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Remarks on asymptotic regularity and fixed points

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Abstract. Asymptotic regularity allows to provide simple proofs of Banach's theorem and Kannan's theorem. Using asymptotic regularity and Kannan's type conditions we generalize these results, in particular, the Banach contraction principle (see Theorem 2.6 and Corollary 2.10). Further, we discuss the analogous results for monotone mappings on preordered metric spaces, where a preordered binary relation is weaker than a partial order. Next, we will prove a random version of the presented deterministic fixed-point theorems.

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1. Banach and Kannan theorems

A sequence $\{x_n\}$ in a metric space (X, d) is *asymptotically regular* if

$$d(x_n, x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty \tag{1}$$

The condition (1) does not guarantee a convergence of the sequence $\{x_n\}$. Notice that we cannot deduce, that a subsequence $\{x_{n_k}\}_k$ of an asymptotically regular sequence $\{x_n\}$ is also asymptotically regular. For example, consider the sequence $\{x_n = \sum_{i=1}^{n} \frac{1}{i}, n \geqslant 1\}$ in the Euclidean space $\mathbb{R}$.

Let $T : X \to X$ be a mapping. For a initial point $x_0 \in X$, define a sequence of iterates $x_{n+1} = Tx_n = T^{n+1}x_0$, $n = 0, 1, 2, \ldots$, and the resulting sequence $\{x_n\}$ is called the sequence of *successive approximations* of T.

Hillam [28] proved:

Theorem 1.1. *Let T be a continuous map of $[0, 1]$ into $[0, 1]$. The sequence $\{x_n = T^n x\}$ of successive approximations of T converges to a fixed point of T if and only if (1) holds.*

Smart [44] showed that this result does not extend beyond one-dimensional case:

Example 1.2. There is a continuous mapping T of the closed unit disc in the Euclidean plane such that the origin and points on the unit circle are fixed points and every other point x satisfies $d(T^n x, T^{n+1}x) \to 0$ but $\{T^n x\}$ is not convergent, see [44] for details.

The following observation is trivial:

Lemma 1.3. *If T is a continuous map of X into X and if $d(T^n x, T^{n+1}x) \to 0$, then any limit point p of the set $\{T^n x\}$ is a fixed point of T.*

Proof. If $T^{n_k}x \to p \in X$, then

$$p = \lim_{k\to\infty} T^{n_k}x = \lim_{k\to\infty} T^{n_k+1}x = T\left(\lim_{k\to\infty} T^{n_k}x\right) = Tp. \qquad \square$$

Thus, the continuity of a mapping $T : X \to X$ and the fact that the sequence of successive approximations $\{T^n x\}$ satisfies (1) does not guarantee the existence of a fixed point. For a guarantee that there is a (unique) fixed point, additional assumptions are needed.

Banach's contraction principle [4] is remarkable in its simplicity, because the contractive condition on the mapping is simple and easy to test, because it requires only a complete metric space for its setting, and because it finds almost canonical applications in the theory of differential and integral equations. In this part, we will give an elementary proof of this result exposing (1), for other proofs see [17, Chapter 2], [18].

Let us recall a few facts.

Definition 1.4. Let (X, d) be a metric space and let $T : X \to X$ be a mapping. T is called a *contraction* if there exists a fixed constant $0 \leqslant L < 1$ such that

$$d(Tx, Ty) \leqslant L \cdot d(x, y) \quad \text{for all} \quad x, y \in X. \tag{2}$$

Each contraction is a continuous mapping.

There are many mappings of this type.

Example 1.5. Let $X = [a, b]$ be with usual metric and $T : X \to X$ be a continuous mapping such that T is differentiable at every $x \in (a, b)$ such that $|T'(x)| \leqslant L < 1$. Then, by the mean value theorem, if $x, y \in X$, there is a point c between x and y such that

$$|Tx - Ty| = |T'(c)| \cdot |x - y| \leqslant L \cdot |x - y|.$$

Here is an elementary proof of Banach's contraction principle.

Theorem 1.6 (Banach contraction principle). *Let (X, d) be a complete metric space, then each contraction $T : X \to X$ has a unique fixed point $p \in X$, and $T^n x \to p$ for each $x \in X$.*

Proof. Choose $x_0 \in X$ arbitrarily and define a sequence $\{x_{n+1} = Tx_n, n = 0, 1, 2, \ldots\}$ of T based on x_0. Since T is a contraction

$$d(x_{n+1}, x_n) \leqslant L \cdot d(x_n, x_{n-1}) \leqslant \cdots \leqslant L^n \cdot d(Tx_0, x_0) \to 0 \quad \text{as} \quad n \to \infty. \tag{3}$$

By triangle rule, for any n and any $k > 0$, we have

$$\begin{aligned} d(x_{n+k+1}, x_{n+1}) &\leqslant L \cdot d(x_{n+k}, x_n) \\ &\leqslant L \cdot \{d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+1}) + d(x_{n+1}, x_n)\}, \end{aligned}$$

so by (3)

$$d(x_{n+k+1}, x_{n+1}) \leqslant \frac{L}{1-L} \cdot \{d(x_{n+k}, x_{n+k+1}) + d(x_{n+1}, x_n)\} \to 0 \text{ as } n \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in a complete metric space X and there exists $p \in X$ such that $x_n \to p \in X$. Because T is continuous and $x_{n+1} = Tx_n$, it follows that $p = Tp$. Suppose q is another fixed point of T. Then

$$0 < d(p, q) = d(Tp, Tq) \leqslant L \cdot d(p, q) < d(p, q),$$

a contradiction. Hence T has unique fixed point $p \in X$. Because

$$d(T^n x, p) = d(T^n x, T^n p) \leqslant L^n \cdot d(x, p) \to 0 \quad \text{as} \quad n \to \infty,$$

we have $T^n x \to p$ for any $x \in X$. □

Remark 1.7. The following trivial fact is noteworthy in that the mapping T is not even assumed to be continuous:

Let (X, d) be a complete metric space and $T : X \to X$ be a mapping for which T^N is contraction for some positive integer $N > 1$, then T has a unique fixed point.

Not only contractions guarantee the existence of a unique fixed point and the possibility of its approximation. In 1968, Kannan [32] established the following theorem, see [25].

Theorem 1.8. *If T is a map of the complete metric space (X, d) into itself and if there exists $0 \leqslant K < \frac{1}{2}$ satisfying*

$$d(Tx, Ty) \leqslant K \cdot \{d(x, Tx) + d(y, Ty)\} \quad \text{for all} \quad x, y \in X, \tag{4}$$

then T has a unique fixed point $p \in X$, and $T^n x \to p$ for each $x \in X$.

Kannan's theorem is important because Subrahmanyam [46] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space (X, d) is complete if and only if every mapping satisfying (4) on X with constant $K < \frac{1}{2}$ has a fixed point. Contractions do not have this property; Connell [11] gave an example of metric space X such that X is not complete and every contraction on X has a fixed point.

Here is an elementary proof of Kannan's theorem.

