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Tracially amenable actions and purely infinite crossed products

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Abstract
We introduce the notion of tracial amenability for actions of discrete groups on unital, tracial C*-algebras, as a weakening of amenability where all the relevant approximations are done in the uniform trace norm. We characterize tracial amenability with various equivalent conditions, including topological amenability of the induced action on the trace space. Our main result concerns the structure of crossed products: for groups containing the free group F_2, we show that outer, tracially amenable actions on simple, unital, $\mathcal{Z}$-stable C*-algebras always have purely infinite crossed products. Finally, we give concrete examples of tracially amenable actions of free groups on simple, unital AF-algebras.

1 Introduction

The theory of amenable actions on C*-algebras is an important tool for studying approximation properties of crossed products. Already initiated in [2], this topic has recently received a lot of attention after it gained new impetus in [9] (see also [1, 4, 8, 32]). As recently established in [32], the notion of amenability is equivalent to the

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so-called *quasicentral approximation property* (QAP) from [8]. The QAP is a versatile tool making powerful averaging techniques accessible, with evidence being the classification of amenable, outer actions on Kirchberg algebras [19] or the equivariant $\mathcal{O}_2$-absorption theorem [40].

The motivation for this paper originates from Elliott's classification program, a long-time endeavour in the theory of C*-algebras aiming to classify nuclear C*-algebras by K-theoretic and tracial data. Elliott's program is now considered to be essentially completed, with unital, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebras satisfying the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet [38] being classified up to isomorphism by their Elliott invariant (see [48] for an exhaustive bibliography on the matter). In the rest of this introduction, and although this terminology is not standard, algebras satisfying all these assumptions will be called *classifiable*. In view of this recent progress, an important further step is to identify prominent classes of C*-algebras that satisfy the assumptions of the classification theorem. This problem has attracted considerable attention in connection to crossed products, and most of the work in the literature has focused on actions of discrete (and usually amenable) groups on C*-algebras that are either abelian or simple. This work deals with the latter setting.

If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a discrete group G on a simple, unital C*-algebra A, then the crossed product $A \rtimes_r G$ is simple whenever α is outer (see [24]). If A is moreover nuclear, then $A \rtimes_r G$ is nuclear if and only if α is amenable (see [2]). Whether $A \rtimes_r G$ satisfies the UCT is a very subtle question: the answer is always "yes" if A satisfies the UCT, and G is torsion-free with the Haagerup property (see [21, 30]), but the problem is equivalent to the UCT question for torsion groups (see [5, Example 23.15.12 (d)]. Therefore, modulo the UCT, classifiability of $A \rtimes_r G$ for an amenable, outer action on a simple, unital, nuclear C*-algebra reduces to proving $\mathcal{Z}$-stability for $A \rtimes_r G$.

The problem of establishing $\mathcal{Z}$-stability for $A \rtimes_r G$ when A is simple and nuclear has been largely investigated in the case where G is amenable and A is $\mathcal{Z}$-stable. In this case, $\mathcal{Z}$-stability of $A \rtimes_r G$ is conjectured to always hold (see [43, Conjecture A]), and this has been verified in full generality if A is purely infinite ([42]), and under various degrees of generality when A is stably finite and its trace space $T(A)$ is a Bauer simplex with finite dimensional boundary ([15, 28, 29, 39, 49]). Further progress has been recently obtained in [14], where $\mathcal{Z}$-stability of $A \rtimes_r G$ is obtained in a number of cases where the boundary of $T(A)$ is compact but not necessarily finite dimensional. On the other hand, $\mathcal{Z}$-stability may fail for actions of nonamenable groups; see [16, Theorem B].

In this paper, we study amenable actions of *nonamenable* groups on simple C*-algebras, inspired by the results in [13] for commutative C^*-algebras. Our original motivation was to show that the crossed product of an outer, amenable action of a nonamenable group on a simple, unital, nuclear, $\mathcal{Z}$-stable C*-algebra is automatically purely infinite and simple. For groups containing the free group F_2, this follows from Corollary 3.15. As it turns out, the assumptions on both the algebra and the action can be weakened significantly. For once, we do not need to assume A to be nuclear or even $\mathcal{Z}$-stable (see Corollary 3.15 for the minimal set of assumptions). More important

for this work is the fact that amenability of α is also stronger than necessary. This observation led us to identify and isolate the following notion:

Definition A (Definition 2.2) An action $\alpha\colon G \to \mathrm{Aut}(A)$ of a countable, discrete group G on a separable, unital, tracial C*-algebra A is said to be *tracially amenable* if there is a sequence $(\xi_n)_{n\in\mathbb{N}}$ of finitely supported functions $\xi_n\colon G \to A$ with $\|\xi_n\| \leq 1$ such that

$$\lim_{n\to\infty} \|\xi_n a - a\xi_n\|_{2,u} = \lim_{n\to\infty} \|\langle \xi_n, \xi_n\rangle - 1\|_{2,u} = \lim_{n\to\infty} \left\|\widetilde{\alpha}_g(\xi_n) - \xi_n\right\|_{2,u} = 0$$

for all $a \in A$ and all $g \in G$.

In the above definition, we denote by $\widetilde{\alpha}$ the diagonal action on $\ell^2(G, A)$ with left translation on G and α on A, and we denote by $\|\cdot\|_{2,u}$ the so-called *uniform trace norm* on $\ell^2(G, A)$; see the comments before Lemma 2.1. Thanks to the characterizations obtained in [32], the usual notion of amenability is obtained by replacing the uniform trace norm by the usual Hilbert C*-module-norm on $\ell^2(G, A)$. Since $\|\cdot\|_{2,u} \leq \|\cdot\|$, it follows that any amenable action is tracially amenable.

Section 2 is devoted to obtaining several characterizations of tracial amenability, inspired by the work done in [32, Theorem 3.2, Theorem 4,4]. We reproduce some of them below:

Theorem B (Theorem 2.5) *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a countable, discrete, exact group G on a separable, unital, tracial C*-algebra A. The following are equivalent:*

(1) The action α is tracially amenable.
(2) There is a sequence $(\xi_n)_{n\in\mathbb{N}}$ of finitely supported functions $\xi_n\colon G \to A$ with $\|\xi_n\| \leq 1$ such that

$$\lim_{n\to\infty} \|\langle \xi_n, \xi_n\rangle - 1\|_{2,u} = \lim_{n\to\infty} \left\|\widetilde{\alpha}_g(\xi_n) - \xi_n\right\|_{2,u} = 0$$

for all $g \in G$.
(3) The induced action $\alpha^\omega\colon G \to \mathrm{Aut}(A^\omega \cap \iota(A)')$ on the tracial ultrapower is amenable.
(4) The induced action $G \curvearrowright \overline{T(A)}$ is topologically amenable.
(5) The induced action $G \curvearrowright \overline{\partial_e T(A)}$ is topologically amenable.
(6) The induced action $\alpha^{**}_{\mathrm{fin}}\colon G \to \mathrm{Aut}(A^{**}_{\mathrm{fin}})$ is von Neumann-amenable.

Some of the above conditions look similar to analogous characterizations of amenability on C*-algebras, while some do not admit a counterpart in that setting. For example, the difference between Definition A and item (2) above is that we do not require approximate centrality in (2). Also, (4) and (5) do not have analogues in the setting of amenable actions, since amenability of $G \curvearrowright S(A)$ does not imply amenability of α.

We study the structure of crossed products in Sect. 3. For groups containing the free group F_2, we show that tracially amenable actions give rise to purely infinite crossed products:

Theorem C *(Theorem 3.15) Let G be a countable, discrete group containing the free group F_2, let A be a simple, separable, unital, stably finite, nuclear, $\mathcal{Z}$-stable C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a tracially amenable, outer action. Then $A \rtimes_r G$ is a unital, simple, purely infinite C*-algebra.*

As mentioned before, the requirements on A can be weakened, and we in particular do not need to assume A to be either nuclear or $\mathcal{Z}$-stable (see the statement of Corollary 3.15 for the precise assumptions on A). The condition on the group can also be relaxed, and it suffices to assume that G has what we call *weak paradoxical towers*; see Definition 3.10. To obtain Theorem C, we show in Proposition 3.9 that it suffices to prove that any action as in the statement satisfies what Bosa, Perera, Wu and Zacharias call *dynamical strict comparison* (Definition 3.8). That this is the case is shown in Theorem 3.14, by exploiting the tension between tracial amenability of α and the existence of weak paradoxical towers in G.

Theorem C is new even if α is amenable. In this setting, $A \rtimes_r G$ is nuclear and therefore a Kirchberg algebra, so in particular $\mathcal{O}_\infty$-stable (and, therefore, also $\mathcal{Z}$-stable). If, moreover, G is torsion-free and has the Haagerup property (for example, $G = F_n$; see [7, Definition 12.2.1]), and A satisfies the UCT, then $A \rtimes_r G$ also satisfies the UCT and thus is completely determined by its K-theory by [33]; see Corollary 3.13. Thus, the assumptions of outerness and amenability on α do in fact guarantee classifiability of the crossed product. There is, however, a drawback:

Problem D Are there any amenable actions of nonamenable groups on simple, unital, stably finite C*-algebras?

The above problem has recently attracted a fair amount of attention, and to the best of our knowledge it remains open. If one drops unitality of the algebra, an example has been constructed in [41].

The fact that Problem D remains open highlights another advantage of focusing on *tracially* amenable actions throughout: namely, we can construct several examples of actions satisfying the assumptions of Theorem C:

Example E (Example 2.10; Example 2.11) For every $n \geq 2$, there exist outer, tracially amenable actions of the free group F_n on stably finite, classifiable C*-algebras, including actions on simple, unital AF-algebras.

We do not know if the actions that we construct here are amenable.

2 Tracially amenable actions

Amenable actions of discrete groups on C*-algebras were introduced in [2] and later extended to locally compact groups in [9]. In this section, we define a tracial analogue of amenability for actions of discrete groups (Definition 2.2). The motivation to study this notion is two-fold. On the one hand, tracial amenability allows us to obtain strong structural results for the crossed product; see Subsection 3. On the other hand, and unlike for amenable actions, it is possible to construct many tracially amenable actions of nonamenable groups on unital, simple, stably finite C*-algebras, by means of classification results (see Subsection 2.1).

