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Hesitant Pythagorean fuzzy interaction aggregation operators and their application in multiple attribute decision-making

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Abstract

The aim of this paper is to develop hesitant Pythagorean fuzzy interaction aggregation operators based on the hesitant fuzzy set, Pythagorean fuzzy set and interaction between membership and non-membership. The new operation laws can overcome shortcomings of existing operation laws of hesitant Pythagorean fuzzy values. Several new hesitant Pythagorean fuzzy interaction aggregation operators have been developed including the hesitant Pythagorean fuzzy interaction weighted averaging operator, the hesitant Pythagorean fuzzy interaction weighted geometric averaging operator and the generalized hesitant Pythagorean fuzzy interaction weighted averaging operator. Using the Bonferroni mean, some hesitant Pythagorean fuzzy interaction Bonferroni mean operators have been developed including the hesitant Pythagorean fuzzy interaction Bonferroni mean operator, the hesitant Pythagorean fuzzy interaction weighted Bonferroni mean (HPFIWBM) operator, the hesitant Pythagorean fuzzy interaction geometric Bonferroni mean operator and the hesitant Pythagorean fuzzy interaction geometric weight Bonferroni mean (HPFIGWBM) operator. Some properties have been studied. A new multiple attribute decision-making method based on the HPFIWBM operator and the HPFIGWBM operator has been presented. Numerical example is presented to illustrate the new method.

Keywords Hesitant fuzzy set · Pythagorean fuzzy set · Multiple attribute decision-making · Bonferroni mean

Introduction

Fuzzy decision-making has been studied and applied extensively [1–3]. Pythagorean fuzzy set [4,5] is the extension of intuitionistic fuzzy set [6]. In intuitionistic fuzzy set, the sum of membership and non-membership is no more than 1, while in Pythagorean fuzzy set, the square sum of membership and non-membership is no more than 1. Hence, Pythagorean fuzzy set has larger feasible region than that of intuitionistic fuzzy set. Thus, it is more powerful and flexible in modeling fuzzy and uncertain information. Pythagorean fuzzy set has been studied and applied extensively [7–21]. Some aggregation operators have been developed in Pythagorean fuzzy environment. Liang et al. [22] proposed the Pythagorean fuzzy Bonferroni mean operator and the weighted Pythagorean fuzzy Bonferroni mean operator. Zhang et al. [23] presented some general-

ized Pythagorean fuzzy Bonferroni mean operator and the generalized Pythagorean fuzzy Bonferroni geometric mean operator. Yang and Pang [24] developed some Pythagorean fuzzy interaction Maclaurin symmetric mean operators. Rahman et al. [25] defined some interval-valued Pythagorean fuzzy aggregation operators including the interval-valued Pythagorean fuzzy weighted geometric operator, the interval-valued Pythagorean fuzzy ordered weighted geometric operator, and the interval-valued Pythagorean fuzzy hybrid geometric operator. Wei and Lu defined some Pythagorean fuzzy power aggregation operators in [26] and presented some dual hesitant Pythagorean fuzzy aggregation operators in [27]. Garg presented the generalized Pythagorean fuzzy Einstein weighted average operator and the generalized Pythagorean fuzzy Einstein ordered weighted average operator in [28] and developed the Pythagorean fuzzy geometric interactive aggregation operators using Einstein operations in [29]. Du et al. [30] defined interval-valued Pythagorean fuzzy linguistic variable set and defined interval-valued Pythagorean fuzzy linguistic ordered weighted averaging operator and generalized interval-valued Pythagorean fuzzy linguistic ordered weighted average operator. Wei [31] defined some

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Pythagorean fuzzy interaction aggregation operators and some Pythagorean fuzzy interaction geometric aggregation operators. Some multiple attribute decision-making methods in Pythagorean fuzzy environment have been developed. Zhang and Xu [32] extended the TOPSIS method to accommodate Pythagorean fuzzy values. Ren et al. [33] proposed Pythagorean fuzzy TODIM approach. Chen [34] presented Pythagorean fuzzy VIKOR methods based on the generalized Pythagorean fuzzy distance measure. Pythagorean fuzzy set has been extended to accommodate interval values [35,36], linguistic arguments [37,38], probabilistic information [39], etc.

Hesitant fuzzy set [40] is the extension of fuzzy set and intuitionistic fuzzy set. In hesitant fuzzy set, each membership may include several possible values. Hesitant fuzzy set has been extended to accommodate intuitionistic fuzzy set [41], linguistic arguments [42], linguistic intuitionistic fuzzy values [43–47]. Hesitant Pythagorean fuzzy sets was defined [48] and some hesitant Pythagorean fuzzy Hamacher aggregation operators have been developed including the hesitant Pythagorean fuzzy Hamacher weighted average operator, hesitant Pythagorean fuzzy Hamacher weighted geometric operator. Khan et al. [49] proposed maximizing deviation method for Pythagorean hesitant fuzzy numbers in which information about attribute weights is incomplete. Garg [50] defined some hesitant Pythagorean fuzzy weighted aggregation operators and hesitant Pythagorean fuzzy geometric aggregation operators. But in real decision-making process, there are still cases that can not be dealt with using existing methods. For example, in evaluating some car, the experts gave evaluation values of power, noise and speed as {(0.9, 0.0)}, {(0.6, 0.5)}, {(0.8, 0.3), (0.7, 0.4)}. If the attribute weight vector is (0.25, 0.35, 0.40). Then, using the existing operation laws of hesitant Pythagorean fuzzy values, the weighted averaging values can be calculated as {(0.7904, 0), (0.7541, 0)}. There is only 0 non-membership and the other two non-memberships are not 0, but they have no effect on the final results. To overcome this shortcoming, we propose some interaction operation laws for hesitant Pythagorean fuzzy values by considering interaction between membership and non-membership. Then, we first develop some aggregation operators including the hesitant Pythagorean fuzzy interaction weighted averaging (HPFIWA) operator, hesitant Pythagorean fuzzy interaction weighted geometric averaging (HPFIWGA) operator and the generalized hesitant Pythagorean fuzzy interaction weighted geometric averaging (GHPFIWA) operator. The Bonferroni mean was first introduced by Bonferroni [51], which can capture inter-relationship among arguments to be aggregated. Yager [52] provided an interpretation of Bonferroni mean as involving a product of each argument with the average of the other arguments. Beliakov et al. [53] developed generalized Bonferroni mean. Beliakov and James [54] extended the generalized Bonferroni mean to intuitionistic fuzzy environment. Zhu and Xu [55] proposed the hesitant fuzzy Bonferroni mean operator. Yang et al. [56] developed the Pythagorean fuzzy interaction partitioned Bonferroni mean operator. But Bonferroni mean for hesitant Pythagorean values considering interaction between membership and non-membership has not been studied yet. Yang et al. [57] proposed q-rung orthopair fuzzy partitioned Bonferroni mean operators. To model interaction among hesitant Pythagorean fuzzy values and interaction between membership and non-membership at the same time, we develop some hesitant Pythagorean fuzzy interaction Bonferroni mean operator including the hesitant Pythagorean fuzzy interaction Bonferroni mean (HPFIBM) operator, the hesitant Pythagorean fuzzy interaction weighted Bonferroni mean (HPFIWBM) operator, the hesitant Pythagorean fuzzy interaction geometric Bonferroni mean (HPFIGBM) operator and the hesitant Pythagorean fuzzy interaction weighted geometric Bonferroni mean (HPFIWGBM) aggregation operator.

The objective of the paper is to develop some hesitant Pythagorean fuzzy interaction Bonferroni mean operators. To do so, the structure of the paper is as follows. In "Preliminaries", some basic concepts on Pythagorean fuzzy set, hesitant fuzzy set have been reviewed. Some interaction operational laws for hesitant Pythagorean fuzzy values have been defined and some properties have been studied. In "Hesitant Pythagorean fuzzy interaction aggregation operators", some hesitant Pythagorean fuzzy interaction aggregation operators have been defined. In "Hesitant Pythagorean fuzzy interaction Bonferroni mean operators", some hesitant Pythagorean fuzzy interaction Bonferroni mean operators have been proposed. In "An approach to Pythagorean fuzzy multiple attribute decision-making based on new interaction aggregation operators", a new multiple attribute decision-making method based on the HPFIWBM operator and the HPFIWGBM operator has been presented. In "An illustrative example", numerical example is presented to illustrate the new method. Conclusions are presented in the final section.

Preliminaries

Definition 1 [40] Let X be a fixed set. A hesitant fuzzy set (HFS) H on X in terms of a function that when applied to X returns a subset of [0, 1],

$$H = \{\langle x, h_H(x)\rangle | x \in X\}, \tag{1}$$

where $h_H(x)$ is a set of values in [0,1], denoting the possible membership degrees of element $x \in X$ to set H. For convenience, $h_H(x)$ is called a hesitant fuzzy element (HFE).

Definition 2 [4] Let X be a fixed set. A Pythagorean fuzzy set P on X can be represented as follows

$$P = \{\langle x, (\mu_P(x), \nu_P(x)) \rangle \mid x \in X\}, \qquad (2)$$

where $\mu_P(x) : X \to [0, 1]$ is the membership function and $\nu_P(x) : X \to [0, 1]$ is the non-membership function. For each $x \in X$, it satisfies the following condition $0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$. $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}$ is the indeterminacy degree of x to X. For simplicity, $(\mu_P(x), \nu_P(x))$ is called a Pythagorean fuzzy number (PFN), denoted by (μ_P, ν_P), where $\mu_P, \nu_P \in [0, 1], \pi_P = \sqrt{1 - (\mu_P)^2 - (\nu_P)^2}$ and $0 \leq (\mu_P)^2 + (\nu_P)^2 \leq 1$.

Definition 3 [32] Let $\alpha = (\mu_\alpha, \nu_\alpha)$, $\alpha_1 = (\mu_1, \nu_1)$ and $\alpha_2 = (\mu_2, \nu_2)$ be three PFNs, the operations are as follows

(1) $\alpha_1 \oplus \alpha_2 = (\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2 \mu_2^2}, \nu_1 \nu_2)$,

(2) $\alpha_1 \otimes \alpha_2 = (\mu_1 \mu_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2 \nu_2^2})$,

(3) $k\alpha = (\sqrt{1 - (1 - \mu^2)^k}, (\nu)^k), k \geq 0$,

(4) $\alpha^k = (\mu^k, \sqrt{1 - (1 - \nu^2)^k}), k \geq 0$.

Definition 4 [48,50] Let X be a fixed set. A hesitant Pythagorean fuzzy set $\tilde{P}$ on X can be represented as follows

$$\tilde{P} = \{\langle x, (\tilde{h}_P(x), \tilde{g}_P(x)) \rangle \mid x \in X\}, \qquad (3)$$

where $\tilde{h}_P(x) = \{\mu_i\}$ is the set of all the possible memberships of element $x \in X$ and $\tilde{g}_P(x) = \{\nu_i\}$ is the set of all the possible non-memberships of element $x \in X$, $\mu_i \in [0, 1], \nu_i \in [0, 1]$. Let $\mu^+ = \max\{\mu_i\}, \nu^+ = \max\{\nu_i\}, (\mu^+)^2 + (\nu^+)^2 \leq 1$. For convenience, $(\tilde{h}_P(x), \tilde{g}_P(x))$ is called a hesitant Pythagorean fuzzy element (HPFE).

Definition 5 [50] Let $\tilde{f} = (\tilde{h}, \tilde{g})$, $\tilde{f}_1 = (\tilde{h}_1, \tilde{g}_1)$, $\tilde{f}_2 = (\tilde{h}_2, \tilde{g}_2)$ be three HPFEs, $\lambda > 0$. The hesitant Pythagorean fuzzy operation can be defined as

(1)

$$\tilde{f}_1 \oplus \tilde{f}_2 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, \nu_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, \nu_{2k_2} \in \tilde{g}_2} \times \left\{ \left(\sqrt{\mu_{1k_1}^2 + \mu_{2k_2}^2 - \mu_{1k_1}^2 \mu_{2k_2}^2}, \nu_{1k_1} \nu_{2k_2} \right) \right\},$$

(2)

$$\tilde{f}_1 \otimes \tilde{f}_2 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, \nu_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, \nu_{2k_2} \in \tilde{g}_2} \times \left\{ \left(\mu_{1k_1} \mu_{2k_2}, \sqrt{\nu_{1k_1}^2 + \nu_{2k_2}^2 - \nu_{1k_1}^2 \nu_{2k_2}^2} \right) \right\},$$

(3) $\lambda \tilde{f} = \bigcup_{\mu_k \in \tilde{h}, \nu_k \in \tilde{g}} \left\{ \left(\sqrt{1 - (1 - \mu_k^2)^\lambda}, (\nu_k^2)^\lambda \right) \right\}$,

(4) $\tilde{f}^\lambda = \bigcup_{\mu_k \in \tilde{h}, \nu_k \in \tilde{g}} \left\{ \left((\mu_k)^\lambda, \sqrt{1 - (1 - \nu_k^2)^\lambda} \right) \right\}$.