Proof. Choose $x_0 \in X$ arbitrarily and define a sequence $\{x_{n+1} = Tx_n, n = 0, 1, 2, \ldots\}$. By (4),

$$d(x_{n+1}, x_n) \leqslant \frac{K}{1-K} \cdot d(x_n, x_{n-1}) \leqslant \cdots \leqslant \left(\frac{K}{1-K}\right)^n \cdot d(Tx_0, x_0) \to 0,$$

as $n \to \infty$. By triangle rule, for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &\leqslant (K+1) \cdot \{d(x_n, x_{n+1}) + d(x_{m+1}, x_m)\} \to 0 \ \text{ as } \ m > n \to \infty. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a complete metric space X and there exists $p \in X$ such that $x_n \to p \in X$. Since,

$$\begin{aligned} d(p, Tp) &\leqslant d(p, T^{n+1}x) + d(T^{n+1}x, Tp) \\ &\leqslant d(p, T^{n+1}x) + K \cdot \{d(T^n x, T^{n+1}x) + d(p, Tp)\}, \end{aligned}$$

so

$$d(p, Tp) \leqslant \frac{1}{1-K} \cdot d(p, T^{n+1}x) + \frac{K}{1-K} \cdot d(T^n x, T^{n+1}x) \to 0 \text{ as } n \to \infty.$$

Hence, $p = Tp$. Suppose q is another fixed point of T. Then

$$0 < d(p, q) = d(Tp, Tq) \leqslant K \cdot \{d(p, Tp) + d(q, Tq\} = 0,$$

a contradiction. Hence T has unique fixed point $p \in X$. From (4), we have $\lim_{n\to\infty} d(T^n x, p) = 0$ for any $x \in X$. □

Remark 1.9. Theorem 1.8 remains true when (4) is replaced by

$$d(Tx, Ty) \leqslant K \cdot \{d(x, Ty) + d(y, Tx)\} \text{ for all } x, y \in X. \tag{5}$$

The proof is analogous. For more information on mappings satisfying (5), see [5,10,39] and references therein.

Obviously conditions (2) and (4) are independent. Condition (4) is neither stronger nor weaker than the contraction mappings. In particular, the mapping satisfying (4) need not be continuous. In the following examples, the spaces are with the usual metrics.

Example 1.10. Mapping $Tx = 0$ for $x \leqslant 2$ and $Tx = -\frac{1}{2}$ for $x > 2$, satisfies (4) with $K = \frac{1}{5}$, and T is not continuous.

Example 1.11. Contraction $Tx = \frac{x}{3}$, $x \in [0, 1]$, not satisfied (4) with $K < \frac{1}{2}$, take $x = 0$ and $y = 1$. If T is a contraction with $L < \frac{1}{3}$, then T satisfies (4) with $K < \frac{1}{2}$.

Example 1.12. The condition (2) with $L = 1$, does not imply the existence of a fixed point. The mapping $Tx = x + 1$ for $x \in \mathbb{R}$ is fixed point free. The condition (4) with $K = \frac{1}{2}$, does not imply the existence of a fixed point. Take the unit circle S on the Euclidean plane and $Tz = -z$, $z \in S$.

There are many generalizations of Theorem 1.6 and Theorem 1.8, and unification of conditions (2) and (4), see [5,14,31,39], and references therein. The literature of this subject is extensive.

Conclusion. In this part, we have presented elementary proofs of Banach's theorem and Kannan's theorem on a fixed point.

2. Asymptotic regularity, continuity and fixed points

We know many conditions that guarantee the existence of a fixed point, see [2,17,26], and references therein. In this part, we present a very simple situation when the mapping T not only satisfies some conditions of Kannan's type, but it is also continuous and asymptotically regular (as in the Banach theorem).

We recall, asymptotic regularity is a fundamentally important concept in metric fixed point theory, see [2, Chapter IX], and [17, Chapter 9]. It was formally introduced by Browder and Petryshym [7].

Definition 2.1. A mapping T of a metric space (X, d) into itself is said to be *asymptotically regular* if

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0 \text{ for all } x \in X.$$

Obviously, if a mapping $T : X \to X$ is a contraction or satisfy (4) or (5) with $K < \frac{1}{2}$, then T is asymptotically regular. Asymptotic regularity is also satisfied by other mappings. But already the asymptotic regularity and nonexpansiveness (i.e. $d(Tx, Ty) \leqslant d(x, y)$ for all x, y), more generally, continuity, are independent.

Example 2.2. Let $B = \{x \in \mathbb{R}^2 : \|x\| \leqslant 1\}$ be the closed unit disc in the Euclidean plane and let T be an anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates. Then T is nonexpansive with the origin as the only fixed point and T is not asymptotically regular. Moreover, the sequence defined by $\{x_{n+1} = Tx_n,\ x_0 = (1, 0)\}$ does not converge to zero.

Example 2.3. The mapping $Tx = 1 - x$, $0 \leqslant x \leqslant 1$, is continuous, is not a contraction and does not satisfy the condition (4), take $x = 0$ and $y = 1$. T has a unique fixed point $\frac{1}{2}$, and $d(T^n(0), T^{n+1}(0)) \not\to 0$.

By an *averaged mapping* we mean one of the form $T_\lambda = (1 - \lambda)I + \lambda T$, where $0 < \lambda < 1$ and I is the identity operator. When T is nonexpansive, so is T_λ and both have the same fixed point set, but T_λ has more much felicitous asymptotic behavior than the original mapping.

Ishikawa [30] proved the following theorem with no restrictions on the geometry of the Banach space!

Theorem 2.4. *If C is a nonempty bounded closed convex subset of a Banach space X and $T : C \to C$ is nonexpansive, then the mapping T_λ is asymptotically regular for each $\lambda \in (0, 1)$.*

It is known [17] that a nonexpansive mapping $T : C \to C$, acting on weakly compact convex subsets of uniformly convex Banach spaces, has a fixed point. Lin [34] gave an example an asymptotically regulate Lipschitzian mapping acting on a weakly compact convex subset of the Hilbert space l^2 which has no fixed point.

Asymptotically regular mappings were studied in many papers, in different contexts, for instance [3,8,13,16,19–23,41,48].

In 1974, De Blasi [12] proved the following theorem, see [14].

Theorem 2.5. *Let C be a nonempty weakly closed subset of a Hilbert space. Suppose that $T : C \to C$ is continuous, asymptotically regular and satisfies*

$$\|Tx - Ty\| \leqslant \|x - Tx\| + \|y - Ty\| \text{ for all } x, y \in C.$$

Then T has a unique fixed point $p \in C$ and $T^n x \to p$ for each $x \in C$.

Now, we prove the following new theorem, which is an extension of previous results.

Theorem 2.6. *If (X, d) is a complete metric space and $T : X \to X$ is a continuous asymptotically regular mapping and if there exists $0 \leqslant M < 1$ and $0 \leqslant K < +\infty$ satisfying*

$$d(Tx, Ty) \leqslant M \cdot d(x, y) + K \cdot \{d(x, Tx) + d(y, Ty)\} \text{ for all } x, y \in X, \tag{6}$$

then T has a unique fixed point $p \in X$ and $T^n x \to p$ for each $x \in X$.