Fix a unital C*-algebra A and a discrete group G. For an action $\alpha\colon G \to \mathrm{Aut}(A)$, we denote by $\ell^2(G, A)$ the completion of $C_c(G, A)$ with respect to the norm $\|\cdot\|$ induced by the A-valued inner product given by

$$\langle \xi, \eta \rangle = \sum_{g \in G} \xi(g)^* \eta(g)$$

for all $\xi, \eta \in C_c(G, A)$. We equip $\ell^2(G, A)$ with the A-bimodule structure given by pointwise multiplication and the diagonal G-action $\widetilde{\alpha}$ given by $\widetilde{\alpha}_g(\xi)(h) = \alpha_g(\xi(g^{-1}h))$, for all $\xi \in \ell^2(G, A)$, and all $g, h \in G$.

Recall that a tracial state (often called simply "trace" in this work) on A is a continuous linear functional $\tau\colon A \to \mathbb{C}$ satisfying $\tau(1) = 1$, $\tau(A_+) \subseteq [0, \infty)$ and $\tau(ab) = \tau(ba)$ for all $a, b \in A$. We write $T(A)$ for the compact convex space of all traces on A. Given a trace $\tau \in T(A)$ and $a \in A$, we denote by $\|a\|_{2,\tau} = \tau(a^*a)^{\frac{1}{2}}$ the associated 2-*seminorm* on A. We also denote by $\|\cdot\|_{2,\tau}$ the induced seminorm on $\ell^2(G, A)$, given by $\|\xi\|_{2,\tau} = \tau(\langle \xi, \xi \rangle)^{\frac{1}{2}}$, for all $\xi \in \ell^2(G, A)$. In order to see that this is indeed a seminorm, note that $\|\xi\|_{2,\tau}$ agrees with the norm of the element $\xi \otimes \xi_\tau \in \ell^2(G, A) \otimes_{\pi_\tau} H_\tau$ where $(\pi_\tau, H_\tau, \xi_\tau)$ is the GNS construction of τ. The *uniform* 2-*seminorm* on A (respectively, on $\ell^2(G, A)$) is given by $\|\cdot\|_{2,u} = \sup_{\tau \in T(A)} \|\cdot\|_{2,\tau}$. The following observation, which is a variant of the Cauchy-Schwarz inequality, will be used repeatedly.

Lemma 2.1 *Let A be a unital C*-algebra and let $\tau \in T(A)$. Then:*

(1) For every $a \in A$, we have $|\tau(a)| \leq \|a\|_{2,\tau} \leq \|a\|$.
(2) For every $\xi, \eta \in \ell^2(G, A)$ we have $\|\langle \xi, \eta \rangle\|_{2,\tau} \leq \|\xi\| \|\eta\|_{2,\tau} \leq \|\xi\| \|\eta\|$.

Part (1) is immediate. For part (2), by [25, Proposition 1.1] we have

$$\langle \xi, \eta \rangle^* \langle \xi, \eta \rangle \leq \|\langle \xi, \xi \rangle\| \langle \eta, \eta \rangle.$$

Applying τ and taking the square roots gives the first inequality, while the second one is immediate.

The following definition is inspired by the *quasicentral approximation property* (QAP) from [8, Definition 3.1], which was shown to be equivalent to amenability for actions on C*-algebras in [32, Theorem 3.2]. In view of this equivalence, we directly present the tracial analogue of [8, Definition 3.1] under the name of tracial amenability.

Definition 2.2 An action $\alpha\colon G \to \mathrm{Aut}(A)$ of a discrete group G on a unital, C*-algebra A with $T(A) \neq \emptyset$ is called *tracially amenable* if for all finite subsets $F \subseteq A$ and $K \subseteq G$, and for every $\varepsilon > 0$, there exists $\xi \in C_c(G, A)$ satisfying

(1) $\|\xi\| \leq 1$ (in the Hilbert C*-module-norm of $\ell^2(G, A)$);
(2) $\|\xi a - a\xi\|_{2,u} < \varepsilon$, for all $a \in F$;
(3) $\|\langle \xi, \xi \rangle - 1\|_{2,u} < \varepsilon$;
(4) $\left\|\widetilde{\alpha}_g(\xi) - \xi\right\|_{2,u} < \varepsilon$, for all $g \in K$.

Remark 2.3 Note that conditions (3) and (4) in Definition 2.2 can be equivalently replaced by the condition $\|\langle \xi, \widetilde{\alpha}_g(\xi)\rangle - 1\|_{2,u} < \varepsilon$ for all $g \in K$. If G is countable and discrete, and A is separable, Definition 2.2 is furthermore equivalent to the existence of a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $C_c(G, A)$ such that the terms in items (2)-(4) converge to zero along $n \to \infty$, for every $a \in A$ and $g \in G$.

The definition of the QAP in [8, Definition 3.1] differs from Definition 2.2 in that all estimates are formulated in the C*-norm, rather than in the tracial seminorm. Since the C*-norm dominates $\|\cdot\|_{2,u}$, it follows from [32, Theorem 3.2] that every amenable action is tracially amenable.

Our next goal is to obtain several characterizations of tracial amenability; see Theorem 2.5. For this, we need some preparation. Given an action $\alpha\colon G \to \mathrm{Aut}(A)$, a function $\theta\colon G \to A$ is said to be of *positive type* with respect to α if for every finite set $K \subseteq G$, the matrix $(\alpha_g(\theta(g^{-1}h)))_{g,h\in K} \in M_{|K|}(A)$ is positive. We reproduce here two results from [2] in a way that will be more convenient for us later. The center of a C*-algebra A is denoted $Z(A)$.

Theorem 2.4 [2, Theorem 3.3, Theorem 4.9] *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. Suppose that for every finite subset $K \subseteq G$ and every $\varepsilon > 0$, there exists a finitely supported positive type function $\theta\colon G \to Z(A)$ (with respect to α) satisfying $\theta(e) \leq 1$ and $\|\theta(g) - 1\| < \varepsilon$ for all $g \in K$. Then α is amenable. If A is commutative, the converse holds as well.*

Given a unital C*-algebra A, its tracial state space $T(A)$ is a convex compact topological space, and we denote by $\partial_e T(A)$ the set of its extremal points.

For a fixed free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, we denote by A^ω the quotient of $\ell^\infty(\mathbb{N}, A)$ by the ideal of sequences $(a_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\omega} \|a_n\|_{2,u} = 0$. This is the *tracial ultrapower* of A. There exists a natural unital homomorphism $\iota\colon A \to A^\omega$, which maps each element in A to the class of the corresponding constant sequence. Note that the map ι is not necessarily injective, unless traces separate positive elements (which is always the case if A is simple and tracial). We denote by $A^\omega \cap \iota(A)'$ the C*-algebra consisting of all elements in A^ω that commute with (the images of) all constant sequences. Any action $\alpha\colon G \to \mathrm{Aut}(A)$ canonically induces an action on A^ω and also on $A^\omega \cap \iota(A)'$, by acting coordinatewise. With a slight abuse of notation, we shall denote both actions by α^ω.

The *finite part* A^{**}_{fin} of the bidual A^{**} of a C*-algebra A can be identified with the weak closure of A with respect to the sum of all GNS-representations for all traces. Namely,

$$A^{**}_{\mathrm{fin}} = \left(\bigoplus\nolimits_{\tau\in T(A)} \pi_\tau\right)(A)'' \subseteq \mathcal{B}\left(\bigoplus\nolimits_{\tau\in T(A)} H_\tau\right).$$

Every action α on A canonically extends to an action $\alpha^{**}_{\mathrm{fin}}$ on A^{**}_{fin}.

An action $\alpha\colon G \to \mathrm{Aut}(M)$ of a discrete group G on a von Neumann algebra M is called *von Neumann-amenable* if there exists a net $(\xi_\lambda)_{\lambda\in\Lambda}$ in $C_c(G, Z(M))$ such that $\langle \xi_\lambda, \xi_\lambda\rangle = 1$, for all $\lambda \in \Lambda$, and the net $(\langle \xi_\lambda, \widetilde{\alpha}_g(\xi_\lambda)\rangle)_{\lambda\in\Lambda}$ converges to $1 \in Z(M)$ pointwise on G in the ultraweak topology (see [4, Theorem 1.1]).

In the following result we give several conditions that are equivalent to tracial amenability of an action. Some of the conditions below look similar to analogous characterizations of amenability obtained in (and, indeed, motivated by) [32, Theorem 3.2, Theorem 4,4]. Others do not have a counterpart in that setting and reveal phenomena that are very special to tracial amenability. We discuss this in more detail after the proof of Theorem 2.5; see also Remark 2.8.

Theorem 2.5 *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital, separable C^*-algebra A with $T(A) \neq \emptyset$. Consider the following conditions:*

(1) The action α is tracially amenable.
(2) For $\varepsilon > 0$ and any finite set $K \subseteq G$, there exists $\xi \in C_c(G, A)$ such that

(a) $\|\xi\| \leq 1$;
(b) $\|\langle \xi, \xi\rangle - 1\|_{2,u} < \varepsilon$;
(c) $\|\widetilde{\alpha}_g(\xi) - \xi\|_{2,u} < \varepsilon$ for all $g \in K$.

(3) For any separable, unital C^*-algebra C with an action $\gamma\colon G \to \mathrm{Aut}(C)$, there is an equivariant, unital, completely positive map $\Phi\colon (C, \gamma) \to (A^\omega \cap \iota(A)', \alpha^\omega)$.
(4) There is a sequence of functions $\theta_n\colon G \to A^\omega \cap \iota(A)'$ of finite support and of positive type with respect to α^ω, which satisfy $\theta_n(e) \leq 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|\theta_n(g) - 1\| = 0$, for all $g \in G$.
(5) The induced action $\alpha^\omega\colon G \to \mathrm{Aut}(A^\omega \cap \iota(A)')$ is amenable.
(6) The induced action $G \curvearrowright \overline{T(A)}$ is topologically amenable.
(7) The induced action $G \curvearrowright \overline{\partial_e T(A)}$ is topologically amenable.
(8) The induced action $\alpha^{**}_{\mathrm{fin}}\colon G \to \mathrm{Aut}(A^{**}_{\mathrm{fin}})$ is von Neumann-amenable.

