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Pointwise mutipliers of Orlicz function spaces and factorization

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Abstract In the paper we find representation of the space of pointwise multipliers between two Orlicz function spaces, which appears to be another Orlicz space and the formula for the Young function generating this space is given. Further, we apply this result to find necessary and sufficient conditions for factorization of Orlicz function spaces.

Keywords Orlicz spaces · Pointwise multipliers · Factorization

Mathematics Subject Classification 46E30 · 46B42

1 Introduction

Given two Orlicz spaces $L^{\varphi_1}, L^{\varphi}$ over the same measure space, the space of pointwise multipliers $M(L^{\varphi_1}, L^{\varphi})$ is the space of all functions x, such that $xy \in L^{\varphi}$ for each $y \in L^{\varphi_1}$, equipped with the operator norm. The problem of identifying such spaces was investigated by many authors, starting from Shragin [14], Ando [1], O'Neil [11] and Zabreiko–Rutickii [16], who gave a number of partial answers.

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These investigations were continued in number of directions and results were presented in different forms. One of them is the following result from Maligranda–Nakaii paper [8], which states that if for two given Young functions φ, φ_1 there is a third one φ_2 satisfying

$$\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}, \tag{1.1}$$

then

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_2}. \tag{1.2}$$

This result, however, neither gives any information when such a function φ_2 exists, nor says anything how to find it. Further, it was proved in [5, Cor. 6.2] that condition (1.1) is necessary for a wide class of φ, φ_1 functions satisfying some additional properties, but at the same time Example 7.8 from [5] ensures that in general it is not a case, i.e. there are functions φ, φ_1 such that no Young function φ_2 satisfies (1.1), while $M(L^{\varphi_1}, L^{\varphi}) = L^{\infty}$, which is also Orlicz space generated by the function φ_2 defined as $\varphi_2(t) = 0$ for $0 \le t \le 1$ and $\varphi_2(t) = \infty$ for $1 < t$. In particular, these functions do not satisfy (1.1), although (1.2) holds.

On the other hand, there is a natural candidate for a function φ_2 satisfying

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_2}.$$

Such a function is the following generalization of Young conjugate function (a kind of generalized Legendre transform considered also in convex analysis, for example in [15]) defined for two Orlicz functions φ, φ_1 as

$$\varphi \ominus \varphi_1(t) = \sup_{s>0}\{\varphi(st) - \varphi_1(s)\}.$$

The function $\varphi \ominus \varphi_1$ is called to be conjugate to φ_1 with respect to φ.

Also in [5] this construction was compared with condition (1.1) and it happens that very often $\varphi_2 = \varphi \ominus \varphi_1$ satisfies (1.1) (cf. [5, Thm. 7.9]), but once again Example 7.8 from [5] shows that $\varphi_2 = \varphi \ominus \varphi_1$ need not satisfy (1.1). In that example, anyhow, there holds $L^{\infty} = L^{\varphi \ominus \varphi_1}$, so that $M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}$. Therefore, it is natural to expect that in general

$$M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}, \tag{1.3}$$

as was already conjectured in [5]. In fact, such theorem was stated for Orlicz N-functions by Maurey in [10], but his proof depends heavily on the false conjecture, that the construction $\varphi \ominus \varphi_1$ enjoys involution property, i.e. $\varphi \ominus (\varphi \ominus \varphi_1) = \varphi_1$ (see Example 7.12 in [5] for counterexample).

On the other hand, the formula (1.3) was already proved for Orlicz sequence spaces by Djakov and Ramanujan in [4], where they used a slightly modified construction $\varphi \ominus \varphi_1$ (the supremum is taken only over $0 < s \le 1$). This modification appeared to be appropriate for sequence case, because then only behaviour of Young functions for small arguments is important, while cannot be used for function spaces. Anyhow, we will borrow some ideas from [4].

In our main Theorem 1 we prove that (1.3) holds in full generality for Orlicz function spaces, as well over finite and infinite nonatomic measure. Then we use this result to find that $\varphi_2 = \varphi \ominus \varphi_1$ satisfies (1.1) if and anly if L^{φ_1} factorizes L^{φ}, which completes the discussion from [6].

