{\omega^-} \|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)} \left\| |v|^{\omega(x)} \right\|_{L^{(\gamma_1(x))'}(\Omega)} \\
&\quad - \frac{2\lambda t^{q^-}}{q^-} \|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)} \left\| |v|^{q(x)} \right\|_{L^{(\gamma_2(x))'}(\Omega)} \\
&\geq \frac{t^{p^+}}{p^+} \|v\|^{p^+} - \frac{2\lambda t^{\omega^-}}{\omega^-} \|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)} \|v\|^{\hat{\omega}}_{L^{r(x)}(\Omega)} \\
&\quad - \frac{2\lambda t^{q^-}}{q^-} \|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)} \|v\|^{\hat{q}}_{L^{r(x)}(\Omega)} \\
&\geq \frac{t^{p^+}}{p^+} \|v\|^{p^+} - \frac{2\lambda t^{\omega^-}}{\omega^-} C_5 \big(\alpha_k \|v\|\big)^{\hat{\omega}} - \frac{2\lambda t^{q^-}}{q^-} C_2 \big(\alpha_k \|v\|\big)^{\hat{q}} \\
&= \frac{t^{p^+}}{p^+} - \frac{2\lambda t^{\omega^-}}{\omega^-} C_5 {\alpha_k}^{\hat{\omega}} - \frac{2\lambda t^{q^-}}{q^-} C_2 {\alpha_k}^{\hat{q}},
\end{aligned}$$

where $C_5 = \|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)}$, $C_2 = \|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)}$. For sufficiently large k, if we take $\frac{2C_2{\alpha_k}^{\hat{q}}}{q^-} < \frac{1}{2\lambda p^+}$,we obtain

$$I_\lambda(tv) \geq \frac{t^{p^+}}{2p^+} - \frac{2\lambda t^{\omega^-}}{\omega^-} C_5 {\alpha_k}^{\hat{\omega}}. \tag{20}$$

Choosing $\rho_k = \left(\frac{2\lambda}{\omega^-} C_5 2q^+ {\alpha_k}^{\hat{\omega}}\right)^{\frac{1}{p^+ - \omega^-}}$, we note that $\rho_k \to 0$ since $p^+ > \omega^-$ and $\alpha_k \to 0$, as $k \to \infty$. When $t = \rho_k$, we have

$$I_\lambda(tv) \geq \left(\frac{1}{2p^+} - \frac{1}{2q^+}\right)\rho_k^{p^+}.$$

Since $p^+ < q^+$, this implies $I_\lambda(u) \geq 0$, namely, $\inf\limits_{u\in Z_k, \|u\|=\rho_k} I_\lambda(u) \geq 0$. Thus, (1) is proved. Let $v \in Y_k$ with $\|v\| = 1$. Taking $u = tv,\ 0 < t < 1$, and using $(F5)$ and Proposition 2.6, we get

$$\begin{aligned} I_\lambda(tv) &= \int_\Omega \frac{1}{p(x)}\left|\nabla\Delta tv\right|^{p(x)} dx - \lambda\int_\Omega F(x,tv)dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega |\nabla\Delta v|^{p(x)}\,dx - \frac{\lambda t^{\eta^+}}{\eta^+}\int_\Omega |a_3(x)|\,|v|^{\eta(x)}dx - \frac{\lambda t^{l^+}}{l^+}\int_\Omega |b_3(x)|\,|v|^{l(x)}dx \\ &\leq \frac{t^{p^-}}{p^-}\|v\|^{p^-} - \frac{\lambda t^{\eta^+}}{\eta^+}\inf_{v\in Y_k,\|v\|=1}\int_\Omega |a_3(x)|\,|v|^{\eta(x)}dx \\ &\quad - \frac{\lambda t^{l^+}}{l^+}\inf_{v\in Y_k,\|v\|=1}\int_\Omega |b_3(x)|\,|v|^{l(x)}dx \\ &= \frac{t^{p^-}}{p^-} - \frac{\lambda t^{\eta^+}}{\eta^+}\inf_{v\in Y_k,\|v\|=1}\int_\Omega |a_3(x)|\,|v|^{\eta(x)}dx \\ &\quad - \frac{\lambda t^{l^+}}{l^+}\inf_{v\in Y_k,\|v\|=1}\int_\Omega |b_3(x)|\,|v|^{l(x)}dx. \end{aligned}$$