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8). If moreover G is countable and exact then all of the above conditions are equivalent.

For the proof we need the following well-known lemmas.

Lemma 2.6 *Let A and B be unital C^*-algebras, let $\pi : A \to B$ be a surjective homomorphism, and let F be a finite set. Denote the induced map $\ell^2(F, A) \to \ell^2(F, B)$ again by π. Given $\eta \in \ell^2(F, B)$, there exists $\xi \in \ell^2(F, A)$ such that $\pi(\xi) = \eta$ and $\|\xi\| = \|\eta\|$.*

Proof Note that there is an isometric embedding $\ell^2(F, A) \hookrightarrow M_{|F|}(A)$ given by

$$(a_1, \ldots, a_n) \mapsto \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix},$$

and an analogous embedding $\ell^2(F, B) \hookrightarrow M_{|F|}(B)$. By a standard functional calculus argument (see, for example, [47, Lemma 17.3.3]), there is an element $\xi' \in M_{|F|}(A)$ satisfying $\pi \otimes \mathrm{id}_{M_{|F|}}(\xi') = \eta$ and $\|\xi'\| = \|\eta\|$. We write $p = \mathrm{diag}(1, 0, \ldots, 0) \in M_{|F|}(A)$. Then $\xi := \xi' p \in \ell^2(F, A)$ has the desired properties. □

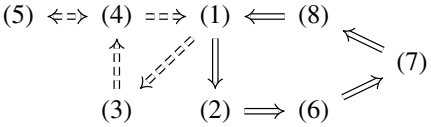
Lemma 2.7 *Let K be a compact convex set and denote by $\mathrm{Aff}_c(K) \subseteq C(K)$ the set of all continuous affine functions $K \to \mathbb{R}$. Then the restriction of the weak topology of $C(K)$ to $\mathrm{Aff}_c(K)$ coincides with the topology of pointwise convergence.*

Proof This follows from the fact that for every $f \in \mathrm{Aff}_c(K)$ and any Radon probability measure μ on K, we have

$$\int_K f d\mu = f(\beta(\mu)),$$

where $\beta(\mu) \in K$ denotes the barycenter of μ (see [44, Lemma IV.6.3]).

Proof of Theorem 2.5 We prove the following set of implications where, in the implications labeled by dashed arrows, we additionally assume that G is countable and exact:



(1) $\Rightarrow$ (3). Fix a state $\varphi \in S(C)$. Since G is countable and A is separable, we can find a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $C_c(G, A)$ satisfying

(a.1) $\|\xi_n\| \leq 1$ for all $n \in \mathbb{N}$;
(a.2) $\lim_{n\to\infty} \|\xi_n a - a\xi_n\|_{2,u} = 0$ for all $a \in A$;
(a.3) $\lim_{n\to\infty} \|\langle \xi_n, \xi_n\rangle - 1\|_{2,u} = 0$;
(a.4) $\lim_{n\to\infty} \|\widetilde{\alpha}_g(\xi_n) - \xi_n\|_{2,u} = 0$ for all $g \in G$.

We define a map $\widetilde{\Phi} = (\Phi_n)_{n\in\mathbb{N}} \colon C \to \ell^\infty(\mathbb{N}, A)$ by setting

$$\Phi_n(c) := \sum_{g\in G} \varphi(\gamma_{g^{-1}}(c))\xi_n(g)^*\xi_n(g),$$

for $n \in \mathbb{N}$ and $c \in C$. (Note that $\sup_{n\in\mathbb{N}} \|\Phi_n\| \leq 1$ by (a.1) above, so that the resulting map $\widetilde{\Phi}$ does indeed take values in $\ell^\infty(\mathbb{N}, A)$.) Denote by $\Phi \colon C \to A^\omega$ the composition of $\widetilde{\Phi}$ with the canonical quotient map $\ell^\infty(\mathbb{N}, A) \to A^\omega$. We claim that Φ is an equivariant, unital, completely positive map, and that its image is contained in $A^\omega \cap \iota(A)'$. The fact that Φ is unital follows from the equality $\|\Phi_n(1) - 1\|_{2,u} = \|\langle \xi_n, \xi_n\rangle - 1\|_{2,u}$ together with (a.3). As each Φ_n is completely positive, the same is true for $\widetilde{\Phi}$ and thus also for Φ. To see that Φ takes values in $A^\omega \cap \iota(A)'$, fix $a \in A$ and $c \in C$. Given $d \in C$, denote by $T_d \in \mathcal{L}(\ell^2(G, A))$ the operator given by pointwise multiplication with the function $g \mapsto \varphi(\gamma_{g^{-1}}(d))$, and note that $\|T_d\| \leq \|d\|$. Using the Cauchy-Schwarz inequality (see part (2) of Lemma 2.1) and $\|\xi_n\| \leq 1$ at the last step, we get

$$\begin{aligned}
\big\|a\Phi_n(c) - \Phi_n(c)a\big\|_{2,u} &= \big\|a\langle \xi_n, T_c\xi_n\rangle - \langle \xi_n, T_c\xi_n\rangle a\big\|_{2,u} \\
&= \big\|\langle \xi_n a^*, T_c\xi_n\rangle - \langle T_{c^*}\xi_n, \xi_n a\rangle\big\|_{2,u} \\
&\leq \big\|\langle \xi_n a^* - a^*\xi_n, T_c\xi_n\rangle\big\|_{2,u} + \big\|\langle T_{c^*}\xi_n, a\xi_n - \xi_n a\rangle\big\|_{2,u} \\
&\leq \|c\|\big(\|\xi_n a^* - a^*\xi_n\|_{2,u} + \|a\xi_n - \xi_n a\|_{2,u}\big),
\end{aligned}$$

with the latter term converging to zero as $n \to \infty$, by (a.2). To see that Φ is equivariant, fix $g \in G$ and $c \in C$. Using again the Cauchy-Schwarz inequality and $\|\xi_n\| = \|\widetilde{\alpha}_g(\xi_n)\| \leq 1$ at the last step, we obtain

$$\begin{aligned}
&\big\|\alpha_g(\Phi_n(c)) - \Phi_n(\gamma_g(c))\big\|_{2,u} \\
&= \Big\| \sum_{h\in G} \varphi(\gamma_{h^{-1}g}(c))\Big(\alpha_g(\xi_n(g^{-1}h)^*\xi_n(g^{-1}h)) - \xi_n(h)^*\xi_n(h)\Big)\Big\|_{2,u} \\
&= \Big\| \langle \widetilde{\alpha}_g(\xi_n), T_{\gamma_g(c)}\widetilde{\alpha}_g(\xi_n)\rangle - \langle \xi_n, T_{\gamma_g(c)}\xi_n\rangle \Big\|_{2,u} \\
&\leq 2\|c\|\|\xi_n - \widetilde{\alpha}_g(\xi_n)\|_{2,u},
\end{aligned}$$

with the latter term converging to zero as $n \to \infty$, by (a.4). This proves (3).

(3) $\Rightarrow$ (4). Recall that since G is countable and exact, there is an amenable action of G on a compact, metrizable space X (see [7, Theorem 5.1.7]). By Theorem 2.4, there exists a sequence of finitely supported positive type functions $\theta_n \colon G \to C(X)$ with respect to α satisfying $\theta_n(e) \leq 1$ and $\lim_{n\to\infty} \|\theta_n(g) - 1\| = 0$ for all $g \in G$. If $\Psi \colon C(X) \to A^\omega \cap \iota(A)'$ is an equivariant, unital, completely positive map, then $\Psi \circ \theta_n \colon G \to A^\omega \cap \iota(A)'$, for $n \in \mathbb{N}$, gives a sequence of positive type functions with respect to α^ω with the desired properties.

(4) $\Rightarrow$ (5). We prove amenability of the action $\alpha^\omega \colon G \to \mathrm{Aut}(A^\omega \cap \iota(A)')$ by showing that it satisfies the QAP, which can be stated as our Definition 2.2 once every occurrence of the tracial norm is replaced with the C*-norm on A^ω (see [8, Definition 3.1]), and then resorting to [32, Theorem 3.2]. Fix a finite subset $K \subseteq G$ and $\varepsilon > 0$. By assumption, there is a finitely supported positive type function $\theta \colon G \to A^\omega \cap \iota(A)'$ satisfying $\theta(e) \leq 1$ and $\|\theta(g) - 1\| < \varepsilon$ for all $g \in K$. By [2, Proposition 2.5], there exists $\eta \in \ell^2(G, A^\omega \cap \iota(A)')$ with $\theta(g) = \Big\langle \eta, \widetilde{\alpha^\omega_g}(\eta)\Big\rangle$ for all $g \in G$. By slightly perturbing η (and therefore θ), we can assume that η has finite support. This almost gives the QAP for α^ω, except for the condition of almost centrality, which follows from a standard reindexing argument. To show this, fix a finite set $F \subseteq A^\omega \cap \iota(A)'$. Applying Lemma 2.6 to the quotient map $\ell^\infty(\mathbb{N}, A) \to A^\omega$, we can find a sequence $(\eta_n)_{n\in\mathbb{N}}$ in $C_c(G, A) \subseteq \ell^2(G, A)$ that represents η and satisfies $\|\eta_n\| \leq 1$ and $\mathrm{supp}(\eta_n) \subseteq \mathrm{supp}(\eta)$ for every $n \in \mathbb{N}$. Since we have $\|\theta(g) - 1\| < \varepsilon$ for all $g \in K$, we can find $(b_n^{(g)})_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A)$, for $g \in K$, satisfying $\lim_{n\to\omega} \|b_n^{(g)}\|_{2,u} = 0$ for all $g \in K$, and

$$\|\langle \eta_n, \widetilde{\alpha}_g(\eta_n)\rangle - 1 + b_n^{(g)}\| \leq \varepsilon \tag{2.1}$$

for all $n \in \mathbb{N}$ and $g \in K$. Now choose a dense sequence $(a_n)_{n\in\mathbb{N}}$ in A, and for each $x \in F$ find a representative $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, A)$. Using that the $(\eta_n)_{n\in\mathbb{N}}$ have uniformly finite support (contained in $\mathrm{supp}(\eta)$) and that each $(\eta_n(g))_{n\in\mathbb{N}}$, for $g \in \mathrm{supp}(\eta)$, represents the element $\eta(g) \in A^\omega \cap \iota(A)'$, we can find inductively an increasing sequence $(k(n))_{n\in\mathbb{N}}$ of natural numbers satisfying

(b.1) $\|x_n\eta_{k(n)} - \eta_{k(n)}x_n\|_{2,u} < \frac{1}{n}$, for all $n \in \mathbb{N}$;
(b.2) $\|a_j\eta_{k(n)} - \eta_{k(n)}a_j\|_{2,u} < \frac{1}{n}$, for all $n \in \mathbb{N}$ and $j = 1, \ldots, n$;
(b.3) $\|b^{(g)}_{k(n)}\|_{2,u} < \frac{1}{n}$, for all $n \in \mathbb{N}$ and $g \in K$.