2 Notation and preliminaries

Let $L^0 = L^0(\Omega, \Sigma, \mu)$ be the space of all classes of equivalence (with respect to equality μ-a.e.) of μ-measurable, real valued functions on Ω, where (Ω, Σ, μ) is a σ-finite complete measure space. A Banach space $X \subset L^0$ is called the *Banach ideal space* if it satisfies the so called ideal property, i.e. $x \in L^0$, $y \in X$ with $|x| \leq |y|$ implies $x \in X$ and $\|x\|_X \leq \|y\|_X$ (here $|x| \leq |y|$ means that $|x(t)| \leq |y(t)|$ for μ-a.e. $t \in \Omega$), and it contains a *weak unit*, i.e. a function $x \in X$ such that $x(t) > 0$ for μ-a.e. $t \in \Omega$. When (Ω, Σ, μ) is purely nonatomic measure space, the respective space is called *Banach function space* (abbreviation B.f.s.), while in case of $\mathbb{N}$ with counting measure we shall speak about *Banach sequence space*. A Banach ideal space X satisfies the *Fatou property* when given a sequence $(x_n) \subset X$, satisfying $x_n \uparrow x$ μ-a.e. and $\sup_n \|x_n\|_X < \infty$, there holds $x \in X$ and $\|x\|_X \leq \sup_n \|x_n\|_X$.

Writing $X = Y$ for two B.f.s. we mean that they are equal as set, but norms are just equivalent. Recall also that for two Banach ideal spaces X, Y over the same measure space, the inclusion $X \subset Y$ is always continuous, i.e. there is $c > 0$ such that $\|x\|_Y \leq c\|x\|_X$ for each $x \in X$.

Given two Banach ideal spaces X, Y over the same measure space (Ω, Σ, μ), the space of pointwise multipliers from X to Y is defined as

$$M(X, Y) = \{y \in L^0 : xy \in Y \text{ for all } y \in X\}$$

with the natural operator norm

$$\|y\|_{M(X,Y)} = \sup_{\|x\|_X \leq 1} \|xy\|_Y.$$

When there is no risk of confusion we will just write $\|\cdot\|_M$ for the norm of $M(X, Y)$. A space of pointwise multipliers may be trivial, for example for nonatomic measure space $M(L^p, L^q) = \{0\}$ when $1 \leq p < q$, and therefore it need not be a Banach function space in the sense of above definition. Anyhow, it is a Banach space with the ideal property (see for example [9]). To provide some intuition for multipliers let us recall that $M(L^p, L^q) = L^r$ when $p > q \geq 1$, $1/p + 1/r = 1/q$ and $M(X, L^1) = X'$, where X' is the Köthe dual of X (see [5,6,9] for more examples).

A function $\varphi : [0, \infty) \to [0, \infty]$ will be called a *Young function* if it is convex, non-decreasing and $\varphi(0) = 0$. We will need the following parameters

$$a_\varphi = \sup\{t \geq 0 : \varphi(t) = 0\} \text{ and } b_\varphi = \sup\{t \geq 0 : \varphi(t) < \infty\}.$$

A Young fuction φ is called *Orlicz function* when $b_\varphi = \infty$. For a Young function φ by φ^{-1} we understand the right-continuous inverse defined as $\varphi^{-1}(v) = \inf\{u \geq 0 : \varphi(u) > v\}$ for $v \geq 0$.

Let φ be a Young function. The *Orlicz space* L^φ is defined as

$$L^\varphi = \{x \in L^0 : I_\varphi(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where the modular I_φ is given by

$$I_\varphi(x) = \int_\Omega \varphi(|x|)d\mu$$

and the Luxemburg–Nakano norm is defined as

$$\|x\|_\varphi = \inf\left\{\lambda > 0 : I_\varphi\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

We point out here that the function $\varphi \equiv 0$ is excluded from the definition of Young functions, but we allow $\varphi(u) = \infty$ for each $u > 0$ and understand that in this case $L^\varphi = \{0\}$.

We will often use the following relation between norm and modular. For $x \in L^\varphi$

$$\|x\|_\varphi \leq 1 \Rightarrow I_\varphi(x) \leq \|x\|_\varphi\,, \tag{2.1}$$

(see for example [7]).