Proof. Choose $x_0 \in X$ arbitrarily and define a sequence $\{x_{n+1} = Tx_n, n = 0, 1, 2, \ldots\}$. According to asymptotic regularity, by triangle rule and (6), we get for any n and any $k > 0$,

$$\begin{aligned} d(x_{n+k}, x_n) &\leqslant d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leqslant d(x_{n+k}, x_{n+k+1}) + M \cdot d(x_{n+k}, x_n) \\ &\quad + K \cdot \{d(x_{n+k}, x_{n+k+1}) + d(x_n, x_{n+1})\} + d(x_{n+1}, x_n), \end{aligned}$$

so

$$(1 - M) \cdot d(x_{n+k}, x_n) \leqslant (K + 1) \cdot \{d(x_{n+k}, x_{n+k+1}) + d(x_n, x_{n+1})\} \to 0,$$

as $n \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in complete space X. There exists $p \in X$ such that $x_n \to p$. Because T is continuous and $x_{n+1} = Tx_n$, it follows that $p = Tp$. Suppose q is another fixed point of T. Then

$$\begin{aligned} 0 < d(p, q) = d(Tp, Tq) &\leqslant M \cdot d(p, q) + (K + 1) \cdot \{d(p, Tp) + d(q, Tq)\} \\ &= M \cdot d(p, q) < d(p, q), \end{aligned}$$

a contradiction. Hence T has unique fixed point $p \in X$. Because

$$\begin{aligned} d(T^n x, p) = d(T^n x, T^n p) &\leqslant d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+1} p) \\ &\leqslant d(T^n x, T^{n+1} x) + M \cdot d(T^n x, T^n p) \\ &\quad + K \cdot d(T^n x, T^{n+1} x), \end{aligned}$$

so

$$(1 - M) \cdot d(T^n x, p) \leqslant (K + 1) \cdot d(T^n x, T^{n+1} x) \to 0 \text{ as } n \to \infty.$$

This shows that $T^n x \to p$ for any $x \in X$. □

Remark 2.7. If $M \geqslant 0$, $K \geqslant 0$ and $M + 2K < 1$, then assumptions of continuity and asymptotic regularity are not necessary for the thesis to hold. If $0 \leqslant M < 1$ and $0 \leqslant K < 1$, then the continuity assumption is not necessary for the thesis to hold, see [25, 39].

Remark 2.8. Theorem 2.4 remains true when condition (6) is replaced by

$$d(Tx,Ty) \leqslant M \cdot d(x,y)+K \cdot \{d(y,Tx)+d(x,Ty)\} \text{ for all } x,y \in X, \quad (7)$$

see [5,10,49].

Example 2.9. Let $X = [0,1] \cup [\frac{3}{2}, \frac{5}{3}]$ with the usual metric $d(x,y) = |x-y|$ and $T : X \to X$ be given by $Tx = 0$, if $0 \leqslant x \leqslant 1$ and $Tx = 1$, if $\frac{3}{2} \leqslant x \leqslant \frac{5}{3}$. Then, $T0 = 0$ and

(a) T does not satisfy the Banach theorem, take $x = 1$ and $y = \frac{3}{2}$;
(b) T does not satisfy the Kannan theorem, take $x = 0$ and $y = \frac{3}{2}$;
(c) T is asymptotically regular;
(d) T is continuous;
(e) T satisfies (6) with $K = 2$ and any $0 \leqslant M < 1$.
Indeed, if $x, y \in [0,1]$ or $x, y \in [\frac{3}{2}, \frac{5}{3}]$, then $d(Tx,Ty) = 0$, when the condition (6) is obviously satisfied. If $x \in [0,1]$ and $y \in [\frac{3}{2}, \frac{5}{3}]$, then $d(Tx,Ty) = 1$ and $d(x,y) \geqslant \frac{1}{2}$, $d(x,Tx) + d(y,Ty) \geqslant x + y - 1 \geqslant \frac{1}{2}$. Therefore,

$$d(Tx,Ty) \leqslant M \cdot d(x,y) + 2 \cdot \{d(x,Tx) + d(y,Ty)\} \text{ for all } x,y \in X,$$

and any $0 \leqslant M < 1$.

When $M = 0$, then from Theorem 2.6 we have a significant extension of Banach's theorem in a new direction:

Corollary 2.10. *Let (X,d) be a complete metric space and $T : X \to X$ be a continuous and asymptotically regular mapping satisfying (4) with $0 \leqslant K < +\infty$ (especially, $K \geqslant 1$), then T has a unique fixed point $p \in X$ and $T^n x \to p$ for each $x \in X$.*

Remark 2.11. Note that each contraction with constant $L < 1$ satisfies (4) with constant $K = \frac{L}{1-L}$. Indeed, for all $x, y \in X$,

$$d(Tx,Ty) \leqslant L \cdot d(x,y) \leqslant L \cdot \{d(x,Tx) + d(Tx,Ty) + d(Ty,y)\},$$

so

$$d(Tx,Ty) \leqslant \frac{L}{1-L} \cdot \{d(x,Tx) + d(y,Ty)\}.$$

Therefore, all contractions satisfy the assumptions of Corollary 2.10.

Example 2.12. Let $T : [0, \frac{\pi}{2}] \to \mathbb{R}$ be defined by $Tx = \cos x$. T is not a contraction. Suppose there exists $L \in (0,1)$ such that

$$\left| \frac{\cos x - \cos y}{x-y} \right| \leqslant L \quad \text{for all} \quad x \neq y.$$

Letting $y \to x$, we get $|\sin x| \leqslant L$ for all $x, y \in [0, \frac{\pi}{2}]$, which is false. For $x, y \in [0, \frac{\pi}{2}]$ we have

$$\begin{aligned} |\cos x - \cos y| &= \left| -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2} \right| \leqslant |x-y| \\ &\leqslant |x-1| + |y-1| \leqslant |x - \cos x| + |y - \cos y|. \end{aligned}$$

Obviously T is continuous and if $x_n = T^n x = \cos x_{n-1}$, $x_0 \in [0, \frac{\pi}{2}]$, then

$$|T^{n+1}x - T^n x| = |\cos x_n - \cos x_{n-1}| = |\sin c_n| \cdot |x_n - x_{n-1}|,$$

where a point c_n is between x_n and x_{n-1}, so $|\sin c_n| < 1$ for $n = 1, 2, \ldots$ Hence T is an asymptotically regular mapping. Corollary 2.10 guarantees that T has a unique fixed point $p \in [0, \frac{\pi}{2}]$ and $p = \lim_{n\to\infty} x_n$, where $x_{n+1} = \cos x_n$, $x_0 \in [0, \frac{\pi}{2}]$. Approximate solution of the equation $x = \cos x$ is $p \approx 0.739$.

