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On pseudo-involutions, involutions and quasi-involutions in the group of almost Riordan arrays

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Abstract
The group of almost Riordan arrays contains the group of Riordan arrays as a subgroup. In this note, we exhibit examples of pseudo-involutions, conditions under which we define an involution, and methods of constructing quasi-involutions in the group of almost Riordan arrays.

Keywords Riordan arrays · Almost Riordan arrays · Involution · Pseudo-involution · Quasi-involution · Quasi-transitional matrix · Quasi-compression

1 Introduction

Since the initial paper [14] defining the Riordan group, there has been interest in studying involutions, pseudo-involutions and quasi-involutions [6–8,10] associated to this group. The idea of Riordan matrices with an extra column has origins from the expression of the LDU decomposition for certain symmetric Toeplitz-plus-Hankel matrix [1]. In this note, we take a look at involutions, pseudo-involutions and quasi-involutions associated to the related group of almost Riordan arrays [2].

In this first section, we recall the definition of the Riordan group, and the definition of the group of almost Riordan arrays (of order 1, initially). We then proceed to look at almost Riordan pseudo-involutions of order one and two, and to conditions that allow us to define involutions in the almost Riordan group. In the last section, we present two methods of building a form of quasi-involutions of the almost Riordan group.

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We define $\mathcal{F}_r$ to be the set of formal power series of order r, where

$$\mathcal{F}_r = \{a_r x^r + a_{r+1}x^{r+1} + a_{r+2}x^{r+2} + \cdots | a_i \in R\}$$

where R is a suitable ring with unit (which we shall denote by 1). In the sequel, we shall take $R = \mathbb{Z}$. The Riordan group over R is then given by the semi-direct product

$$\mathcal{R} = \mathcal{F}_0 \rtimes \mathcal{F}_1.$$

To an element $\big(g(x), f(x)\big) \in \mathcal{R}$ we associate the R-matrix with (n, k)-th element

$$T_{n,k} = [x^n]g(x)f(x)^k.$$

This is an invertible lower-triangular matrix. For $g(x) \in \mathcal{F}_0$ and $f(x) \in \mathcal{F}_1$, we shall also use the notation $\big(g(x), f(x)\big)$ to represent the matrix that begins

$$\begin{pmatrix} g_0 & 0 & 0 & 0 \\ g_1 & g_0 f_1 & 0 & 0 \\ g_2 & g_0 f_2 + g_1 f_1 & g_0 f_1^2 & 0 \\ g_3 & g_0 f_3 + g_1 f_2 + g_2 f_1 & 2g_0 f_1 f_2 + g_1 f_1^2 & g_0 f_1^3 \end{pmatrix}.$$

Example 1 The Riordan array $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ has associated matrix equal to the binomial matrix (Pascal's triangle) $B = \left(\binom{n}{k}\right)$. We have

$$\begin{aligned} T_{n,k} &= [x^n]\frac{1}{1-x}\frac{x^k}{(1-x)^k} \\ &= [x^{n-k}]\frac{1}{(1-x)^{k+1}} \\ &= [x^{n-k}]\binom{-(k+1)}{j}(-1)^j x^j \\ &= [x^{n-k}]\binom{k+1+j-1}{j}x^j \\ &= \binom{k+n-k}{n-k} \\ &= \binom{n}{k}, \end{aligned}$$

and we write

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We shall confine ourselves to the subgroup

$$\mathcal{R}^{(1)} = \mathcal{F}_0^{(1)} \rtimes \mathcal{F}_1^{(1)},$$

where

$$g(x) \in \mathcal{F}_0^{(1)} \leftrightarrow g_0 = 1,$$

and

$$f(x) \in \mathcal{F}_1^{(1)} \leftrightarrow f_1 = 1.$$

This ensures that all elements of $\big(g(x), f(x)\big)^{-1}$ are in R, where

$$\big(g(x), f(x)\big)^{-1} = \left(\frac{1}{g\big(\bar{f}(x)\big)}, \bar{f}(x)\right).$$

Here, $\bar{f}(x)$ is the compositional inverse of $f(x)$. The multiplication in the Riordan group is specified by

$$\big(g(x), f(x)\big) \cdot \big(u(x), v(x)\big) = \Big(g(x)u\big(f(x)\big), v\big(f(x)\big)\Big).$$

The element $I = (1, x)$ is the identity for multiplication. The *fundamental theorem of Riordan arrays* [14] says that if $h(x) \in R[[x]]$ is a column vector, then from the product of the Riordan matrix $\big(g(x), f(x)\big)$ and $h(x)$, we get the column vector

$$\big(g(x), f(x)\big) \cdot h(x) = g(x)h\big(f(x)\big).$$

We can interpret this in terms of the matrix interpretation of $(g(x), f(x))$ as follows. It says that the generating function of elements of the infinite vector produced by multiplying the vector whose elements are given by the expansion of $h(x)$ (that is, the vector $(h_0, h_1, h_2, \ldots)$ by the matrix $(T_{n,k})$ is given by $g(x)h\big(f(x)\big)$.

1.1 Almost Riordan arrays

Let $a(x) \in \mathcal{F}_0$ be the generating function of the initial column of an infinite lower triangular matrix that its submatrix after the second column follows the Riordan structure we described above. Such a matrix then begins

$$\begin{pmatrix} a_0 & 0 & 0 & 0 \\ a_1 & g_0 & 0 & 0 \\ a_2 & g_1 & g_0 f_1 & 0 \\ a_3 & g_2 & g_0 f_2 + g_1 f_1 & 1 \end{pmatrix}.$$

These matrices are called almost Riordan arrays of the first order, and we have shown [2] that if we define the set

$$a\mathcal{R}(1) = \left\{ \left(a(x) \middle| g(x), f(x) \right) \middle| a(x) \in \mathcal{F}_0, \left(g(x), f(x) \right) \in \mathcal{R} \right\},$$

then $a\mathcal{R}(1)$ is also a group, called the *group of almost Riordan arrays* (of order 1).