Given two Orlicz functions φ, φ_1, the conjugate function $\varphi \ominus \varphi_1$ of φ_1 with respect to φ is defined by

$$\varphi \ominus \varphi_1(u) = \sup_{0\leq s}\{\varphi(su) - \varphi_1(s)\},$$

for $u \geq 0$. Since we need to deal with Young functions, one may be confused by possibility of appearance of indefinite symbol $\infty - \infty$ in the above definition, when $b_\varphi, b_{\varphi_1} < \infty$. To avoid such a situation we understand that for Young functions φ, φ_1 the conjugate function $\varphi \ominus \varphi_1$ is defined as

$$\varphi \ominus \varphi_1(u) = \begin{cases} \sup_{0\leq s<\infty}\{\varphi(su) - \varphi_1(s)\}, & \text{when } b_{\varphi_1} = \infty, \\ \sup_{0<s<b_{\varphi_1}}\{\varphi(su) - \varphi_1(s)\}, & \text{when } b_{\varphi_1} < \infty \text{ and } \varphi_1(b_{\varphi_1}) = \infty, \\ \sup_{0<s\leq b_{\varphi_1}}\{\varphi(su) - \varphi_1(s)\}, & \text{when } b_{\varphi_1} < \infty \text{ and } \varphi_1(b_{\varphi_1}) < \infty. \end{cases}$$

Notice that it is just a natural generalization of conjugate function in a sense of Young, i.e. when $\varphi(u) = u$ we get the classical conjugate function φ_1^* to φ_1. Of course, functions φ, φ_1 and $\varphi \ominus \varphi_1$ satisfy the generalized Young inequality, i.e.

$$\varphi(uv) \leq \varphi \ominus \varphi_1(u) + \varphi_1(v)$$

for each $u, v \geq 0$.

We will also need the following construction.

Definition 1 For two Young functions φ, φ_1 and $0 < a < b_{\varphi_1}$ we define

$$\varphi \ominus_a \varphi_1(u) = \sup_{0 \le s \le a} \{\varphi(su) - \varphi_1(s)\}, u \ge 0.$$

Such defined function $\varphi \ominus_a \varphi_1$ enjoys the following elementary properties.

Lemma 2 *Let φ, φ_1 be two Young functions.*

(i) $\varphi \ominus_a \varphi_1$ *is Young function for each* $0 < a < b_{\varphi_1}$.
(ii) *For each* $t \ge 0$ *there holds*

$$\lim_{a \to b_{\varphi_1}^-} \varphi \ominus_a \varphi_1(u) = \varphi \ominus \varphi_1(u).$$

Remark 3 Notice that dilations of Young functions do not change Orlicz spaces, i.e. when φ is a Young function and ψ is defined by $\psi(u) = \varphi(au)$ for some $a > 0$, then $L^\varphi = L^\psi$. It gives a reason to expect that dilating φ, φ_1 results in dilation of $\varphi \ominus \varphi_1$. In fact, let φ, φ_1 be Young functions and put $\psi(u) = \varphi(au)$, $\psi_1(u) = \varphi_1(bu)$. Then

$$\psi \ominus \psi_1(u) = \sup_{0<s}(\varphi(aus) - \varphi_1(bs)) = \sup_{0<s}(\varphi(aus/b) - \varphi_1(s)) = \varphi \ominus \varphi_1(au/b).$$

Moreover, if $b_\varphi = b_{\varphi_1} < \infty$, then supremum in the definition of $\varphi \ominus \varphi_1$ is attained for each $u < 1$, i.e. for each $u < 1$ there is $0 < s < b_{\varphi_1}$ such that $\varphi \ominus \varphi_1(u) = \varphi(us) - \varphi_1(s)$. In particular, $b_{\varphi \ominus \varphi_1} = 1$.

Remark 4 Let us also recall that a fundamental function f_φ of an Orlicz space L^φ is defined for $0 \le t \le \mu(\Omega)$ as $f_\varphi(t) = \|\chi_A\|_\varphi$, where $\mu(A) = t$. Notice that it is well defined, since $\|\chi_A\|_\varphi$ does not depend on particular choice of measurable set $A \subset \Omega$ with $\mu(A) = t$ (in general Orlicz spaces belong to the class of the so called rearrangement invariant spaces—see for example [2] for respective definitions). Further, it is well known that f_φ is given by the formula $f_\varphi(t) = \frac{1}{\varphi^{-1}(1/t)}$, for $0 < t < \mu(\Omega)$ and $f_\varphi(0) = 0$. In particular, the fundamental function of L^φ is right-continuous at 0 if and only if $b_\varphi = \infty$, or equivalently, $b_\varphi = \infty$ if and only if for each $\varepsilon > 0$ there is $\delta > 0$ such that if $A \in \Sigma$, $\mu(A) < \delta$ then $\|\chi_A\|_\varphi < \varepsilon$.

