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ISRM SUGGESTED METHOD

# Mohr–Coulomb Failure Criterion

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**List of Symbols**

| | |
|---|---|
| $a$ | $(m-1)/(m+1)$ |
| $b$ | $1/(m+1)$ |
| $c$ | Cohesion |
| $C_0$ | Uniaxial compressive strength |
| $m$ | $(1+\sin\phi)/(1-\sin\phi)$ |
| $S_0$ | Inherent shear strength (cohesion) |
| $T$ | Uniaxial tensile strength |
| $T_0$ | Theoretical MC uniaxial tensile strength |
| $\phi$ | Angle of internal friction |
| $\mu = \tan\phi$ | Coefficient of internal friction |
| $\sigma$ | Normal stress on plane |
| $\tau$ | Shear stress on plane |
| $\sigma_1, \sigma_2, \sigma_3$ | Principal stresses, with no regard to order |
| $\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}, \sigma_{\mathrm{III}}$ | Major, intermediate, minor principal stresses |
| $\sigma_{\mathrm{m}}$ | $(\sigma_{\mathrm{I}}+\sigma_{\mathrm{III}})/2$ |
| $\tau_{\mathrm{m}}$ | $(\sigma_{\mathrm{I}}-\sigma_{\mathrm{III}})/2$ |
| $\sigma_{\mathrm{I}}^*$ | $C_0 - mT$ |
| $\sigma_{\mathrm{III}}^*$ | $-T$ |

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## 1 Description

The Mohr–Coulomb (MC) failure criterion is a set of linear equations in principal stress space describing the conditions for which an isotropic material will fail, with any effect from the intermediate principal stress $\sigma_{\mathrm{II}}$ being neglected. MC can be written as a function of (1) major $\sigma_{\mathrm{I}}$ and minor $\sigma_{\mathrm{III}}$ principal stresses, or (2) normal stress $\sigma$ and shear stress $\tau$ on the failure plane (Jaeger and Cook 1979). When all principal stresses are compressive, experiments demonstrate that the criterion applies reasonably well to rock, where the uniaxial compressive strength $C_0$ is much greater than the uniaxial tensile strength $T$, e.g. $C_0/T > 10$; some modification is needed when tensile stresses act, because the (theoretical) uniaxial tensile strength $T_0$ predicted from MC is not measured in experiments. The MC criterion can be considered as a contribution from Mohr and Coulomb (Nadai 1950). Mohr's condition is based on the assumption that failure depends only on $\sigma_{\mathrm{I}}$ and $\sigma_{\mathrm{III}}$, and the shape of the failure envelope, the loci of $\sigma$, $\tau$ acting on a failure plane, can be linear or nonlinear (Mohr 1900). Coulomb's condition is based on a linear failure envelope to determine the critical combination of $\sigma$, $\tau$ that will cause failure on some plane (Coulomb 1776). A linear failure criterion with an intermediate stress effect was described by Paul (1968) and implemented by Meyer and Labuz (2012).

## 2 Background

Coulomb, in his investigations of retaining walls (Heyman 1972), proposed the relationship