We recall that the product in the group of almost Riordan arrays (of first order), for $a(x), g(x), b(x), u(x) \in \mathcal{F}_0$ and $f(x), v(x) \in \mathcal{F}_1$, is given by

$$\left(a(x)\middle|g(x), f(x)\right) \cdot \left(b(x)\middle|u(x), v(x)\right) = \left(\left(a(x)\middle|g(x), f(x)\right) b(x) \middle| g(x) u\left(f(x)\right), v\left(f(x)\right) \right),$$

where the fundamental theorem of almost Riordan arrays [2], for $h_(x) \in \mathcal{F}_0$, tells us that

$$\left(a(x)\middle|g(x), f(x)\right) \cdot h(x) = h_0 a(x) + x g(x) \tilde{h}\left(f(x)\right),$$

with

$$\tilde{h}(x) = \frac{h(x) - h_0}{x}.$$

The inverse in $a\mathcal{R}(1)$ [2] is given by

$$\left(a(x)\middle|g(x), f(x)\right)^{-1} = \left(a^*(x) \middle| \frac{1}{g\left(\bar{f}(x)\right)}, \bar{f}(x) \right),$$

where

$$a^*(x) = \left(1\middle| - g(x), f(x)\right)^{-1} a(x),$$

with

$$\left(1\middle|g(x), f(x)\right)^{-1} = \left(1 \middle| \frac{1}{g\left(\bar{f}(x)\right)}, \bar{f}(x) \right).$$

To the almost Riordan array $\left(a(x)\middle|g(x), f(x)\right)$, we associate the matrix M with

$$\begin{aligned} M_{n,k} &= [x^{n-1}]g(x)f(x)^{k-1}, \quad n,k \geq 1, \\ M_{n,0} &= a_n, \\ M_{0,k} &= a_0 0^k. \end{aligned}$$

Example 2 (*Chebyshev polynomials of the first kind*) The Chebyshev polynomials of the first kind $T_n(x)$ [11] are defined by $T_0(x) = 1$, $T_1(x) = x$, and

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n \geq 2.$$

Their generating function is given by

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-tx}{1-2xt+t^2}.$$

By its form, this is not the bivariate generating function of a Riordan array. It is, however, defined by an almost Riordan array of the first order.

Proposition 1 *The coefficient array $t_{n,k}$ of the Chebyshev polynomials of the first kind, where*

$$T_n(x) = \sum_{k=0}^{n} t_{n,k}x^k,$$

is the lower-triangular matrix defined by the almost Riordan array

$$\left(\frac{1}{1+x^2}\middle|\frac{1-x^2}{(1+x^2)^2}, \frac{2x}{1+x^2}\right).$$

This array begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & 5 & 0 & -20 & 0 & 16 & 0 & 0 \\ -1 & 0 & 18 & 0 & -48 & 0 & 32 & 0 \\ 0 & -7 & 0 & 56 & 0 & -112 & 0 & 64 \end{pmatrix}.$$

Proof We use the "dummy" parameter t to define our arrays, reserving x for the polynomial coefficient. The bivariate generating function of the almost Riordan array

$\left(\frac{1}{1+t^2}\middle|\frac{1-t^2}{(1+t^2)^2}, \frac{2t}{1+t^2}\right)$ is given by

$$\left(\frac{1}{1+t^2}\middle|\frac{1-t^2}{(1+t^2)^2}, \frac{2t}{1+t^2}\right) \cdot \frac{1}{1-tx},$$

which we evaluate using the fundamental theorem of almost Riordan arrays of first order. We let

$$h(x) = \frac{2t}{1+t^2} \Longrightarrow \tilde{h}(x) = \frac{x}{1-tx}.$$

Then,

$$\begin{aligned}\left(\frac{1}{1+t^2}\middle|\frac{1-t^2}{(1+t^2)^2}, \frac{2t}{1+t^2}\right) \cdot \frac{1}{1-tx} &= \frac{1}{1+t^2} + \frac{t(1-t^2)}{(1+t^2)^2}\tilde{h}\left(\frac{2t}{1+t^2}\right)\\ &= \frac{1-tx}{1-2xt+t^2}.\end{aligned}$$

□

We can identify the normal subgroup of elements of the form $\left(a(x)\middle|1, x\right)$ with $\mathcal{F}_0$, and hence, we have

$$a\mathcal{R}(1) = \mathcal{F}_0 \rtimes \mathcal{R}.$$

For instance, we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 1 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 & 1 & 0 & 0 \\ a_5 & 0 & 0 & 0 & 0 & 1 & 0 \\ a_6 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 & 0 & 0 \\ 0 & 4 & 6 & 4 & 1 & 0 & 0 \\ 0 & 5 & 10 & 10 & 5 & 1 & 0 \\ 0 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 2 & 1 & 0 & 0 & 0 & 0 \\ a_3 & 3 & 3 & 1 & 0 & 0 & 0 \\ a_4 & 4 & 6 & 4 & 1 & 0 & 0 \\ a_5 & 5 & 10 & 10 & 5 & 1 & 0 \\ a_6 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix}.$$

While all matrices in this note are of infinite extent for $n, k \geq 0$ (except for the last section), we display only indicative truncations.

2 Pseudo-involutions in the group of almost Riordan arrays

We let $\bar{I} = (1, -x)$. We have $\bar{I}^2 = I$. We say that $\big(g(x), f(x)\big) \in \mathcal{R}$ is an *pseudo-involution* in the Riordan group if $\Big(\big(g(x), f(x)\big)\bar{I}\Big)^2 = I$ [13]. The matrix corresponding to $\bar{I}$ is the diagonal matrix with diagonal $(1, -1, 1, -1, \dots)$. In the current section, we present pseudo-involutions which are almost Riordan arrays of order 1 and order 2.

2.1 Pseudo-involutions in the group of almost Riordan arrays of first order

We shall say that $\big(a(x)\big|g(x), f(x)\big) \in a\mathcal{R}(1)$ is a pseudo-involution in the group of almost Riordan arrays (of order 1) if $(M\bar{I})^2 = I$, where M is the matrix associated to $\big(a(x)\big|g(x), f(x)\big)$. Equivalently, we require that

$$\bar{I}M\bar{I} = M^{-1}. \tag{1}$$

We then have the following proposition.