Example 2.13. Let $X = \mathbb{R}$ be with the usual metric and let T be defined as follows: $T0 = 0$ and $Tx = \frac{x}{2} \sin \frac{1}{x}$ if $x \neq 0$. Taking $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$, we obtain

$$|Tx - Ty| = \frac{8}{3\pi} > \frac{4}{3\pi} = |x - y|.$$

Taking $x = 0$ and $y = \frac{2}{\pi}$, we have

$$\left|T0 - T\left(\frac{2}{\pi}\right)\right| = \frac{1}{\pi} = |0 - T0| + \left|\frac{2}{\pi} - T\left(\frac{2}{\pi}\right)\right|,$$

so there is no universal constant $K < 1$ satisfying (4), therefore Corollary 2.10 is also some extension of Theorem 3.1 from [25]. On the other hand,

$$|Tx - Ty| \leqslant |Tx| + |Ty| \leqslant \frac{|x|}{2} + \frac{|y|}{2} = \left|x - \frac{x}{2}\right| + \left|y - \frac{y}{2}\right| \leqslant |x - Tx| + |y - Ty|$$

for all $x, y \in \mathbb{R}$. Obviously T is asymptotically regular and continuous, therefore, all assumptions of the Corollary 2.10 are fulfilled.

Remark 2.14. For clarity of this presentation we omit discussion in $b-$metric spaces (see [33]) and $G-$metric spaces (see [1]) and consideration of semigroups [20].

Conclusion. In this part, we presented a new extension of Banach's theorem with examples.

3. Fixed point theorems in preordered sets

An interplay between the order and metrical structure of the space turned out to be very fruitful. In Refs. [36,37], we find an analogue of Banach theorem in partially ordered sets, further extensions are contained in [24,42,43]. In all these works, the mapping considered are monotone. For such mappings one of the fundamental results in fixed-point theory is the classical Knaster–Tarski theorem (also known as the Abian–Brown theorem), see [26], [38]. Recently, Espínola and Wiśnicki [15] studied the problem whether the classical Kirk's theorem for nonexpansive mappings (see [17]) still holds for monotone-nonexpansive mappings. They proved in some partially ordered sets a general theorem which guarantees the existence of a fixed point for monotone mappings (which need not be either monotone-nonexpansive nor continuous), and which does not impose any conditions on the Banach space.

An interesting reference with many applications of the fixed point theory of monotone mappings is [9].

In this section, we extend Corollary 2.10 on preordered metric spaces, where a preordered binary relation is weaker than a partial order. The key feature in this theorem is that the Kannan's type condition on the map is only assumed to hold on elements that are comparable but not on the entire set on which they are defined, see Example 3.9.

Definition 3.1. Let $X \neq \emptyset$ be a set. Binary relation $\preccurlyeq$ on X is

(a) *reflexive* if $x \preccurlyeq x$ for all $x \in X$,
(b) *transitive* if $x \preccurlyeq z$ for all $x, y, z \in X$ such that $x \preccurlyeq y$ and $y \preccurlyeq z$.

A reflexive and transitive relation on X is a *preordered* on X. In such case $(X, \preccurlyeq)$ is a *preordered space*. Write $x \prec y$ when $x \preccurlyeq y$ and $x \neq y$. We will say that $x, y \in X$ are *comparable* whenever $x \preccurlyeq y$ or $y \preccurlyeq x$.

Example 3.2. Let $\preccurlyeq$ be the binary relation on $\mathbb{R}$ given by

$$x \preccurlyeq y \;\Leftrightarrow\; (x = y \text{ or } x < y \leqslant 0).$$

Then $\preccurlyeq$ is a partial order (and so preordered) on $\mathbb{R}$, but it is different from $\leqslant$.

Definition 3.3. A *preordered metric space* is a triple $(X, d, \preccurlyeq)$ where (X, d) is a metric space and $\preccurlyeq$ is a preordered on X.

One of the most important hypothesis that we shall use in this section is the monotonicity of the involved mappings.

Definition 3.4. Let $\preccurlyeq$ be a binary relation on X. A map $T : X \to X$ is *monotone* if $Tx \preccurlyeq Ty$ whenever $x \preccurlyeq y$.

The following result is the extension of Corollary 2.10 to Kannan's type mappings on preordered metric spaces.

Theorem 3.5. *Let $(X, d, \preccurlyeq)$ be a preordered metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(i) *(X, d) is complete,*
(ii) *T is monotone,*
(iii) *T is continuous,*
(iv) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,*
(v) *T is asymptotically regular, i.e. $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for all $x \in X$,*
(vi) *for all $x, y \in X$ with $x \preccurlyeq y$,*

$$d(Tx, Ty) \leqslant K \cdot \{d(x, Tx) + d(y, Ty)\} \text{ for some } 0 \leqslant K < +\infty. \tag{8}$$

Then there exists a fixed point of T, and it is unique, say u, if

$$\forall_{(x,y)\in X\times X} \exists_{w\in X} \; (x \preccurlyeq w \text{ and } y \preccurlyeq w). \tag{9}$$

Moreover, for each $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, the sequence $\{T^n x_0\}$ of iterates converges to u.

Proof. Let $x_0 \in X$ be a point satisfying (iv), that is, $x_0 \preccurlyeq Tx_0$. We define a sequence $\{x_n\} \subset X$ as follows

$$x_n = Tx_{n-1}, \; n \geqslant 1. \tag{10}$$

Regarding that T is a monotone mapping together with (10) we have

$$x_0 \preccurlyeq Tx_0 = x_1 \text{ implies } x_1 = Tx_0 \preccurlyeq Tx_1 = x_2.$$

Inductively, we obtain

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \cdots \preccurlyeq x_{n-1} \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots$$

Now, by triangle rule and asymptotic regularity, for $m > n$, we get

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &\leqslant d(x_n, x_{n+1}) + K \cdot \{d(x_n, x_{n+1}) + d(x_m, x_{m+1})\} + d(x_m, x_{m+1}) \\ &= (K+1) \cdot \{d(x_n, x_{n+1}) + d(x_m, x_{m+1})\} \to 0 \end{aligned}$$

as $m > n \to \infty$. This implies that $\{x_n\}$ is a Cauchy sequence in X. From the completeness of X there exists $u \in X$ such that $x_n \to u$. Because T is continuous and $x_{n+1} = Tx_n$, it follows that $u = Tu$.

To prove uniqueness, we assume that $v \in X$ is another fixed point of T such that $u \neq v$. By hypothesis, there exists $w \in X$ such that $u \preccurlyeq w$ and $v \preccurlyeq w$.

Let $\{w_n = Tw_{n-1}\}$ be the sequence of successive approximations of T based on $w_0 = w$. As T is monotone, $v = Tv \preccurlyeq Tw = w_1$ and $u = Tu \preccurlyeq Tw = w_1$. By induction, $v \preccurlyeq w_n$ and $u \preccurlyeq w_n$ for all $n \geqslant 0$.

Case 1. If $v = w_{n_0}$ for some $n_0 \geqslant 0$, then $v = Tv = Tw_{n_0} = w_{n_0+1}$ and by induction, $w_n = v$ for all $n \geqslant n_0$, so $w_n \to v$.