$$|\tau| = S_0 + \sigma \tan\phi \quad (1)$$

where $S_0$ is the inherent shear strength, also known as

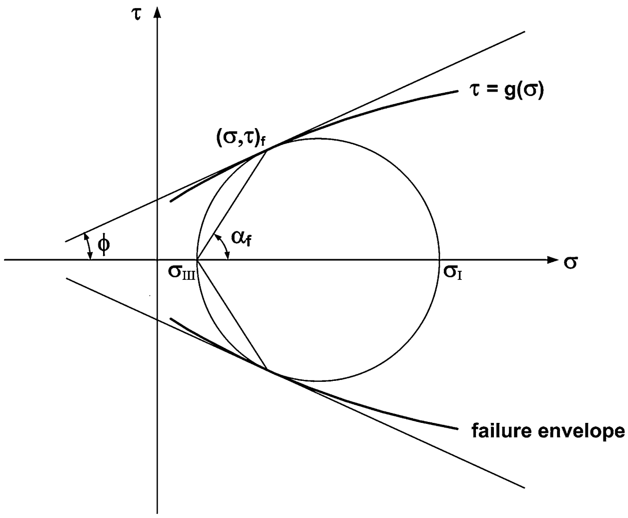


**Fig. 1** Mohr diagram and failure envelopes

cohesion $c$, and $\phi$ is the angle of internal friction, with the coefficient of internal friction $\mu = \tan\,\phi$. The criterion contains two material constants, $S_0$ and $\phi$, as opposed to one material constant for the Tresca criterion (Nadai 1950). The representation of Eq. (1) in the Mohr diagram is a straight line inclined to the $\sigma$-axis by the angle $\phi$ (Fig. 1). By constructing a Mohr circle tangent to the line (a stress state associated with failure) and using trigonometric relations, the alternative form of Eq. (1) in terms of principal stresses is obtained:

$$(\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}}) = (\sigma_{\mathrm{I}} + \sigma_{\mathrm{III}}) \sin\phi + 2S_0 \cos\phi \tag{2}$$

One form of Mohr's failure criterion is

$$\tau_{\mathrm{m}} = f(\sigma_{\mathrm{m}}) \tag{3}$$

where $\tau_{\mathrm{m}} = (\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}})/2$, $\sigma_{\mathrm{m}} = (\sigma_{\mathrm{I}} + \sigma_{\mathrm{III}})/2$. Knowing the relationship given by Eq. (3), the Mohr envelope can be constructed on the $\sigma$, $\tau$ plane (Fig. 1), and failure occurs if the stress state at failure, the circle of diameter $(\sigma_{\mathrm{I}} - \sigma_{\mathrm{III}})$, is tangent to the failure envelope, $\tau = g(\sigma)$. Thus, from Eq. (2), Coulomb's criterion is equivalent to the assumption of a linear Mohr envelope.

Coulomb's and Mohr's criteria are notable in that an effect of $\sigma_{\mathrm{m}}$, the mean stress in the $\sigma_{\mathrm{I}}$, $\sigma_{\mathrm{III}}$ plane, is considered, which is important for materials such as rock and soil; i.e., experiments on geomaterials demonstrate that $\tau_{\mathrm{m}}$ at failure increases with $\sigma_{\mathrm{m}}$. However, the additional claim that the point of tangency of the critical stress circle with the failure envelope, as constructed on the Mohr diagram, represents the normal and shear stresses $(\sigma,\ \tau)_{\mathrm{f}}$ on the failure plane with normal inclined to $\sigma_{\mathrm{I}}$ at an angle $\alpha_{\mathrm{f}}$ is not always observed in experiments. Nonetheless, Mohr's criterion allows for a curved shape of the failure envelope, and this nonlinear behavior is exhibited by many rock types (Jaeger and Cook 1979).

## 3 Formulation

With no order implied by the principal stresses $\sigma_1$, $\sigma_2$, $\sigma_3$, the MC criterion can be written as

$$\pm\frac{\sigma_1-\sigma_2}{2} = a\frac{\sigma_1+\sigma_2}{2} + b,\ \pm\frac{\sigma_2-\sigma_3}{2} = a\frac{\sigma_2+\sigma_3}{2} + b,\ \pm\frac{\sigma_3-\sigma_1}{2} = a\frac{\sigma_3+\sigma_1}{2} + b \tag{4}$$

where $a = \frac{m-1}{m+1}$, $m = \frac{C_0}{T_0} = \frac{1+\sin\phi}{1-\sin\phi}$, $b = \frac{1}{m+1}$, $C_0 = \frac{m}{m+1}$, $T_0 = \frac{C_0}{2}(1-\sin\phi)$, and $0 \le a < 1$. $T_0$ is the theoretical MC uniaxial tensile strength (Fig. 2a) that is not observed in experiments; rather, a much lower strength $T$ is measured ($\sigma_{\mathrm{I}} = 0$, $\sigma_{\mathrm{III}} = -T$), with the failure plane being normal to $\sigma_{\mathrm{III}}$. $C_0$ is the theoretical MC uniaxial compressive strength (Fig. 2a) that is usually close to the measured value (so another symbol is not introduced).