3 Multipliers of Orlicz function spaces

Since now on we are interested only in Orlicz function spaces, so that the underlying measure space (Ω, Σ, μ) is understood to be purely nonatomic for all spaces below and to the end of the paper.

Lemma 5 *Let φ, φ_1 be Young functions such that $b_\varphi < \infty$ and $b_{\varphi_1} = \infty$. Then*

$$M(L^{\varphi_1}, L^\varphi) = \{0\}.$$

Proof The proof follows immediately from Proposition 3.2 in [5], since under our assumptions $L^{\varphi_1} \not\subset L^\infty$ but $L^\varphi \subset L^\infty$. □

Lemma 6 *Let φ, φ_1 be Young functions and $b_\varphi < \infty$. Then*

$$M(L^{\varphi_1}, L^\varphi) \subset L^\infty.$$

Proof Suppose that $M(L^{\varphi_1}, L^\varphi) \not\subset L^\infty$. Then there exists $0 \le y \in M(L^{\varphi_1}, L^\varphi)$ such that $\|y\|_M = 1$ and for each $n > 0$

$$\mu(\{t \in \Omega : y(t) \ge n\}) > 0.$$

Denote $A_n = \{t \in \Omega : y(t) \ge n\}$ for $n \in \mathbb{N}$. Then $\left\|n\chi_{A_n}\right\|_M \le 1$ and for A_{n_0} chosen in such a way that $\mu(A_{n_0}) < \infty$, it follows

$$\|y\|_M \ge \|n\chi_{A_n}\|_M \ge \frac{n}{\|\chi_{A_{n_0}}\|_{\varphi_1}}\|\chi_{A_n}\chi_{A_{n_0}}\|_\varphi = \frac{n}{\|\chi_{A_{n_0}}\|_{\varphi_1}}\|\chi_{A_n}\|_\varphi \ge \frac{nb_\varphi^{-1}}{\|\chi_{A_{n_0}}\|_{\varphi_1}},$$

for each $n > n_0$. This contradiction shows that $M(L^{\varphi_1}, L^\varphi) \subset L^\infty$. □

We are in a position to prove the main theorem.

Theorem 1 *Let φ, φ_1 be Young functions. Then*

$$M(L^{\varphi_1}, L^\varphi) = L^{\varphi \ominus \varphi_1}.$$

Proof The inclusion

$$L^{\varphi \ominus \varphi_1} \subset M(L^{\varphi_1}, L^\varphi) \tag{3.1}$$

is well known (see [1,5,8] or [11]) and follows from equivalence of generalized Young inequality and inequality $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \lesssim \varphi^{-1}$. For the completeness of presentation we present the proof which employs the generalized Young inequality directly. If $\varphi \ominus \varphi_1(u) = \infty$ for each $u > 0$ then $L^{\varphi \ominus \varphi_1} = \{0\}$ and inclusion trivially holds. Suppose $L^{\varphi \ominus \varphi_1} \ne \{0\}$, i.e. $\varphi \ominus \varphi_1(u) < \infty$ for some $u > 0$. Let $y \in L^{\varphi \ominus \varphi_1}$ and $x \in L^{\varphi_1}$ be such that

$$\|y\|_{\varphi \ominus \varphi_1} \le \frac{1}{2} \text{ and } \|x\|_{\varphi_1} \le \frac{1}{2}.$$