Proposition 2 *The almost Riordan matrix* $\left(\frac{1+x(r-1)}{1-x}\middle|\frac{1}{(1-x)^2}, \frac{x}{1-x}\right)$ *is a pseudo-involution in the group of almost Riordan arrays.*

We note that the corresponding matrix M coincides with the binomial matrix $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$, except that the initial column $(1, 1, 1, \dots)^T$ has been replaced by $(1, r, r, r, \dots)^T$. For $r \neq 1$, this is not a Riordan matrix, but an almost Riordan matrix. It begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ r & 1 & 0 & 0 & 0 & 0 & 0 \\ r & 2 & 1 & 0 & 0 & 0 & 0 \\ r & 3 & 3 & 1 & 0 & 0 & 0 \\ r & 4 & 6 & 4 & 1 & 0 & 0 \\ r & 5 & 10 & 10 & 5 & 1 & 0 \\ r & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix}.$$

Proof We must show that the initial column of the inverse matrix is $(1, -r, r, -r, \dots)^T$, with generating function $1 - \frac{rx}{1+x}$. The "interior" elements are taken care of by the fact that the Riordan array $\left(\frac{1}{(1-x)^2}, \frac{x}{1-x}\right)$ is a pseudo-involution in the Riordan group. Now since

$$\big(a(x)\big|g(x), f(x)\big)^{-1} = \left(a^*(x)\middle|\frac{1}{g\big(\bar{f}(x)\big)}, \bar{f}(x)\right), \tag{2}$$

we must calculate

$$a^*(x) = \big(1\big| - g(x), f(x)\big)^{-1} a(x),$$

where

$$a(x)=\frac{1+x(r-1)}{1-x},\quad g(x)=\frac{1}{(1-x)^2},\quad \text{and } f(x)=\frac{x}{1-x}.$$

We have

$$\begin{aligned}
a^*(x) &= \big(1\big|-g(x),\, f(x)\big)^{-1}\cdot a(x)\\
&= \left(1\left|-\frac{1}{g\big(\bar{f}(x)\big)},\, \bar{f}(x)\right.\right)\cdot a(x)\\
&= \left(1\left|-\frac{1}{(1+x)^2},\, \frac{x}{1+x}\right.\right)\frac{1+x(r-1)}{1-x}\\
&= a_0.1+x\left(\frac{-1}{(1+x)^2}\right)\frac{r}{1-\frac{x}{1+x}}\\
&= 1-\frac{x}{(1+x)^2}\frac{r(1+x)}{1+x-x}\\
&= 1-\frac{rx}{1+x},
\end{aligned}$$

and the inverse matrix 2 begins

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
-r & 1 & 0 & 0 & 0 & 0 & 0\\
r & -2 & 1 & 0 & 0 & 0 & 0\\
-r & 3 & -3 & 1 & 0 & 0 & 0\\
r & -4 & 6 & -4 & 1 & 0 & 0\\
-r & 5 & -10 & 10 & -5 & 1 & 0\\
r & -6 & 15 & -20 & 15 & -6 & 1
\end{pmatrix},$$

which satisfies Eq. (1). □

We shall use the notation $A_r=\left(\frac{1+x(r-1)}{1-x}\left|\frac{1}{(1-x)^2},\frac{x}{1-x}\right.\right)$. We then have the following lemma.

Lemma 1

$$A_r^p=\left(\frac{1+px(r-1)}{1-px}\left|\frac{1}{(1-px)^2},\frac{x}{1-px}\right.\right).$$

Proof We deal with the Riordan array part first. Thus, we must show that

$$\begin{aligned}
M &= \left(\frac{1}{(1-x)^2},\frac{x}{1-x}\right)\cdot\left(\frac{1}{1-(p-1)x)^2},\frac{x}{1-(p-1)x}\right)\\
&= \left(\frac{1}{(1-px)^2},\frac{x}{1-px}\right).
\end{aligned}$$

We have

$$\begin{aligned} M &= \left(\frac{1}{(1-x)^2}, \frac{x}{1-x}\right) \cdot \left(\frac{1}{1-(p-1)x)^2}, \frac{x}{1-(p-1)x}\right) \\ &= \left(\frac{1}{(1-x)^2} \frac{1}{(1-(p-1)x/(1-x))^2}, \frac{x/(1-x)}{1-(p-1)x/(1-x)}\right) \\ &= \left(\frac{1}{(1-x-(p-1)x)^2}, \frac{x}{1-x-(p-1)x}\right) \\ &= \left(\frac{1}{(1-px)^2}, \frac{1}{1-px}\right). \end{aligned}$$

Next, we must show that

$$A(x) = \left(\frac{1+x(r-1)}{1-x} \middle| \frac{1}{(1-x)^2}, \frac{x}{1-x}\right) \cdot \frac{1+(p-1)x(r-1)}{1-(p-1)x} = \frac{1+px(r-1)}{1-px}.$$

We have

$$\begin{aligned} A(x) &= \left(\frac{1+x(r-1)}{1-x} \middle| \frac{1}{(1-x)^2}, \frac{x}{1-x}\right) \cdot \frac{1+(p-1)x(r-1)}{1-(p-1)x} \\ &= \frac{1+x(r-1)}{1-x} + \frac{x}{(1-x)^2} \left(\frac{1}{x}\left(\frac{1+(p-1)x(r-1)}{1-(p-1)x} - 1\right)\right)\left(\frac{x}{1-x}\right) \\ &= \frac{1+x(r-1)}{1-x} + \frac{x}{(1-x)^2} \left(\frac{(p-1)r}{1-(p-1)x}\right)\left(\frac{x}{1-x}\right) \\ &= \frac{1+x(r-1)}{1-x} + \frac{x}{(1-x)^2} \frac{(p-1)r}{1-(p-1)x/(1-x)} \\ &= \frac{1+x(r-1)}{1-x} + \frac{x}{1-x} \frac{(p-1)r}{1-x-(p-1)x} \\ &= \frac{1+x(r-1)}{1-x} + \frac{x}{1-x} \frac{(p-1)r}{1-p)x} \\ &= \frac{1+px(r-1)}{1-px}. \end{aligned}$$

This proves the Lemma. □

Thus, A_r^p coincides with M^p, where M^p is defined as

$$M^p = \left(\frac{1}{1-px}, \frac{x}{1-px}\right) \tag{3}$$

except that the initial column $(1, p, p^2, p^3, \ldots)^T$ is replaced by $(1, rp, rp^2, rp^3, \ldots)$. This leads us to the following proposition.