The shape of the failure surface in principal stress space is dependent on the form of the failure criterion: linear functions map as planes and nonlinear functions as curvilinear surfaces. As shown in Fig. 2b, the six equations in (4) are represented by six planes that intersect one another along six edges, defining a hexagonal pyramid. Also presented in Fig. 2b is the failure surface on the equipressure ($\sigma_1 + \sigma_2 + \sigma_3 =$ constant) or $\pi$-plane perpendicular to the hydrostatic axis, where MC can be described as an irregular hexagon with sides of equal length (Shield 1955). Isotropy requires threefold symmetry because an interchange of $\sigma_1$, $\sigma_2$, $\sigma_3$ should not influence the failure surface for an isotropic material. Note that, the failure surface need only be given in any one of the 60° regions (Fig. 2b).

Consider the transformation from principal stress space ($\sigma_1$, $\sigma_2$, $\sigma_3$) to the Mohr diagram ($\sigma$, $\tau$). Although the radial distance from the hydrostatic axis to the stress point is proportional to the deviatoric stress, a point in principal stress space does not directly indicate the value of shear stress on a plane. However, each point on the failure surface in principal stress space corresponds to a Mohr circle tangent to the failure envelope (Fig. 2a). For the particular case where $\sigma_2$ is the intermediate principal stress in the order $\sigma_1 \ge \sigma_2 \ge \sigma_3$, the failure surface is given by the side $ACD$ of the hexagonal pyramid (Fig. 2b). The principal stresses at point $D$ represent the stress state for a triaxial compression test $(\sigma_1,\ \sigma_2 = \sigma_3)_D$, and point $D$ is given by circle $D$ in the Mohr diagram. Similarly, for point $C$ with principal stresses $(\sigma_3,\ \sigma_1 = \sigma_2)_C$ associated with a triaxial extension test, Mohr circle $C$ depicts the stress state. Points $D$ and $C$ can be viewed as the extremes of the intermediate stress variation, and the normal and shear stresses corresponding to failure are given by points $D_{\mathrm{f}}$ and $C_{\mathrm{f}}$. Points lying on the line $CD$ (Fig. 2b) will be represented by circles between $C$ and $D$ (Fig. 2a).