In the following, we present hesitant Pythagorean fuzzy interaction operation laws as follows.

Definition 6 Let $\tilde{f} = (\tilde{h}, \tilde{g})$, $\tilde{f}_1 = (\tilde{h}_1, \tilde{g}_1)$, $\tilde{f}_2 = (\tilde{h}_2, \tilde{g}_2)$ be three HPFEs, $\lambda > 0$. The hesitant Pythagorean fuzzy interaction operations can be defined as

(1)

$$\tilde{f}_1 \oplus \tilde{f}_2 = z \bigcup_{\mu_{1k_1} \in \tilde{h}_1, \nu_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, \nu_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left(\sqrt{\mu_{1k_1}^2 + \mu_{2k_2}^2 - \mu_{1k_1}^2 \mu_{2k_2}^2}, \right. \right.$$
$$\times \left. \left. \sqrt{\nu_{1k_1}^2 + \nu_{2k_2}^2 - \nu_{1k_1}^2 \nu_{2k_2}^2 - \mu_{1k_1}^2 \nu_{2k_2}^2 - \nu_{1k_1}^2 \mu_{2k_2}^2} \right) \right\},$$

(2)

$$\tilde{f}_1 \otimes \tilde{f}_2 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, \nu_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, \nu_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left((\mu_{1k_1}^2 + \mu_{2k_2}^2 - \mu_{1k_1}^2 \mu_{2k_2}^2 \right. \right.$$
$$\left. -\nu_{1k_1}^2 \mu_{2k_2}^2 - \mu_{1k_1}^2 \nu_{2k_2}^2)^{1/2}, \sqrt{\nu_{1k_1}^2 + \nu_{2k_2}^2 - \nu_{1k_1}^2 \nu_{2k_2}^2} \right) \right\},$$

(3)

$$\lambda \tilde{f} = \bigcup_{\mu_k \in \tilde{h}, \nu_k \in \tilde{g}} \left\{ \left(\sqrt{1 - (1 - \mu_k^2)^\lambda}, \right. \right.$$
$$\times \left. \left. \sqrt{(1 - \mu_k^2)^\lambda - (1 - (\mu_k^2 + \nu_k^2))^\lambda} \right) \right\},$$

(4)

$$\tilde{f}^\lambda = \bigcup_{\mu_k \in \tilde{h}, \nu_k \in \tilde{g}} \left\{ \left(\sqrt{(1 - \nu_k^2)^\lambda - (1 - (\mu_k^2 + \nu_k^2))^\lambda}, \right. \right.$$
$$\times \left. \left. \sqrt{1 - (1 - \nu_k^2)^\lambda} \right) \right\}.$$

The $\tilde{f}_1 \oplus \tilde{f}_2$ and $\tilde{f}_1 \otimes \tilde{f}_2$ can be rewritten as follows

$$\tilde{f}_1 \oplus \tilde{f}_2 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, \nu_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, \nu_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left(\sqrt{1 - (1 - \mu_{1k_1}^2)(1 - \mu_{2k_2}^2)}, \right. \right.$$
$$\times \left. \left. \sqrt{(1 - \mu_{1k_1}^2)(1 - \mu_{2k_2}^2) - (1 - (\mu_{1k_1}^2 + \nu_{1k_1}^2))(1 - (\mu_{2k_2}^2 + \nu_{2k_2}^2))} \right) \right\},$$

$$\tilde{f}_1 \otimes \tilde{f}_2 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, \nu_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, \nu_{2k_2} \in \tilde{g}_2}$$

$$\times \left\{ \left(\left((1 - v_{1k_1}^2)(1 - v_{2k_2}^2) - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))(1 - (\mu_{2k_2}^2 + v_{2k_2}^2)) \right)^{1/2}, \right. \right.$$
$$\left. \left. \times \sqrt{1 - (1 - v_{1k_1}^2)(1 - v_{2k_2}^2)} \right) \right\}.$$

The results of above operations are still HPFEs. The proofs of (1) and (3) are given as follows and others can be proved similarly.

Proof

$$\tilde{f}_1 \oplus \tilde{f}_2 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left(\sqrt{\mu_{1k_1}^2 + \mu_{2k_2}^2 - \mu_{1k_1}^2 \mu_{2k_2}^2}, \right. \right.$$
$$\left. \left. \times \sqrt{v_{1k_1}^2 + v_{2k_2}^2 - v_{1k_1}^2 v_{2k_2}^2 - \mu_{1k_1}^2 v_{2k_2}^2 - v_{1k_1}^2 \mu_{2k_2}^2} \right) \right\},$$

$$\left(\sqrt{\mu_{1k_1}^2 + \mu_{2k_2}^2 - \mu_{1k_1}^2 \mu_{2k_2}^2} \right)^2$$
$$+ ((v_{1k_1}^2 + v_{2k_2}^2 - v_{1k_1}^2 v_{2k_2}^2 - \mu_{1k_1}^2 v_{2k_2}^2 v_{1k_1}^2 \mu_{2k_2}^2)^{1/2})^2$$
$$= \mu_{1k_1}^2 + \mu_{2k_2}^2 - \mu_{1k_1}^2 \mu_{2k_2}^2 + v_{1k_1}^2 + v_{2k_2}^2$$
$$- v_{1k_1}^2 v_{2k_2}^2 - \mu_{1k_1}^2 v_{2k_2}^2 - v_{1k_1}^2 \mu_{2k_2}^2$$
$$= 1 - (1 - \mu_{1k_1}^2)(1 - \mu_{2k_2}^2)$$
$$+ (1 - \mu_{1k_1}^2)(1 - \mu_{2k_2}^2) - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2)) *$$
$$\times (1 - (\mu_{2k_2}^2 + v_{2k_2}^2))$$
$$= 1 - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))(1 - (\mu_{2k_2}^2 + v_{2k_2}^2)).$$

Since $0 \le \mu_{1k_1}^2 + v_{1k_1}^2 \le 1$, $0 \le \mu_{2k_2}^2 + v_{2k_2}^2 \le 1$, then $0 \le (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))(1 - (\mu_{2k_2}^2 + v_{2k_2}^2)) \le 1$ and $0 \le 1 - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))(1 - (\mu_{2k_2}^2 + v_{2k_2}^2)) \le 1$. Hence, $\tilde{f}_1 \oplus \tilde{f}_2$ is still an HPFE.

(3)

$$\lambda \tilde{f} = \bigcup_{\mu_k \in \tilde{h}, v_k \in \tilde{g}}$$
$$\times \left\{ \left(\sqrt{1 - (1 - \mu_k^2)^\lambda}, \right. \right.$$
$$\left. \left. \times \sqrt{(1 - \mu_k^2)^\lambda - (1 - (\mu_k^2 + v_k^2))^\lambda} \right) \right\},$$

$$\left(\sqrt{1 - (1 - \mu_k^2)^\lambda} \right)^2$$
$$+ \left(\sqrt{(1 - \mu_k^2)^\lambda - (1 - (\mu_k^2 + v_k^2))^\lambda} \right)^2$$
$$= 1 - (1 - \mu_k^2)^\lambda + (1 - \mu_k^2)^\lambda - (1 - (\mu_k^2 + v_k^2))^\lambda$$
$$= 1 - (1 - (\mu_k^2 + v_k^2))^\lambda.$$

Since $0 \le \mu_k^2 + v_k^2 \le 1$, $0 \le (1 - (\mu_k^2 + v_k^2))^\lambda \le 1$, $0 \le 1 - (1 - (\mu_k^2 + v_k^2))^\lambda \le 1$. Then, the $\lambda \tilde{f}$ is still an HPFE.

Theorem 1 *Let $\tilde{f} = (\tilde{h}, \tilde{g})$, $\tilde{f}_1 = (\tilde{h}_1, \tilde{g}_1)$ and $\tilde{f}_2 = (\tilde{h}_2, \tilde{g}_2)$ be three HPFEs, then we have*

(1) $\tilde{f}_1 \oplus \tilde{f}_2 = \tilde{f}_2 \oplus \tilde{f}_1$,
(2) $\tilde{f}_1 \otimes \tilde{f}_2 = \tilde{f}_2 \otimes \tilde{f}_1$,
(3) $k(\tilde{f}_1 \oplus \tilde{f}_2) = k\tilde{f}_1 \oplus k\tilde{f}_2$,
(4) $k(\tilde{f}_1 \otimes \tilde{f}_2) = k\tilde{f}_1 \otimes k\tilde{f}_2$,
(5) $(k_1 + k_2)\tilde{f} = k_1\tilde{f} \oplus k_2\tilde{f}$,
(6) $\tilde{f}^{k_1} \otimes \tilde{f}^{k_2} = \tilde{f}^{(k_1+k_2)}$.

Proof We only prove (1), (3) and (5), and others can be proved similarly.

(1)

$$\tilde{f}_1 \oplus \tilde{f}_2 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left(\sqrt{\mu_{1k_1}^2 + \mu_{2k_2}^2 - \mu_{1k_1}^2 \mu_{2k_2}^2}, (v_{1k_1}^2 \right. \right.$$
$$\left. \left. + v_{2k_2}^2 - v_{1k_1}^2 v_{2k_2}^2 - \mu_{1k_1}^2 v_{2k_2}^2 - v_{1k_1}^2 \mu_{2k_2}^2)^{1/2} \right) \right\}$$
$$= \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left(\sqrt{\mu_{2k_2}^2 + \mu_{1k_1}^2 - \mu_{2k_2}^2 \mu_{1k_1}^2}, (v_{2k_2}^2 \right. \right.$$
$$\left. \left. + v_{1k_1}^2 - v_{2k_2}^2 v_{1k_1}^2 - \mu_{2k_2}^2 v_{1k_1}^2 - v_{2k_2}^2 \mu_{1k_1}^2)^{1/2} \right) \right\}$$
$$= \tilde{f}_2 \oplus \tilde{f}_1.$$

(3)

$$\tilde{f}_1 \oplus \tilde{f}_2 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left(\sqrt{1 - (1 - \mu_{1k_1}^2)(1 - \mu_{2k_2}^2)}, \right. \right.$$
$$\left. \left. \times \sqrt{(1 - \mu_{1k_1}^2)(1 - \mu_{2k_2}^2) - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))(1 - (\mu_{2k_2}^2 + v_{2k_2}^2))} \right) \right\}$$
$$= \{(\mu_{\tilde{f}_1 \oplus \tilde{f}_2}, v_{\tilde{f}_1 \oplus \tilde{f}_2})\}.$$
$$k(\tilde{f}_1 \oplus \tilde{f}_2) = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left(\sqrt{1 - (1 - \mu_{1k_1}^2)^k (1 - \mu_{2k_2}^2)^k}, \right. \right.$$
$$\left. \left. \times \sqrt{(1 - \mu_{1k_1}^2)^k (1 - \mu_{2k_2}^2)^k - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))^k (1 - (\mu_{2k_2}^2 + v_{2k_2}^2))^k} \right) \right\},$$
$$k\tilde{f}_1 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1}$$
$$\times \left\{ \left(\sqrt{1 - (1 - \mu_{1k_1}^2)^k}, \sqrt{(1 - \mu_{1k_1}^2)^k - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))^k} \right) \right\}$$
$$= \bigcup \left\{ (\mu_{k\tilde{h}_1}, v_{k\tilde{h}_1}) \right\},$$
$$k\tilde{f}_2 = \bigcup_{\mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left(\sqrt{1 - (1 - \mu_{2k_2}^2)^k}, \right. \right.$$

$$\times \sqrt{(1 - \mu_{2k_2}^2)^k - (1 - (\mu_{2k_2}^2 + v_{2k_2}^2))^k}\Big)\Big\}$$

$$= \bigcup \Big\{\big(\mu_{k\tilde{h}_2}, v_{k\tilde{h}_2}\big)\Big\},$$

$$k\tilde{f}_1 \oplus k\tilde{f}_2 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2}$$

$$\times \Big\{\Big(\sqrt{1 - (1 - \mu_{1k_1}^2)^k (1 - \mu_{2k_2}^2)^k}, ((1 - \mu_{1k_1}^2)^k (1 - \mu_{2k_2}^2)^k - (1 - (\mu_{1k_1}^2$$