Then generalized Young inequality gives

$$I_\varphi(yx) \le I_{\varphi \ominus \varphi_1}(y) + I_{\varphi_1}(x) \le 1.$$

Consequently $yx \in L^\varphi$ and $\|yx\|_\varphi \le 1$. Therefore, $L^{\varphi \ominus \varphi_1} \subset M(L^{\varphi_1}, L^\varphi)$ and

$$\|y\|_M \le 4\,\|y\|_{\varphi \ominus \varphi_1}.$$

To prove the second inclusion it is enough to indicate a constant $c > 0$ such that for each simple function $y \in M(L^{\varphi_1}, L^\varphi)$ there holds

$$\|y\|_{\varphi \ominus \varphi_1} \le c\|y\|_M. \tag{3.2}$$

In fact, it follows directly from the Fatou property of both $L^{\varphi\ominus\varphi_1}$ and $M(L^{\varphi_1}, L^{\varphi})$ spaces (it is elementary fact that $M(X, Y)$ has the Fatou property when Y has so). Let $0 \leq y \in M(L^{\varphi_1}, L^{\varphi})$ and $0 \leq y_n \uparrow y$ μ-a.e., where y_n are simple functions. Then, by (3.2),

$$\|y_n\|_{\varphi\ominus\varphi_1} \leq c\|y_n\|_M \to c\|y\|_M$$

and so the Fatou property of $L^{\varphi\ominus\varphi_1}$ implies $y \in L^{\varphi\ominus\varphi_1}$ and $\|y\|_{\varphi\ominus\varphi_1} \leq c\|y\|_M$.

The proof of (3.2) will be divided into four cases, depending on finiteness of b_φ and b_{φ_1}.

Consider firstly the most important case $b_\varphi = b_{\varphi_1} = \infty$. Let $0 \leq y \in M(L^{\varphi_1}, L^{\varphi})$ be a simple function of the form $y = \sum_k a_k \chi_{B_k}$ and such that $\|y\|_M \leq \frac{1}{2}$. We will show that for each $a > 1$

$$I_{\varphi\ominus_a\varphi_1}(y) \leq 1.$$

Let $a > 1$ be arbitrary. For each a_k there exists $b_k \geq 0$ such that

$$\varphi(a_k b_k) = \varphi \ominus_a \varphi_1(a_k) + \varphi_1(b_k).$$

This is, for $x = \sum_k b_k \chi_{B_k}$, there holds $\varphi(xy) = \varphi \ominus_a \varphi_1(x) + \varphi_1(y)$. Note that from definition of $\varphi \ominus_a \varphi_1$ we have $x(t) \leq a$ for each $t \in \Omega$. Further, since $b_{\varphi_1} = \infty$, there exists $t_a > 0$ such that $\|\chi_A\|_{\varphi_1} \leq \frac{1}{a}$ for each $A \subset \Omega$ with $\mu(A) < t_a$ (see Remark 4). Suppose $\mu(\Omega) = \infty$. Since (Ω, Σ, μ) is σ-finite and atomless, we can divide Ω into a sequence of pairwise disjoint sets (A_n) with $\mu(A_n) = t_a$ for each $n \in \mathbb{N}$ and $\Omega = \bigcup A_n$. In the case of $\mu(\Omega) < \infty$ the sequence (A_n) may be chosen finite and such that $\mu(A_n) = \delta \leq t_a$ for each $n = 1, \ldots, N$ with $\Omega = \bigcup A_n$.

In any case, for each A_n we have

$$\|yx\chi_{A_n}\|_\varphi \leq \|y\|_M \|x\chi_{A_n}\|_{\varphi_1} \leq \frac{a}{2}\|\chi_{A_n}\|_{\varphi_1} \leq \frac{1}{2},$$

because $\mu(A_n) \leq t_a$ and $x(t) \leq a$ for $t \in \Omega$. In consequence, using inequality $\varphi_1(x) \leq \varphi(yx)$, we have for each A_n

$$I_{\varphi_1}(x\chi_{A_n}) \leq I_\varphi(yx\chi_{A_n}) \leq \|yx\chi_{A_n}\|_\varphi \leq \frac{1}{2}. \tag{3.3}$$

Define now

$$x_n = \sum_{k=1}^{n} x\chi_{A_k}.$$

We claim that $I_{\varphi_1}(x_n) \leq \frac{1}{2}$ for each n. It will be shown by induction. For $n = 1$ it comes from (3.3). Let $n > 1$ and suppose

$$I_{\varphi_1}(x_{n-1}) \leq \frac{1}{2}.$$

It follows

$$I_{\varphi_1}(x_n) = I_{\varphi_1}(x_{n-1}) + I_{\varphi_1}(x\chi_{A_n}) \leq 1,$$

thus $\|x_n\|_{\varphi_1} \leq 1$. Moreover, inequality

$$\|yx_n\|_{\varphi} \leq \frac{1}{2}\,\|x_n\|_{\varphi_1} \leq \frac{1}{2}$$

together with $\varphi_1(x) \leq \varphi(yx)$ imply

$$I_{\varphi_1}(x_n) \leq I_{\varphi}(yx_n) \leq \|yx_n\|_{\varphi} \leq \frac{1}{2}.$$

It means we proved the claim and can proceed with the proof.