Then the sequence $\xi := (\eta_{k(n)})_{n\in\mathbb{N}}$ gives rise to a contraction in $\ell^2(G, A^\omega \cap \iota(A)')$ of finite support satisfying $x\xi = \xi x$ for all $x \in F$. Moreover,

$$\left\|\langle \xi, \widetilde{\alpha_g^\omega}(\xi)\rangle - 1\right\| \leq \sup_{n\in\mathbb{N}} \left\|\langle \eta_{k(n)}, \widetilde{\alpha}_g(\eta_{k(n)})\rangle - 1 + b_{k(n)}^{(g)}\right\| \overset{(2.1)}{\leq} \varepsilon,$$

for all $g \in K$. This proves (5).

(5) $\Rightarrow$ (4). This follows from associating to an element $\xi \in \ell^2(G, A^\omega \cap \iota(A)')$ of finite support, the finitely supported positive type function $\theta_\xi : G \to A^\omega \cap \iota(A)'$ given by

$$\theta_\xi(g) = \left\langle \xi, \widetilde{\alpha_g^\omega}(\xi)\right\rangle$$

for all $g \in G$, and using that amenability of α^ω is equivalent to the QAP.

(4) $\Rightarrow$ (1). Let $F \subseteq A$ and $K \subseteq G$ be finite sets and $\varepsilon > 0$. As in the proof of (4) $\Rightarrow$ (5), we can find an element $\eta \in C_c(G, A^\omega \cap \iota(A)')$ satisfying $\|\eta\| \leq 1$ and

$$\left\|\left\langle \eta, \widetilde{\alpha_g^\omega}(\eta)\right\rangle - 1\right\| < \varepsilon, \tag{2.2}$$

for all $g \in K$, and a sequence $(\eta_n)_{n\in\mathbb{N}}$ in $C_c(G, A)$ representing η that satisfies $\|\eta_n\| \leq 1$ and $\mathrm{supp}(\eta_n) \subseteq \mathrm{supp}(\eta)$ for every $n \in \mathbb{N}$. We can thus find a large enough $n_0 \in \mathbb{N}$ so that

(c.1) $\|a\eta_{n_0} - \eta_{n_0}a\|_{2,u} < \varepsilon$, for all $a \in F$;
(c.2) $\|\langle \eta_{n_0}, \widetilde{\alpha}_g(\eta_{n_0})\rangle - 1\|_{2,u} \leq \varepsilon$, for all $g \in K$.

This proves (1) (see Remark 2.3).

(1) $\Rightarrow$ (2). This is immediate.

(2) $\Rightarrow$ (6). Fix a finite set $K \subseteq G$ and $\varepsilon > 0$. Choose an element $\xi \in C_c(G, A)$ satisfying $\|\xi\| \leq 1$ and

$$\|\langle \xi, \widetilde{\alpha}_g(\xi)\rangle - 1\|_{2,u} < \varepsilon,$$

for $g \in K$. We define a positive type function $\theta\colon G \to C(T(A))$ by

$$\theta(g)(\tau) := \tau(\langle \xi, \widetilde{\alpha}_g(\xi)\rangle)$$

for all $g \in G$ and all $\tau \in T(A)$. Using part (1) of Lemma 2.1 at the second step, we get

$$\begin{aligned}\|\theta(g) - 1\| &= \sup_{\tau\in T(A)} \left|\tau(\langle \xi, \widetilde{\alpha}_g(\xi)\rangle - 1)\right| \\ &\leq \|\langle \xi, \widetilde{\alpha}_g(\xi)\rangle - 1\|_{2,u} < \varepsilon.\end{aligned}$$

for all $g \in K$. Thus, $G \curvearrowright T(A)$ is topologically amenable by Theorem 2.4.

(6) $\Rightarrow$ (7). This follows by composing the functions $G \to C(T(A))$ coming from amenability of $G \curvearrowright T(A)$ with the equivariant quotient map $C(T(A)) \to C(\overline{\partial_e T(A)})$.

(7) $\Rightarrow$ (8). Denote by $B(\partial_e T(A))$ the C^*-algebra of Borel bounded functions on $\partial_e T(A)$. By [31, Theorem 3], there is a unital homomorphism $\theta\colon B(\partial_e T(A)) \to Z(A_{\mathrm{fin}}^{**})$. We claim that θ is equivariant. Given $\tau \in T(A)$, identify τ with its unique normal extension to A_{fin}^{**} and write μ_τ for the unique probability measure on $\partial_e T(A)$ whose barycenter is τ. The map $\theta\colon B(\partial_e T(A)) \to Z(A_{\mathrm{fin}}^{**})$ obtained in [31, Theorem 3] satisfies

$$\tau(\theta(f)) = \int_{\partial_e T(A)} f(\lambda)\, d\mu_\tau(\lambda), \tag{2.3}$$

for every $f \in B(\partial_e T(A))$ and every $\tau \in T(A)$. Condition (2.3) uniquely determines θ, since traces on A (or rather their normal extensions to A_{fin}^{**}) are in bijective correspondence with the normal states on $Z(A_{\mathrm{fin}}^{**})$, via the restriction map (see [6, Theorem III.2.5.7] and [6, Theorem III.2.5.14]). In what follows, α^*, α^{**} and $\alpha_{\mathrm{fin}}^{**}$ denote the actions induced by α respectively on $T(A)$, $B(\partial_e T(A))$ and A_{fin}^{**}. In order to verify that θ is equivariant, it is thus sufficient to check that $\theta_g := (\alpha_{\mathrm{fin}}^{**})_g \circ \theta \circ \alpha_{g^{-1}}^{**}$ also verifies equality (2.3) for every $g \in G$. To prove this, notice first that, given $\tau \in T(A)$ and $g \in G$, by uniqueness of the boundary measure $\mu_{\tau\circ\alpha_g}$ we have that

$$\mu_{\tau\circ\alpha_g} = \mu_\tau \circ \alpha_g^*. \tag{2.4}$$

Fix $f \in B(\partial_e T(A))$ and $\tau \in T(A)$. Then

$$\begin{aligned}
\tau(\theta_g(f)) &= (\tau \circ \alpha_g)(\theta(f \circ \alpha_g^*)) \\
&\overset{(2.3)}{=} \int_{\partial_e T(A)} f(\alpha_g^*(\lambda))\, d\mu_{\tau\circ\alpha_g}(\lambda) \\
&\overset{(2.4)}{=} \int_{\partial_e T(A)} f(\alpha_g^*(\lambda))\, d\mu_\tau(\alpha_g^*(\lambda)) \\
&= \int_{\partial_e T(A)} f(\sigma)\, d\mu_\tau(\sigma).
\end{aligned}$$

This proves equivariance of θ. Composing θ with the unital, equivariant inclusion of $C\left(\overline{\partial_e T(A)}\right)$ into $B(\partial_e T(A))$, one obtains a unital, equivariant homomorphism $C\left(\overline{\partial_e T(A)}\right) \to Z(A_{\mathrm{fin}}^{**})$. Since the action on $\overline{\partial_e T(A)}$ is assumed to be amenable, we conclude that the action on A_{fin}^{**} is von Neumann-amenable (see, for example, [9, Lemma 3.21]).

(8) $\Rightarrow$ (1). Fix finite sets $F \subseteq A$ and $K \subseteq G$ and $\varepsilon > 0$. We may assume without loss of generality that F consists of contractions. Throughout the proof, we will identify A with its natural (not necessarily isomorphic) copy inside A_{fin}^{**}. This is possible thanks to Lemma 2.6 and the fact that $\|\cdot\|_{2,u}$ coincides on A and its copy inside A_{fin}^{**}.

We claim that for every $\delta > 0$ and a finite subset $T \subseteq T(A)$, there is $\eta \in C_c(G, A)$ satisfying $\|\eta\| \leq 1$ and

$$1 - \|\eta\|_{2,\tau}^2 + \sum_{g \in K} \|\eta - \widetilde{\alpha}_g(\eta)\|_{2,\tau}^2 + \sum_{a \in F} \|a\eta - \eta a\|_{2,\tau}^2 < \delta,$$

for all $\tau \in T$.