$$+ v_{1k_1}^2))^k (1 - (\mu_{2k_2}^2 + v_{2k_2}^2))^k)^{1/2}\Big)\Big\}$$

$$= k(\tilde{f}_1 \oplus \tilde{f}_2).$$

(5)

$$(k_1 + k_2)\tilde{f} = \bigcup_{\mu_k \in \tilde{h}, v_k \in \tilde{g}}$$

$$\times \Big\{\Big(\sqrt{1 - (1 - \mu_k^2)^{k_1 + k_2}}, ((1 - \mu_k^2)^{k_1 + k_2}$$

$$- (1 - (\mu_k^2 + v_k^2))^{k_1 + k_2})^{1/2}\Big)\Big\},$$

$$k_1 \tilde{f} = \bigcup_{\mu_k \in \tilde{h}, v_k \in \tilde{g}} \Big\{\Big(\sqrt{1 - (1 - \mu_k^2)^{k_1}},$$

$$\times \sqrt{(1 - \mu_k^2)^{k_1} - (1 - (\mu_k^2 + v_k^2))^{k_1}}\Big)\Big\},$$

$$k_2 \tilde{f} = \bigcup_{\mu_k \in \tilde{h}, v_k \in \tilde{g}} \Big\{\Big(\sqrt{1 - (1 - \mu_k^2)^{k_2}},$$

$$\times \sqrt{(1 - \mu_k^2)^{k_2} - (1 - (\mu_k^2 + v_k^2))^{k_2}}\Big)\Big\},$$

$$k_1 \tilde{f} \oplus k_2 \tilde{f} = \bigcup_{\mu_k \in \tilde{h}, v_k \in \tilde{g}} \Big\{\Big(\sqrt{1 - (1 - \mu_k)^{k_1}(1 - \mu_k)^{k_2}},$$

$$\times ((1 - \mu_k^2)^{k_1}(1 - \mu_k^2)^{k_2}$$

$$- (1 - (\mu_k^2 + v_k^2))^{k_1}(1 - (\mu_k^2 + v_k^2))^{k_2})^{1/2}\Big)\Big\}$$

$$= \bigcup_{\mu_k \in \tilde{h}, v_k \in \tilde{g}} \Big\{\Big(\sqrt{1 - (1 - \mu_k)^{k_1 + k_2}},$$

$$\times ((1 - \mu_k^2)^{k_1 + k_2} - (1 - (\mu_k^2 + v_k^2))^{k_1 + k_2})^{1/2}\Big)\Big\}$$

$$= (k_1 + k_2)\tilde{f}.$$

Definition 7 Let $\tilde{f} = (\tilde{h}, \tilde{g})$ be an HPFE. The score of $\tilde{f}$ can be defined as

$$S(\tilde{f}) = \frac{1}{l_{\tilde{h}}} \sum_{\mu_k \in \tilde{h}} \mu_k^2 - \frac{1}{l_{\tilde{g}}} \sum_{v_k \in \tilde{g}} v_k^2. \tag{4}$$

The accuracy function can be defined as

$$A(\tilde{f}) = \frac{1}{l_{\tilde{h}}} \sum_{\mu_k \in \tilde{h}} \mu_k^2 + \frac{1}{l_{\tilde{g}}} \sum_{v_k \in \tilde{g}} v_k^2, \tag{5}$$

where $l_{\tilde{h}}$ is the number of memberships in $\tilde{h}$ and $l_{\tilde{g}}$ is the number of non-memberships in $\tilde{g}$.

Definition 8 Let $\tilde{f}_1$, $\tilde{f}_2$ be two HPFEs. Then if

(1) If $S(\tilde{f}_1) > S(\tilde{f}_2)$, then $\tilde{f}_1 > \tilde{f}_2$,
(2) If $S(\tilde{f}_1) = S(\tilde{f}_2)$, then
    If $A(\tilde{f}_1) > A(\tilde{f}_2)$, then $\tilde{f}_1 > \tilde{f}_2$,
    If $A(\tilde{f}_1) = A(\tilde{f}_2)$, then $\tilde{f}_1 \sim \tilde{f}_2$.

To define distance measure between HPFEs more accurately, the two HPFEs should have the same number of memberships and non-memberships. The HPFEs can be extended according to the risk attitude of decision-makers. If the decision-maker is risk seeking, the largest Pythagorean fuzzy value can be added; if decision-maker is risk averse, the smallest Pythagorean fuzzy value can be added; and if decision- maker is risk neutral, the average value of Pythagorean fuzzy values can be added.

Definition 9 The distance between two extended hesitant Pythagorean fuzzy values $\tilde{f}_1 = (\tilde{h}_1, \tilde{g}_1)$ and $\tilde{f}_2 = (\tilde{h}_2, \tilde{g}_2)$ can be defined as

$$d(\tilde{f}_1, \tilde{f}_2) = \frac{1}{l_{\tilde{h}_1}} \sum (|\mu_{1k_1}^2 - \mu_{2k_2}^2| + |v_{1k_1}^2 - v_{2k_2}^2|)/2, \tag{6}$$

where $\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2$ and $l_{\tilde{h}_1}$ is the number of memberships in the $\tilde{f}_1$.

Hesitant Pythagorean fuzzy interaction aggregation operators

Definition 10 Let $\tilde{f}_i$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. The hesitant Pythagorean fuzzy interaction weighted averaging (HPFIWA) operator can be defined as

$$\text{HPFIWA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \oplus_{j=1}^n w_j \tilde{f}_j. \tag{7}$$

Theorem 2 Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i) = \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i} \{(\mu_{ik_i}, v_{ik_i})\}$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. The aggregated value of the HPFIWA operator is still an HPFE, that is

$$\text{HPFIWA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i} \Big\{\Big(\sqrt{1 - \prod_{i=1}^n (1 - \mu_{ik_i}^2)^{w_i}},$$

$$\times \Big(\prod_{i=1}^n (1 - \mu_{ik_i}^2)^{w_i}$$

$$- \prod_{i=1}^n (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i}\Big)^{1/2}\Big)\Big\}. \tag{8}$$

Proof The theorem can be proved using mathematical induction.

If $n = 2$, $\text{HPFIWA}(\tilde{f}_1, \tilde{f}_2) = w_1 \tilde{f}_1 \oplus w_2 \tilde{f}_2$.

$$w_1 \tilde{f}_1 = \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1} \left\{ \left(\sqrt{1 - (1 - \mu_{1k_1}^2)^{w_1}}, ((1 - \mu_{1k_1}^2)^{w_1} \right. \right.$$
$$\left. \left. - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))^{w_1})^{1/2} \right) \right\},$$

$$w_2 \tilde{f}_2 = \bigcup_{\mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2} \left\{ \left(\sqrt{1 - (1 - \mu_{2k_2}^2)^{w_2}}, ((1 - \mu_{2k_2}^2)^{w_2} \right. \right.$$
$$\left. \left. - (1 - (\mu_{2k_2}^2 + v_{2k_2}^2))^{w_2})^{1/2} \right) \right\},$$

$$w_1 \tilde{f}_1 \oplus w_2 \tilde{f}_2$$
$$= \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1, \mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2}$$
$$\times \left\{ \left((1 - (1 - \mu_{1k_1}^2)^{w_1} (1 - \mu_{2k_2}^2)^{w_2})^{1/2}, \right. \right.$$
$$\times ((1 - \mu_{1k_1}^2)^{w_1} (1 - \mu_{2k_2}^2)^{w_2} - (1 - (\mu_{1k_1}^2 + v_{1k}^2))^{w_1}$$
$$\left. \left. (1 - (\mu_{2k_2}^2 + v_{2k_2}^2))^{w_2})^{1/2} \right) \right\}$$
$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i} \left\{ \left(\left(1 - \prod_{i=1}^{2} (1 - \mu_{ik_i}^2)^{w_i} \right)^{1/2}, \right. \right.$$
$$\times \left(\prod_{i=1}^{2} (1 - \mu_{ik_i}^2)^{w_i} \right.$$
$$\left. \left. - \prod_{i=1}^{2} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i})^{1/2} \right) \right\}.$$

Suppose Eq. (8) holds for $n = l$, that is

$$\text{HPFIWA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_l) = \oplus_{i=1}^{l} w_i \tilde{f}_i$$
$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i} \left\{ \left(\left(1 - \prod_{i=1}^{l} (1 - \mu_{ik_i}^2)^{w_i} \right)^{1/2}, \right. \right.$$
$$\times \left(\prod_{i=1}^{l} (1 - \mu_{ik_i}^2)^{w_i} \right.$$
$$\left. \left. - \prod_{i=1}^{l} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i} \right)^{1/2} \right) \right\}.$$

If $n = l+1$, using the interaction operation laws of hesitant Pythagorean fuzzy value, we can get
$\text{HPFIWA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_l, \tilde{f}_{l+1}) = (\oplus_{j=1}^{l} w_j \tilde{f}_j) \oplus (w_{l+1} \tilde{f}_{l+1})$,

$$w_{l+1} \tilde{f}_{l+1} = \bigcup_{\mu_{l+1k_{l+1}} \in \tilde{h}_{l+1}, v_{l+1k_{l+1}} \in \tilde{g}_{l+1}}$$
$$\times \left((1 - (1 - \mu_{l+1k_{l+1}}^2)^{w_{l+1}})^{1/2}, ((1 - \mu_{l+1k_{l+1}}^2)^{w_{l+1}} \right.$$
$$\left. - (1 - (\mu_{l+1k_{l+1}}^2 + v_{l+1k_{l+1}}^2))^{w_{l+1}})^{1/2} \right),$$

$$\text{HPFIWA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_l, \tilde{f}_{l+1})$$
$$= (\oplus_{j=1}^{l} w_j \tilde{f}_j) \oplus (w_{l+1} \tilde{f}_{l+1})$$
$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}$$
$$\times \left\{ \left(\sqrt{1 - \prod_{i=1}^{l} (1 - \mu_{ik_i}^2)^{w_i}}, \left(\prod_{i=1}^{l} (1 - \mu_{ik_i}^2)^{w_i} \right. \right. \right.$$
$$\left. \left. \left. - \prod_{i=1}^{l} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i})^{1/2} \right) \right\} \oplus$$
$$\bigcup_{\mu_{l+1k_{k+1}} \in \tilde{h}_{l+1}, v_{l+1k_{l+1}} \in \tilde{g}_{l+1}}$$
$$\times \left\{ \left(\sqrt{1 - (1 - \mu_{l+1k_{l+1}}^2)^{w_{l+1}}}, ((1 - \mu_{l+1k_{l+1}}^2)^{w_{l+1}} \right. \right.$$
$$\left. \left. - (1 - (\mu_{l+1k_{l+1}}^2 + v_{l+1k_{l+1}}^2))^{w_{l+1}})^{1/2} \right) \right\}$$
$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}$$
$$\times \left\{ \left(\sqrt{1 - \prod_{i=1}^{l} (1 - \mu_{ik_i}^2)^{w_i} (1 - \mu_{l+1k_{l+1}}^2)^{w_{l+1}}}, \right. \right.$$
$$\times \left(\prod_{i=1}^{l} (1 - \mu_{ik_i}^2)^{w_i} (1 - \mu_{l+1k_{l+1}}^2)^{w_{l+1}} \right.$$
$$- \prod_{i=1}^{l} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i} *$$
$$\left. \left. \times (1 - (\mu_{l+1k_{l+1}}^2 + v_{l+1k_{l+1}}^2))^{w_{l+1}})^{1/2} \right) \right\}$$
$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}$$
$$\times \left\{ \left(\sqrt{1 - \prod_{i=1}^{l+1} (1 - \mu_{ik_i}^2)^{w_i}}, \left(\prod_{i=1}^{l+1} (1 - \mu_{ik_i}^2)^{w_i} \right. \right. \right.$$
$$\left. \left. \left. - \prod_{i=1}^{l+1} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i} \right)^{1/2} \right) \right\}.$$

By mathematical induction, Eq. (8) holds for all n. Moreover, for each (μ, v) in the HPFIWA operator, we have $\mu^2 + v^2 = 1 - \prod_{i=1}^{n} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i}$. Since $0 \leq \mu_{ik_i}^2 + v_{ik_i}^2 \leq 1$, we have $0 \leq \mu^2 + v^2 \leq 1$. Then, the aggregated result of the HPFIWA operator is still an HPFE.