Clearly, $x_n \uparrow x$ μ-a.e., thus from the Fatou property of L^{φ_1} we obtain that $x \in L^{\varphi_1}$ and

$$\|x\|_{\varphi_1} \leq \sup_n \|x_n\|_{\varphi_1} \leq 1.$$

Finally, inequalities $\varphi \ominus_a \varphi_1(y) \leq \varphi(yx)$ and $\|yx\|_{\varphi} \leq \frac{1}{2}\|x\|_{\varphi_1} \leq \frac{1}{2}$ give

$$I_{\varphi\ominus_a\varphi_1}(y) \leq I_{\varphi}(yx) \leq \|yx\|_{\varphi} \leq \frac{1}{2}.$$

Applying the Fatou Lemma we obtain

$$I_{\varphi\ominus\varphi_1}(y) = \int \varphi \ominus \varphi_1(y)d\mu \leq \liminf_{a\to\infty} \int \varphi \ominus_a \varphi_1(y)d\mu \leq \frac{1}{2}.$$

In consequence $y \in L^{\varphi\ominus\varphi_1}$ with $\|y\|_{\varphi\ominus\varphi_1} \leq 1$. This gives also constant for inclusion, i.e.

$$\|y\|_{\varphi\ominus\varphi_1} \leq 2\|y\|_M,$$

when $y \in M(L^{\varphi_1}, L^{\varphi})$.

Let us consider the second case, this is $b_{\varphi} = \infty$ and $b_{\varphi_1} < \infty$. Without loss of generality we can assume that $b_{\varphi_1} > 1$ (see Remark 3). Let $0 \leq y \in M(L^{\varphi_1}, L^{\varphi})$ be a simple function satisfying $\|y\|_M \leq \frac{1}{2b_{\varphi_1}}$. Notice that $b_{\varphi} = \infty$ with $b_{\varphi_1} < \infty$

imply that $b_{\varphi \ominus \varphi_1} = \infty$. Moreover, as before, there exists a simple function x such that $0 < x(t) \leq b_{\varphi_1}$ for each $t \in \Omega$ and

$$\varphi(yx) = \varphi \ominus \varphi_1(y) + \varphi_1(x)$$

(see Remark 3). As before, we can find $t_0 > 0$ such that $\mu(A) < t_0$ implies $\|\chi_A\|_{\varphi_1} \leq 1$. Selecting the sequence (A_n) like previously, but with $\mu(A_n) \leq t_0$ for each A_n, we obtain

$$\left\| yx\chi_{A_n} \right\|_{\varphi} \leq \frac{b_{\varphi_1}}{2b_{\varphi_1}} \|\chi_{A_n}\|_{\varphi_1} \leq \frac{1}{2}.$$

Define further

$$x_n = \sum_{k=1}^{n} x\chi_{A_k}.$$

Then it may be proved by the same induction as before, that $I_{\varphi_1}(x_n) \leq \frac{1}{2}$ for each n. Following respective steps from previous case we get

$$\|y\|_{\varphi \ominus \varphi_1} \leq 2b_{\varphi_1} \|y\|_M .$$

Let now $b_\varphi, b_{\varphi_1} < \infty$. We can assume that $b_{\varphi_1} = b_\varphi = 1$ (see Remark 3). From Lemma 6 it follows that there exists a constant $c \geq 1$ such that for each $y \in M(L^{\varphi_1}, L^{\varphi})$ we have

$$\|y\|_\infty \leq c\|y\|_M.$$

Let $0 \leq y \in M(L^{\varphi_1}, L^{\varphi})$ be a simple function and $\|y\|_M \leq \frac{1}{4c}$. We have $y(t) \leq \frac{1}{4} \leq \frac{b_{\varphi \ominus \varphi_1}}{2}$ (cf. Remark 3) for almost every $t \in \Omega$, therefore $\varphi \ominus \varphi_1(y(t)) < \infty$. Consequently, we can choose a simple function x satisfying

