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Strongly Proper Connected Coloring of Graphs

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Abstract. We study a new variant of *connected coloring* of graphs based on the concept of *strong* edge coloring (every color class forms an *induced* matching). In particular, an edge-colored path is *strongly proper* if its color sequence does not contain identical terms within a distance of at most two. A *strong proper connected* coloring of G is the one in which every pair of vertices is joined by at least one strongly proper path. Let $\mathrm{spc}(G)$ denote the least number of colors needed for such coloring of a graph G. We prove that the upper bound $\mathrm{spc}(G) \leq 5$ holds for any 2-connected graph G. On the other hand, we demonstrate that there are 2-connected graphs with arbitrarily large girth satisfying $\mathrm{spc}(G) \geq 4$. Additionally, we prove that graphs whose cycle lengths are divisible by 3 satisfy $\mathrm{spc}(G) \leq 3$. We also consider briefly other connected colorings defined by various restrictions on color sequences of connecting paths. For instance, in a *nonrepetitive connected coloring* of G, every pair of vertices should be joined by a path whose color sequence is *nonrepetitive*, that is, it does not contain two adjacent identical blocks. We demonstrate that 2-connected graphs are 15-colorable, while 4-connected graphs are 6-colorable, in the connected nonrepetitive sense. A similar conclusion with a finite upper bound on the number of colors holds for a much wider variety of connected colorings corresponding to fairly general properties of sequences. We end the paper with some open problems of concrete and general nature.

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1. Introduction

Let G be a simple connected graph with colored edges. A path P in G is *proper* if no two consecutive edges of P have the same color. An edge coloring of G (not neccessarily proper in the usual sense) is called a *proper connected coloring* if every pair of distinct vertices is joined by at least one proper path. The least number of colors needed for such a coloring of a graph G is denoted by $\mathrm{pc}(G)$.

In the proper edge coloring of a graph, in a traditional sense, each pair of edges with a common vertex is colored differently, so every path is proper. Hence, by the well-known theorem of Vizing [20],

$$\mathrm{pc}(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

It is also easy to see that for every tree T, we have $\mathrm{pc}(T) = \chi'(T) = \Delta(T)$. However, if G is a 2-connected graph,[1] then we already have $\mathrm{pc}(G) \leq 3$, and this is tight. This somewhat surprising fact was proved by Borozan, Fujita, Gerek, Magnant, Manoussakis, Montero, and Tuza in [3] (see also [14]), where the proper connected coloring was introduced, in analogy to the well-studied topic of *rainbow connected coloring*, invented by Chartrand, Johns, McKeon, and Zhang in [6] (see [15]). In the later concept, as you can imagine, the point is that each pair of vertices should be connected by a *rainbow* path, i.e., one in which no two edges have the same color. In a similar way, one may investigate a variety of coloring concepts involving various restrictions on *color patterns* allowable on connecting paths (see [4,5,7]).

In the present paper, we consider a new variant of connected coloring, inspired by the concept of *strong edge coloring* of graphs, invented by Fouquet and Joliviet [10], and, independently, by Erdős and Nešetřil [9]. In the strong edge coloring of a graph G, every color class should form an *induced* matching, which means that not only every pair of adjacent edges is colored differently, but also every pair of edges adjacent to some other common edge should be differently colored (see [8] for a recent survey on this topic). In particular, in a strong edge coloring of a path, any sub-path with at most three edges must be rainbow. We will call such paths *strongly proper*. Analogously, an edge-colored graph is called *strongly proper connected* if any two vertices are connected by at least one strongly proper path. The least number of colors needed for such a coloring of a graph G is denoted by $\mathrm{spc}(G)$ and referred to as the *strong proper connection number* of G.

In much the same way as for the traditional edge coloring, we have the trivial bound $\mathrm{spc}(G) \leq \chi'_s(G)$, where $\chi'_s(G)$ is the *strong chromatic index* of G, defined naturally as the least number of colors needed for a strong edge coloring of G. However, this parameter is more mysterious than its classical archetype $\chi'(G)$. In particular, a long-standing conjecture by Erdős and Nešetřil [9] states that $\chi'_s(G) \leq \frac{5}{4}\Delta(G)^2$. This bound is tight (if true), as is demonstrated by the family of blowups of C_5. Currently best general result, obtained by Hurley, de

[1] A graph G is *k-connected* if it cannot be disconnected by deleting $k-1$ vertices. Similarly, G is *k-edge-connected* if it cannot be disconnected by deleting $k-1$ edges.

Joannis de Verclos, and Kang [11], states that the bound $\chi'_s(G) \leq 1.772\Delta(G)^2$ holds for sufficiently large $\Delta(G)$ (see [8] for a survey of many other results toward the Erdős-Nešetřil conjecture).

In this paper, we prove a finite upper bound on $\mathrm{spc}(G)$ for 2-connected graphs. Our main result reads as follows.

Theorem 1. *Every* 2*-connected graph* G *satisfies* $\mathrm{spc}(G) \leq 5$.

We do not know if this upper bound is tight. Clearly, $\mathrm{spc}(G) \geq 3$ for any graph of diameter at least 3. Curiously, it is not so easy to produce examples of 2-connected graphs demanding four colors. However, we provide a general construction of a family of graphs with $\mathrm{spc}(G) = 4$ and arbitrarily large girth. Moreover, this family contains infinitely many bipartite graphs.