If the weight vector is taken as $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)$, the HPFIWA operator reduces to the hesitant Pythagorean fuzzy interaction averaging (HPFIA) operator as follows

$$\text{HPFIA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \frac{1}{n} \oplus_{j=1}^{n} \tilde{f}_j$$
$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i} \left\{ \left(\sqrt{1 - \prod_{i=1}^{n} (1 - \mu_{ik_i}^2)^{\frac{1}{n}}}, \right. \right.$$
$$\left. \left. \times \left(\prod_{i=1}^{n} (1 - \mu_{ik_i}^2)^{\frac{1}{n}} - \prod_{i=1}^{n} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\frac{1}{n}} \right)^{1/2} \right) \right\}.$$

Theorem 3 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. If all the HPFEs reduces to $\tilde{f} = (\tilde{h}, \tilde{g})$, the HPFIWA operator reduces to the following form*

$$\text{HPFIWA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \tilde{f}.$$

Theorem 4 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Let $\tilde{f}^+ = (1, 0)$, $\tilde{f}^- = (0, 1)$, then*

$$\tilde{f}^- \leq \text{HPFIWA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) \leq \tilde{f}^+.$$

Example 1 Let $\tilde{f}_1 = \{(0.9, 0.0)\}$, $\tilde{f}_2 = \{(0.6, 0.5)\}$, $\tilde{f}_3 = \{(0.8, 0.3), (0.7, 0.4)\}$ and $W = (0.25, 0.35, 0.40)$. Using the HPFIWA operator, we can get HPFIWA$(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = \{(0.7904, 0.5649), (0.7541, 0.5969)\}$.

Definition 11 Let $\tilde{f}_i$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. The hesitant Pythagorean fuzzy interaction weighted geometric averaging (HPFIWGA) operator can be defined as

$$\text{HPFIWGA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \otimes_{j=1}^n \tilde{f}_j^{w_j}. \tag{9}$$

Theorem 5 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Then, the aggregated result of the HPFIWGA operator is still an HPFE, which has the following form*

$$
\begin{aligned}
&\text{HPFIWGA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)\\
&= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}\\
&\quad \times \left\{ \left(\sqrt{\prod_{i=1}^n (1 - v_{ik_i}^2)^{w_i} - \prod_{i=1}^n (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i}}, \right.\right.\\
&\quad \left.\left. \times \sqrt{1 - \prod_{i=1}^n (1 - v_{ik_i}^2)^{w_i}} \right) \right\}.
\end{aligned}
\tag{10}
$$

Proof If $n = 2$, HPFIWGA$(\tilde{f}_1, \tilde{f}_2) = \tilde{f}_1^{w_1} \otimes \tilde{f}_2^{w_2}$.

$$
\begin{aligned}
\tilde{f}_1^{w_1} &= \bigcup_{\mu_{1k_1} \in \tilde{h}_1, v_{1k_1} \in \tilde{g}_1}\\
&\quad \times \left\{ \left(\sqrt{(1 - v_{1k_1}^2)^{w_1} - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))^{w_1}}, \right.\right.\\
&\quad \left.\left. \times \sqrt{1 - (1 - v_{1k_1}^2)^{w_1}} \right) \right\},\\
\tilde{f}_2^{w_2} &= \bigcup_{\mu_{2k_2} \in \tilde{h}_2, v_{2k_2} \in \tilde{g}_2}\\
&\quad \times \left\{ \left(\sqrt{(1 - v_{2k_2}^2)^{w_2} - (1 - (\mu_{2k_2}^2 + v_{2k_2}^2))^{w_2}}, \right.\right.\\
&\quad \left.\left. \times \sqrt{1 - (1 - v_{2k_2}^2)^{w_2}} \right) \right\}.
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{f}_1^{w_1} \otimes \tilde{f}_2^{w_2}\\
&= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}\\
&\quad \times \left\{ \left(((1 - v_{1k_1}^2)^{w_1}(1 - v_{2k_2}^2)^{w_2} - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))^{w_1} \right.\right.\\
&\quad \times (1 - (\mu_{2k_2}^2 + v_{2k_2}^2))^{w_2})^{1/2},\\
&\quad \left.\left. \times \sqrt{1 - (1 - v_{1k_1}^2)^{w_1}(1 - v_{2k_2}^2)^{w_2}} \right) \right\}\\
&= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}\\
&\quad \times \left\{ \left(\sqrt{\prod_{i=1}^2 (1 - v_{ik_i}^2)^{w_i} - \prod_{i=1}^2 (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i}}, \right.\right.\\
&\quad \left.\left. \times \sqrt{1 - \prod_{i=1}^2 (1 - v_{ik_i}^2)^{w_i}} \right) \right\}.
\end{aligned}
$$

Equation (10) holds for $n = 2$. If Eq. (10) is established for $n = l$, i.e.

$$
\begin{aligned}
\otimes_{i=1}^l \tilde{f}_i^{w_i} &= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}\\
&\quad \times \left\{ \left(\left(\prod_{i=1}^l (1 - v_{ik_i}^2)^{w_i} - \prod_{i=1}^l (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i} \right)^{1/2}, \right.\right.\\
&\quad \left.\left. \times \sqrt{1 - \prod_{i=1}^l (1 - v_{ik_i}^2)^{w_i}} \right) \right\}.
\end{aligned}
$$

Then for $n = l + 1$, $\otimes_{i=1}^{l+1} \tilde{f}_i^{w_i} = \left(\otimes_{i=1}^l \tilde{f}_i^{w_i} \right) \otimes \left(\tilde{f}_{l+1}^{w_{l+1}} \right)$. By Eq.(10), we have

$$
\begin{aligned}
\tilde{f}_{l+1}^{w_{l+1}} &= \bigcup_{\mu_{l+1k_{l+1}} \in \tilde{h}_{l+1}, v_{l+1l_{l+1}} \in \tilde{g}_{l+1}}\\
&\quad \times \left\{ \left(((1 - v_{l+1k_{l+1}}^2)^{w_{l+1}} \right.\right.\\
&\quad - (1 - (\mu_{l+1k_{l+1}}^2 + v_{l+1k_{l+1}}^2))^{w_{l+1}})^{1/2},\\
&\quad \left.\left. \times \sqrt{1 - (1 - v_{l+1k_{l+1}}^2)^{w_{l+1}}} \right) \right\}.
\end{aligned}
$$

$$
\begin{aligned}
&\otimes_{i=1}^{l+1} \tilde{f}_i^{w_i} = \left(\otimes_{i=1}^l \tilde{f}_i^{w_i} \right) \otimes \left(\tilde{f}_{l+1}^{w_{l+1}} \right)\\
&= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}\\
&\quad \times \left\{ \left(\left(1 - \left(1 - \prod_{i=1}^l (1 - v_{ik_i}^2)^{w_i} \right) \right) \right.\right.\\
&\quad \times (1 - (1 - (1 - v_{l+1k_{l+1}}^2)^{w_{l+1}k_{l+1}}))\\
&\quad - \left(1 - \left(1 - \prod_{i=1}^l \left(1 - \left(\mu_{ik_i}^2 + v_{ik_i}^2 \right) \right)^{w_i} \right) \right)\\
&\quad \left.\left. \times \left(1 - \left(1 - (1 - (\mu_{l+1k_{l+1}}^2 + v_{l+1k_{l+1}}^2))^{w_{l+1}} \right) \right) \right)^{1/2}, \right.
\end{aligned}
$$

$$\times \left(1 - \left(1 - \left(1 - \prod_{i=1}^{l}(1 - v_{ik_i}^2)^{w_i}\right)\right) *\right.$$

$$\left.\times \left(1 - \left(1 - (1 - v_{l+1k_{l+1}}^2)^{w_{k+1}}\right)\right)\right)^{1/2}\right)\right\}$$

$$= \bigcup_{\mu_{ik_i}\in\tilde{h}_i, v_{ik_i}\in\tilde{g}_i} \left\{\left(\left(\left(\prod_{i=1}^{l}(1 - v_{ik_i}^2)^{w_i}(1 - v_{l+1k_{l+1}}^2)^{w_{l+1}}\right.\right.\right.\right.$$

$$-\prod_{i=1}^{l}(1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i}$$

$$\left.(1 - (\mu_{l+1k_{l+1}}^2 + v_{l+1k_{l+1}}^2))^{w_{l+1}}\right)^{1/2},$$

$$\left.\times \left(1 - \prod_{i=1}^{k}(1 - v_i^2)^{w_i} * (1 - v_{k+1}^2)^{w_{k+1}}\right)^{1/2}\right)\right\}$$

$$= \bigcup_{\mu_{ik_i}\in\tilde{h}_i, v_{ik_i}\in\tilde{g}_i} \left\{\left(\left(\left(\prod_{i=1}^{l+1}(1 - v_{ik_i}^2)^{w_i}\right.\right.\right.\right.$$

$$\left.-\prod_{i=1}^{l+1}(1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i}\right)^{1/2},$$

$$\left.\times \sqrt{1 - \prod_{i=1}^{k+1}(1 - v_{ik_i}^2)^{w_i}}\right)\right\}.$$

Then, Eq. (10) holds for $n = k + 1$. Therefore, using mathematical induction on n, Eq. (10) holds for all n. Moreover, for each (μ, v) in the HPFIWGA operator, $\mu^2 + v^2 = 1 - \prod_{i=1}^{l+1}(1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i}$. Since $0 \leq \mu_{ik_i}^2 + v_{ik_i}^2 \leq 1$, hence $0 \leq \mu^2 + v^2 \leq 1$. The aggregated result of the HPFIWGA operator is still an HPFE. □

Theorem 6 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. If all the HPFEs reduces to $\tilde{f} = (\tilde{h}, \tilde{g})$, the HPFIWGA operator reduces to the following form*

$$HPFIWGA(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \tilde{f}.$$

Theorem 7 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Let $\tilde{f}^+ = (1, 0)$, $\tilde{f}^- = (0, 1)$, then*

$$\tilde{f}^- \leq HPFIWGA(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) \leq \tilde{f}^+.$$

If the weight vector is taken as $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, the HPFIWGA operator reduces to the hesitant Pythagorean fuzzy interaction geometric averaging (HPFIGA) operator as follows

$$HPFIGA(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \otimes_{j=1}^{n} \tilde{f}_j^{\frac{1}{n}}$$

$$= \bigcup_{\mu_{ik_i}\in\tilde{h}_i, v_{ik_i}\in\tilde{g}_i}$$

$$\times \left\{\left(\sqrt{\prod_{i=1}^{n}(1 - v_{ik_i}^2)^{\frac{1}{n}} - \prod_{i=1}^{n}(1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\frac{1}{n}}},\right.\right.$$

$$\left.\left.\times \sqrt{1 - \prod_{i=1}^{n}(1 - v_{ik_i}^2)^{\frac{1}{n}}}\right)\right\}.$$

Example 2 Suppose the Pythagorean fuzzy values are the same as that in Example 1. Using the HPFIWGA operator, we can get HPFIWGA($\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$) = {(0.7678, 0.3595), (0.7289, 0.3959)}.

Definition 12 Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. The generalized hesitant Pythagorean fuzzy interaction weighted averaging (GHPFIWA) operator can be defined as

$$GHPFIWA(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \left(\oplus_{j=1}^{n} w_j \tilde{f}_j^{\lambda}\right)^{1/\lambda}. \quad (11)$$

Theorem 8 *Let $\tilde{f}_i$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Then*

$$GHPFIWA_\lambda(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \bigcup_{\mu_{ik_i}\in\tilde{h}_i, v_{ik_i}\in\tilde{g}_i}$$

$$\times \left\{\left(\left(1 - \left(\prod_{i=1}^{n}(1 - ((1 - v_{ik_i}^2)^{\lambda}\right.\right.\right.\right.$$

$$-(1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\lambda})^{w_i} - \prod_{i=1}^{n}(1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\lambda w_i}\right)^{1/\lambda}$$

$$-\left(\prod_{i=1}^{n}(1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i}\right)^{1/2}, \left(1 - (1 - \left(\prod_{i=1}^{n}(1-\right.\right.$$

$$\times((1 - v_{ik_{ik_i}}^2)^{\lambda} - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\lambda})^{w_i}$$

$$\left.\left.\left.-\prod_{i=1}^{n}(1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\lambda w_j})^{1/\lambda}\right)^{1/2}\right)\right\}. \quad (12)$$