$$\varphi(yx) = \varphi \ominus \varphi_1(y) + \varphi_1(x).$$

Then $x(t) \leq b_\varphi = 1$ for each $t \in \Omega$. Further, we can find $t_0 > 0$ so that inequality

$$\|\chi_A\|_{\varphi_1} \leq 2$$

is fulfilled for each A with $\mu(A) \leq t_0$, just because $\lim_{t \to 0^+} f_\varphi(t) = b_\varphi = 1$. Choosing a sequence (A_n) as in previous cases we get

$$\|yx\chi_{A_n}\|_\varphi \leq \frac{1}{4c} \|\chi_{A_n}\|_{\varphi_1} \leq \frac{1}{2}.$$

Once again we can show by induction that for each $x_n = \sum_{k=1}^{n} x\chi_{A_k}$ there holds $I_{\varphi_1}(x_n) \leq \frac{1}{2}$. Therefore $\|x_n\|_{\varphi_1} \leq 1$ and, by the Fatou property of L^{φ_1}, $\|x\|_{\varphi_1} \leq 1$. It follows

$$\|yx\|_{\varphi} \leq 1$$

and by inequality $\varphi \ominus \varphi_1(y) \leq \varphi(yx)$ we obtain

$$I_{\varphi \ominus \varphi_1}(y) \leq I_{\varphi}(yx) \leq \|yx\|_{\varphi} \leq 1.$$

In consequence

$$\|y\|_{\varphi \ominus \varphi_1} \leq 4c\|y\|_M.$$

Finally, there left the trivial case of $b_{\varphi} < \infty$, $b_{\varphi_1} = \infty$ to consider. However, Lemma 5 with the embedding (3.1) give

$$L^{\varphi \ominus \varphi_1} = M(L^{\varphi_1}, L^{\varphi}) = \{0\}$$

and the proof is finished. □

4 Factorization

Recall that given two B.f.s. X, Y over the same measure space, we say that X factorizes Y when

$$X \odot M(X, Y) = Y,$$

where

$$X \odot M(X, Y) = \{z \in L^0 : z = xy \text{ for some } x \in X, y \in M(X, Y)\}.$$

The idea of such factorization goes back to Lozanovskii, who proved that each B.f.s. factorizes L^1. For more informations on factorization and its importance we send a reader to papers [3,6] and [13] which are devoted mainly to this subject.

Also in [6] one may find a discussion on factorization of Orlicz spaces (and even more general Calderón–Lozanovskii spaces). Having in hand our representation $M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi \ominus \varphi_1}$ we are able to complete this discussion by proving sufficient and necessary conditions for factorization in terms of respective Young functions.

We say that equivalence $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ holds for all [large] arguments when there are constants $c, C > 0$ such that

$$c\varphi^{-1}(u) \leq \varphi_1^{-1}(u)\varphi_2^{-1}(u) \leq C\varphi^{-1}(u)$$

for all $u \geq 0$ [for some $u_0 > 0$ and all $u > u_0$].

Theorem 2 *Let φ, φ_1 be two Young functions. Then L^{φ_1} factorizes L^{φ}, i.e.*

$$L^{\varphi_1} \odot M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi}$$

if and only if

(i) equivalence $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \approx \varphi^{-1}$ is satisfied for all arguments when $\mu(\Omega) = \infty$.
(ii) equivalence $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \approx \varphi^{-1}$ is satisfied for large arguments when $\mu(\Omega) < \infty$,

Proof In the light of Theorem 1

$$L^{\varphi_1} \odot M(L^{\varphi_1}, L^{\varphi}) = L^{\varphi_1} \odot L^{\varphi \ominus \varphi_1}.$$

Therefore L^{φ_1} factorizes L^{φ} if and only if $L^{\varphi_1} \odot L^{\varphi \ominus \varphi_1} = L^{\varphi}$. The latter, however, is equivalent with $\varphi_1^{-1}(\varphi \ominus \varphi_1)^{-1} \approx \varphi^{-1}$ for all, or for large arguments, depending on Ω, as proved in Corollary 6 from [6]. □

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