Proposition 3 A_r^p *is a pseudo-involution in the group of almost Riordan matrices.*

Proof We wish to show that the initial column of $(A_r^p)^{-1}$ is given by $(1, -rp, rp^2, -rp^3, \ldots)$, with generating function $1 - \frac{prx}{1+px}$. We have in this case that

$$\begin{aligned}
a^*(x) &= \left(1 \middle| -\frac{1}{(1-px)^2}, \frac{x}{1-px}\right)^{-1} \frac{1+px(r-1)}{1-px} \\
&= \left(1 \middle| -\frac{1}{(1+px)^2}, \frac{x}{1+px}\right) \frac{1+px(r-1)}{1-px} \\
&= a_0.1 + x\left(\frac{-1}{(1+px)^2}\right) \frac{\dfrac{1+\dfrac{1}{1+px}(r-1)}{1-p\dfrac{1}{1+px}} - 1}{\dfrac{x}{1+px}} \\
&= 1 - \frac{1}{1+px}\big(1+px+px(r-1)-1\big) \\
&= 1 - \frac{prx}{1+px}.
\end{aligned}$$

□

Corollary 1 *For each $r \neq 1$, the matrix A_r generates a subgroup of pseudo-involutions in the group of almost Riordan arrays of first order.*

We note that we have

$$A_r^p \cdot A_s^q = \left(1 + \frac{prx}{1-px} + sx\left(\frac{1}{1-(p+q)x} - \frac{1}{1-px}\right) \middle| \frac{1}{(1-(p+q)x)^2}, \frac{x}{1-(p+q)x}\right).$$

Thus, $A_r^p \cdot A_s^q \neq A_s^q \cdot A_r^p$ in general.

We end this section by noting that if $\big(g(x), f(x)\big)$ is a pseudo-involution in the Riordan group, then $\big(1\big|g(x), f(x)\big)$ is a pseudo-involution in the group of almost Riordan arrays. We have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ r & 1 & 0 & 0 & 0 & 0 \\ r & 2 & 1 & 0 & 0 & 0 \\ r & 3 & 3 & 1 & 0 & 0 \\ r & 4 & 6 & 4 & 1 & 0 \\ r & 5 & 10 & 10 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ r & 1 & 0 & 0 & 0 & 0 \\ r & 0 & 1 & 0 & 0 & 0 \\ r & 0 & 0 & 1 & 0 & 0 \\ r & 0 & 0 & 0 & 1 & 0 \\ r & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 & 0 \\ 0 & 4 & 6 & 4 & 1 & 0 \\ 0 & 5 & 10 & 10 & 5 & 1 \end{pmatrix},$$

where the last matrix $\left(1\middle|\frac{1}{(1-x)^2}, \frac{x}{1-x}\right)$ is a pseudo-involution. Taking inverses, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -r & 1 & 0 & 0 & 0 & 0 \\ r & -2 & 1 & 0 & 0 & 0 \\ -r & 3 & -3 & 1 & 0 & 0 \\ r & -4 & 6 & -4 & 1 & 0 \\ -r & 5 & -10 & 10 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & 1 & 0 & 0 \\ 0 & -4 & 6 & -4 & 1 & 0 \\ 0 & 5 & -10 & 10 & -5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -r & 1 & 0 & 0 & 0 & 0 \\ -r & 0 & 1 & 0 & 0 & 0 \\ -r & 0 & 0 & 1 & 0 & 0 \\ -r & 0 & 0 & 0 & 1 & 0 \\ -r & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.2 Pseudo-involutions in the group of almost Riordan arrays of the second order

It is possible [2] to extend the definition of $a\mathcal{R}(1)$ to derive the group $a\mathcal{R}(2)$ of almost Riordan arrays of order 2. The elements of this group are of the form

$$\big(a(x), b(x)\big|g(x), f(x)\big)$$

where $a_0 = 1$, $b_0 = 1$, $g_0 = 1$ and $f_0 = 0$, $f_1 = 1$. The fundamental theorem for this group states that

$$\big(a(x), b(x)\big|g(x), f(x)\big) \cdot h(x) = h_0 a(x) + h_1 x b(x) + x^2 g(x) \tilde{\tilde{h}}\big(f(x)\big),$$

where

$$\tilde{\tilde{h}}(x) = \frac{h(x) - h_0 - h_1 x}{x^2}.$$

Following [2], we can define the inverse of a 4-tuple as follows.

$$\big(a(x), b(x)\big|g(x), f(x)\big)^{-1} = \left(a^{**}(x), b^*(x)\middle|\frac{1}{g\big(\bar{f}(x)\big)}, \bar{f}(x)\right), \tag{4}$$

where

$$b^*(x) = \big(1\big| - g(x), f(x)\big)^{-1} \cdot b(x) = \left(1\middle| -\frac{1}{g\big(\bar{f}(x)\big)}, \bar{f}(x)\right) \cdot b(x), \tag{5}$$

and

$$a^{**}(x) = \big(1, -b(x)\big|-g(x), f(x)\big)^{-1} \cdot a(x) = \left(1, -b^*(x)\middle| -\frac{1}{g\big(\bar{f}(x)\big)}, \bar{f}(x)\right) \cdot a(x). \tag{6}$$

Example 3 We define an element of $a\mathcal{R}(2)$ by $\left(\frac{1+x}{1-x}, \frac{1+x}{1-x}\,\middle|\, \frac{1}{(1-x)^2}, \frac{x}{1-x}\right)$. The corresponding matrix M is then defined by

$$M_{n,k} = [x^n]x^k \frac{1+x}{1-x}, \quad k < 2;$$
$$M_{n,k} = [x^n]\frac{x^k}{(1-x)^k}, \quad k \geq 2.$$

Thus, the matrix M begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 3 & 3 & 1 & 0 & 0 \\ 2 & 2 & 4 & 6 & 4 & 1 & 0 \\ 2 & 2 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}.$$

The inverse of this matrix then begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 0 & 0 & 0 \\ 2 & -2 & 3 & -3 & 1 & 0 & 0 \\ -2 & 2 & -4 & 6 & -4 & 1 & 0 \\ 2 & -2 & 5 & -10 & 10 & -5 & 1 \end{pmatrix}.$$

Note that we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 & 1 & 0 \\ 0 & 0 & 4 & 6 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 3 & 3 & 1 & 0 \\ 2 & 2 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

That gives us the following proposition.

Proposition 4 *The element* $\left(\frac{1+x}{1-x}, \frac{1+x}{1-x}\,\middle|\, \frac{1}{(1-x)^2}, \frac{x}{1-x}\right)$ *of* $a\mathcal{R}(2)$ *is a pseudo-involution.*

Proof We have

$$b^*(x) = \left(1\,\middle|\, -\frac{1}{(1-x)^2}, \frac{x}{1-x}\right)^{-1} \frac{1+x}{1-x}$$

$$
\begin{aligned}
&= \left(1 \middle| -\frac{1}{(1+x)^2}, \frac{x}{1+x}\right) \frac{1+x}{1-x} \\
&= 1 - \frac{x}{(1+x)^2} \frac{2}{1-\frac{x}{1+x}} \\
&= 1 - \frac{2}{1+x} = \frac{1-x}{1+x},
\end{aligned}
$$

which expands to give $1 - 2, 2, -2, \ldots$.