From $1 < \eta^- \leq \eta^+ < p^- < p^+ < l^- \leq l^+ < q^-$, we can derive that there exists a $\delta_k \in (0, \rho_k)$ such that when $t = \delta_k$, $I_\lambda(tv) < 0$. Thus, $d_k = \max\limits_{u\in Y_k,\|u\|=\delta_k} I_\lambda(u) < 0$.
In summary, (2) is proved.
Since $Y_k \cap Z_k \neq \emptyset$ and $\delta_k < \rho_k$, we have

$$f_k := \inf_{u\in Z_k,\|u\|\leq\rho_k} I_\lambda(u) \leq d_k := \max_{u\in Y_k,\|u\|=\delta_k} I_\lambda(u) < 0.$$

By (20), for $v \in Z_k$ with $\|v\| = 1$, let $u = tv$ and $0 \leq t \leq \rho_k$, we can deduce

$$\begin{aligned} I_\lambda(u) =& I_\lambda(tv) \geq \frac{t^{p^+}}{2p^+} - \frac{2\lambda t^{\omega^-}}{\omega^-}C_5\left(\alpha_k\right)^{\hat{\omega}} \geq -\frac{2\lambda t^{\omega^-}}{\omega^-}C_5\left(\alpha_k\right)^{\hat{\omega}} \\ &\geq -\frac{2\lambda {\rho_k}^{\omega^-}}{\omega^-}C_5\left(\alpha_k\right)^{\hat{\omega}} \geq -\frac{2\lambda}{\omega^-}C_5\left(\alpha_k\right)^{\hat{\omega}}. \end{aligned}$$

So $\inf\limits_{u\in Z_k,\|u\|\leq\rho_k} I_\lambda(u) \to 0, k \to \infty$. Hence, (3) is proved. □

Example 3.4 Denote the functional

$$I(u) = \int_\Omega \frac{1}{p(x)}\left|\nabla\Delta u\right|^{p(x)} dx - \lambda\int_\Omega \frac{|a_1(x)|}{\omega(x)}|u|^{\omega(x)}dx - \mu\int_\Omega \frac{|b_1(x)|}{q(x)}|u|^{q(x)}dx.$$

Suppose $(H1)$, $(H2)$ and $(F1)$ hold. Then the problem

$$\begin{cases} -\Delta^3_{p(x)}u = \lambda\left|a_1(x)\right|\left|u\right|^{\omega(x)-2}u + \mu\left|b_1(x)\right|\left|u\right|^{q(x)-2}u, & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega, \end{cases} \tag{21}$$

has the following result:

(1) If $\mu > 0$, then for any $\lambda \in \mathbb{R}$, the equation (21) possesses a set of solutions $(\pm u_k)_{k=1}^{\infty}$ such that $I_\lambda(u_k) \to \infty$, as $k \to \infty$.
(2) If $\lambda > 0$, then for any $\mu \in \mathbb{R}$, the equation (21) possesses a set of solutions $(\pm v_k)_{k=1}^{\infty}$ such that $I_\lambda(v_k) < 0$, when $k \to \infty$, $I_\lambda(v_k) \to 0$.

Proof **Step one**. $I(u)$ satisfies $(PS)_c^*$-condition for all $c > 0$.