Proof If $n = 2$, $GHPFIWA_\lambda(\tilde{f}_1, \tilde{f}_2) = \left(w_1\tilde{f}_1^\lambda \oplus w_2\tilde{f}_2^\lambda\right)^{1/\lambda}$,

$$\tilde{f}_1^{w_1} = \bigcup_{\mu_{1k_1}\in\tilde{h}_1, v_{1k_1}\in\tilde{g}_1}$$

$$\times \left\{\left(\sqrt{(1 - v_{1k_1}^2)^{w_1} - (1 - (\mu_{1k_1}^2 + v_{1k_1}^2))^{w_1}},\right.\right.$$

$$\left.\left.\times \sqrt{1 - (1 - v_{1k_1}^2)^{w_1}}\right)\right\},$$

$$\tilde{f}_2^{w_2} = \bigcup_{\mu_{2k_2}\in\tilde{h}_2, v_{2k_2}\in\tilde{g}_2}$$

$$\times \left\{\left(\sqrt{(1 - v_{2k_2}^2)^{w_2} - (1 - (\mu_{2k_2}^2 + v_{2k_2}^2))^{w_2}},\right.\right.$$

$$\left.\left.\times \sqrt{1 - (1 - v_{2k_2}^2)^{w_2}}\right)\right\}.$$

$$\tilde{f}_1^{w_1} \otimes \tilde{f}_2^{w_2}$$

$$= \bigcup_{\mu_{ik_i}\in\tilde{h}_i, v_{ik_i}\in\tilde{g}_i}$$

$$\times \left\{\left(((1 - v_{1k_1}^2)^{w_1}(1 - v_{2k_2}^2)^{w_2} - (1 - (\mu_{1k_1}^2\right.\right.$$

$$+v_{1k_1}^2))^{w_1}(1-(\mu_{2k_2}^2+v_{2k_2}^2))^{w_2})^{1/2},$$

$$\times\sqrt{1-(1-v_{1k_1}^2)^{w_1}(1-v_{2k_2}^2)^{w_2}}\Bigg)\Bigg\}$$

$$=\bigcup_{\mu_{ik_i}\in\tilde{h}_i,v_{ik_i}\in\tilde{g}_i}$$

$$\times\Bigg\{\Bigg(\sqrt{\prod_{i=1}^2(1-v_{ik_i}^2)^{w_i}-\prod_{i=1}^2(1-(\mu_{ik_i}^2+v_{ik_i}^2))^{w_i}},$$

$$\times\sqrt{1-\prod_{i=1}^2(1-v_{ik_i}^2)^{w_i}}\Bigg)\Bigg\}.$$

Hence, Eq. (12) holds for $n=2$. If Eq. (12) is established for $n=l$, i.e.

$$\left(\oplus_{i=1}^l w_i\tilde{f}_i^\lambda\right)^{1/\lambda}=\bigcup_{\mu_{ik_i}\in\tilde{h}_i,v_{ik_i}\in\tilde{g}_i}$$

$$\times\Bigg\{\Bigg(\Bigg(1-\Bigg(\prod_{i=1}^l\Big(1-((1-v_{ik_i}^2)^\lambda$$

$$-(1-(\mu_{ik_i}^2+v_{ik_i}^2))^\lambda)^{w_i}$$

$$-\prod_{i=1}^l(1-(\mu_{ik_i}^2+v_{ik_i}^2))^{\lambda w_i}\Bigg)\Bigg)^{1/\lambda}$$

$$-\prod_{i=1}^l\Big(1-(\mu_{ik_i}^2+v_{ik_i}^2))^{w_i}\Big)^{1/2},$$

$$\times\Bigg(1-\Bigg(1-\Bigg(\prod_{i=1}^l(1-((1-v_{ik_i}^2)^\lambda-(1-(\mu_{ik_i}^2+v_{ik_i}^2)))^\lambda\Bigg)^{w_i}$$

$$\times-\prod_{i=1}^l(1-(\mu_{ik_i}^2+v_{ik_i}^2))^{\lambda w_i}\Bigg)^{1/\lambda}\Bigg)\Bigg)^{1/2}\Bigg)\Bigg\}.\qquad\square$$

Then $n=l+1$,

$$\left(\oplus_{i=1}^{l+1}w_i\tilde{f}_i^\lambda\right)^{1/\lambda}$$

$$=\bigcup_{\mu_{ik_i}\in\tilde{h}_i,v_{ik_i}\in\tilde{g}_i}\left((\oplus_{i=1}^l w_i\tilde{f}_i^\lambda)\oplus(w_{l+1}\tilde{f}_{l+1}^\lambda)\right)^{1/\lambda}$$

$$\tilde{f}_{l+1}^\lambda=\bigcup_{\mu_{ik_i}\in\tilde{h}_i,v_{ik_i}\in\tilde{g}_i}$$

$$\times\Bigg\{\Bigg(((1-v_{l+1k_{l+1}}^2)^\lambda-(1-(\mu_{l+1k_{l+1}}^2$$

$$+v_{l+1k_{l+1}}^2))^\lambda)^{1/2},\sqrt{1-(1-v_{l+1k_{l+1}}^2)^\lambda}\Bigg)\Bigg\},$$

$$w_{l+1}\tilde{f}_{l+1}^\lambda=\bigcup_{\mu_{ik_i}\in\tilde{h}_i,v_{ik_i}\in\tilde{g}_i}$$

$$\times\Bigg\{\Bigg(\Big(1-(1-((1-v_{l+1k_{l+1}}^2)^\lambda-(1-(\mu_{l+1k_{l+1}}^2$$

$$+v_{l+1k_{l+1}}^2))^\lambda))^{w_{l+1}}\Big)^{1/2},\Big((1-((1-v_{l+1k_{l+1}}^2)^\lambda$$

$$-(1-(\mu_{l+1k_{l+1}}^2+v_{l+1k_{l+1}}^2))^\lambda))^{w_{l+1}}$$

$$-(1-(\mu_{l+1k_{l+1}}^2+v_{l+1k_{l+1}}^2))^{w_{l+1}\lambda}\Big)^{1/2}\Bigg)\Bigg\},$$

$$\oplus_{i=1}^{l+1}w_i\tilde{f}_i^\lambda=\left(\oplus_{i=1}^l w_i\tilde{f}_i^\lambda\right)\oplus\left(w_{l+1}\alpha_{l+1}^\lambda\right)$$

$$=\bigcup_{\mu_{ik_i}\in\tilde{h}_i,v_{ik_i}\in\tilde{g}_i}$$

$$\times\Bigg\{\Bigg(\Bigg(1-\prod_{i=1}^l(1-((1-v_{ik_i}^2)^\lambda$$

$$-(1-(\mu_{ik_i}^2+v_{ik_i}^2))^\lambda))^{w_i}*((1-v_{l+1k_{l+1}}^2)^\lambda$$

$$-(1-(\mu_{l+1k_{l+1}}^2+v_{l+1k_{l+1}}^2))^\lambda)^{w_{l+1}}\Bigg)^{1/2},$$

$$\times\Bigg(\prod_{i=1}^l(1-((1-v_{ik_i}^2)^\lambda-(1-(\mu_{ik_i}^2+v_{ik_i}^2))^\lambda))^{w_i}$$

$$\times((1-v_{l+1k_{l+1}}^2)^\lambda-(1-(\mu_{l+1k_{l+1}}^2+v_{l+1k_{l+1}}^2))^\lambda)^{w_{l+1}}$$

$$-\prod_{i=1}^l(1-(\mu_{ik_i}^2+v_{ik_i}^2))^{\lambda w_i}$$

$$\times(1-(\mu_{l+1k_{l+1}}^2+v_{l+1k_{l+1}}^2))^\lambda)^{1/2}\Bigg)\Bigg\}$$

$$=\bigcup_{\mu_{ik_i}\in\tilde{h}_i,v_{ik_i}\in\tilde{g}_i}$$

$$\times\Bigg\{\Bigg(\Bigg(1-\prod_{i=1}^{l+1}(1-((1-v_{ik_i}^2)^\lambda$$

$$-(1-(\mu_{ik_i}^2+v_{ik_i}^2))^\lambda))^{w_i}\Bigg)^{1/2},$$

$$\times\Bigg(\prod_{i=1}^{l+1}(1-((1-v_{ik_i}^2)^\lambda-(1-(\mu_{ik_i}^2+v_{ik_i}^2))^\lambda)^{w_i})$$

$$-\prod_{i=1}^{l+1}(1-(\mu_{ik_i}^2+v_{ik_i}^2))^{\lambda w_i}\Bigg)^{1/2}\Bigg)\Bigg\}.$$

$$\left(\oplus_{i=1}^{l+1}w_i\tilde{f}_i^\lambda\right)^{1/\lambda}=\bigcup_{\mu_{ik_i}\in\tilde{h}_i,v_{ik_i}\in\tilde{g}_i}$$

$$\times\Bigg\{\Bigg(\Bigg(\Bigg(1-\Bigg(\prod_{i=1}^{l+1}(1-((1-v_{ik_i}^2)^\lambda$$

$$-(1-(\mu_{ik_i}^2+v_{ik_i}^2))^\lambda))^{w_i}\Bigg)$$

$$-\prod_{i=1}^{l+1}(1-(\mu_{ik_i}^2+v_{ik_i}^2))^{\lambda w_i}\Bigg)^{1/\lambda}$$

$$-\prod_{i=1}^{k+1}(1-(\mu_{ik_i}^2+v_{ik_i}^2))^{w_i}\Bigg)^{1/2},$$

$$\times\Bigg(1-\Bigg(1-\Bigg(\prod_{i=1}^{l+1}(1-((1-v_{ik_i}^2)^\lambda$$

$$-(1-(\mu_{ik_i}^2+v_{ik_i}^2))^\lambda))^{w_i}\Bigg)$$

$$-\prod_{i=1}^{l+1}(1-(\mu_{ik_i}^2+v_{ik_i}^2))^{\lambda w_i}\Bigg)\Bigg)^{1/\lambda}\Bigg)^{1/2}\Bigg)\Bigg\}.$$

Then Eq. (12) holds for $n=l+1$. Hence, Eq. (12) holds for all n from mathematical induction. Moreover, for each (μ,v) in the GHPFIWA operator,

$$\mu^2 + v^2 = 1 - \prod_{i=1}^{k+1} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\lambda w_i}))^{1/\lambda},$$

Since $0 \le \mu_{ik_i}^2 + v_{ik_i}^2 \le 1$, then $0 \le \mu^2 + v^2 \le 1$. The aggregated result of the GHPFIWA operator is still an HPFE.

Theorem 9 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. If all the HPFEs reduce to $\tilde{f} = (\tilde{h}, \tilde{g})$, the GHPFIWA operator reduces to the following form*

$$GHPFIWA(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \tilde{f}.$$

Theorem 10 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Let $\tilde{f}^+ = (1, 0)$, $\tilde{f}^- = (0, 1)$, then*

$$\tilde{f}^- \le GHPFIWA(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) \le \tilde{f}^+.$$

If the weight vector is taken as $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, the GHPFIWA$_\lambda$ operator reduces to the generalized Pythagorean fuzzy interaction averaging (GHPFIA) operator as follows

$$GHPFIA(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \left(\oplus_{j=1}^n \frac{1}{n} \tilde{f}_j^\lambda \right)^{1/\lambda}$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}$$

$$\times \left\{ \left(\left(\left(1 - \left(\prod_{i=1}^{k+1} (1 - ((1 - v_{ik_i}^2)^\lambda \right. \right. \right. \right. \right.$$

$$\left. - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^\lambda))^{\frac{1}{n}} \right)$$

$$- \prod_{i=1}^{k+1} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\frac{\lambda}{n}} \right)^{1/\lambda}$$

$$- \prod_{i=1}^{k+1} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\frac{1}{n}} \right)^{1/2},$$

$$\times \left(1 - \left(1 - \left(\prod_{i=1}^{k+1} (1 - ((1 - v_{ik_i}^2)^\lambda \right. \right. \right.$$

$$\left. - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^\lambda)^{\frac{1}{n}} \right)$$

$$\left. \left. \left. - \prod_{i=1}^{k+1} (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{\frac{\lambda}{n}} \right) \right)^{1/\lambda} \right)^{1/2} \right\}.$$

Example 3 Suppose the Pythagorean fuzzy values are the same as that in Example 1. Using the GHPFIWA operator, we can get GHPFIWA$(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = \{(0.5590, 0.1800),$ $(0.4870, 0.2125)\}$.

Hesitant Pythagorean fuzzy interaction Bonferroni mean operators

The Bonferroni mean aggregation operator was defined by Bonferroni [51] in 1950. It was generalized by Yager [52] and others.