Case 2. If $v \prec w_n$ for all $n \geqslant 0$, then

$$\begin{aligned} d(v, w_{n+1}) = d(Tv, Tw_n) &\leqslant K \cdot \{d(v, Tv) + d(w_n, Tw_n)\} \\ &= K \cdot \{d(v, Tv) + d(T^n w, T^{n+1} w)\} \to 0, \end{aligned}$$

as $n \to \infty$, by asymptotic regularity. Hence $w_n \to v$.

Thus $w_n \to v$ and $w_n \to u$. The uniqueness of the limit concludes that $u = v$, so T has a unique fixed point. □

Remark 3.6. Theorem 3.5 remains true when condition (6) is satisfied (with $0 \leqslant M < 1$ and $0 \leqslant K < +\infty$) in place of condition (8). Then in *Case 2* we have an estimate:

$$\begin{aligned} d(v, w_{n+1}) &= d(Tv, Tw_n) \\ &\leqslant M \cdot d(v, w_n) + K \cdot \{d(v, Tv) + d(w_n, Tw_n)\} \\ &\leqslant M \cdot \{d(v, w_{n+1}) + d(w_{n+1}, w_n)\} + K \cdot \{d(v, Tv) + d(w_n, Tw_n)\} \end{aligned}$$

so

$$(1 - M) \cdot d(v, w_{n+1}) \leqslant M \cdot d(w_{n+1}, w_n) + K \cdot \{d(v, Tv) + d(w_n, Tw_n)\} \to 0$$

as $n \to \infty$. Hence $w_n \to v$. Then, we act as in the proof of Theorem 3.5.

After the appearance of the Ran and Reurings' result [37], Nieto and Rodríguez-López [36] changed the continuity of the mapping T with the condition nondecreasing regularity (Definition 3.7). Now, we exchanged the continuity of the mapping T with the condition nondecreasing regularity and we obtain in preordered metric spaces an analogue of [25, Theorem 3.1].

Definition 3.7. Let (X,d) be a metric space, let $A \subset X$ be a nonempty subset and let $\preccurlyeq$ be a binary relation on X. Then triple $(A,d,\preccurlyeq)$ is said to be *nondecreasing regular* if for all sequence $\{x_n\} \subset A$ such that $\{x_n\} \to x \in A$ and $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$, we have that $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.

Theorem 3.8. *Let $(X,d,\preccurlyeq)$ be a preordered metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

- (i) *(X,d) is complete,*
- (ii) *T is monotone,*
- (iii) *$(X,d,\preccurlyeq)$ is nondecreasing regular,*
- (iv) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,*
- (v) *T is asymptotically regular, i.e. $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for all $x \in X$,*
- (vi) *for all $x,y \in X$ with $x \preccurlyeq y$,*

$$d(Tx,Ty) \leqslant K \cdot \{d(x,Tx) + d(y,Ty)\} \text{ for some } 0 \leqslant K < 1.$$

Then there exists a fixed point of T, and it is unique, say u, if (9) is satisfied. Moreover, for each $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, the sequence $\{T^n x_0\}$ of iterates converges to u.

Proof. Following the proof of Theorem 3.5, we have a monotone (nondecreasing) sequence $\{x_n = Tx_{n-1}\}$ which is convergent to $u \in X$. Due to (iii), we have $x_n \preccurlyeq u$ for all $n \geqslant 1$. Now, we show that u is a fixed point of T. Fix an $\varepsilon > 0$. Since $T^n x_0 \to u$, given $\frac{\varepsilon}{2} > 0$, there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$,

$$d(T^n x_0, u) < \frac{\varepsilon}{2}(1-K) \quad \text{and} \quad d(T^n x_0, T^{n+1}x_0) < \frac{\varepsilon}{2}(1-K).$$

Taking $n \geqslant n_1$ and using that $T^n x_0 \preccurlyeq u$ for all $n \in \mathbb{N}$, we get

$$\begin{aligned} d(Tu,u) &\leqslant d(Tu, T^{n+1}x_0) + d(T^{n+1}x_0, u) \\ &\leqslant K \cdot \{d(u,Tu) + d(T^n x_0, T^{n+1}x_0)\} + d(T^{n+1}x_0, u), \end{aligned}$$

so

$$\begin{aligned} d(Tu,u) &\leqslant \frac{K}{1-K} d(T^n x_0, T^{n+1}x_0) + \frac{1}{1-K} d(T^{n+1}x_0, u) \\ &< \frac{1}{1-K} d(T^n x_0, T^{n+1}x_0) + \frac{1}{1-K} d(T^{n+1}x_0, u) < \varepsilon. \end{aligned}$$

In consequence, since $\varepsilon > 0$ is arbitrary, $d(Tu,u) = 0$. Hence $u = Tu$. Uniqueness of u can be observed as in the proof of Theorem 3.5. □

Observe that condition (4) with $0 \leqslant K < 1$, see [25, Theorem 3.1], is slightly stronger than condition (vi) of Theorem 3.8, which only requires the inequality for comparable points, that is, for all $x,y \in X$ such that $x \preccurlyeq y$ or $y \preccurlyeq x$.

Example 3.9. Let $X = [-1,1]$ be endowed with the metric $d(x,y) = |x-y|$ for all $x,y \in X$. Consider on X the partial order

$$x \preccurlyeq y \iff (x = y \text{ or } x < y \leqslant 0).$$

Define $T : X \to X$ by $Tx = -\frac{x}{4}$ for $-1 \leqslant x \leqslant 0$ and $Tx = \frac{9}{10}x$ for $0 < x \leqslant 1$. Obviously mapping T is asymptotically regular. Let $x, y \in X$ be such that $x \preccurlyeq y$. If $x = y$, then

$$d(Tx, Ty) = 0 \leqslant \frac{1}{3} \cdot \{d(x, Tx) + d(y, Ty)\}$$

trivially holds. Assume that $x \neq y$. Then $x < y \leqslant 0$. Hence

$$d(Tx, Ty) \leqslant \frac{1}{4}(|x| + |y|) \quad \text{and} \quad d(x, Tx) + d(y, Ty) \geqslant \frac{3}{4}(|x| + |y|).$$

Thus

$$d(Tx, Ty) \leqslant \frac{1}{3} \cdot \{d(x, Tx) + d(y, Ty)\}$$

for all $x, y \in [-1, 0]$. Hence (vi) holds. However, condition (4) with $0 \leqslant K < 1$ is false in this case because if $x = 0$ and $y = 1$, then $d(T(0), T(1)) = \frac{9}{10} > \frac{1}{10} = d(0, T(0)) + d(1, T(1))$.

In the next example, we have shown that if condition (9) in Theorems 3.5 and 3.8 fails, it is possible to find examples of functions T with more than one fixed point.