(*i*) Let $\{u_n\} \subset X$ be a $(PS)_c^*$-sequence, i.e., $u_n \in Y_n,\ \ I(u_n) \to c,\ \ (I|_{Y_n})'(u_n) \to 0,\ \ n \to \infty.$ Without loss of generality, we assume that $\lambda > 0,\ \mu > 0$. Let's assume that $\|u_n\| > 1$, for sufficiently large n. Using Proposition 2.1, Proposition 2.6, Proposition 2.7 and $\left\langle (I\,|_{Y_n})'(u_n), u_n\right\rangle \geq -\left\|(I\,|_{Y_n})'(u_n)\right\|_{X^*}\|u_n\|_X$, when $\lambda \geq 0$, we have

$$\begin{aligned}
c+1 \geq I(u_n) &= \int_\Omega \frac{1}{p(x)}\left|\nabla\Delta u_n\right|^{p(x)}dx - \lambda\int_\Omega \frac{\left|a_1(x)\right|}{\omega(x)}|u_n|^{\omega(x)}dx \\
&\quad - \mu\int_\Omega \frac{\left|b_1(x)\right|}{q(x)}|u_n|^{q(x)}dx \\
&\geq \frac{1}{p^+}\int_\Omega \left|\nabla\Delta u_n\right|^{p(x)}dx - \frac{\lambda}{\omega^-}\int_\Omega \left|a_1(x)\right||u_n|^{\omega(x)}dx - \frac{\mu}{q^-}\int_\Omega \left|b_1(x)\right||u_n|^{q(x)}dx \\
&= \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\int_\Omega \left|\nabla\Delta u_n\right|^{p(x)}dx - \lambda\left(\frac{1}{\omega^-} - \frac{1}{q^-}\right)\int_\Omega \left|a_1(x)\right||u_n|^{\omega(x)}dx \\
&\quad + \frac{1}{q^-}\left\langle I'(u_n), u_n\right\rangle \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\|u_n\|^{p^-} - 2\lambda\left(\frac{1}{\omega^-} - \frac{1}{q^-}\right)\left\|a_1(x)\right\|_{L^{\gamma_1(x)}(\Omega)}\left\||u_n|^{\omega(x)}\right\|_{L^{(\gamma_1(x))'}(\Omega)} \\
&\quad + \frac{1}{q^-}\left\langle (I\,|_{Y_n})'(u_n), u_n\right\rangle \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\|u_n\|^{p^-} - 2\lambda\left(\frac{1}{\omega^-} - \frac{1}{q^-}\right)\left\|a_1(x)\right\|_{L^{\gamma_1(x)}(\Omega)}\|u_n\|^{\hat{\omega}}_{L^{r(x)}(\Omega)} \\
&\quad + \frac{1}{q^-}\left\langle (I\,|_{Y_n})'(u_n), u_n\right\rangle \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\|u_n\|^{p^-} - 2C\lambda\left(\frac{1}{\omega^-} - \frac{1}{q^-}\right)\left\|a_1(x)\right\|_{L^{\gamma_1(x)}(\Omega)}\|u_n\|^{\omega^+} \\
&\quad - \frac{1}{q^-}\left\|(I\,|_{Y_n})'(u_n)\right\|_{X^*}\|u_n\|_X .
\end{aligned}$$

Since $\omega^+ < p-$, $(I|_{Y_n})'(u_n) \to 0$, as $n \to \infty$, we prove the boundedness of $\{u_n\}$.
(*ii*) Consider a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, where $u_n \rightharpoonup u$. Denoted $X = \overline{\bigcup_n Y_n}$, and take $\{v_n\} \subset Y_n$ such that $v_n \to u$. Then

$$\begin{aligned}\lim_{n\to\infty}\left\langle I'(u_n), u_n-u\right\rangle &= \lim_{n\to\infty}\left\langle I'(u_n), u_n-v_n\right\rangle+\lim_{n\to\infty}\left\langle I'(u_n), v_n-u\right\rangle\\ &=\lim_{n\to\infty}\left\langle (I\mid_{Y_n})'(u_n), u_n-v_n\right\rangle\\ &=0.\end{aligned}$$