Set $M = A_{\mathrm{fin}}^{**}$ and $\gamma = \alpha_{\mathrm{fin}}^{**}$. Since γ is von Neumann amenable, there exists a net $(\xi_\lambda)_{\lambda \in \Lambda}$ of finitely supported contractions in $\ell^2(G, Z(M))$ such that

$$\langle \xi_\lambda, \xi_\lambda \rangle \to 1, \quad \text{and} \quad \langle \xi_\lambda - \widetilde{\gamma}_g(\xi_\lambda), \xi_\lambda - \widetilde{\gamma}_g(\xi_\lambda) \rangle \to 0$$

pointwise on G in the ultraweak topology. Since each trace $\tau \in T$ extends to a normal state on M, there exists $\lambda \in \Lambda$ such that, with $\xi := \xi_\lambda$, we have

$$\tau(1 - \langle \xi, \xi \rangle) < \tfrac{\delta}{9} \ \text{ and } \ \sum_{g \in K} \tau(\langle \xi - \widetilde{\gamma}_g(\xi), \xi - \widetilde{\gamma}_g(\xi) \rangle) < \tfrac{\delta}{9},$$

for all $\tau \in T$. By an application of Kaplansky's density theorem for Hilbert modules (see [9, Lemma 4.5]), there exists $\eta \in C_c(G, A) \subseteq \ell^2(G, A)$ with $\|\eta\| \leq 1$ satisfying $\tau(\langle \eta - \xi, \eta - \xi \rangle)^{\frac{1}{2}} < \frac{\delta}{9(|F|+|K|)}$, for all $\tau \in T$. Using the triangle inequality, one gets

$$\begin{aligned} 1 - \|\eta\|_{2,\tau}^2 = \tau(1 - \langle \eta, \eta \rangle) < \tfrac{\delta}{3}, \quad \text{and} \\ \sum_{g \in K} \|\eta - \widetilde{\alpha}_g(\eta)\|_{2,\tau}^2 = \sum_{g \in K} \tau(\langle \eta - \widetilde{\alpha}_g(\eta), \eta - \widetilde{\alpha}_g(\eta) \rangle) < \tfrac{\delta}{3}, \end{aligned}$$

for all $\tau \in T$. Since ξ takes values in the center of M, then η can be chosen so that $\sum_{a \in F} \|a\eta - \eta a\|_{2,\tau}^2 < \frac{\delta}{3}$, for all $\tau \in T$. This proves the claim.

Given $\eta \in C_c(G, A)$, with $\|\eta\| \leq 1$, set $Q(\eta) := \langle \eta, \eta \rangle$ and

$$D(\eta) := 1 - Q(\eta) + \sum_{g \in K} Q(\eta - \widetilde{\alpha}_g(\eta)) + \sum_{a \in F} Q(a\eta - \eta a).$$

Note that the map $\widehat{D(\eta)} \colon T(A) \to [0, \infty)$ given by $\widehat{D(\eta)}(\tau) = \tau(D(\eta))$, for all $\tau \in T(A)$, is continuous and affine. By the claim above, 0 is in the closure of the set

$$Z := \left\{ \widehat{D(\eta)} \colon \eta \in C_c(G, A), \|\eta\| \leq 1 \right\} \subseteq C(T(A))$$

with respect to the topology of pointwise convergence. Since the elements of Z are continuous, affine functions, it follows from Lemma 2.7 that 0 is also in the weak closure of Z. Thus, by the Hahn–Banach Theorem, there are $\eta_1, \ldots, \eta_n \in C_c(G, A)$ with $\|\eta_j\| \leq 1$ for all $j = 1, \ldots, n$, and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{j=1}^n \lambda_j = 1$, such that

$$\sum_{j=1}^{n} \lambda_j \widehat{D(\eta_j)}(\tau) \leq \varepsilon^2 \tag{2.5}$$

for all $\tau \in T(A)$. Assuming without loss of generality that G is infinite, and replacing each η_j with an appropriate right translate, we can assume that the supports of $\widetilde{\alpha}_g(\eta_i)$ and $\widetilde{\alpha}_h(\eta_j)$ are disjoint whenever $i \neq j$ and $g, h \in K \cup \{1\}$. Set

$$\xi := \sum_{j=1}^{n} \lambda_j^{\frac{1}{2}} \eta_j \in C_c(G, A).$$

Using that $\langle \eta_i, \eta_j \rangle \leq \delta_{i,j}$ for $i, j = 1, \ldots, n$, one checks that $\|\xi\| \leq 1$, since

$$\|\xi\|^2 \leq \sum_{j=1}^{n} \lambda_j \|\eta_j\|^2 \leq 1,$$

and that $D(\xi) = \sum_{j=1}^{n} \lambda_j D(\eta_j)$. Hence

$$\sup_{\tau \in T(A)} \tau(D(\xi)) = \sup_{\tau \in T(A)} \sum_{j=1}^{n} \lambda_j \tau(D(\eta_j)) \overset{(2.5)}{\leq} \varepsilon^2,$$

which implies

(1) $\|\xi a - a\xi\|_{2,u} \leq \varepsilon$, for all $a \in F$;
(2) $\|\langle \xi, \xi \rangle - 1\|_{2,u} \leq \varepsilon$;
(3) $\left\|\widetilde{\alpha}_g(\xi) - \xi\right\|_{2,u} \leq \varepsilon$, for all $g \in K$.

This proves (1). □

As mentioned before, some of the conditions in Theorem 2.5 resemble similar characterizations of amenability for actions on C*-algebras. For example, the equivalence between (1) and (5) is the tracial version of the fact that amenability of an action on A is equivalent to amenability of the induced action on the norm-central sequence algebra $A_\omega \cap A'$; see [32, Theorem 4.4], while the equivalence between (1) and (8) is the tracial version of the fact that amenability of an action on A is equivalent to von Neumann amenability of the induced action on A^{**}; see [32, Theorem 3.2] and [1, Lemma 6.5]. On the other hand, (2), (6) and (7) do not have analogous statements for amenable actions.

The authors would like to thank Siegfried Echterhoff and Rufus Willett for pointing out to them the following observation.

Remark 2.8 The analogue of (1) ⇔ (6) in Theorem 2.5, where one replaces tracial amenability with amenability, and $T(A)$ with the state space $S(A)$, is not true. While amenability of an action $\alpha\colon G \to \mathrm{Aut}(A)$ always implies amenability of the induced action on the state space by [32, Proposition 3.5], the converse fails. An easy example can be constructed from any amenable action $G \curvearrowright X$ of a nonamenable group G on

a compact space X by considering the associated inner action β on $C(X) \rtimes G$. As an inner action, β is not amenable as it induces the trivial action on $Z((C(X) \rtimes G)^{**})$. However, β induces an amenable action on $S(C(X) \rtimes G)$. Indeed, this follows from Theorem 2.4 and the equivariance of the unital, completely positive map $\iota\colon C(X) \to C(S(C(X) \rtimes G))$ given by $\iota(f)(\varphi) := \varphi(f)$, for all $f \in C(X)$ and $\varphi \in S(C(X) \rtimes G)$. For the same reason, there is no analogue of (1) $\Leftrightarrow$ (2) for amenable actions.

The key tool that allows to obtain these equivalences in the tracial setting is [31, Theorem 3].

Given an amenable action $\alpha\colon G \to \mathrm{Aut}(A)$ of a discrete group on a unital C*-algebra, it is not hard to show that there exists an invariant state on A if and only if G is amenable. The next lemma, which will be needed in Sect. 3, shows that the tracial counterpart of this statement also holds.

Lemma 2.9 *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a tracially amenable action of a discrete group on a unital C*-algebra with $T(A) \neq \emptyset$. Then G is amenable if and only if A has an invariant trace.*

Proof If G is amenable, then it is well-known that the affine action $G \curvearrowright T(A)$ has a fixed point. Conversely, assume that $\tau \in T(A)$ is G-invariant. Then the Dirac measure on $T(A)$ associated to τ is also G-invariant. Since $G \curvearrowright T(A)$ is amenable by (1)$\Rightarrow$(6) of Theorem 2.5 (the proof of this implication applies verbatim also if A is not separable), it follows from [13, Lemma 2.2] that G is amenable. □

2.1 Examples

In this subsection, we use our characterizations of tracial amenability from Theorem 2.5 (particularly (7)), together with results from the classification programme for simple, nuclear C*-algebras, to construct a wide class of examples of tracially amenable actions on unital, simple, stably finite C*-algebras. Since we lift automorphisms from the Elliott invariant to the C*-algebra in question, we work with free groups in order to guarantee that we get a group action. Recall that the *Elliott invariant* of a unital C*-algebra A is given by

$$\mathrm{Ell}(A) = ((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A), \rho_A);$$

see [18, Definition 2.4] for its definition.

Example 2.10 Let $n \in \mathbb{N}$, let $F_n = \langle g_1, \ldots, g_n \rangle$ be the free group on n generators, and let $\theta\colon F_n \curvearrowright X$ be an amenable action on a compact, metric space (for example, let X be the Gromov boundary ∂F_n). By [18, Theorem 14.8] (see also [12, Theorem 2.8], or [10]), there is a stably finite, classifiable C*-algebra A with

$$\mathrm{Ell}(A) \cong ((\mathbb{Z}, \mathbb{Z}_+, 1), \{0\}, \mathrm{Prob}(X), \rho_A),$$

where $\rho_A\colon K_0(A) \times T(A) \to \mathbb{R}$ is the (uniquely determined) pairing map given by $\rho_A(n[1_A], \tau) = n$ for all $n \in \mathbb{Z}$ and all $\tau \in T(A)$. Let $\widehat{\theta}\colon F_n \curvearrowright \mathrm{Prob}(X)$ be

the action induced by θ. For each $j = 1, \ldots, n$, the triple $(\mathrm{id}_{K_0(A)}, \mathrm{id}_{K_1(A)}, \widehat{\theta_{g_j}})$ is an automorphism of $\mathrm{Ell}(A)$ in the sense of [18, Definition 2.4], and thus by [18, Theorem 29.8] (see also [12, Theorem 2.7], or [10]) there exists an automorphism $\alpha_j \in \mathrm{Aut}(A)$ such that $\mathrm{Ell}(\alpha_j) = (\mathrm{id}_{K_0(A)}, \mathrm{id}_{K_1(A)}, \widehat{\theta_{g_j}})$. Using the universal property of F_n, we obtain an action $\alpha\colon F_n \to \mathrm{Aut}(A)$ whose induced action on $\partial_e T(A) \cong X$ is conjugate to θ. Tracial amenability for α then follows directly from the equivalence between (1) and (7) in Theorem 2.5.

In the above construction there is a significant amount of freedom in the choice of the Elliott invariant of the C*-algebra A. The following variant of the construction above was suggested to the authors by Mikael Rørdam, and it has the advantage of readily producing actions on unital, simple AF-algebras.