Definition 13 Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. For any $p, q \ge 0$ with $p + q > 0$, the hesitant Pythagorean fuzzy interaction Bonferroni mean (HPFIBM) aggregation operator can be defined as

$$HPFIBM^{p,q}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \left(\frac{1}{n(n-1)} \oplus_{i,j=1,i\ne j}^n (\tilde{f}_i^p \otimes \tilde{f}_j^q) \right)^{\frac{1}{p+q}}. \quad (13)$$

Theorem 11 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Then, the aggregated result of the HPFIBM operator is still an HPFE, which has the following form*

$$HPFIBM^{p,q}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \left(\frac{1}{n(n-1)} \oplus_{i,j=1,i\ne j}^n (\tilde{f}_i^p \otimes \tilde{f}_j^q) \right)^{\frac{1}{p+q}}$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i, \mu_{jk_j} \in \tilde{h}_j, v_{jk_j} \in \tilde{g}_j}$$

$$\times \left\{ \left(\left(\left(1 - \prod_{i,j=1,i\ne j}^n (1 - (1 - v_{ik_i}^2)^p (1 - v_{jk_j}^2)^q \right. \right. \right. \right.$$

$$\left. - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q \right)^{\frac{1}{n(n-1)}}$$

$$+ \prod_{i,j=1,i\ne j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p$$

$$\left. \times (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}$$

$$- \prod_{i,j=1,i\ne j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p *$$

$$\left. \times (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)(p+q)}} \right)^{1/2},$$

$$\times \left(1 - \left(1 - \prod_{i,j=1,i\ne j}^n (1 - (1 - v_{ik_i}^2)^p * \right. \right.$$

$$\times (1 - v_{jk_j}^2)^q - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p$$

$$(1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}}$$

$$+ \prod_{i,j=1,i\ne j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p$$

$$\left. \left. (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right)^{1/2} \right\}.$$

Proof Since

$$\tilde{f}_i^p = \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}$$

$$\times \left\{ \left(\sqrt{(1 - v_{ik_i}^2)^p - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p}, \right. \right.$$

$$\left. \left. \times \sqrt{1 - (1 - v_{ik_i}^2)^p} \right) \right\},$$

$$\tilde{f}_j^q = \bigcup_{\mu_{jk_j} \in \tilde{g}_j, v_{jk_j} \in \tilde{h}_j}$$

$$\times \left\{ \left(\sqrt{(1 - v_{jk_j}^2)^q - (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q},\right.\right.$$

$$\left.\left.\sqrt{1 - (1 - v_{jk_j}^2)^q} \right) \right\},$$

$$\tilde{f}_i^p \otimes \tilde{f}_j^q = \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i, \mu_{jk_j} \in \tilde{h}_j, v_{jk_j} \in \tilde{g}_j}$$

$$\times \left\{ \left(((1 - v_{ik_i}^2)^p (1 - v_{jk_j}^2)^q \right.\right.$$

$$\left. - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{1/2}, \right.$$

$$\left.\left. \times \sqrt{1 - (1 - v_{ik_i}^2)^p (1 - v_{jk_j}^2)^q} \right) \right\},$$

$$\oplus_{i,j=1, i \neq j}^n (\tilde{f}_i^p \otimes \tilde{f}_j^q)$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i, \mu_{jk_j} \in \tilde{h}_j, v_{jk_j} \in \tilde{g}_j}$$

$$\times \left\{ \left(\left(1 - \prod_{i,j=1, i \neq j}^n (1 - (1 - v_{ik_i}^2)^p (1 - v_{jk_j}^2)^q \right.\right.\right.$$

$$\left.\left. + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q) \right)^{1/2}, \right.$$

$$\times \left(\prod_{i,j=1, i \neq j}^n (1 - (1 - v_{ik_i}^2)^p (1 \right.$$

$$- v_{jk_j}^2)^q - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)$$

$$\left.\left.\left. - \prod_{i,j=1, i \neq j}^n (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q \right)^{1/2} \right) \right\},$$

$$\frac{1}{n(n-1)} \oplus_{i,j=1, i \neq j}^n (\tilde{f}_i^p \otimes \tilde{f}_j^q)$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{g}_i, v_{ik_i} \in \tilde{h}_i, \mu_{jk_j} \in \tilde{g}_j, v_{jk_j} \in \tilde{h}_j}$$

$$\times \left\{ \left(\left(1 - \prod_{i,j=1, i \neq j}^n (1 - (1 - v_{ik_i}^2)^p (1 - v_{jk_j}^2)^q \right.\right.\right.$$

$$\left.\left. + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{1/2}, \right.$$

$$\times \left(\prod_{i,j=1, i \neq j}^n (1 - (1 \right.$$

$$- v_{ik_i}^2)^p (1 - v_{jk_j}^2)^q$$

$$- (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}}$$

$$\left.\left. - \prod_{i,j=1, i \neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p \right.\right.$$

$$\left.\left.\left. \times (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{1/2} \right) \right\},$$

$$\times \left(\frac{1}{n(n-1)} \oplus_{i,j=1, i \neq j}^n (\tilde{f}_i^p \otimes \tilde{f}_j^q) \right)^{\frac{1}{p+q}}$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{g}_i, v_{ik_i} \in \tilde{h}_i, \mu_{jk_j} \in \tilde{g}_j, v_{jk_j} \in \tilde{h}_j}$$

$$\times \left\{ \left(\left(\left(1 - \prod_{i,j=1, i \neq j}^n (1 - (1 - v_{ik_i}^2)^p (1 - v_{jk_j}^2)^q - \right.\right.\right.\right.$$

$$\times (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}}$$

$$+ \prod_{i,j=1, i \neq j}^n ((1 - (\mu_{ik_i}^2$$

$$\left. + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}$$

$$- \prod_{i,j=1, i \neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p *$$

$$\left. \times (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)(p+q)}} \right)^{1/2},$$

$$\times \left(1 - \left(1 - \prod_{i,j=1, i \neq j}^n (1 - (1 - v_{ik_i}^2)^p * \right.\right.$$

$$\times (1 - v_{jk_j}^2)^q - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 -$$

$$\times (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}}$$

$$+ \prod_{i,j=1, i \neq j}^n ((1 - (\mu_{ik_i}^2 q + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2$$

$$\left.\left.\left. + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right)^{1/2} \right) \right\}.$$

Moreover, for each (μ, v) in the aggregated result of the HPFIBM operator, we have

$$\mu^2 + v^2 = 1 - \prod_{i,j=1, i \neq j}^n ((1 - (\mu_{ik_i}^2$$

$$+ v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)(p+q)}}.$$

Since $0 \leq \mu_{ik_i}^2 + v_{ik_i}^2 \leq 1$ and $0 \leq \mu_{jk_j}^2 + v_{jk_j}^2 \leq 1$, hence $0 \leq \mu^2 + v^2 \leq 1$. Then, the aggregated result of the HPFIBM operator is still an HPFE. $\square$

Theorem 12 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. If all the HPFEs reduce to $\tilde{f} = (\tilde{h}, \tilde{g})$, the HPFIBM operator reduces to the following form*

$$HPFIBM(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \tilde{f}.$$

Theorem 13 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Let $\tilde{f}^+ = (1, 0)$, $\tilde{f}^- = (0, 1)$, then*

$$\tilde{f}^- \leq HPFIBM(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) \leq \tilde{f}^+.$$

Example 4 Suppose the Pythagorean fuzzy values are the same as that in Example 1. Using the HPFIBM operator, we can get HPFIBM($\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$) = {(0.7931, 0.3155), (0.7823, 0.3234), (0.7701, 0.3515), (0.7576, 0.3607)}.

Definition 14 Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. For any $p, q \geq 0$ with $p + q > 0$, the hesitant Pythagorean fuzzy interaction weighted Bonferroni mean (HPFIWBM) aggregation operator can be defined as

$$\text{HPFIWBM}^{p,q}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \left(\frac{1}{n(n-1)} \oplus_{i,j=1,i\neq j}^n ((w_i \tilde{f}_i)^p \otimes (w_j \tilde{f}_j)^q) \right)^{\frac{1}{p+q}},$$

$$(14)$$

where $(w_1, w_2, \ldots, w_n)$ is the weight vector satisfying $w_i \geq 0$, $\sum_{i=1}^n w_i = 1$.

Theorem 14 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Then, the aggregated result of the HPFIWBM operator is still an HPFE, which has the following form*

$$\text{HPFIWBM}^{p,q}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \left(\frac{1}{n(n-1)} \oplus_{i,j=1,i\neq j}^n ((w_i \tilde{f}_i)^p \otimes (w_j \tilde{f}_j)^q) \right)^{\frac{1}{p+q}}$$

$$= \bigcup_{\mu_{ik_i}\in\tilde{h}_i, v_{ik_i}\in\tilde{g}_i, \mu_{jk_j}\in\tilde{h}_j, v_{jk_j}\in\tilde{g}_j}$$

$$\times \Bigg\{ \Bigg(\Bigg(\Big(1 - \prod_{i,j=1,i\neq j}^n (1 - (1 - (1 - \mu_{ik_i}^2)^{w_i} + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i})^p (1 - (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^{w_j})^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i p} (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^{w_j q})^{\frac{1}{n(n-1)}}$$

$$+ \prod_{i,j=1,i\neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i p} (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^{w_j q})^{\frac{1}{n(n-1)}} \Big)^{\frac{1}{p+q}}$$

$$- \prod_{i,j=1,i\neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i p} (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^{w_j q})^{\frac{1}{n(n-1)(p+q)}} \Bigg)^{1/2},$$

$$\times \Bigg(1 - \Big(1 - \prod_{i,j=1,i\neq j}^n (1 - (1 - (1 - \mu_{ik_i}^2)^{w_i} + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i})^p (1 - (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^{w_j})^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i p} (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^{w_j q})^{\frac{1}{n(n-1)}}$$

$$\times \prod_{i,j=1,i\neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i p} (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^{w_j q})^{\frac{1}{n(n-1)}} \Big)^{\frac{1}{p+q}} \Bigg)^{1/2} \Bigg) \Bigg\}.$$

Definition 15 Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. For any $p, q \geq 0$ with $p + q > 0$, the hesitant Pythagorean fuzzy interaction geometric Bonferroni mean (HPFIGBM) aggregation operator can be defined as

$$\text{HPFIGBM}^{p,q}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \frac{1}{p+q} \otimes_{i,j=1,i\neq j}^n (p\tilde{f}_i \oplus q\tilde{f}_j)^{\frac{1}{n(n-1)}} \qquad (15)$$

Theorem 15 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Then*

$$\text{HPFIGBM}^{p,q}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \frac{1}{p+q} \otimes_{i,j=1,i\neq j}^n \left(p\tilde{f}_i \oplus q\tilde{f}_j \right)^{\frac{1}{n(n-1)}}$$

$$= \bigcup_{\mu_{ik_i}\in\tilde{h}_i, v_{ik_i}\in\tilde{g}_i, \mu_{jk_j}\in\tilde{h}_j, v_{jk_j}\in\tilde{g}_j}$$

$$\times \Bigg\{ \Bigg(\Big(1 - \Big(1 - \prod_{i,j=1,i\neq j}^n (1 - (1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}}$$

$$+ \prod_{i,j=1,i\neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \Big)^{\frac{1}{p+q}} \Big)^{1/2}$$

$$\times \Bigg(\Big(1 - \prod_{i,j=1,i\neq j}^n (1 - (1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}}$$

$$+ \prod_{i,j=1,i\neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \Big)^{\frac{1}{p+q}}$$

$$- \prod_{i,j=1,i\neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)(p+q)}} \Big)^{1/2} \Bigg) \Bigg\}.$$