By Lemma 3.6, it follows that I' is a mapping of type (S_+), so $u_n \to u$. From $I(u) \in C^1(X, R)$, we have $I'(u_n) \to I'(u)$. For any $\omega_k \in Y_k$, when $n \geq k$, we have

$$\begin{aligned}\left\langle I'(u), \omega_k\right\rangle &= \left\langle I'(u)-I'(u_n), \omega_k\right\rangle+\left\langle I'(u_n), \omega_k\right\rangle\\ &=\left\langle I'(u)-I'(u_n), \omega_k\right\rangle+\left\langle (I\mid_{Y_n})'(u_n), \omega_k\right\rangle.\end{aligned}$$

In the limit of *n*, we have $\langle I'(u), \omega_k\rangle = 0,$ for any $\omega_k \in Y_k$. Since$X = \overline{\bigcup_k Y_k}$, $I'(u) = 0$. In this way we prove that *I* satisfies $(PS)_c^*$-condition for any $c \in \mathbb{R}$.

step two. If $\mu > 0$, then for any $\lambda \in \mathbb{R}$, $I(u)$ satisfies the remaining conditions of the Fountain theorem, i.e., for sufficiently large *k* there exists $\rho_k > \delta_k > 0$ such that

(*i*) $b_k := \inf\{I(u) | u \in Z_k,\ \|u\| = \delta_k\} \to \infty,$ as $k \to \infty,$

(*ii*) $a_k := \max\{I(u) | u \in Y_k,\ \|u\| = \rho_k\} \leq 0.$

(*i*)Let $u \in Z_k$ with $\|u\| \geq 1$, By Propositions 2.1, Propositions 2.6 and Propositions 2.7, we have

$$\begin{aligned}I(u) &= \int_\Omega \frac{1}{p(x)}\left|\nabla\Delta u\right|^{p(x)}dx - \lambda\int_\Omega \frac{|a_1(x)|}{\omega(x)}|u|^{\omega(x)}dx - \mu\int_\Omega \frac{|b_1(x)|}{q(x)}|u|^{q(x)}dx\\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{|\lambda|}{\omega^-}\int_\Omega |a_1(x)|\,|u|^{\omega(x)}dx - \frac{\mu}{q^-}\int_\Omega |b_1(x)|\,|u|^{q(x)}dx\\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{2|\lambda|}{\omega^-}\|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)}\left\||u|^{\omega(x)}\right\|_{L^{(\gamma_1(x))'}(\Omega)}\\ &\quad - \frac{2\mu}{q^-}\|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)}\left\||u|^{q(x)}\right\|_{L^{(\gamma_2(x))'}(\Omega)}\\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{2|\lambda|}{\omega^-}\|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)}\|u\|^{\hat{\omega}}_{L^{r(x)}(\Omega)} - \frac{2\mu}{q^-}\|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)}\|u\|^{\hat{q}}_{L^{r(x)}(\Omega)}\\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{2|\lambda|}{\omega^-}C_5(\alpha_k\|u\|)^{\hat{\omega}} - \frac{2\mu}{q^-}C_2(\alpha_k\|u\|)^{\hat{q}},\end{aligned}$$

where $C_5 = \|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)}$, $C_2 = \|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)}$. We can choose $\frac{2}{\omega^-}C_5{\alpha_k}^{\hat{\omega}} < \frac{1}{2|\lambda|p^+}$ since $\alpha_k \to 0$, as $k \to \infty$. Thus,

$$I(u) \geq \frac{1}{2p^+}\|u\|^{p^-} - \frac{2\mu}{q^-}C_2(\alpha_k\|u\|)^{\hat{q}}.$$

Choosing $\delta_k = \left(\frac{2\mu}{q^-}C_2 2q^+\alpha_k^{\hat{q}}\right)^{\frac{1}{p^- - \hat{q}}}$. Then $\delta_k \to \infty$ since $p^- < \hat{q}$ and $\alpha_k \to 0$, as $k \to \infty$. If $u \in Z_k$ with $\|u\| = \delta_k \to \infty$, as $k \to \infty$ and $(H1)$ holds, we deduce that

$$I\left(u\right)\geq\left(\frac{1}{2p^{+}}-\frac{1}{2q^{+}}\right)\delta_{k}{}^{p^{-}}\rightarrow\infty,\ \text{as}\ k\rightarrow\infty.$$