Example 3.10. Let $X = \{(1, 0), (0, 1)\} \subset \mathbb{R}^2$, and consider the order

$$(x, y) \preccurlyeq (z, t) \iff (x \leqslant z \ \text{ and } \ y \leqslant t).$$

Thus, $(X, \preccurlyeq)$ is a partially ordered set, whose different elements are not comparable. The metric space (X, d) with the Euclidean distance is a complete metric space. The identity map $T(x, y) = (x, y)$ is trivially continuous, asymptotically regular, non-decreasing and condition

$$d(T(x, y), T(z, t)) \leqslant K \cdot \{d((x, y), T(x, y)) + d((z, t), T(z, t))\},$$

holds for any $0 \leqslant K < +\infty$, since elements in X are only comparable to themselves. Moreover, $(1, 0) \preccurlyeq T(1, 0) = (1, 0)$. In this case, there are two fixed points in X. Hypotheses in Theorem 3.5 hold. Theorem 3.8 is also applicable since if $\{(x_n, y_n)\} \subset X$ is a monotone (nondecreasing) sequence converging to $(x, y) \in X$, then necessarily $(x_n, y_n)\}$ is a constant sequence and $(x_n, y_n) = (x, y)$ for all $n \in \mathbb{N}$, so the limit (x, y) is an upper bound for all the terms in the sequence.

This shows that conditions in Theorems 3.5 and 3.8 do not imply uniqueness of the fixed point.

In this example, condition (9) does not hold, since given two different elements in X, there is no upper bound of them. In this case, T may have more than one fixed point.

Conclusion. In this section, we discussed the extension of Banach's theorem in preordered metric spaces.

4. Random fixed-point theorems

An interesting aspect of the nonlinear analysis is to randomize deterministic fixed-point theorems of nonlinear mappings. The study of fixed-point theorems for random operators was initiated by the Prague school of probability research. The first results were studied in 1955–1956 by Špaček and Hanš in the context of Fredholm integral equations with random kernel, see for instance [45]. In a separable metric space, random fixed-point theorems for contraction mappings were proved by Hanš [27] (for some set-valued mappings see [40]).

In many cases, the mathematical models or equations used to describe phenomena in biology, physics, engineering contain certain parameters whose values are unknown. Then, it is more realistic to consider such equations as random operator equations. These equations are much more difficult to handle mathematically than deterministic equations [6].

It has been shown that when the underlying measurable space (Ω, Σ) is a Suslin family (see [47] for definitions), a deterministic fixed-point theorem may, in general, correspond to a random fixed-point theorem. However, it is unknown if the same is true when the measurable space (Ω, Σ) is not a Suslin family.

Nieto et al. [35] proved the random version in partially ordered metric spaces of the classical Banach contraction principle. In this section, we will prove some random fixed point theorems for single-valued operators which are asymptotically regular and satisfies some Kannan's type conditions.

Let (Ω, Σ) be a measurable space with Σ a σ-algebra of subsets of Ω. For a metric space (X, d), we denote by $CL(X)$ the family of all nonempty closed subsets of X.

Definition 4.1. A set-valued operator $F : \Omega \to 2^X$ is called Σ-*measurable* if for any open subset B of X, the set $F^{-1}(B) = \{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\}$ belongs to Σ.

Definition 4.2. A measurable (single-valued) operator $x : \Omega \to X$ is called a *selector* for a measurable set-valued operator $F : \Omega \to 2^X$ if $x(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

Definition 4.3. A mapping $T : \Omega \times X \to X$ is called a *random operator* if for each $x \in X$, the map $T(\cdot, x) : \Omega \to X$ is measurable.

Definition 4.4. A measurable operator $x : \Omega \to X$ is said to be a *random fixed point* of random operator $T : \Omega \times X \to X$ if $T(\omega, x(\omega)) = x(\omega)$ for all $\omega \in \Omega$.

Equivalently, it is a measurable selection for the set-valued map $Fix\ T : \Omega \to 2^X$ defined by $Fix\ T(\omega) = \{x \in X : T(\omega, x) = x\}$.

We recall, a random mapping $T : \Omega \times X \to X$ is said to be continuous if for each fixed $\omega \in \Omega$, the map $T(\omega, \cdot) : X \to X$ has this particular property.

We will list the following results related to the concept of measurability.

Theorem 4.5 [29]. *Let (Ω, Σ) be a measurable space, X be a separable metric space and Y a metric space. If $T : \Omega \times X \to Y$ is measurable in $\omega \in \Omega$ and* *continuous in* $x \in X$*, respectively, and if* $x : \Omega \to X$ *is measurable, then* $T(\cdot, x(\cdot)) : \Omega \to X$ *is a measurable.*

Theorem 4.6 [47]. *Let* (Ω, Σ) *be a measurable space,* (Y, d) *be a Polish space (i.e. complete and separable metric space) and* $F : \Omega \to CL(Y)$ *a measurable map. Then* F *has a measurable selection.*

Definition 4.7. Let (Ω, Σ) be a measurable space and (X, d) be a metric space. A random operator $T : \Omega \times X \to X$ is said to be *asymptotically regular* if for each fixed $\omega \in \Omega$,

$$\lim_{n\to\infty} d(T^n(\omega, x), T^{n+1}(\omega, x)) = 0$$

for all $x \in X$. Here $T^n(\omega, x)$ is the value at x of the nth iterate of the map $T(\omega, \cdot)$, i.e. $T^n(\omega, x) = T(\omega, T^{n-1}(\omega, x))$.

The following result is the randomization of Theorem 2.6.

Theorem 4.8. *Let* (Ω, Σ) *be a measurable space and* (X, d) *be a Polish space. If* $T : \Omega \times X \to X$ *is a random continuous operator which is asymptotically regular, and exists functions* $M : \Omega \to [0, 1)$ *and* $K : \Omega \to [0, \infty)$ *such that for each* $\omega \in \Omega$*,*

$$\begin{aligned} &d(T(\omega, x), T(\omega, y)) \\ &\quad \leqslant M(\omega) \cdot d(x, y) + K(\omega) \cdot \{d(x, T(\omega, x)) + d(y, T(\omega, y))\} \end{aligned} \tag{11}$$

for all $x, y \in X$*, then* T *has a unique random fixed point.*

(We do not assume the measurability of the functions $M(\cdot)$ and $K(\cdot)$.)

Proof. Fix a measurable function $x_0 : \Omega \to X$. If for each $\omega \in \Omega$, $T(\omega, x_0(\omega)) = x_0(\omega)$, then x_0 is a random fixed point of T. Suppose that, for some $\omega \in \Omega$, $T(\omega, x_0(\omega)) \neq x_0(\omega)$. We define the sequence

$$y_0(\omega) = x_0(\omega) \text{ and } y_n(\omega) = T(\omega, y_{n-1}(\omega)) = T^n(\omega, x_0(\omega)),$$

for all $\omega \in \Omega$ and integers $n \geqslant 1$. Using (11), by triangle rule and asymptotic regularity, we get for $m > n$,

$$\begin{aligned} &d(y_n(\omega), y_m(\omega)) \\ &\quad \leqslant d(y_n(\omega), y_{n+1}(\omega)) + d(y_{n+1}(\omega), y_{m+1}(\omega)) + d(y_{m+1}(\omega), y_m(\omega)) \\ &\quad \leqslant d(y_n(\omega), y_{n+1}(\omega)) + M(\omega) \cdot d(y_n(\omega), y_m(\omega)) \\ &\qquad + K(\omega) \cdot \{d(y_n(\omega), y_{n+1}(\omega)) + d(y_m(\omega), y_{m+1}(\omega))\} + d(y_{m+1}(\omega), y_m(\omega)), \end{aligned}$$