Example 2.11 Let $\theta\colon F_n \curvearrowright X$ be an amenable action on the Cantor space X. By [35, Proposition 1.4.5], there is a unital AF-algebra A with

$$(K_0(A), K_0(A)_+, [1]) \cong (C(X, \mathbb{Q}^{\mathrm{d}}), C(X, \mathbb{Q}^{\mathrm{d}})_{++} \cup \{0\}, 1),$$

where $\mathbb{Q}^{\mathrm{d}}$ denotes the rational numbers with the discrete topology, and $C(X, \mathbb{Q}^{\mathrm{d}})_{++}$ denotes the strictly positive continuous functions $X \to \mathbb{Q}^{\mathrm{d}}$. Since $(K_0(A), K_0(A)_+)$ is simple in the sense of [35, Definition 1.5.1], it follows from [35, Corollary 1.5.4] that A is simple.

Use Elliott's classification of AF-algebras (in the form stated in [35, Theorem 1.3.3 (ii)]) to lift the automorphisms of $(K_0(A), K_0(A)_+, [1])$ induced by the homeomorphisms $\theta_{g_1}, \ldots, \theta_{g_n}$ to automorphisms $\alpha_{g_1}, \ldots, \alpha_{g_n}$ of A. We again obtain an action $\alpha\colon F_n \to \mathrm{Aut}(A)$. Since there is a natural continuous affine homeomorphism $T(A) \cong \mathrm{Prob}(X)$ by [35, Proposition 1.5.5], it follows that the action that α induces on $T(A)$ is conjugate to θ. Hence α is tracially amenable by Theorem 2.5.

3 Purely infinite crossed products

In this section, we investigate the internal structure of reduced crossed products by tracially amenable actions. For a large class of nonamenable groups, we show that outer, tracially amenable actions on simple, unital, $\mathcal{Z}$-stable C*-algebras produce simple, purely infinite reduced crossed products. Our proof proceeds in two steps. The first one is Proposition 3.9, where we show in great generality that dynamical strict comparison is sufficient to conclude pure infiniteness of the reduced crossed product. The second step is Theorem 3.14, the main result of this work, where we show that tracial amenability implies dynamical strict comparison for a vast class of actions of nonamenable groups on simple C*-algebras.

3.1 From dynamical comparison to pure infiniteness of crossed products

We begin by recalling some preliminaries about Cuntz comparison, and refer the reader to [3, 17, 45] for modern introductions to the subject.

Let A be a C*-algebra. For positive elements $a, b \in A_+$, we say that a is *Cuntz subequivalent* to b, written $a \precsim b$, if there exists a sequence $(r_n)_{n\in\mathbb{N}}$ in A such that $\lim_{n\to\infty} \|r_n b r_n^* - a\| = 0$.

For a unital C*-algebra A, we denote by $QT(A)$ the set of (everywhere defined) normalized (2-)quasitraces on A ([45, Definition 6.7]). We will be mainly interested in C*-algebras where $QT(A) = T(A)$, which is the case, for instance, whenever A is exact ([20]).

Given $\tau \in QT(A)$, there exists a unique extension of τ to a lower semicontinuous (unbounded) quasitrace on $A \otimes \mathcal{K}$. With a slight abuse of notation, we denote such extension again by τ. For $a \in (A \otimes \mathcal{K})_+$ and $\tau \in QT(A)$, set

$$d_\tau(a) := \lim_{n\to\infty} \tau(a^{1/n}) \in [0, \infty].$$

The following lemma, which was pointed out to us by Sam Evington, will be needed in the proof of Theorem 3.14. For $\varepsilon > 0$, we denote by $f_\varepsilon \colon [0, \infty) \to [0, \infty)$ the function given by $f_\varepsilon(t) = \max\{0, t - \varepsilon\}$ for $t \in [0, \infty)$. For $a \in A_+$, we will usually write $(a - \varepsilon)_+$ for $f_\varepsilon(a)$.

Lemma 3.1 *Let A be a unital C*-algebra, and let $\gamma, \varepsilon > 0$.*

(1) For every $f \in C([0, 1])$ there exists $\delta > 0$ such that for all contractions $a, b \in A_+$ and all $\tau \in T(A)$ satisfying $\|a - b\|_{2,\tau} < \delta$, we have

$$\|f(a) - f(b)\|_{2,\tau} < \gamma.$$

(2) There is $\delta > 0$ such that for all contractions $a, b \in A_+$ and all $\tau \in T(A)$ satisfying $\|a - b\|_{2,\tau} < \delta$, we have

$$d_\tau((a - \varepsilon)_+) < d_\tau(b) + \gamma.$$

Proof (1). Suppose that there is $n \in \mathbb{N}$ such that $f(t) = t^n$ for all $t \in [0, 1]$. Then

$$\|f(a) - f(b)\|_{2,\tau} \leq \|a - b\|_{2,\tau} \sum_{j=0}^{n-1} \|a\|^j \|b\|^{n-1-j} \leq n\|a - b\|_{2,\tau}.$$

Thus, in this case we may take $\delta = \frac{\gamma}{n}$. The case where f is a polynomial can be deduced from this, and the general case follows using a density argument.

(2). Let $g_\varepsilon \in C([0, 1])$ be the function defined as $g_\varepsilon(t) := \min\{\varepsilon^{-1}t, 1\}$, for all $t \in [0, 1]$. Using part (1), find $\delta > 0$ such that whenever $a, b \in A_+$ are contractions satisfying $\|a - b\|_{2,\tau} < \delta$, then $\|g_\varepsilon(a) - g_\varepsilon(b)\|_{2,\tau} < \gamma$. By part (1) of Lemma 2.1, we have

$$|\tau(g_\varepsilon(a) - g_\varepsilon(b))| \leq \|g_\varepsilon(a) - g_\varepsilon(b)\|_{2,\tau} < \gamma. \tag{3.1}$$

Denote by μ the measure on the spectrum of a induced by the trace $\tau|_{C^*(1,a)}$, so that

$$d_\tau((a - \varepsilon)_+) = \mu\big(\{x \in \mathrm{sp}(a) \colon f_\varepsilon(x) > 0\}\big) = \mu((\varepsilon, 1]). \tag{3.2}$$

Using that $g_\varepsilon = 1$ on $(\varepsilon, 1]$ at the second step, we get

$$d_\tau((a-\varepsilon)_+) \overset{(3.2)}{=} \mu((\varepsilon,1]) \leq \int_{[0,1]} g_\varepsilon d\mu = \tau(g_\varepsilon(a)) \overset{(3.1)}{\leq} \tau(g_\varepsilon(b)) + \gamma \leq d_\tau(b) + \gamma,$$

as desired. □

For a compact convex set K, we denote by $\mathrm{LAff}(K)_+$ (respectively, $\mathrm{LAff}(K)_{++}$) the set of lower semicontinuous positive (respectively, strictly positive) affine functions on K.

Definition 3.2 Let A be a unital C*-algebra. For $a \in (A \otimes \mathcal{K})_+$, we define its *rank function* to be the map $\mathrm{rk}(a)\colon QT(A) \to [0,\infty]$ given by

$$\mathrm{rk}(a)(\tau) = d_\tau(a)$$

for all $\tau \in QT(A)$.

Given a simple, unital C^*-algebra A and $a \in (A \otimes \mathcal{K})_+ \backslash \{0\}$, it is always true that $d_\tau(a) > 0$, for every $\tau \in QT(A)$. As $\mathrm{rk}(a)$ is a lower semicontinuous map on a compact set, it attains its minimum. It follows that $\mathrm{rk}(a) \in \mathrm{LAff}(QT(A))_{++}$, for every $a \in (A \otimes \mathcal{K})_+ \setminus \{0\}$. We say that *all ranks are realized* in A if the rank map $\mathrm{rk}\colon (A \otimes \mathcal{K})_+ \backslash \{0\} \to \mathrm{LAff}(QT(A))_{++}$ is surjective.

One of the assumptions of Theorem 3.14 is that all ranks are realized for the coefficient C*-algebra. This is in fact a very mild assumption, which is automatic in many cases of interest. This is for example the case for nonelementary, simple, separable, unital C*-algebras of stable rank one by [46, Theorem 8.11]. Moreover, it is an open question of Nate Brown whether all ranks are realized in every nonelementary, unital, simple, separable, stably finite C*-algebra (see [46, Question 1.1]).

We will need the notion of strict comparison. Although this is most commonly defined using tracial states, we choose to do it using normalized quasitraces since this version is more compatible with the Cuntz semigroup outside of the exact setting.

Definition 3.3 A unital, simple C*-algebra A is said to have *strict comparison* if we have $a \precsim b$ whenever $a, b \in (A \otimes \mathcal{K})_+ \backslash \{0\}$ satisfy $\mathrm{rk}(a) < \mathrm{rk}(b)$.

Remark 3.4 When A is a simple, unital C*-algebra, and $QT(A) = \emptyset$, then strict comparison is equivalent to the fact that $1 \precsim a$ for all nonzero $a \in (A \otimes \mathcal{K})_+$. This, in turn, is equivalent to pure infiniteness of A (see [23, Definition 4.1] and [11]).

The following definition originates from an ongoing project by Bosa, Perera, Wu and Zacharias, and it provides a noncommutative generalization of the notion of dynamical subequivalence from [22, Definition 3.2].

Definition 3.5 Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a countable, discrete group G on a C*-algebra A, and let $a, b \in A_+$.

(1) Write $a \precsim_0 b$ if for any $\varepsilon > 0$ there are $\delta > 0$, $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$, and positive elements $x_1, \ldots, x_n \in M_\infty(A)$, such that

$$(a-\varepsilon)_+ \precsim \oplus_{j=1}^n \alpha_{g_j}(x_j) \quad \text{and} \quad \oplus_{j=1}^n x_j \precsim (b-\delta)_+.$$

(2) Write $a \precsim_G b$ if there are $x_1, \ldots, x_n \in M_\infty(A)_+$ with

$$a = x_1 \precsim_0 x_2 \precsim_0 \ldots \precsim_0 x_n = b.$$

The need for (2) stems for the fact that, unlike for $\precsim$, it is not clear whether the relation $\precsim_0$ is transitive.