This completes the proof of (i).
(ii) Let $v\in Y_k$ with $\|v\|=1$, and $t>1$, we have

$$\begin{aligned}I\left(tv\right)&=\int_{\Omega}\frac{1}{p(x)}\left|\nabla\Delta tv\right|^{p(x)}dx-\lambda\int_{\Omega}\frac{\left|a_{1}\left(x\right)\right|}{\omega\left(x\right)}\left|tv\right|^{\omega(x)}dx-\mu\int_{\Omega}\frac{\left|b_{1}\left(x\right)\right|}{q\left(x\right)}\left|tv\right|^{q(x)}dx\\&\leq\frac{t^{p^{+}}}{p^{-}}\|v\|^{p^{+}}+\left|\lambda\right|t^{\omega^{+}}\int_{\Omega}\frac{\left|a_{1}\left(x\right)\right|}{\omega\left(x\right)}\left|v\right|^{\omega(x)}dx-\mu t^{q^{-}}\int_{\Omega}\frac{\left|b_{1}\left(x\right)\right|}{q\left(x\right)}\left|v\right|^{q(x)}dx\\&\leq\frac{t^{p^{+}}}{p^{-}}+\left|\lambda\right|t^{\omega^{+}}\sup_{v\in Y_{k},\|v\|=1}\int_{\Omega}\frac{\left|a_{1}\left(x\right)\right|}{\omega\left(x\right)}\left|v\right|^{\omega(x)}dx\\&\quad-\mu t^{q^{-}}\inf_{v\in Y_{k},\|v\|=1}\int_{\Omega}\frac{\left|b_{1}\left(x\right)\right|}{q\left(x\right)}\left|v\right|^{q(x)}dx.\end{aligned}$$

Since $1<\omega^{-}\leq\omega^{+}<p^{-}<p^{+}<q^{-}\leq q^{+}$, there exists $\rho_k>\delta_k$ such that for $t=\rho_k$ and $u=tv$, we have $I\left(u\right)=I\left(tv\right)\leq 0$. This completes the proof of (ii).

Step three. If $\lambda>0$, then for any $\mu\in\mathbb{R}$, $I\left(u\right)$ satisfies the remaining conditions of the dual fountain theorem, i.e., for any $k\geq k_0$ there exists $\rho_k>\delta_k>0$ such that (i) $c_k:=\inf\left\{I(u):\ u\in Z_k,\ \|u\|=\rho_k\right\}\geq 0$,

(ii) $d_k:=\max\left\{I\left(u\right):\ u\in Y_k,\ \|u\|=\delta_k\right\}<0$,

(iii) $f_k:=\inf\left\{I\left(u\right):\ u\in Z_k,\ \|u\|\leq\rho_k\right\}\rightarrow 0,\ k\rightarrow\infty$.

(i)Let $v\in Z_k$ be such that $\|v\|=1$. Taking $u=tv,\ 0<t<1$, and applying Proposition 2.1, Proposition 2.6 and Proposition 2.7, we have