so

$$\begin{aligned} &(1 - M(\omega)) \cdot d(y_n(\omega), y_m(\omega)) \\ &\quad \leqslant (K(\omega) + 1) \cdot \{d(y_n(\omega), y_{n+1}(\omega)) + d(y_m(\omega), y_{m+1}(\omega))\} \to 0, \end{aligned}$$

as $m > n \to \infty$. Hence $\{y_n(\omega)\}_n$ is a Cauchy sequence for every $\omega \in \Omega$. Since X is a complete space there exists $y_*(\omega) \in X$, $\omega \in \Omega$, such that $y_*(\omega) = \lim_{n\to\infty} y_n(\omega)$. Since $y_0(\cdot)$ is measurable, then $y_1(\cdot)$ is measurable. Hence, by induction, we can easily prove that for each $n \in \mathbb{N}$, the function $\omega \to y_n(\omega)$ is measurable. The mapping $y_* : \Omega \to X$ is the pointwise limit of measurable mappings, so it is measurable.

Now, we show that y_* is a random fixed point of T, i.e. $y_*(\omega) = T(\omega, y_*(\omega))$, $\omega \in \Omega$. It is clear, for each $\omega \in \Omega$, by asymptotic regularity, we get

$$\lim_{n\to\infty} d(y_n(\omega), y_{n+1}(\omega)) = \lim_{n\to\infty} d(T^n(\omega, x_0(\omega)), T^{n+1}(\omega, x_0(\omega))) = 0,$$

and because $T(\omega, \cdot)$ is continuous,

$$d(y_*(\omega), T(\omega, y_*(\omega))) = \lim_{n\to\infty} d(y_n(\omega), T(\omega, y_n(\omega))) = 0.$$

Thus,

$$y_*(\omega) = T(\omega, y_*(\omega)) \text{ for each } \omega \in \Omega.$$

The uniqueness of the random fixed point follows from the uniqueness of $y_*(\omega)$ for every $\omega \in \Omega$. Suppose $z_*(\omega) = T(\omega, z_*(\omega))$ is another random fixed point of T, then by (11),

$$\begin{aligned} 0 < d(y_*(\omega), z_*(\omega)) &= d(T(\omega, y_*(\omega)), T(\omega, z_*(\omega))) \\ &\leqslant M(\omega) \cdot d(y_*(\omega), z_*(\omega)) \\ &\quad + K(\omega) \cdot \{d(y_*(\omega), T(\omega, y_*(\omega))) + d(z_*(\omega), T(\omega, z_*(\omega)))\} \\ &= M(\omega) \cdot d(y_*(\omega), z_*(\omega)) < d(y_*(\omega), z_*(\omega)), \end{aligned}$$

a contradiction. Hence $y_*(\omega) = z_*(\omega)$. □

Corollary 4.9. *Let (Ω, Σ) be a measurable space and (X, d) be a Polish space. If $T : \Omega \times X \to X$ is a random operator, and exists a function $M : \Omega \to [0, 1)$ such that for each $\omega \in \Omega$,*

$$d(T(\omega, x), T(\omega, y)) \leqslant M(\omega) \cdot d(x, y)$$

for all $x, y \in X$, then T has a unique random fixed point.

Corollary 4.10. *Let (Ω, Σ) be a measurable space and (X, d) be a Polish space. If $T : \Omega \times X \to X$ is a random continuous operator which is asymptotically regular, and exists a function $K : \Omega \to [0, \infty)$ such that for each $\omega \in \Omega$,*

$$d(T(\omega, x), T(\omega, y)) \leqslant K(\omega) \cdot \{d(x, T(\omega, x)) + d(y, T(\omega, y))\}$$

for all $x, y \in X$, then T has a unique random fixed point.

Now, we establish a random version of some fixed-point theorem in preordered metric spaces.

Theorem 4.11. *Let (Ω, Σ) be a measurable space, $(X, d, \preccurlyeq)$ be a Polish preordered metric space and let $T : \Omega \times X \to X$ be a mapping. Suppose that the following conditions hold:*

(i) *T is a continuous random operator,*
(ii) *for each $\omega \in \Omega$, the function $T(\omega, \cdot)$ is monotone operator, i.e.*

$$(x, y \in X \text{ and } x \preccurlyeq y) \implies T(\omega, x) \preccurlyeq T(\omega, y),$$

(iii) *there exists a random variable $x_0 : \Omega \to X$ with*

$$x_0(\omega) \preccurlyeq T(\omega, x_0(\omega)) \text{ or } x_0(\omega) \succcurlyeq T(\omega, x_0(\omega)) \text{ for each } \omega \in \Omega,$$

(iv) *T is asymptotically regular,*

(v) *there exists a function* $K : \Omega \to [0, \infty)$ *such that for each* $\omega \in \Omega$,

$$d(T(\omega, x), T(\omega, y)) \leqslant K(\omega) \cdot \{d(x, T(\omega, x)) + d(y, T(\omega, y))\}$$

for every comparable $x, y \in X$, *i.e.* $x \preccurlyeq y$ *or* $y \preccurlyeq x$.

Then there exists a random variable $x : \Omega \to X$ *which is a random fixed point of* T, *and it is unique if*

for every $x, y \in X$, *there exists* $z \in X$ *that is comparable to* x *and* y.

Proof. Fix a measurable function $x_0 : \Omega \to X$. If for each $\omega \in \Omega$, $T(\omega, x_0(\omega)) = x_0(\omega)$, then x_0 is a random fixed point of T. Suppose that, for some $\omega \in \Omega$, $T(\omega, x_0(\omega)) \neq x_0(\omega)$. We define a sequence

$$y_0(\omega) = x_0(\omega) \text{ and } y_n(\omega) = T(\omega, y_{n-1}(\omega)) = T^n(\omega, x_0(\omega)),$$

for all $\omega \in \Omega$ and integers $n \geqslant 1$. From the conditions (ii) and (iii), we have

$$y_0(\omega) = x_0 \preccurlyeq T(\omega, x_0(\omega)) = y_1(\omega)$$

implies

$$y_1(\omega) = T(\omega, y_0(\omega)) \preccurlyeq T(\omega, y_1(\omega)) = y_2(\omega).$$

Inductively, we obtain

$$y_0(\omega) \preccurlyeq y_1(\omega) \preccurlyeq y_2(\omega) \preccurlyeq \cdots \preccurlyeq y_{n-1}(\omega) \preccurlyeq y_n(\omega) \preccurlyeq y_{n+1}(\omega) \preccurlyeq \cdots,$$

or in the second case

$$y_0(\omega) \succcurlyeq y_1(\omega) \succcurlyeq y_2(\omega) \succcurlyeq \cdots \succcurlyeq y_{n-1}(\omega) \succcurlyeq y_n(\omega) \succcurlyeq y_{n+1}(\omega) \succcurlyeq \cdots$$