Remark 3.6 Assume that there are $n \in \mathbb{N}$, elements $g_1, \ldots, g_n \in G$, and positive elements $x_1, \ldots, x_n \in M_\infty(A)$ such that

$$a \precsim \oplus_{j=1}^n \alpha_{g_j}(x_j) \quad \text{and} \quad \oplus_{j=1}^n x_j \precsim b.$$

Then $a \precsim_0 b$. This follows from double application of [34, Proposition 2.4].

Remark 3.7 One can check that $a \precsim_G b$ entails $a \precsim b$ in $A \rtimes_r G$, by putting together [34, Proposition 2.4], [45, Proposition 2.18], and the fact that the action on A is unitarily implemented inside $A \rtimes_r G$.

The notion of dynamical strict comparison below is due to Bosa, Perera, Wu and Zacharias, as part of their ongoing work on almost elementariness extending the ideas and techniques developed in [22] to the noncommutative setting.

Definition 3.8 An action $\alpha\colon G \to \mathrm{Aut}(A)$ of a countable, discrete group G on a unital C*-algebra A is said to have *dynamical strict comparison* if we have $a \precsim_G b$ for all $a, b \in (A \otimes \mathcal{K})_+ \setminus \{0\}$ satisfying $d_\tau(a) < d_\tau(b)$ for all invariant quasitraces $\tau \in QT(A)$.

In the definition above, we deliberately allow for the possibility that there are no invariant quasitraces, in which case dynamical strict comparison is equivalent to the statement that $a \precsim_G b$ for all nonzero $a, b \in (A \otimes \mathcal{K})_+$.

Note that when α is trivial, dynamical strict comparison for α is equivalent to strict comparison for A. The following is a generalization of [27, Theorem 1.1] to the noncommutative setting (see also [13, Theorem 2.8]). Recall that an action $\alpha\colon G \to \mathrm{Aut}(A)$ is said to be *outer* if α_g is not inner for all $g \in G \setminus \{1\}$.

Proposition 3.9 *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an outer action of a countable, discrete group G on a unital, simple C*-algebra A. Assume that α has dynamical strict comparison and no invariant quasitraces. Then $A \rtimes_r G$ is simple and purely infinite.*

Proof Simplicity follows from [24, Theorem 3.1]. Let $a \in A \rtimes_r G$ be a nonzero positive element. We will show that $1 \precsim a$ in $A \rtimes_r G$, which implies that $A \rtimes_r G$ is purely infinite by Remark 3.4. By [37, Lemma 3.2] (see also [14, Lemma 4.2]) there exists a nonzero element $b \in A_+$ such that $b \precsim a$. In the absence of invariant quasitraces, dynamical strict comparison gives $1 \precsim_G b$. By Remark 3.7, we have that $1 \precsim b \precsim a$ in $A \rtimes_r G$, as desired. □

3.2 Groups with weak paradoxical towers

Next, we introduce the notion of weak paradoxical towers, based on ideas from [13]. This is a technical assumption which allows us to link tracial amenability to strict comparison. The class of groups having weak paradoxical towers is quite large, and it for example includes all groups containing the free group F_2; see Corollary 3.13.

Definition 3.10 Let $n \in \mathbb{N}$. A group G is said to have *weak n-paradoxical towers* if for every $m \in \mathbb{N}$, there exist a finite subset $D \subseteq G$ with $|D| \geq m$, subsets $K_1, \dots K_n \subseteq G$, and group elements $g_1, \dots, g_n \in G$ such that

(1) For each $j = 1, \dots, n$, the sets $\{dK_j\}_{d \in D}$ are pairwise disjoint;
(2) $\bigcup_{j=1}^{n} g_j K_j = G$.

A group is said to have *weak paradoxical towers* if it has weak n-paradoxical towers for some $n \in \mathbb{N}$.

We make the following observation:

Lemma 3.11 *Let G be a group with weak paradoxical towers. Then G is not amenable.*

Using elementary arguments, one can show that all nonabelian free groups have weak 2-paradoxical towers; see [13, Proposition 3.2] for a stronger result. More generally, groups that have paradoxical towers in the sense of [13, Definition 3.1] have weak paradoxical towers in our sense, and in particular all the groups considered in [13, Section 4] have weak paradoxical towers. The class of groups we consider here is really larger than the one considered in [13]: for example $F_2 \times \mathbb{Z}$ does not have paradoxical towers by [13, Remark 3.7], but it has *weak* paradoxical towers by Corollary 3.13 below. The key difference is that weak paradoxical towers can be induced from subgroups:

Proposition 3.12 *Let G be a group and $H \subseteq G$ a subgroup with weak n-paradoxical towers for some $n \in \mathbb{N}$. Then G has weak n-paradoxical towers.*

Proof Let $m \in \mathbb{N}$. Since H has weak n-paradoxical towers, there are a finite set $D \subseteq H$ with $|D| \geq m$, subsets $L_1, \dots, L_n \subseteq H$, and elements $g_1, \dots, g_n \in H$, such that the sets $\{dL_j\}_{d \in D}$ are pairwise disjoint for every $j = 1, \dots, n$, and such that $H = \bigcup_{j=1}^{n} g_j L_j$. Let $S \subseteq G$ be a set satisfying $G = \bigsqcup_{s \in S} Hs$, and set $K_j := \bigsqcup_{s \in S} L_j s$, $j = 1, \dots, n$. One readily checks that $\bigcup_{j=1}^{n} g_j K_j = G$, and that the sets $\{dK_j\}_{d \in D}$ are pairwise disjoint, for every $j = 1, \dots, n$. □

A combination of Proposition 3.12 with [13, Proposition 3.2] gives the following corollary.

Corollary 3.13 *Every group containing F_2 has weak paradoxical towers.*

3.3 Establishing dynamical comparison from tracial amenability

We are now ready to prove the main result of this section. As before, given $\varepsilon > 0$ we denote by $f_\varepsilon \colon [0, \infty) \to [0, \infty)$ the function $f_\varepsilon(t) := \max\{0, t - \varepsilon\}$ for $t > 0$,

and for a positive element a we write $(a-\varepsilon)_+$ for $f_\varepsilon(a)$. Since functional calculus commutes with automorphisms, for $\varphi \in \mathrm{Aut}(A)$ we have $\varphi((a-\varepsilon)_+) = (\varphi(a)-\varepsilon)_+$.

Theorem 3.14 *Let G be a countable, discrete group with weak paradoxical towers, let A be a unital, simple, separable C^*-algebra with $QT(A) = T(A) \neq \emptyset$, with strict comparison, and for which all ranks are realized. Then, any tracially amenable action $\alpha : G \to \mathrm{Aut}(A)$ has dynamical strict comparison.*

Proof Since G has weak paradoxical towers, it is not amenable by Lemma 3.11 and thus has no invariant traces by Lemma 2.9. In particular, dynamical strict comparison for α amounts to showing that $1 \precsim_G a$ for every nonzero $a \in A_+$. Fix such a. Since $\mathrm{rk}(a)$ is a strictly positive, lower semicontinuous function on the compact space $T(A)$, it attains its infimum and thus there is $m \in \mathbb{N}$ such that

$$m \cdot \mathrm{rk}(a) \geq 1. \tag{3.3}$$

Let $n \in \mathbb{N}$ be such that G has weak n-paradoxical towers. Fix a finite subset $D \subseteq G$ with $\frac{3n}{|D|-1}(2n^2+1) < \frac{1}{m}$, and use weak paradoxical towers for G to find $K_1, \ldots, K_n \subseteq G$ and $g_1, \ldots, g_n \in G$ such that

(a1) $\{dK_j\}_{d\in D}$ are pairwise disjoint for all $j = 1, \ldots, n$;
(a2) $\bigcup_{j=1}^n g_j K_j = G$.

Let $0 < \varepsilon < \frac{1}{18n|D|}$. Apply part (2) of Lemma 3.1 to find $\delta > 0$ such that for all positive contractions $s, t \in A$ satisfying $\|s-t\|_{2,u} < \delta$, we have

$$d_\tau((s-\varepsilon)_+) < d_\tau(t) + \varepsilon, \tag{3.4}$$

for all $\tau \in T(A)$. Apply now part (1) of Lemma 3.1, find $0 < \gamma \leq \varepsilon$, such that for all positive contractions $s, t \in A$ satisfying $\|s-t\|_{2,u} < \gamma$, we have

$$\left\| \left(s - \tfrac{1}{2n}\right)_+ - \left(t - \tfrac{1}{2n}\right)_+ \right\|_{2,u} < \delta. \tag{3.5}$$

By tracial amenability of α, there is $\xi \in C_c(G, A)$ satisfying

(b1) $\|\xi\| \leq 1$;
(b2) $\|\langle \xi, \xi\rangle - 1\|_{2,u} < \varepsilon$;
(b3) $\|\widetilde{\alpha}_g(\xi) - \xi\|_{2,u} < \frac{\gamma}{2}$, for all $g \in D \cup \{g_1, \ldots, g_n\}$.

For $j = 1, \ldots, n$, set

$$c_j = \sum_{g\in K_j} \xi(g)^*\xi(g), \quad b_j = \left(c_j - \tfrac{1}{2n}\right)_+, \quad \text{and} \quad b := b_1 \oplus \cdots \oplus b_n.$$

For an element $x \in A$ and $\ell \in \mathbb{N}$, we will denote by $x^{\oplus \ell} \in M_\ell(A)$ the ℓ-fold direct sum of x with itself. We will prove that $1 \precsim_G a$ in two steps.

Claim 3.14.1 *We have* $1 \precsim_0 (b-\varepsilon)_+^{\oplus 3n}$.