We then obtain that

$$
\begin{aligned}
a^{**}(x) &= \left(1, -\frac{1-x}{1+x} \middle| -\frac{1}{(1+x)^2}, \frac{x}{1+x}\right) \frac{1+x}{1-x} \\
&= 1 - 2x\frac{1-x}{1+x} - \frac{x^2}{(1+x)^2} \frac{2(1+x)}{1+x-x} \\
&= \frac{1-x}{1+x}.
\end{aligned}
$$

Thus,

$$
M^{-1} = \left(\frac{1-x}{1+x}, \frac{1-x}{1+x} \middle| \frac{1}{(1+x)^2}, \frac{x}{1+x}\right).
$$

We can now show that

$$
M^{-1} = \bar{I} \cdot M \cdot \bar{I}.
$$

Thus, we have

$$
\begin{aligned}
I &= M \cdot M^{-1} \\
&= M \cdot \bar{I} \cdot M \cdot \bar{I} \\
&= (M \cdot \bar{I}) \cdot (M \cdot \bar{I}) \\
&= (M \cdot \bar{I})^2.
\end{aligned}
$$

Thus, M is a pseudo-involution. □

It is possible to extend this result to higher orders. For instance, we can consider the almost Riordan array of third order defined by $\left(\frac{1+x}{1-x}, \frac{1+x}{1-x}, \frac{1+x}{1-x} \middle| \frac{1}{(1-x)^2}, \frac{x}{1-x}\right)$. The corresponding matrix has

$$
\begin{aligned}
M_{n,k} &= [x^n] x^k \frac{1+x}{1-x}, \quad k < 3; \\
M_{n,k} &= [x^n] \frac{x^k}{(1-x)^{k-1}}, \quad k \geq 3,
\end{aligned}
$$

and begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 3 & 3 & 1 & 0 & 0 \\ 2 & 2 & 2 & 4 & 6 & 4 & 1 & 0 \\ 2 & 2 & 2 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}.$$

The inverse of this matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & -2 & 1 & 0 & 0 & 0 & 0 \\ 2 & -2 & 2 & -2 & 1 & 0 & 0 & 0 \\ -2 & 2 & -2 & 3 & -3 & 1 & 0 & 0 \\ 2 & -2 & 2 & -4 & 6 & -4 & 1 & 0 \\ -2 & 2 & -2 & 5 & -10 & 10 & -5 & 1 \end{pmatrix}.$$

3 Involutions in the group of almost Riordan arrays

Let $R = \big(g(x), f(x)\big)$ be an ordinary Riordan array of order 2, which means that $R^2 = I$. We call such a Riordan array R an *involution*. Multiplying R with itself, we get

$$R^2 = I \Leftrightarrow \big(g(x), f(x)\big) \cdot \big(g(x), f(x)\big) = (1, x),$$

and

$$\Big(g(x)g\big(f(x)\big), f\big(f(x)\big)\Big) = (1, x), \tag{7}$$

which gives us $f\big(f(x)\big) = x$, and $g\big(f(x)\big) = \frac{1}{g(x)}$.

Searching for involutions among the almost Riordan matrices, we suppose that we have the involution $\big(a(x)\big|g(x), f(x)\big)$, so we want

$$\big(a(x)\big|g(x), f(x)\big) \cdot \big(a(x)\big|g(x), f(x)\big) = (1|1, x)$$

which becomes

$$\Big(\big(a(x)\big|g(x), f(x)\big) \cdot a(x)\Big|g(x) \cdot g\big(f(x)\big), f\big(f(x)\big)\Big) = (1|1, x). \tag{8}$$

The same conditions as in the case of the involutions of (7) are satisfied for the internal generating functions $g(x)$ and $f(x)$ of (8), while for the initial column on the left, we have that

$$\begin{aligned}\big(a(x)\big|g(x), f(x)\big)\cdot a(x) &= a_0 a(x) + xg(x)\frac{a\big(f(x)\big)-a_0}{f(x)}\\ &= a_0 a(x) + \frac{xg(x)a\big(f(x)\big)}{f(x)} - \frac{a_0 xg(x)}{f(x)}\\ &= a_0\left(a(x) - \frac{xg(x)}{f(x)}\right) + \frac{xg(x)a\big(f(x)\big)}{f(x)},\end{aligned} \tag{9}$$

which has to be equal to 1. According to the definition of an almost Riordan array, the generating function of the initial column $a(x)$ is a power series in $\mathcal{F}_0$, while also $g(x) \in \mathcal{F}_0$, and $f(x) \in \mathcal{F}_1$.

A first result is the following.

Proposition 5 *If $\big(g(x), f(x)\big)$ is an involution in $\mathcal{R}$, then $\big(1|g(x), f(x)\big)$ is an involution in $a\mathcal{R}(1)$.*

Proof We have $a(x) = 1$ and thus

$$a_0\left(a(x) - \frac{xg(x)}{f(x)}\right) + \frac{xg(x)a\big(f(x)\big)}{f(x)} = 1 - \frac{xg(x)}{f(x)} + \frac{xg(x)}{f(x)} = 1,$$

as required. □

By choosing $a(x) = \frac{xg(x)}{f(x)}$, we get a power series in $\mathcal{F}_0$, and Eq. (9) becomes

$$\begin{aligned}a_0\left(a(x) - \frac{xg(x)}{f(x)}\right) + \frac{xg(x)a\big(f(x)\big)}{f(x)} &= \frac{xg(x)a\big(f(x)\big)}{f(x)}\\ &= \frac{xg(x)\dfrac{f(x)g\big(f(x)\big)}{f\big(f(x)\big)}}{f(x)}\end{aligned}$$

Applying the conditions of (7), we have that this is equal to 1. Hence, we have proven the following proposition.

Proposition 6 *Let $\big(g(x), f(x)\big)$ be an ordinary Riordan involution, then the almost Riordan array $\big(a(x)\big|g(x), f(x)\big)$ is also an involution if $a(x) = \frac{xg(x)}{f(x)}$.*

For the Riordan family of subgroups which are solely defined by their second generating function f,

$$H[r,s,p] = \left\{\left(\left(\frac{f(z)}{z}\right)^r (f'(z))^s \left(\frac{f(z)-1}{z-1}\right)^p, f(z)\right)\middle|(r,s,p)\in\mathbb{Q}^3, f(z)\in\mathbb{F}_1\right\},$$

we only need the condition $f = \bar{f}$, as the g function depends on f [4].