$$\begin{aligned}I\left(tv\right)&=\int_{\Omega}\frac{1}{p(x)}\left|\nabla\Delta tv\right|^{p(x)}dx-\lambda\int_{\Omega}\frac{\left|a_{1}\left(x\right)\right|}{\omega\left(x\right)}\left|tv\right|^{\omega(x)}dx-\mu\int_{\Omega}\frac{\left|b_{1}\left(x\right)\right|}{q\left(x\right)}\left|tv\right|^{q(x)}dx\\&\geq\frac{t^{p^{+}}}{p^{+}}\int_{\Omega}\left|\nabla\Delta v\right|^{p(x)}dx-\frac{\lambda t^{\omega^{-}}}{\omega^{-}}\int_{\Omega}\left|a_{1}(x)\right|\left|v\right|^{\omega(x)}dx\\&\quad-\frac{\left|\mu\right|t^{q^{-}}}{q^{-}}\int_{\Omega}\left|b_{1}(x)\right|\left|v\right|^{q(x)}dx\\&\geq\frac{t^{p^{+}}}{p^{+}}\|v\|^{p^{+}}-\frac{2\lambda t^{\omega^{-}}}{\omega^{-}}\|a_{1}(x)\|_{L^{\gamma_{1}(x)}(\Omega)}\left\|\left|v\right|^{\omega(x)}\right\|_{L^{(\gamma_{1}(x))'}(\Omega)}\\&\quad-\frac{2\left|\mu\right|t^{q^{-}}}{q^{-}}\|b_{1}(x)\|_{L^{\gamma_{2}(x)}(\Omega)}\left\|\left|v\right|^{q(x)}\right\|_{L^{(\gamma_{2}(x))'}(\Omega)}\\&\geq\frac{t^{p^{+}}}{p^{+}}\|v\|^{p^{+}}-\frac{2\lambda t^{\omega^{-}}}{\omega^{-}}\|a_{1}(x)\|_{L^{\gamma_{1}(x)}(\Omega)}\|v\|_{L^{r(x)}(\Omega)}^{\hat{\omega}}\\&\quad-\frac{2\left|\mu\right|t^{q^{-}}}{q^{-}}\|b_{1}(x)\|_{L^{|\mu|_{2}(x)}(\Omega)}\|v\|_{L^{r(x)}(\Omega)}^{\hat{q}}\\&\geq\frac{t^{p^{+}}}{p^{+}}\|v\|^{p^{+}}-\frac{2\lambda t^{\omega^{-}}}{\omega^{-}}C_{4}\left(\alpha_{k}\|v\|\right)^{\hat{\omega}}-\frac{2\left|\mu\right|t^{q^{-}}}{q^{-}}C_{2}\left(\alpha_{k}\|v\|\right)^{\hat{q}}\\&=\frac{t^{p^{+}}}{p^{+}}-\frac{2\lambda t^{\omega^{-}}}{\omega^{-}}C_{5}\alpha_{k}{}^{\hat{\omega}}-\frac{2\left|\mu\right|t^{q^{-}}}{q^{-}}C_{2}\alpha_{k}{}^{\hat{q}},\end{aligned}$$

where $C_5 = \|a_1(x)\|_{L^{\gamma_1(x)}(\Omega)}$, $C_2 = \|b_1(x)\|_{L^{\gamma_2(x)}(\Omega)}$. When k is large enough, we take $\frac{2C_2(\alpha_k)^{\hat{q}}}{q^-} < \frac{1}{2|\mu|p^+}$ to obtain

$$I(tv) \geq \frac{t^{p^+}}{2p^+} - \frac{2\lambda t^{\omega^-}}{\omega^-} C_5 {\alpha_k}^{\hat{\omega}}. \tag{22}$$

Choosing $\rho_k = \left(\frac{2\lambda}{\omega^-} C_5 2q^+ {\alpha_k}^{\hat{\omega}}\right)^{\frac{1}{p^+ - \omega^-}}$, then $\rho_k \to 0$ since $p^+ > \omega^-$ and $\alpha_k \to 0$, as $k \to \infty$. When $t = \rho_k$, we have

$$I(tv) \geq \left(\frac{1}{2p^+} - \frac{1}{2q^+}\right) \rho_k^{p^+}.$$

Thus, $I_\lambda(u) \geq 0$ since $p^+ < q^+$, namely, $\inf\limits_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq 0$. In summary, (*i*) is proved.