Using triangle rule and asymptotic regularity, we get for $m > n$,

$$\begin{aligned} & d(y_n(\omega), y_m(\omega)) \\ & \quad \leqslant d(y_n(\omega), y_{n+1}(\omega)) + d(y_{n+1}(\omega), y_{m+1}(\omega)) + d(y_{m+1}(\omega), y_m(\omega)) \\ & \quad = d(y_n(\omega), y_{n+1}(\omega)) + d(T(\omega, y_n(\omega)), T(\omega, y_m(\omega))) + d(y_{m+1}(\omega), y_m(\omega)) \\ & \quad \leqslant (K(\omega) + 1) \cdot \{d(y_n(\omega), y_{n+1}(\omega)) + d(y_m(\omega), y_{m+1}(\omega)) \to 0, \end{aligned}$$

as $m > n \to \infty$. So $\{y_n(\omega)\}_n$ is a Cauchy sequence for every $\omega \in \Omega$. Since X is a complete space there exists $y_*(\omega) \in X$, $\omega \in \Omega$, such that $y_*(\omega) = \lim_{n \to \infty} y_n(\omega)$. Since $y_0(\cdot)$ is measurable, then $y_1(\cdot)$ is measurable. Hence, by induction, we can easily prove that for each $n \in \mathbb{N}$, the function $\omega \to y_n(\omega)$ is measurable. The mapping $y_* : \Omega \to X$ is the pointwise limit of measurable mappings, so it is measurable.

Now, we show that y_* is a random fixed point of T, i.e. $y_*(\omega) = T(\omega, y_*(\omega))$, $\omega \in \Omega$. It is clear, for each $\omega \in \Omega$, by asymptotic regularity, we get

$$\lim_{n \to \infty} d(y_n(\omega), y_{n+1}(\omega)) = \lim_{n \to \infty} d(T^n(\omega, x_0(\omega)), T^{n+1}(\omega, x_0(\omega))) = 0,$$

and because $T(\omega, \cdot)$ is continuous,

$$d(y_*(\omega), T(\omega, y_*(\omega))) = \lim_{n \to \infty} d(y_n(\omega), T(\omega, y_n(\omega))) = 0.$$

Thus,

$$y_*(\omega) = T(\omega, y_*(\omega)) \quad \text{for each} \quad \omega \in \Omega.$$

It remains for us to show that y_* is the unique random fixed point of T. We prove that, if we take any random variable $\overline{x}_0 : \Omega \to X$ and we define the sequence

$$\overline{y}_0(\omega) = \overline{x}_0(\omega) \quad \text{and} \quad \overline{y}_n(\omega) = T(\omega, \overline{y}_{n-1}(\omega)) = T^n(\omega, \overline{x}_0(\omega)),$$

for all $\omega \in \Omega$ and integers $n \geqslant 1$, we get $\overline{y}_n(\omega) \to y_*(\omega)$, as $n \to \infty$, for every $\omega \in \Omega$, where y_* is the random fixed point of T obtained in the previous part of the proof.

If $\overline{x}_0(\omega)$ is comparable to $x_0(\omega)$ for every $\omega \in \Omega$, it is obvious, since $T(\omega, \overline{x}_0(\omega))$ is comparable to $T(\omega, x_0(\omega))$ for every $\omega \in \Omega$, so that $\overline{y}_n(\omega)$ is comparable to $y_n(\omega)$ for every $\omega \in \Omega$. Hence, by (v) and asymptotic regularity

$$\begin{aligned} &d(y_n(\omega), \overline{y}_n(\omega)) = d(T(\omega, y_{n-1}(\omega)), T(\omega, \overline{y}_{n-1}(\omega))) \\ &\quad\leqslant K(\omega) \cdot \{d(y_{n-1}(\omega), T(\omega, y_{n-1}(\omega))) + d(\overline{y}_{n-1}(\omega), T(\omega, \overline{y}_{n-1}(\omega)))\} \\ &\quad= K(\omega) \cdot \{d(y_{n-1}(\omega), y_n(\omega)) + d(\overline{y}_{n-1}(\omega), \overline{y}_n(\omega))\} \to 0,\ \omega \in \Omega, \end{aligned}$$

as $n \to \infty$. Therefore

$$d(\overline{y}_n(\omega), y_*(\omega)) \leqslant d(\overline{y}_n(\omega), y_n(\omega)) + d(y_n(\omega), y_*(\omega)) \to 0,$$

as $n \to \infty$, and $\overline{y}_n(\omega) \to y_*(\omega)$, as $n \to \infty$, for every $\omega \in \Omega$.

On the other hand, for an arbitrary random variable $\overline{x}_0 : \Omega \to X$ then, for each $\omega \in \Omega$, there exists $z(\omega) \in X$ that is comparable to $x_0(\omega)$ and $\overline{x}_0(\omega)$ simultaneously, then if we define the sequence

$$z_0(\omega) = z(\omega) \text{ and } z_n(\omega) = T(\omega, z_{n-1}(\omega)) = T^n(\omega, z(\omega)),$$

for all $\omega \in \Omega$ and integers $n \geqslant 1$, then $y_n(\omega)$ is comparable to $z_n(\omega)$, for every $\omega \in \Omega$, and $\overline{y}_n(\omega)$ is comparable to $z_n(\omega)$, for every $\omega \in \Omega$. Therefore

$$\begin{aligned} &d(y_n(\omega), \overline{y}_n(\omega)) \leqslant d(y_n(\omega), z_n(\omega)) + d(z_n(\omega), \overline{y}_n(\omega)) \\ &\quad= d(T(\omega, y_{n-1}(\omega)), T(\omega, z_{n-1}(\omega))) + d(T(\omega, z_{n-1}(\omega)), T(\omega, \overline{y}_{n-1}(\omega))) \\ &\quad\leqslant K(\omega) \cdot \{d(y_{n-1}(\omega), y_n(\omega)) + d(z_{n-1}(\omega), z_n(\omega))\} \\ &\qquad+K(\omega) \cdot \{d(z_{n-1}(\omega), z_n(\omega)) + d(\overline{y}_{n-1}(\omega), \overline{y}_n(\omega))\} \to 0,\ \omega \in \Omega, \end{aligned}$$

as $n \to \infty$, which proves that $\overline{y}_n(\omega) \to y_*(\omega)$, as $n \to \infty$, for every $\omega \in \Omega$. This proves the theorem. □

Remark 4.12. Theorem 4.11 remains true when condition (v) is replaced by

(v′) *there exists functions* $M : \Omega \to [0, 1)$ *and* $K : \Omega \to [0, \infty)$ *such that for each* $\omega \in \Omega$,

$$\begin{aligned} &d(T(\omega, x), T(\omega, y)) \\ &\quad\leqslant M(\omega) \cdot d(x, y) + K(\omega) \cdot \{d(x, T(\omega, x)) + d(y, T(\omega, y))\} \end{aligned}$$

for every comparable $x, y \in X$, *i.e.* $x \preccurlyeq y$ *or* $y \preccurlyeq x$.

Conclusion. In this part, we presented the extension of Banach's theorem in stochastic situations.

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