Corollary 2 *Let* $H_f[\rho,\sigma,\pi] = \left(\left(\frac{f(x)}{x}\right)^{\rho}\left((f'(x))^{\sigma}\left(\frac{f(x)-1}{x-1}\right)^{\pi}, f(x)\right)\right.$ *be a Riordan involution, then the almost Riordan array*

$$\left(\left(\frac{f(x)}{x}\right)^{\rho-1}\left(f'(x)\right)^{\sigma}\left(\frac{f(x)-1}{x-1}\right)^{\pi}\middle|\left(\frac{f(x)}{x}\right)^{\rho}\left(f'(x)\right)^{\sigma}\left(\frac{f(x)-1}{x-1}\right)^{\pi}, f(x)\right)$$

is also an involution.

Example 4 Let $f(x) = -\frac{x}{1+x}$, so $H_f[1,1,1]$ will be $\left(\frac{f(x)}{x}f'(x)\frac{f(x)-1}{x-1}, f(x)\right)$, which corresponds to the matrix

$$\left(\frac{1+2x}{(1+x)^4(1-x)}, -\frac{x}{1+x}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & -3 & -1 & 0 & 0 & 0 & 0 \\ -4 & 2 & 6 & 4 & 1 & 0 & 0 & 0 \\ 10 & 2 & -8 & -10 & -5 & -1 & 0 & 0 \\ -18 & -12 & 6 & 18 & 15 & 6 & 1 & 0 \\ 30 & 30 & 6 & -24 & -33 & -21 & -7 & -1 \end{pmatrix},$$

which is an involution, as $f = \bar{f}$. Now, the almost Riordan matrix which contains $H_f[1,1,1]$ and it is also an involution will be

$$\left(f'(x)\frac{f(x)-1}{x-1}\middle|\frac{f(x)}{x}f'(x)\frac{f(x)-1}{x-1}, f(x)\right) = \left(\frac{1+2x}{(1+x)^3(1-x)}, -\frac{x}{1+x}\right)$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & -3 & -3 & -1 & 0 & 0 & 0 \\ 6 & -4 & 2 & 6 & 4 & 1 & 0 & 0 \\ -8 & 10 & 2 & -8 & -10 & -5 & -1 & 0 \\ 12 & -18 & -12 & 6 & 18 & 15 & 6 & 1 \end{pmatrix}$$

We note that in the general case, the almost Riordan array $\left(\frac{xg(x)}{f(x)}\middle| g(x), f(x)\right)$ is in fact a Riordan array. It coincides with the Riordan array $\left(\frac{xg(x)}{f(x)}, f(x)\right)$. Iterating, we have the following proposition.

Proposition 7 *If* $\big(g(x), f(x)\big)$ *is an involution in* $\mathcal{R}$*, then so is* $\left(\frac{x^n}{f(x)^n}g(x), f(x)\right)$.

Proof Let $\big(g(x), f(x)\big)$ be an involution. That means $f(x) = \bar{f}(x)$ and $g\big(f(x)\big) = \frac{1}{g(x)}$. For the Riordan array $\big(G(x), f(x)\big)$, where $G(x) = \frac{x^n}{f(x)^n}g(x)$, it can be easily shown that

$$G\big(f(x)\big) = \frac{1}{G(x)}.$$

This is so since we have

$$G\big(f(x)\big) = \frac{(f(x))^n}{(f(f(x)))^n \cdot g(x)} = \frac{(f(x))^n}{x^n \cdot g(x)} = \frac{1}{G(x)}.$$

□

We also note that the Riordan array $\left(\frac{x^n}{f(x)^n} g(x),\, f(x)\right)$ can also be written as an almost Riordan array as $\left(\frac{x^n}{f(x)^n} g(x) \middle| \frac{x^n}{f(x)^n} g(x),\, f(x)\right)$.

We now exhibit some simple involutions in the almost Riordan group.

Proposition 8 *The almost Riordan array of first order given by* $\left(\frac{1}{1+x} \middle| -1, x\right)$, *and which begins*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

is an involution.

Proof We must show that $\left(\frac{1}{1+x} \middle| -1, x\right) \cdot \left(\frac{1}{1+x} \middle| -1, x\right) = I$. We have

$$\left(\frac{1}{1+x} \middle| -1, x\right) \cdot \left(\frac{1}{1+x} \middle| -1, x\right) = \left(\left(\frac{1}{1+x} \middle| -1, x\right) \frac{1}{1+x} \middle| 1, x\right).$$

Now for $h(x) = \frac{1}{1+x}$, we have $\tilde{h}(x) = \frac{-1}{1+x}$. Thus, we have

$$\left(\frac{1}{1+x} \middle| -1, x\right) \cdot \frac{1}{1+x} = \frac{1}{1+x} + x \cdot (-1) \cdot \left(\frac{-1}{1+x}\right) = \frac{1+x}{1+x} = 1.$$

Thus, we have

$$\left(\frac{1}{1+x} \middle| -1, x\right)^2 = (1|1, x)$$

as required. □

Example 5 In like fashion, we can show that the almost Riordan array of first order given by

$$\left(\frac{1}{1-x} \middle| -\frac{1+x}{1-x}, -x\right),$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & -1 & 0 & 0 & 0 \\ 1 & -2 & 2 & -2 & 1 & 0 & 0 \\ 1 & -2 & 2 & -2 & 2 & -1 & 0 \\ 1 & -2 & 2 & -2 & 2 & -2 & 1 \end{pmatrix},$$

is an involution.

There is a general construction which allows us to find, for any given almost Riordan array of first order, an almost Riordan array that is an involution.

Proposition 9 *Let $\left(a(x)\middle|g(x), f(x)\right)$ be an almost Riordan array of first order. Then, the almost Riordan array defined by the product*

$$\left(a(x)\middle|g(x), f(x)\right) \cdot \left(a_0 + \frac{x\tilde{a}\left(\bar{f}(-x)\right)}{g\left(\bar{f}(-x)\right)} \middle| \frac{1}{g\left(\bar{f}(-x)\right)}, \bar{f}(-x)\right)$$

is an involution in the group of almost Riordan arrays of first order.