(*ii*) Let $v \in Y_k$ with $\|v\| = 1$. Taking $u = tv,\ 0 < t < 1$, by Proposition 2.6, we get

$$\begin{aligned} I(tv) &= \int_\Omega \frac{1}{p(x)} |\nabla \Delta tv|^{p(x)} dx - \lambda \int_\Omega \frac{|a_1(x)|}{\omega(x)} |tv|^{\omega(x)} dx - \mu \int_\Omega \frac{|b_1(x)|}{q(x)} |tv|^{q(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} \|v\|^{p^-} - \lambda t^{\omega^+} \int_\Omega \frac{|a_1(x)|}{\omega(x)} |v|^{\omega(x)} dx + |\mu| t^{q^-} \int_\Omega \frac{|b_1(x)|}{q(x)} |v|^{q(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} - \lambda t^{\omega^+} \inf_{v \in Y_k, \|v\|=1} \int_\Omega \frac{|a_1(x)|}{\omega(x)} |v|^{\omega(x)} dx \\ &\quad + |\mu| t^{q^-} \sup_{v \in Y_k, \|v\|=1} \int_\Omega \frac{|b_1(x)|}{q(x)} |v|^{q(x)} dx. \end{aligned}$$

From $1 < \omega^- \leq \omega^+ < p^- < p^+ < q^- \leq q^+$, we can derive that there exists a $\delta_k \in (0, \rho_k)$ such that when $t = \delta_k$, $I(tv) < 0$. So $d_k = \max\limits_{u \in Y_k, \|u\| = \delta_k} I(u) < 0$. In summary, (*ii*) is proved.

(*iii*) Since $Y_k \cap Z_k \neq \emptyset$ and $\delta_k < \rho_k$, we have

$$f_k := \inf_{u \in Z_k, \|u\| \leq \rho_k} I(u) \leq d_k := \max_{u \in Y_k, \|u\| = \delta_k} I(u) < 0.$$

By (22), for $v \in Z_k$, $\|v\| = 1$. $u = tv$, and $0 \leq t \leq \rho_k$, we can deduce

$$\begin{aligned} I(u) = I(tv) &\geq \frac{t^{p^+}}{2p^+} - \frac{2\lambda t^{\omega^-}}{\omega^-} C_5 {\alpha_k}^{\hat{\omega}} \geq -\frac{2\lambda t^{\omega^-}}{\omega^-} C_5 {\alpha_k}^{\hat{\omega}} \\ &\geq -\frac{2\lambda {\rho_k}^{\omega^-}}{\omega^-} C_5 {\alpha_k}^{\hat{\omega}} \geq -\frac{2\lambda}{\omega^-} C_5 {\alpha_k}^{\hat{\omega}}. \end{aligned}$$

Thus, $\inf\limits_{u\in Z_k,\|u\|\le\rho_k} I(u)\to 0, k\to\infty$. (*iii*) is proved. □

4 Index of Notations

We use the following notations:

Ω: bounded domain of R^N with smooth boundary $\partial\Omega$,
$\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian operator,
$\Delta^2_{p(x)} := \Delta\left(|\Delta u|^{p(x)-2}\Delta u\right)$ is called the $p(x)$-biharmonic operator,
$C_+\left(\overline{\Omega}\right) := \left\{h(x) : h(x)\in C\left(\overline{\Omega}\right), \min_{x\in\overline{\Omega}} h(x) > 1\right\}$,
$h^+ := \sup_{x\in\overline{\Omega}} h(x), h(x)\in C_+\left(\overline{\Omega}\right), h^- := \inf_{x\in\overline{\Omega}} h(x), h(x)\in C_+\left(\overline{\Omega}\right)$,
$\mathcal{K}^{\check{p}} := \begin{cases} \mathcal{K}^{p^+}, 0<\mathcal{K}\leqslant 1,\\ \mathcal{K}^{p^-}, \mathcal{K}>1,\end{cases}$
$\mathcal{K}^{\hat{p}} := \begin{cases} \mathcal{K}^{p^-}, 0<\mathcal{K}\leqslant 1,\\ \mathcal{K}^{p^+}, \mathcal{K}>1,\end{cases}$
$p_k^*(x) := \begin{cases} \frac{Np(x)}{N-kp(x)}, & p(x)<\frac{N}{k},\\ +\infty, & p(x)\geqslant\frac{N}{k}.\end{cases}$ We denote by $\to$ (resp. $\rightharpoonup$) the strong (resp. weak) convergence.

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