For $j = 1, \ldots, n$ and $h \in D \cup \{g_1, \ldots, g_n\}$, we denote by $1_{hK_j} \in \mathcal{L}_A(\ell^2(G, A))$ the projection onto the elements supported on $hK_j \subseteq G$. Using the Cauchy-Schwarz inequality (see part (2) of Lemma 2.1) for the third step, we obtain

$$\begin{aligned}
&\Big\|\alpha_h(c_j) - \sum_{g\in hK_j} \xi(g)^*\xi(g)\Big\|_{2,u} \\
&= \big\|\langle 1_{hK_j}\widetilde{\alpha}_h(\xi), \widetilde{\alpha}_h(\xi)\rangle - \langle 1_{hK_j}\xi, \xi\rangle\big\|_{2,u} \\
&\leq \big\|\langle 1_{hK_j}\widetilde{\alpha}_h(\xi), \widetilde{\alpha}_h(\xi) - \xi\rangle\big\|_{2,u} + \big\|\langle \xi - \widetilde{\alpha}_h(\xi), 1_{hK_j}\xi\rangle\big\|_{2,u} \\
&\leq 2\,\|\widetilde{\alpha}_h(\xi) - \xi\|_{2,u} < \gamma. \qquad (3.6)
\end{aligned}$$

Combining (3.5) and (3.6), we get

$$\Big\|\alpha_h(b_j) - \Big(\sum_{g\in hK_j} \xi(g)^*\xi(g) - \tfrac{1}{2n}\Big)_+\Big\|_{2,u} < \delta, \qquad (3.7)$$

for all $h \in D \cup \{g_1, \ldots, g_n\}$ and $j = 1, \ldots, n$. Fix $\tau \in T(A)$. For $j \in \{1, \ldots, n\}$, denote by μ_j the measure on the spectrum of $\alpha_{g_j}(c_j)$ corresponding to $\tau|_{C^*\left(1,\alpha_{g_j}(c_j)\right)}$. We have

$$\begin{aligned}
d_\tau(\alpha_{g_j}((b_j-\varepsilon)_+)) &= d_\tau\big(\big(\alpha_{g_j}(c_j) - \varepsilon - \tfrac{1}{2n}\big)_+\big) \\
&= \int_{\left(\varepsilon+\frac{1}{2n},1\right]} 1 d\mu_j \\
&\geq \int_{[0,1]} t d\mu_j - \big(\varepsilon + \tfrac{1}{2n}\big) \\
&= \tau(\alpha_{g_j}(c_j)) - \big(\varepsilon + \tfrac{1}{2n}\big) \\
&\overset{(3.6)}{\geq} \tau\Big(\sum_{g\in g_jK_j} \xi(g)^*\xi(g)\Big) - \big(\varepsilon + \tfrac{1}{2n} + \gamma\big). \qquad (3.8)
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\sum_{j=1}^{n} \tau\Big(\sum_{g\in g_jK_j} \xi(g)^*\xi(g)\Big) &\overset{a2}{\geq} \tau\Big(\sum_{g\in G} \xi(g)^*\xi(g)\Big) \\
&\overset{b2}{\geq} 1 - \varepsilon.
\end{aligned}$$

We conclude that there is $j \in \{1, \ldots, n\}$ for which

$$\tau\Big(\sum_{g\in g_jK_j} \xi(g)^*\xi(g)\Big) \geq \frac{1-\varepsilon}{n} \geq \frac{1}{n} - \varepsilon. \qquad (3.9)$$

Combining (3.8) and (3.9), we obtain

$$\sum_{j=1}^{n} d_\tau(\alpha_{g_j}((b_j-\varepsilon)_+)) \geq \tfrac{1}{n} - \varepsilon - \Big(\varepsilon + \tfrac{1}{2n} + \gamma\Big) \geq \tfrac{1}{2n} - 3\varepsilon > \tfrac{1}{3n},$$

for all $\tau \in T(A)$. Using strict comparison for A, this implies that

$$1 \precsim \oplus_{j=1}^{n} \alpha_{g_j}((b_j-\varepsilon)_+)^{\oplus 3n}. \tag{3.10}$$

As $\oplus_{j=1}^{n}(b_j-\varepsilon)_+^{\oplus 3n} = (b-\varepsilon)_+^{\oplus 3n}$, it follows that $1 \precsim_0 (b-\varepsilon)_+^{\oplus 3n}$, proving Claim 3.14.1.

Claim 3.14.2 *We have* $(b-\varepsilon)_+^{\oplus 3n} \precsim_0 a$.

Fix $\tau \in T(A)$, $d \in D$, and $j \in \{1,\dots,n\}$. Denote by μ the probability measure on the spectrum of $\sum_{g\in dK_j} \xi(g)^*\xi(g)$ corresponding to the restriction of τ to $C^*\big(1, \sum_{g\in dK_j} \xi(g)^*\xi(g)\big)$. Then

$$\begin{aligned} d_\tau(\alpha_d((b_j-\varepsilon)_+)) &\overset{(3.4),(3.7)}{\leq} d_\tau\Big(\Big(\sum_{g\in dK_j} \xi(g)^*\xi(g) - \tfrac{1}{2n}\Big)_+\Big) + \varepsilon \\ &= \int_{\left(\frac{1}{2n},1\right]} 1\, d\mu + \varepsilon \\ &\leq 2n \int_{[0,1]} t\, d\mu + \varepsilon \\ &= 2n\tau\Big(\sum_{g\in dK_j} \xi(g)^*\xi(g)\Big) + \varepsilon. \end{aligned} \tag{3.11}$$

Taking the sum of (3.11) over $j \in \{1,\dots,n\}$ and $d \in D$ yields

$$\begin{aligned} \sum_{d\in D}\sum_{j=1}^{n} d_\tau(\alpha_d((b_j-\varepsilon)_+)) &\leq 2n \sum_{j=1}^{n} \tau\Big(\sum_{d\in D}\sum_{g\in dK_j} \xi(g)^*\xi(g)\Big) + n|D|\varepsilon \\ &\overset{a1,b1}{\leq} 2n^2 + 1. \end{aligned} \tag{3.12}$$

Since all ranks are realized in A, we can find positive elements $b'_d \in M_\infty(A)$ for $d \in D$ satisfying

$$\mathrm{rk}(b'_d) = \frac{3n}{|D|-1}\mathrm{rk}(\alpha_d((b-\varepsilon)_+)). \tag{3.13}$$

Note that $(b-\varepsilon)_+ \neq 0$ by Claim 3.14.1. Using this at the second step, we get

$$3n \cdot d_\tau((b-\varepsilon)_+) = \frac{3n}{|D|}\sum_{d\in D} d_{\tau\circ\alpha_{d^{-1}}}(\alpha_d((b-\varepsilon)_+))$$

$$< \frac{3n}{|D|-1} \sum_{d\in D} d_{\tau\circ\alpha_{d^{-1}}}(\alpha_d((b-\varepsilon)_+)) \overset{(3.13)}{=} \sum_{d\in D} d_\tau(\alpha_{d^{-1}}(b'_d)),$$

for all $d \in D$ and $\tau \in T(A)$. Using strict comparison for A, it follows that

$$(b-\varepsilon)_+^{\oplus 3n} \precsim \oplus_{d\in D}\alpha_{d^{-1}}(b'_d). \tag{3.14}$$

On the other hand, we have

$$\begin{aligned}\sum_{d\in D} d_\tau(b'_d) &\overset{(3.13)}{=} \frac{3n}{|D|-1}\sum_{d\in D} d_\tau(\alpha_d((b-\varepsilon)_+)) \\ &\overset{(3.12)}{\leq} \frac{3n}{|D|-1}(2n^2+1) \\ &\overset{(3.3)}{<} d_\tau(a)\end{aligned}$$

for all $\tau \in T(A)$. Again by strict comparison, this implies that

$$\oplus_{d\in D}\, b'_d \precsim a, \tag{3.15}$$

and therefore

$$(b-\varepsilon)_+^{\oplus 3n} \precsim_0 a.$$

Putting together both claims, we get $1 \precsim_G a$, as required. □

The following is the desired result for crossed products.

Corollary 3.15 *Let G be a countable, discrete group with weak paradoxical towers, let A be a unital, simple, separable C*-algebra with $QT(A) = T(A) \neq \emptyset$, with strict comparison and for which all ranks are realized, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a tracially amenable, outer action. Then $A \rtimes_r G$ is simple and purely infinite.*

Proof The statement follows from Theorem 3.14 and Proposition 3.9. □

Let A be a simple, separable, exact, unital, stably finite C*-algebra. If A has strict comparison and stable rank one, then all ranks are realized on it by [46, Theorem 8.11], and so A satisfies the assumptions of Corollary 3.15. In particular, this is the case if A is $\mathcal{Z}$-stable, by [36, Theorem 4.5 and Theorem 6.7]. Instead of stable rank one, one can equivalently require A to be almost divisible. Indeed, simple, separable, unital, stably finite C*-algebras with strict comparison and almost divisibility have stable rank one by work in progress of Geffen and Winter (see also [26], where the assumption of almost divisibility is replaced by tracial approximate oscillation zero).

We close the paper with an application of our results to classifiable C*-algebras. In the following corollary, we require amenability of the action, as tracial amenability is not sufficient to grant nuclearity of the crossed product.

Corollary 3.16 *Let G be a countable, discrete group with weak paradoxical towers, and let A be a simple, separable, unital, nuclear, stably finite, $\mathcal{Z}$-stable C^*-algebra. Suppose that $\alpha\colon G \to \mathrm{Aut}(A)$ is an amenable, outer action.*[1] *Then $A \rtimes_r G$ is a unital Kirchberg algebra. If A satisfies the UCT and G is torsion-free and satisfies the Haagerup property (for example, $G = F_n$; see [7, Definition 12.2.1]), then $A \rtimes_r G$ also satisfies the UCT, and is therefore completely determined by its K-theory.*

Proof By Corollary 3.15 and the comments after it, $A \rtimes_r G$ is simple, separable, unital and purely infinite. By [2, Theorem 4.5], it is also nuclear. The fact that it satisfies the UCT follows from [21] and [30, Corollary 9.4] (see [38, Corollary 7.2] for the case $G = F_n$). □

We remark that Corollary 3.16 holds also when A is purely infinite, by applying [37, Lemma 3.2] (see also [14, Lemma 4.2]).

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[1] We do not actually know if such an action exists; see Problem D in the introduction.

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