Proof Using the product rule in the group of almost Riordan arrays of first order, we find that the product is equal to $(1|1, x)$. □

The similar result for Riordan arrays is given in [3].

Example 6 We consider the almost Riordan array of first order which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 3 & 1 & 0 & 0 \\ 1 & 5 & 5 & 5 & 4 & 1 & 0 \\ 1 & 8 & 8 & 8 & 8 & 5 & 1 \end{pmatrix}.$$

This is defined by $\left(\frac{1}{1-x}\middle|\frac{1}{1-x-x^2}, x(1+x)\right)$. We have $\bar{f}(x) = xc(-x)$, where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers. Then, $\frac{1}{g\left(\bar{f}(x)\right)} = 1 - x$. We find that the product of the proposition is then given by

$$\left(\frac{1}{1-x}\middle|\frac{1}{1-x-x^2}, x(1+x)\right) \cdot \left(\frac{4+5x+3x^2+x(1+x)\sqrt{1-4x}}{2(2+x)}\middle|1+x, -xc(x)\right).$$

This matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 \\ 2 & -4 & 4 & -1 & 0 & 0 & 0 \\ 0 & -6 & 12 & -6 & 1 & 0 & 0 \\ -6 & -10 & 34 & -24 & 8 & -1 & 0 \\ -30 & -16 & 102 & -86 & 40 & -10 & 1 \end{pmatrix}.$$

The Riordan array part of the above product is given by

$$\left(\frac{1}{1-x-x^2}, x(1+x)\right) \cdot \big(1+x, -xc(x)\big)$$
$$= \left(\frac{1+x+x^2}{1-x-x^2}, -x(1+x)c\big(x(1+x)\big)\right).$$

In order to work out the first term of the resulting almost Riordan array, we set

$$b(x) = \frac{4+5x+3x^2+x(1+x)\sqrt{1-4x}}{2(2+x)} \Longrightarrow \tilde{b} = \frac{(1+x)(3+\sqrt{1-4x})}{2(2+x)}.$$

Then, the first term is given by

$$1 \cdot \frac{1}{1-x} + \frac{x}{1-x-x^2}\tilde{b}\big(x(1+x)\big)$$
$$= \frac{4+x-4x^2-4x^3-5x^4+x(1-x)(1+x+x^2)\sqrt{1-4x-4x^2}}{2(1-x)(1-x-x^2)(2+x+x^2)}.$$

The involution in the group of almost Riordan arrays of first order that we seek is then given by

$$\left(\frac{4+x-4x^2-4x^3-5x^4+x(1-x)(1+x+x^2)\sqrt{1-4x-4x^2}}{2(1-x)(1-x-x^2)(2+x+x^2)}\right|$$
$$\left.\frac{1+x+x^2}{1-x-x^2}, -x(1+x)c\big(x(1+x)\big)\right).$$

4 Quasi-involutions in the group of almost Riordan arrays

We present two methods of constructing almost Riordan quasi-involutions based on quasi-involutions that come from intersections of known Riordan subgroups [9,12], and using their quasi-compressions [5]. First, by adjoining an extra column on the left of a Riordan quasi-involution, and then by replacing the initial column of such matrix.

We consider the quasi-involution [8]

$$\big(g(x), xg(x)\big) = \left(\frac{1-x^2-\sqrt{1-6x^2+x^4}}{2x^2}, \frac{1-x^2-\sqrt{1-6x^2+x^4}}{2x}\right).$$

The corresponding matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 6 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 8 & 0 & 1 & 0 & 0 & 0 \\ 22 & 0 & 30 & 0 & 10 & 0 & 1 & 0 & 0 \\ 0 & 68 & 0 & 48 & 0 & 12 & 0 & 1 & 0 \\ 90 & 0 & 146 & 0 & 70 & 0 & 14 & 0 & 1 \end{pmatrix}.$$

The initial column of this matrix is given by the aerated large Schroeder numbers

$$1, 0, 2, 0, 6, 0, 22, 0, 90, 0, 394, 0, 1806, 0, 8558, \ldots.$$

The inverse of this matrix begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & -6 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & -8 & 0 & 1 & 0 & 0 & 0 \\ -22 & 0 & 30 & 0 & -10 & 0 & 1 & 0 & 0 \\ 0 & -68 & 0 & 48 & 0 & -12 & 0 & 1 & 0 \\ 90 & 0 & -146 & 0 & 70 & 0 & -14 & 0 & 1 \end{pmatrix}.$$

We say that when an aerated matrix has an inverse with the same elements as the original matrix, except that the signs change on alternate nonzero diagonals, then the matrix is a *quasi-involution*. A Riordan array $\big(g(x^2), xg(x^2)\big)$ is a quasi-involution if its inverse is given by $\big(g(-x^2), xg(-x^2)\big)$. This will be the case if and only if the $\Omega_{1,1}$-sequence of the array satisfies $\Omega_{1,1}(x) = \frac{1}{\Omega_{1,1}(-x)}$ [8]. For the matrix above, we have $\Omega_{1,1} = \frac{1+x}{1-x}$.

4.1 Adding a new column

Recently, it has been shown that certain types of quasi-involutions of different levels can be related by

$$W = Q_{2k}.Q_k^{-1}, \tag{10}$$

where Q_{2k} is a $2k$-aerated Riordan quasi-involution, while Q_k is a Riordan matrix which is based on the generating functions of Q_{2k} and holds the quasi-involution property. We name Q_k as the (k-aerated) *quasi-compression* of Q_{2k}, while W is called the *quasi-transitional matrix* [5].

Equation 10 gives us

$$\left(\frac{F(x)}{x}, F(x)\right) = \left(\frac{f(x)}{x}, f(x)\right) . \left(\frac{f^*(x)}{x}, f^*(x)\right)^{-1} \tag{11}$$

where $F(x) = \frac{x}{\sqrt[k]{\sqrt{1-cx^{2k}}+cx^k}}$, $f(x) = \frac{x}{\sqrt[2k]{1-cx^{2k}}}$, and $f^*(x) = \frac{x}{\sqrt[k]{1-cx^k}}$.