Proof

$$p\tilde{f}_i = \bigcup_{\mu_{ik_i}\in\tilde{h}_i, v_{ik_i}\in\tilde{g}_i}$$

$$\times \Big\{ \Big(\sqrt{1 - (1 - \mu_{ik_i}^2)^p}, ((1 - \mu_{ik_i}^2)^p - (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p)^{1/2} \Big) \Big\},$$

$$q\tilde{f}_j = \bigcup_{\mu_{jk_j}\in\tilde{h}_j, v_{jk_j}\in\tilde{g}_j}$$

$$\times \Big\{ \Big(\sqrt{1 - (1 - \mu_{jk_j}^2)^q}, ((1 - \mu_{jk_j}^2)^q - (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{1/2} \Big) \Big\},$$

$$p\tilde{f}_i \oplus q\tilde{f}_j$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_j} \in \tilde{g}_i, \mu_{jk_j} \in \tilde{h}_j, v_{jk_j} \in \tilde{g}_j}$$

$$\times \left\{ \left(\left(\sqrt{1 - (1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q}, \right. \right. \right.$$

$$\times ((1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q - (1$$

$$\left. \left. \left. - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{1/2} \right) \right\}, \right.$$

$$(p\tilde{f}_i \oplus q\tilde{f}_j)^{\frac{1}{n(n-1)}}$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_j} \in \tilde{g}_i, \mu_{jk_j} \in \tilde{h}_j, v_{jk_j} \in \tilde{g}_j}$$

$$\times \left\{ \left(((1 - (1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} - ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}})^{1/2}, \right. \right.$$

$$\times (1 - (1 - (1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}})^{1/2} \Big) \Big\},$$

$$\otimes_{i,j=1,i \neq j}^{n} (p\tilde{f}_i \oplus q\tilde{f}_j)^{\frac{1}{n(n-1)}}$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_j} \in \tilde{g}_i, \mu_{jk_j} \in \tilde{h}_j, v_{jk_j} \in \tilde{g}_j}$$

$$\times \left\{ \left(\left(\prod_{i,j=1,i \neq j}^{n} (1 - (1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right. \right. \right.$$

$$\left. - \prod_{i,j=1,i \neq j}^{n} ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{1/2},$$

$$\times \left(1 - \prod_{i,j=1,i \neq j}^{n} (1 - (1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{1/2} \Big) \right\},$$

$$\frac{1}{p+q} \otimes_{i,j=1,i \neq j}^{n} (p\tilde{f}_i \oplus q\tilde{f}_j)^{\frac{1}{n(n-1)}}$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_j} \in \tilde{g}_i, \mu_{jk_j} \in \tilde{h}_j, v_{jk_j} \in \tilde{g}_j}$$

$$\times \left\{ \left(\left(1 - \left(1 - \prod_{i,j=1,i \neq j}^{n} (1 - (1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} + \prod_{i,j=1,i \neq j}^{n} ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right)^{1/2}, \right. \right.$$

$$\times \left(\left(1 - \prod_{i,j=1,i \neq j}^{n} (1 - (1 - \mu_{ik_i}^2)^p (1 - \mu_{jk_j}^2)^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} + \prod_{i,j=1,i \neq j}^{n} ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right. $$

$$\left. \left. - \prod_{i,j=1,i \neq j}^{n} ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)(p+q)}} \right)^{1/2} \right) \right\}.$$

Moreover, for each (μ, v) in the HPFIGBM operator,

$$\mu^2 + v^2 = 1 - \prod_{i,j=1,i \neq j}^{n} ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^p (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^q)^{\frac{1}{n(n-1)(p+q)}}.$$

Since $0 \leq \mu_{ik_i}^2 + v_{ik_i}^2 \leq 1$ and $0 \leq \mu_{jk_j}^2 + v_{jk_j}^2 \leq 1$, then $0 \leq \mu^2 + v^2 \leq 1$. Hence, the aggregated result of the HPFIGBM operator is still an HPFE. □

Theorem 16 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. If all the HPFEs reduce to $\tilde{f} = (\tilde{h}, \tilde{g})$, the HPFIGBM operator reduces to the following form*

$$HPFIGBM(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \tilde{f}.$$

Theorem 17 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. Let $\tilde{f}^+ = (1, 0)$, $\tilde{f}^- = (0, 1)$, then*

$$\tilde{f}^- \leq HPFIGBM(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) \leq \tilde{f}^+.$$

Example 5 Suppose the Pythagorean fuzzy values are the same as that in Example 1. Using the HPFIGBM operator, we can get HPFIGBM$(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = \{(0.7804, 0.3457), (0.7429, 0.4058), (0.7752, 0.3399), (0.7389, 0.3975)\}$.

Definition 16 Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. For any $p, q \geq 0$ with $p+q > 0$, the hesitant Pythagorean fuzzy interaction geometric weight Bonferroni mean (HPFIGWBM) aggregation operator can be defined as

$$HPFIGWBM^{p,q}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \frac{1}{p+q} \otimes_{i,j=1,i \neq j}^{n} \left(p\tilde{f}_i^{w_i} \oplus q\tilde{f}_j^{w_j} \right)^{\frac{1}{n(n-1)}}, \qquad (16)$$

where $(w_1, w_2, \ldots, w_n)$ is the weight vector satisfying $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$.

Theorem 18 *Let $\tilde{f}_i = (\tilde{h}_i, \tilde{g}_i)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs. $(w_1, w_2, \ldots, w_n)$ is the weight vector with $w_i \geq 0$, $\sum_{i=1}^n w_i = 1$. Then*

$$HPFIGWBM^{p,q}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$
$$= \frac{1}{p+q} \otimes_{i,j=1,i\neq j}^n \left(p\tilde{f}_i^{w_i} \oplus q\tilde{f}_j^{w_j} \right)^{\frac{1}{n(n-1)}}$$
$$= \bigcup_{\mu_{ik_i}\in\tilde{h}_i, v_{ik_i}\in\tilde{g}_i, \mu_{jk_j}\in\tilde{h}_j, v_{jk_j}\in\tilde{g}_j}$$
$$\left\{ \left(\left(1 - \left(1 - \prod_{i,j=1,i\neq j}^n (1 - (1 - (1 - v_{ik_i}^2)^{w_i} \right.\right.\right.\right.$$
$$+ (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i})^p (1 - (1 - v_{jk_j}^2)^{w_j} + (1 - (\mu_{jk_j}^2$$
$$+ v_{jk_j}^2))^{w_j})^q + (1$$
$$- (\mu_{ik_i}^2 + v_{ik_i}^2))^{pw_i} (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^{qw_j})^{\frac{1}{n(n-1)}}$$
$$+ \prod_{i,j=1,i\neq j}^n ((1 - (\mu_{ik_i}^2$$
$$+ v_{ik_i}^2))^{pw_i} (1 - (\mu_{jk_j}^2 + v_{jk_j}^2))^{qw_j})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right)^{1/2},$$
$$\left(\left(1 - \prod_{i,j=1,i\neq j}^n (1 - (1 \right.\right.$$
$$- (1 - v_{ik_i}^2)^{w_i} + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i})^p (1$$
$$- (1 - v_{jk_j}^2)^{w_j} + (1 - (\mu_{jk_j}^2$$
$$+ v_{jk_j}^2))^{w_j})^q + (1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{pw_i} (1$$
$$- (\mu_{jk_j}^2 + v_{jk_j}^2))^{qw_j})^{\frac{1}{n(n-1)}}$$
$$+ \prod_{i,j=1,i\neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{pw_i} (1$$
$$- (\mu_{jk_j}^2 + v_{jk_j}^2))^{qw_j})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} -$$
$$\prod_{i,j=1,i\neq j}^n ((1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{pw_i} (1$$
$$- (\mu_{jk_j}^2 + v_{jk_j}^2))^{qw_j})^{\frac{1}{n(n-1)(p+q)}} \right)^{1/2} \right) \right\}.$$

An approach to Pythagorean fuzzy multiple attribute decision-making based on new interaction aggregation operators

Suppose there is a multiple attribute decision-making problem. $\{A_1, A_2, \ldots, A_m\}$ is the alternative set, $\{C_1, C_2, \ldots, C_n\}$ is the attribute set. The experts evaluate alternatives with respect to attributes with Pythagorean fuzzy values. If they are familiar with the attributes, they can give evaluation values; if they are not familiar with attributes, they can refuse to give any evaluation values. Hence, the hesitant Pythagorean fuzzy decision matrix is formed. The proposed method based on the new hesitant Pythagorean fuzzy interaction aggregation operators is as follows.

Step 1. Decision-makers evaluate alternatives with respect to attributes with Pythagorean fuzzy values and hesitant Pythagorean fuzzy decision matrix is formed as $\tilde{D} = (\tilde{f}_{ij})_{m\times n}$.

Step 2. Calculate alternatives' collective evaluation values using the HPFIWBM operator or the HPFIGWBM operator using the following equations.

$$\tilde{f}_i = HPFIWBM^{p,q}(\tilde{f}_{i1}, \tilde{f}_{i2}, \ldots, \tilde{f}_{in})$$
$$= \left(\frac{1}{n(n-1)} \oplus_{j,l=1,j\neq l}^n ((w_j\tilde{f}_{ij})^p \otimes (w_l\tilde{f}_{il})^q) \right)^{\frac{1}{p+q}}$$
$$= \bigcup_{\mu_{ijk_j}\in\tilde{h}_{ij}, v_{ijk_j}\in\tilde{g}_{ij}, \mu_{ilk_l}\in\tilde{h}_{il}, v_{ilk_l}\in\tilde{g}_{il}}$$
$$\times \left\{ \left(\left(\left(1 - \prod_{j,l=1,j\neq l}^n (1 - (1 - (1 - \mu_{ijk_j}^2)^{w_j} \right.\right.\right.\right.$$
$$+ (1 - (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{w_j})^p (1 - (1 - \mu_{ilk_l}^2)^q$$
$$+ (1 - (\mu_{ilk_l}^2 + v_{ilk_l}^2))^{w_l})^q + (1$$
$$- (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{w_j p} (1 - (\mu_{ilk_l}^2 + v_{ilk_l}^2))^{w_l q})^{\frac{1}{n(n-1)}}$$
$$+ \prod_{j,l=1,j\neq l}^n ((1 - (\mu_{ijk_j}^2$$
$$+ v_{ijk_j}^2))^{w_j p} (1 - (\mu_{ilk_l}^2 + v_{ilk_l}^2))^{w_l q})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}$$
$$- \prod_{j,l=1,j\neq l}^n ((1 - (\mu_{ijk_j}^2$$
$$+ v_{ijk_j}^2))^{w_j p} (1 - (\mu_{ilk_l}^2 + v_{ilk_l}^2))^{w_l q})^{\frac{1}{n(n-1)(p+q)}} \right)^{1/2},$$
$$\times \left(1 - \left(1 - \prod_{j,l=1,j\neq l}^n (1 \right.\right.$$
$$- (1 - (1 - \mu_{ijk_j}^2)^{w_j} + (1 - (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{w_j})^p$$
$$\times (1 - (1 - \mu_{ilk_l}^2)^q + (1$$
$$- (\mu_{ilk_l}^2$$
$$+ v_{ilk_l}^2))^{w_l})^q + (1 - (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{w_j p} (1$$
$$- (\mu_{ilk_l}^2 + v_{iljk_l}^2))^{w_l q})^{\frac{1}{n(n-1)}}$$
$$+ \prod_{j,l=1,j\neq l}^n ((1 - (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{w_j p} (1$$
$$- (\mu_{ilk_l}^2 + v_{ilk_l}^2))^{w_l q})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right)^{1/2} \right) \right\}.$$

$$\tilde{f}_i = HPFIGWBM^{p,q}(\tilde{f}_{i1}, \tilde{f}_{i2}, \ldots, \tilde{f}_{in})$$
$$= \frac{1}{p+q} \otimes_{j,l=1,j\neq l}^n \left(p\tilde{f}_{ij}^{w_j} \oplus q\tilde{f}_{il}^{w_l} \right)^{\frac{1}{n(n-1)}}$$
$$= \bigcup_{\mu_{ijk_j}\in\tilde{h}_{ij}, v_{ijk_j}\in\tilde{g}_{ij}, \mu_{ilk_l}\in\tilde{h}_{il}, v_{ilk_l}\in\tilde{g}_{il}}$$
$$\times \left\{ \left(\left(1 - \left(1 - \prod_{j,l=1,j\neq l}^n (1 - (1 - (1 - v_{ijk_j}^2)^{w_j} \right.\right.\right.\right.$$
$$+ (1 - (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{w_j})^p (1$$

Table 1 Pythagorean fuzzy decision matrix $\tilde{D}$

	C_1	C_2	C_3
A_1	$\{(0.9, 0.2)\}$	$\{(0.7, 0.3)\}$	$\{(0.6, 0.5)\}$
A_2	$\{(0.5, 0.6), (0.6, 0.3)\}$	$\{(0.8, 0.3)\}$	$\{(0.4, 0.5)\}$
A_3	$\{(0.7, 0)\}$	$\{(0.6, 0.2)\}$	$\{(0.6, 0.2), (0.8, 0.2)\}$
A_4	$\{(0.7, 0.4)\}$	$\{(0.8, 0.1), (0.7, 0.4)\}$	$\{(0.5, 0.3)\}$

$$-(1 - v_{ilk_l}^2)^{w_l} + (1 - (\mu_{ilk_l}^2 + v_{ilk_l}^2))^{w_l})^q + (1$$
$$-(\mu_{ijk_j}^2 + v_{ijk_j}^2))^{pw_j}(1 - (\mu_{ilk_l}^2$$
$$+ v_{ilk_l}^2))^{qw_l})^{\frac{1}{n(n-1)}} + \prod_{j,l=1, j \neq l}^n ((1 - (\mu_{ijk_j}^2$$
$$+ v_{ijk_j}^2))^{pw_j}(1 - (\mu_{ilk_l}^2 + v_{ilk_l}^2))^{qw_l})^{\frac{1}{n(n-1)}} \Big)^{\frac{1}{p+q}} \Big)^{1/2},$$
$$\times \Big(\Big(1 - \prod_{j,l=1, j \neq l}^n (1 - (1$$
$$-(1 - v_{ijk_j}^2)^{w_j} + (1 - (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{w_j})^p (1$$
$$-(1 - v_{ilk_l}^2)^{w_l} + (1 - (\mu_{ilk_l}^2$$
$$+ v_{ilk_l}^2))^{w_l})^q + (1 - (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{pw_j}(1$$
$$-(\mu_{ilk_l}^2 + v_{ilk_l}^2))^{qw_l})^{\frac{1}{n(n-1)}}$$
$$+ \prod_{j,l=1, j \neq l}^n ((1 - (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{pw_j}(1 - (\mu_{ilk_l}^2$$
$$+ v_{ilk_l}^2))^{qw_l})^{\frac{1}{n(n-1)}} \Big)^{\frac{1}{p+q}}$$
$$- \prod_{j,l=1, j \neq l}^n ((1 - (\mu_{ijk_j}^2 + v_{ijk_j}^2))^{pw_j}(1 - (\mu_{ilk_l}^2$$
$$+ v_{ilk_l}^2))^{qw_l})^{\frac{1}{n(n-1)(p+q)}} \Big)^{1/2} \Big) \Big\},$$

where $(w_1, w_2, \ldots, w_n)$ is the weight vector of different attributes with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$.