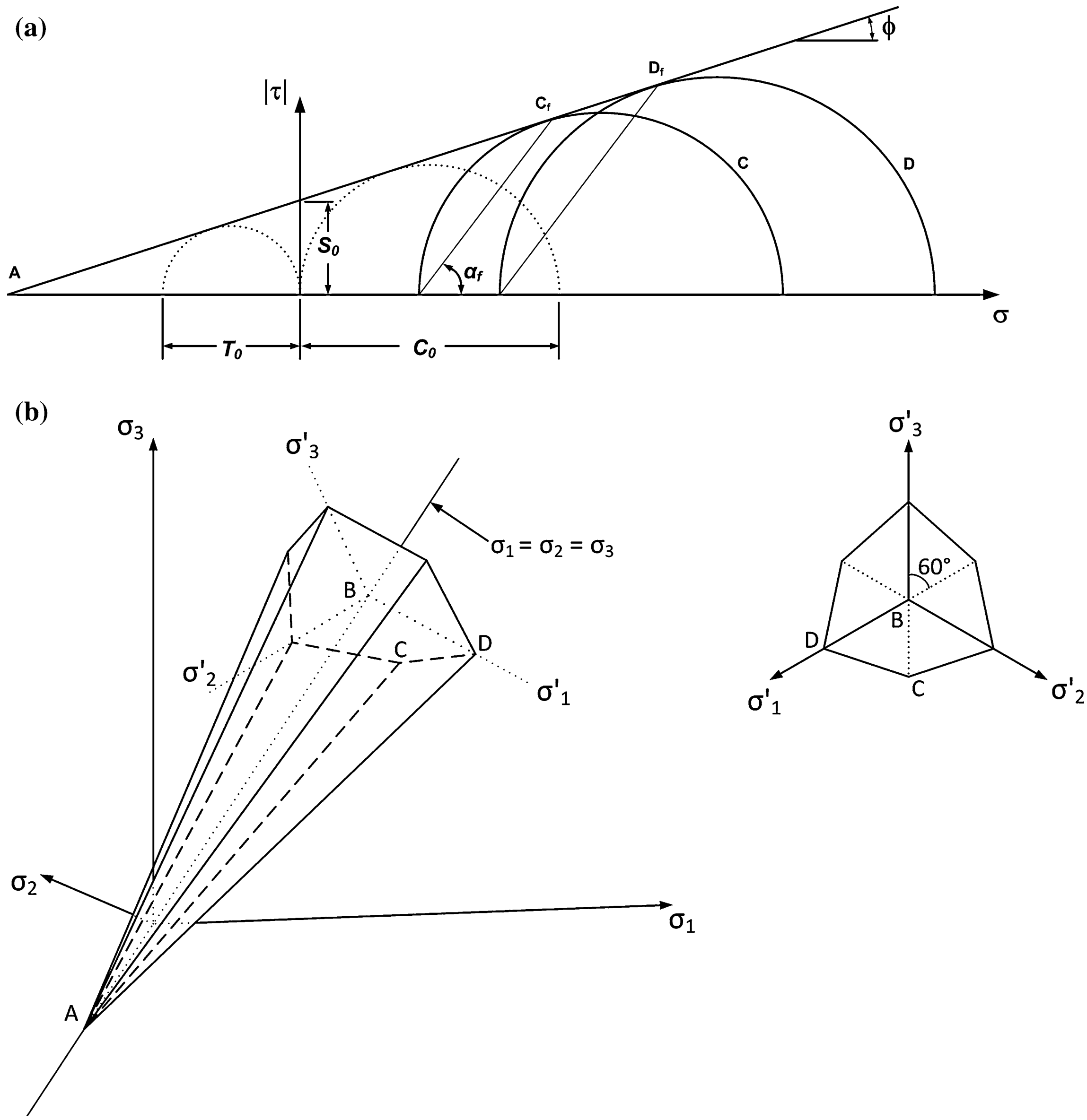


**Fig. 2** Mohr–Coulomb failure criterion: **a** linear envelope in the Mohr diagram; **b** pyramidal surface in principal stress space and cross-section in the equipressure plane

For negative (tensile) values of the minor principal stress, experiments show that the failure plane is perpendicular to $\sigma_{III} = -T$. Indeed, the tensile failure mode is completely different from the shear failure mode that occurs with compressive normal stresses, although failure under uniaxial compression is also different, usually observed as axial splitting (Vardoulakis et al. 1998). To account for tensile failure, Paul (1961) introduced the concept of tension cut-offs and a modified MC failure criterion requiring three material constants: Eq. (3) is valid when

$$\sigma_I > (C_0 - mT) = \sigma_I^* \tag{5}$$

but MC is modified as

$$\sigma_{III} = -T \text{ when } \sigma_I < \sigma_I^* \tag{6}$$

The representation of tension cut-offs on the Mohr diagram is shown in Fig. 3a. Note that, the stress state depicted by the broken circle, defined by $\sigma_I = \sigma_I^* = (C_0 - mT)$, $\sigma_{III}^* = -T$, is not part of the failure envelope. Rather, all Mohr circles with $\sigma_I < \sigma_I^*$ are tangent to the envelope at the point $\sigma_{III}^* = -T$. In principal stress space, the modified MC criterion with tension cut-offs involves the MC pyramid intercepted by a second pyramid with three planes perpendicular to the principal stress axes (Fig. 3b).

## 4 Experimental Data

Typically, laboratory results are evaluated using the MC failure criterion, as axisymmetric loading imposes a representation where the intermediate stress $\sigma_{II}$ is equal to the minor $\sigma_{III}$ or major $\sigma_I$ principal stress. Few tests independently control $\sigma_{II}$ because of experimental challenges, although conventional triaxial compression ($\sigma_1 > \sigma_2 = \sigma_3$) and extension ($\sigma_1 = \sigma_2 > \sigma_3$) tests offer simple approaches to evaluate an influence of the intermediate stress. However, a true triaxial apparatus is needed to investigate stress states between the axisymmetric conditions represented by points C and D in Fig. 2b (Meyer and Labuz 2012).