Now, by adding a column on each of the arrays on the RHS of (11), we have:

$$\left(A(x)\middle|\frac{1}{\sqrt[2k]{1-cx^{2k}}}, \frac{x}{\sqrt[2k]{1-cx^{2k}}}\right) . \left(a(x)\middle|\frac{1}{\sqrt[k]{1+cx^k}}, \frac{x}{\sqrt[k]{1+cx^k}}\right)$$

where $A, a \in \mathbb{F}_0$, and their quasi-transitional matrix becomes

$$\left(A(x) + a\big(f(x)\big) + 1\middle|\frac{1}{\sqrt[k]{\sqrt{1-cx^{2k}}+cx^k}}, \frac{x}{\sqrt[k]{\sqrt{1-cx^{2k}}+cx^k}}\right) \tag{12}$$

We note that $A(x)$ needs to be a $2k$-aerated, and $a(x)$ a k-aerated formal power series, as we see on the following example.

Example 7 Let the Riordan quasi-involution

$$Y = \left(\frac{1}{\sqrt{1-8x^2}}, \frac{x}{\sqrt{1-8x^2}}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 8 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 24 & 0 & 12 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 64 & 0 & 16 & 0 & 1 & 0 & \cdots \\ 160 & 0 & 120 & 0 & 20 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and let

$$\sigma Y = \left(K(x) \middle| \frac{1}{\sqrt{1-8x^2}}, \frac{x}{\sqrt{1-8x^2}} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ B & 0 & 8 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 24 & 0 & 12 & 0 & 1 & 0 & 0 & \cdots \\ \Gamma & 0 & 64 & 0 & 16 & 0 & 1 & 0 & \cdots \\ 0 & 160 & 0 & 120 & 0 & 20 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

be the same matrix with the extra column

$$1, 0, A, 0, B, 0, \Gamma, 0, \Delta, 0, E, 0, Z, 0, H, 0, \Theta, 0, I.. \tag{13}$$

on the left.

We need $\sigma Y \cdot (\sigma Y)^{-1} = I$, where

$$(\sigma Y)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -A & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ B & 0 & -8 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 24 & 0 & -12 & 0 & 1 & 0 & 0 & \cdots \\ -\Gamma & 0 & 64 & 0 & -16 & 0 & 1 & 0 & \cdots \\ 0 & -160 & 0 & 120 & 0 & -20 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which leads us to the equations

$$B = 4A$$

$$\Delta = 4(3\Gamma - 7A)$$

$$Z = 4(5E - 320\Gamma - 3,904A)$$

$$\vdots$$

Hence, the sequence (13) is expressed as the formal power series

$$K(x) = 1 + Ax^2 + 4Ax^4 + \Gamma x^6 + 4(3\Gamma - 7A)x^8 + Ex^{10} + 4(5E - 320\Gamma - 3,904A)x^{12} + \cdots, \tag{14}$$

and we say that the almost-Riordan array

$$\sigma Y = \left(K(x)\middle|\frac{1}{\sqrt{1-8x^2}}, \frac{x}{\sqrt{1-8x^2}}\right)$$

is a quasi-involution in $\alpha\mathcal{R}(1)$.

For the quasi-compression of σY, we take the Pascal-like array

$$\sigma Y^* = \left(k(x)\middle|\frac{1}{1-8x}, \frac{x}{1-8x}\right),$$

where, by working similarly, we get

$$k(x) = 1 + \alpha x + 4\alpha x^2 + \gamma x^3 + (12\gamma + 22\alpha)x^4 + \cdots . \tag{15}$$

And the formal power series $K(x)$ and $k(x)$ from (14) to (15) respectively, are used in (12) to link these two quasi-involutions through Eq. 10.

4.2 Replacing a column

We also define quasi-involutions in $\alpha\mathcal{R}$, by replacing the first column of a given quasi-involution. Again, we have the quasi-involution of the form

$$\left(\frac{f(x)}{x}, f(x)\right) = \left(\frac{1}{\sqrt[2k]{1-cx^{2k}}}, \frac{x}{\sqrt[2k]{1-cx^{2k}}}\right).$$

Since the Riordan array structure of the matrix that we are going to construct starts from the second column, its generating function will be $\frac{f^2(x)}{x}$, while the multiplier function remains the same. So, we have

$$U_{2k} = \left(B(x)\middle|\frac{f(x)}{x}f(x), f(x)\right) = \left(B(x)\middle|\frac{x}{\sqrt[k]{1-cx^{2k}}}, \frac{x}{\sqrt[2k]{1-cx^{2k}}}\right),$$

and

$$U_k^{-1} = \left(b(x)\middle|\frac{x}{\sqrt[k]{(1+cx^k)^2}}, \frac{x}{\sqrt[k]{1+cx^k}}\right),$$

the inverse of the almost-Riordan matrix which is constructed by its quasi-compression. So, their quasi-transitional matrix $W = U_{2k} \cdot U_k^{-1}$ is

$$\left(B(x) + xf(x)^{\frac{k-1}{2}}\left(b\left(\frac{\sqrt{f(x)}}{x}\right) - 1\right)\middle|\frac{xf(x)}{\sqrt[k]{(\sqrt{1-cx^{2k}}+cx^k)^2}}, \frac{x}{\sqrt[k]{\sqrt{1-cx^{2k}}+cx^k}}\right) \tag{16}$$

Example 8 Using the same Riordan quasi-involution as in Example 7, we have the almost-Riordan array

$$\tau Y = \left(\Lambda(z) \middle| \frac{z}{1-8z^2}, \frac{z}{\sqrt{1-8z^2}} \right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ B & 0 & 12 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 64 & 0 & 16 & 0 & 1 & 0 & 0 & \cdots \\ C & 0 & 120 & 0 & 20 & 0 & 1 & 0 & \cdots \\ 0 & 512 & 0 & 192 & 0 & 24 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and working similarly as in Example 7, we get that

$$\Lambda(x) = 1 + Ax + 6Ax^2 + Cx^3 + (280A + 14C)x^4 + \cdots \tag{17}$$

Additionally, the quasi-compression of τY is

$$\tau Y^* = \left(\lambda(x) \middle| \frac{x}{(1-8x)^2}, \frac{x}{\sqrt{1-8x}} \right),$$

where

$$\lambda(x) = 1 + ax + 8ax^2 + cx^3 + (512a - 16c)x^4 + \cdots. \tag{18}$$

Again, the formal power series $\Lambda(x)$ and $\lambda(x)$ from (17) and (18) respectively, are used in (16) to link these two quasi-involutions through Eq. (10).

We note that for the appropriate values of the parameters $A, B, C, D, \ldots$ and $a, b, c, d, \ldots$ of the almost-Riordan arrays with a replaced column, these matrices are equal to their equivalent quasi-involutions of the Riordan group.

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