Step 3. Calculate each alternative's $S(\tilde{f}_i)$ and $A(\tilde{f}_i)$ using the Eqs. (4)–(5).

Step 4. Rank alternatives according to the method in Definition 7.

The new method has the following characteristics: the evaluation values are given as HPFEs, which are more flexible and powerful; interaction between membership and non-membership has been considered; and interaction between arguments to be aggregated has been modeled using the Bonferroni mean operator.

An illustrative example

Suppose there is an investing company wanting to invest a large amount of money (adapted from [57]). They invite several experts to evaluate several possible companies: A_1—an

artificial intelligent company, A_2—an architecture company, A_3—a catering company and A_4—a logistics company. They mainly consider the following attributes: C_1−interest rate, C_2—risk, C_3—growth potential. The new method is used to rank alternatives as follows.

Step 1. The experts evaluate alternatives with respect to attributes in Pythagorean fuzzy values and the decision matrix is formed as $\tilde{D} = (\tilde{f}_{ij})_{4 \times 3}$ (Table 1).

Step 2. Assume the weight vector of attributes is (0.45, 0.35, 0.20). Calculate the collective evaluation values by using the HPFIWBM^{2,2} operator to get

$\tilde{f}_1 = \{(0.5605, 0.2174)\}$, $\tilde{f}_2 = \{(0.4238, 0.3176), (0.4240, 0.2861), (0.4368, 0.2272), (0.4392, 0.2623)\}$, $\tilde{f}_3 = \{(0.4477, 0.1012), (0.4995, 0.0880), (0.4975, 0.0988), (0.5420, 0.0862)\}$, $\tilde{f}_4 = \{(0.4486, 0.2522), (0.4621, 0.2264), (0.4638, 0.2229), (0.4753, 0.1972)\}$.

Step 3. The scores of $\tilde{f}_i$ are calculated as

$$S(\tilde{f}_1) = 0.2670, \quad S(\tilde{f}_2) = 0.1100,$$
$$S(\tilde{f}_3) = 0.2515, \quad S(\tilde{f}_4) = 0.1631.$$

Step 4. The alternatives can be ranked as

$$A_3 > A_1 > A_4 > A_2.$$

The optimal alternative is A_3.

Comparing with other methods

If the hesitant Pythagorean fuzzy interaction weighted averaging (HPFIWA) operator is used in aggregating, we can get $\tilde{f}_1 = \{(0.8449, 0.3123)\}$, $\tilde{f}_2 = \{(0.6979, 0.4759), (0.7228, 0.3785)\}$, $\tilde{f}_3 = \{(0.7864, 0.1734), (0.7111, 0.1670)\}$, $\tilde{f}_4 = \{(0.7683, 0.2955), (0.7216, 0.1670)\}$. The scores of $\tilde{f}_i$ $(i = 1, 2, \ldots, 4)$ can be calculated as $S(\tilde{f}_1) = 0.6163$, $S(\tilde{f}_2) = 0.3199$, $S(\tilde{f}_3) = 0.5331$, $S(\tilde{f}_4) = 0.4334$. Then alternatives can be ranked as $A_1 > A_3 > A_4 > A_2$ and the optimal alternative is A_1. The ranking result is different from that based on the HPFIWBM operator. Though the interaction between membership and non-membership has been considered, interaction between hesitant Pythagorean fuzzy elements has not been considered.

$$\text{HPFIWA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}$$

$$\times \left\{ \left(\sqrt{1 - \prod_{i=1}^{n}(1 - \mu_{ik_i}^2)^{w_i}}, \left(\prod_{i=1}^{n}(1 - \mu_{ik_i}^2)^{w_i} \right. \right. \right.$$

$$\left. \left. \left. - \prod_{i=1}^{n}(1 - (\mu_{ik_i}^2 + v_{ik_i}^2))^{w_i} \right)^{1/2} \right) \right\}. \tag{17}$$

If interaction between membership and non-membership is not considered and we calculate the collective ones using the hesitant Pythagorean fuzzy weighted averaging (HPFWA) operator as follows

$$\text{HPFWA}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \bigcup_{\mu_{ik_i} \in \tilde{h}_i, v_{ik_i} \in \tilde{g}_i}$$

$$\times \left\{ \left(\sqrt{1 - \prod_{i=1}^{n}(1 - \mu_{ik_i}^2)^{w_i}}, \prod_{i=1}^{n} v_{ik_i}^{w_i} \right) \right\}. \tag{18}$$

We can calculate collective evaluation values using the HPFWA operator to get $\tilde{f}_1 = \{(0.8110, 0.2769)\}$, $\tilde{f}_2 = \{(0.6377, 0.4539), (0.6689, 0.3323)\}$, $\tilde{f}_3 = \{(0.6964, 0), (0.6951, 0)\}$, $\tilde{f}_4 = \{(0.6964, 0.3446), (0.7158, 0.2325)\}$. The scores can be calculated as $S(\tilde{f}_1) = 0.5811$, $S(\tilde{f}_2) = 0.2688$, $S(\tilde{f}_3) = 0.4841$, $S(\tilde{f}_4) = 0.4123$. Then, alternatives can be ranked as $A_1 > A_3 > A_4 > A_2$ and the optimal alternative is A_1. From the result, we can see that the aggregated non-membership of A_3 is 0 due to the 0 nonmembership in evaluation values. Though all the other non-memberships are not 0, they have no effect on the final result.

If the TOPSIS method is used to rank alternatives, we first extend decision matrix by adding the minimum Pythagorean fuzzy value (0.4, 0.5) to make all the evaluation values have the same number of membership and non-membership. The hesitant Pythagorean fuzzy positive ideal solution $\tilde{f}^+$ and hesitant Pythagorean fuzzy negative ideal solution $\tilde{f}^-$ can be determined as $\tilde{f}^+ = \Big(\{(0.9, 0.2), (0.4, 0.5)\}, \{(0.8, 0.1),$

$(0.7, 0.4)\}, \{(0.8, 0.1), (0.7, 0.4)\}, \{(0.8, 0.2), (0.6, 0.2)\} \Big),$

$\tilde{f}^- = \Big(\{(0.5, 0.6), (0.6, 0.3)\}, \{(0.6, 0.2), (0.4, 0.5)\}, \{(0.6,$

$0.2), (0.4, 0.5)\}, \{(0.4, 0.5), (0.4, 0.5)\} \Big)$. The weighted distances between alternative evaluation values and $\tilde{f}^+$, $\tilde{f}^-$ can be calculated using the following equations $d_i^+ = w_1 d(\tilde{f}_1^+, \tilde{f}_{i1}) + w_2 d(\tilde{f}_2^+, \tilde{f}_{i2}) + w_3 d(\tilde{f}_3^+, \tilde{f}_{i3})$, $d_i^- = w_1 d(\tilde{f}_1^-, \tilde{f}_{i1}) + w_2 d(\tilde{f}_2^-, \tilde{f}_{i2}) + w_3 d(\tilde{f}_3^-, \tilde{f}_{i3})$. The results are as $d_1^+ = 0.0301$, $d_2^+ = 0.0495$, $d_3^+ = 0.0834$, $d_4^+ = 0.0390$, $d_1^- = 0.1653$, $d_2^- = 0.0289$, $d_3^- = 0.1630$,

$d_4^- = 0.1663$. The closeness coefficients can be calculated by $CC_i = \frac{d_i^-}{d_i^- + d_i^+}$ to get $CC_1 = 0.8458$, $CC_2 = 0.3684$, $CC_3 = 0.6616$, $CC_4 = 0.8101$. Then alternatives can be ranked as $A_1 > A_4 > A_3 > A_2$ and the optimal alternative is A_1.

If interaction between memberships and non-memberships is not considered, the hesitant Pythagorean fuzzy weighted Bonferroni mean (HPFWBM) operator as follows is used in aggregating.

$$\text{HPFWBM}(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$$

$$= \left(\frac{1}{n(n-1)} \oplus_{i,j=1,i \neq j}^{n} ((w_i \tilde{f}_i)^p \otimes (w_j \tilde{f}_j)^q) \right)^{\frac{1}{p+q}}$$

$$= \bigcup_{\mu_{jk_j} \in \tilde{h}_j, v_{jk_j} \in \tilde{g}_j}$$

$$\times \left\{ \left(\sqrt{\left(1 - \prod_{i,j=1,i \neq j}^{n} \left(1 - (1 - (1 - \mu_{jk_j}^2)^{w_j})^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}}, \right. \right.$$

$$\left. \left. \times \sqrt{\left(1 - \left(1 - \prod_{i,j=1,i \neq j}^{n} (1 - (1 - (v_{jk_j}^-)^2)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right)} \right) \right\}. \tag{19}$$

The collective evaluation values can be calculated as $\tilde{f}_1 = \{(0.4713, 0.7377)\}$, $\tilde{f}_2 = \{(0.3533, 0.8181), (0.3886, 0.7650), (0.3540, 0.8065), (0.3889, 0.7585)\}$, $\tilde{f}_3 = \{(0.4066, 0.6147), (0.3806, 0.6147), (0.3998, 0.6147), (0.3694, 0.6147)\}$, $\tilde{f}_4 = \{(0.4275, 0.7107), (0.3944, 0.7517), (0.4263, 0.7264), (0.3923, 0.7753)\}$. The scores can be calculated as $S(\tilde{f}_1) = -0.3221$, $S(\tilde{f}_2) = -0.4820$, $S(\tilde{f}_3) = -0.2257$, $S(\tilde{f}_4) = -0.7377$. The alternatives can be ranked as $A_3 > A_1 > A_4 > A_2$ and the optimal alternative is A_3. Though ranking result is the same as that based on the HPFIWBM operator, but in aggregation process, the effect of non-memberships has been reduced since there is 0 of non-membership in the evaluation process. In other decision-making problems, we may get different ranking results.

The differences between the proposed method and the existing methods have been summarized in Table 2. In a word, our proposed method is based on the hesitant Pythagorean fuzzy values and the Bonferroni mean operator. Moreover, interaction between arguments to be aggregated is considered and interaction between membership and non-membership is also considered.

Conclusions

In this paper, we first define some hesitant Pythagorean fuzzy interaction aggregation laws for HPFEs, and then develop some hesitant Pythagorean interaction aggregation operators. Using the Bonferroni mean operator, we develop

Table 2 The characteristic comparisons of different methods

Methods	Information by Pythagorean fuzzy number	Whether consider the inter-relationships between aggregating arguments
Liang et al. [22]	Yes	Yes
Xu and Yager [58]	No	Yes
Zhang and Xu [32]	Yes	No
Garg [50]	Yes	No
Our proposed method	Yes	Yes

Methods	Whether consider the interactions between membership and non-membership	Whether information by hesitant Pythagorean fuzzy number
Liang et al. [22]	No	No
Xu and Yager [58]	No	No
Zhang and Xu [32]	No	No
Garg [50]	Yes	Yes
Our proposed method	Yes	Yes

some hesitant Pythagorean fuzzy interaction Bonferroni mean operators. Based on the HPFIWBM operator and the HPFIWGBM operator, we propose a new multiple attribute decision-making method. Numerical example is presented to illustrate the new method.

In the future, we will apply the new aggregation operators to other complicated decision problems and we will also develop new interaction aggregation operators for HPFEs.

Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this paper.

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