Various researchers (Mogi 1971, 1974; Takahashi and Koide 1989; Chang and Haimson 2000; Al-Ajmi and Zimmerman 2005) have performed true triaxial testing,

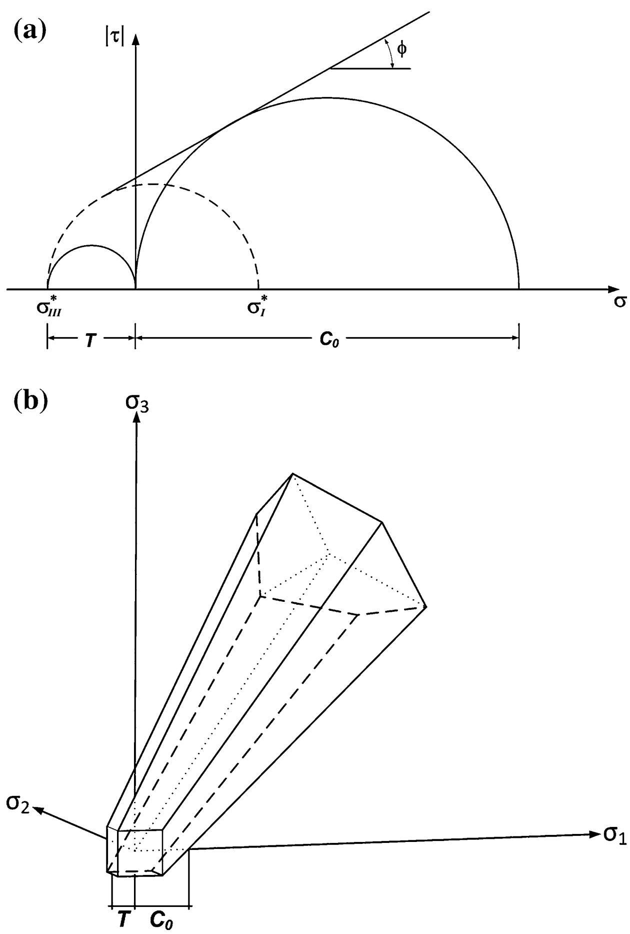


**Fig. 3** Tension cut-offs for the modified Mohr-Coulomb failure criterion: **a** failure envelope in the Mohr diagram; **b** representation in principal stress space

and the intermediate stress effect appears to depend on rock type, although anisotropy and experimental conditions may also influence the results. In fact, anisotropy can cause a reserve intermediate-stress effect, where the friction angle appears larger in compression than extension (Dehler and Labuz 2007). In addition, boundary conditions can play a substantial role in experiments with rock, where a uniform state of stress is a basic assumption of element testing that is often violated (Labuz and Bridell 1993; Paul and Gangal 1967).

Several references can be found dealing with the application of the MC failure criterion (Vutukuri et al. 1974; Andreev 1995; Paterson and Wong 2005). In a treatise on rock properties (Landolt-Börnstein 1982), a chapter by Rummel (pp. 141–238) gives an overview of failure parameters for various types of rock, and Mogi (2007) summarized results on a number of rocks. Generally, it is claimed that MC well describes the stress state at failure over a limited range of mean stress. Statistical treatment of various failure criteria applied to experiments on intact rock can be found in the literature (Colmenares and Zoback 2002; Hoek et al. 2002; Pincus 2000; Al-Ajmi and Zimmerman 2005; Pariseau 2007; Benz and Schwab 2008; Das and Basudhar 2009).

## 5 Advantages and Limitations

The advantages of the MC failure criterion are its mathematical simplicity, clear physical meaning of the material parameters, and general level of acceptance. A limitation surrounds the numerical implementation of a failure criterion containing corners in the $\pi$-plane (Fig. 2b), as opposed to a smooth function, e.g., Drucker-Prager (1952) failure criterion. Deformation analysis requires a flow rule, a relationship between strain increments and stress, such that the flow rule determines the orientation of the strain-increment vector with respect to the yield condition, e.g., normal for an associative flow rule. Thus, the orientation of the strain-increment vectors is unique along the sides of the MC pyramid. However, along the edges of the pyramid (corners in the $\pi$-plane), there is some freedom in the orientation (Drescher 1991).

## 6 Recommendations

Among the various failure criteria available, both linear and nonlinear equations dependent on the major $\sigma_I$ and minor $\sigma_{III}$ principal stresses are attractive because the geometric representation of laboratory data can be either in the principal stress plane or the Mohr diagram, which is often convenient. Triaxial compression and extension testing is suggested as a standard procedure to evaluate an intermediate-stress effect, although true triaxial testing is needed to describe the failure surface between the axi-symmetric stress states. Nonetheless, as a first order approximation to the behaviour of rock, the Mohr–Coulomb failure criterion is recommended when the three principal stresses are compressive and when considering a limited range of mean stress.

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