Theorem 2. *For every* $d \geq 3$*, there exists a* 2*-connected graph* G_d *with girth at least* d*, such that* $\mathrm{spc}(G_d) \geq 4$.

We complement these results by proving that graphs with all cycle lengths divisible by 3 are 3-colorable in a strongly proper connected sense.

Theorem 3. *Let* G *be a* 2*-connected graph. If the length of every cycle in* G *is divisible by* 3*, then* $\mathrm{spc}(G) \leq 3$.

The proof of this result is simple, but it provides a good illustration of the main idea used in the proof Theorem 1. The main tool is based on the ear decomposition of graphs. Recall that an *open ear decomposition* of a graph G is a sequence $P_1, \ldots, P_h$ of subgraphs of G that partition the set of edges of G, where P_1 is a cycle and every P_i, with $2 \leq i \leq h$, is a path that intersects $P_1 \cup \cdots \cup P_{i-1}$ in exactly its endpoints. Each P_i is called an *ear*. We will make use of following well-known result of Whitney [21].

Theorem 4. *(Whitney, 1932) A graph* G *with at least two edges is* 2*-connected if and only if it has an open ear decomposition.*

A paradigm of colorful connectedness can be studied for other types of colorings as well. As postulated by Brause, Jendrol', and Schiermeyer [4], one may fix any property $\mathcal{P}$ of words (sequences), and then investigate the corresponding $\mathcal{P}$*-connected coloring*, defined by the condition that every pair of vertices is joined by a path whose color sequence has the property $\mathcal{P}$. Consider for example the following, particularly intriguing property of sequences, which is close (in some sense) to the one stemming from the strong edge coloring of graphs.

A sequence of the form $c_1c_2\cdots c_nc_1c_2\cdots c_n$ is called a *repetition*. An edge-colored path is *nonrepetitive* if its color sequence does not contain a repetition as a *block*, i.e., a subsequence of *consecutive* terms. An edge-colored graph G is *nonrepetitively connected* if every pair of distinct vertices is joined by at least one nonrepetitive path. Denote by $\mathrm{nrc}(G)$ the least number of colors needed for such a coloring of G.

Is it true that $\mathrm{nrc}(G)$ is finitely bounded for 2-connected graphs? Notice that it is not at all obvious that $\mathrm{nrc}(P)$ is finite for every path P. However, by a 1906 result of Thue [18], every path can be nonrepetitively colored using just *three* colors. Therefore, there is a chance for a finite bound and we prove that this is indeed true.

Theorem 5. *Every* 2*-connected graph* G *satisfies* $\mathrm{nrc}(G) \leq 15$ *and every* 4*-connected graph* G *satisfies* $\mathrm{nrc}(G) \leq 6$*.*

Let us mention that the notion of nonrepetitive coloring of graphs, as introduced by Alon, Hałuszczak, Grytczuk, and Riordan in [1], can be considered more generally, in a way similar to the usual proper coloring of graphs (in both, edge or vertex version). A recent survey by Wood [22] collects many interesting results on this topic.

The proof of Theorem 5 is based on known results on spanning trees in k-connected graphs. It can be easily extended to more general scenarios involving fairly universal properties of words. We discuss these issues in the final section of the paper, where we state a general conjecture on $\mathcal{P}$-connected coloring of graphs. Proofs of all our results are collected in the next section.

2. Proofs of the Results

2.1. 2-Connected Graphs with Cycle Lengths Divisible by 3

We start with a simple proof that 2-connected graphs whose all cycle lengths are divisible by 3 satisfy $\mathrm{spc}(G) \leq 3$.

Proof of Theorem 3. Let G be a 2-connected graph. By Theorem 4, G has an open ear decomposition, which we denote as $ED = (P_1, \ldots, P_h)$. Let G_i be the subgraph of G consisting of the first i ears of ED, that is, $G_i = P_1 \cup \cdots \cup P_i$. For each ear P_i, let s_i and t_i be the endpoints of P_i. First, we present a claim concerning the lengths of P_i and some other paths in G_i.

Claim 1. Let P_i be an ear from ED. Then, the length of P_i and the lengths of all paths between s_i and t_i in the graph G_{i-1} are divisible by 3 (see Fig. 1).

Proof. The statement is obvious for $i = 1$, so assume that $i > 1$. The graph G_{i-1} is 2-connected, so there exist two paths, Q and R, from s_i to t_i, such that $Q \cap R = \{s_i, t_i\}$. Let q and r be the lengths of Q and R, respectively, and let p be the length of P_i. The paths Q and R form a cycle in G_{i-1}, so $q + r$ must be divisible by 3. Moreover, Q and P_i, as well as R and P_i, form a cycle in G_i. Therefore, we have

$$q + r = q + p = r + p \equiv 0 \pmod 3.$$

This implies that $q+r+p = 0 \pmod 3$, and therefore, $q = r = p = 0 \pmod 3$, which completes the proof of the claim. □

Let us define a sequence of oriented graphs $D_0, D_1, \ldots, D_h$, such that D_0 is empty and for each $i > 0$, D_i is obtained from D_{i-1} by adding the path P_i, oriented from s_i to t_i. Let D be the last oriented graph in this sequence, i.e., $D := D_h$. Now, we will color the edges of D with 3 colors. In this coloring, every *directed* path will be strongly proper. It is easy to see that the digraph

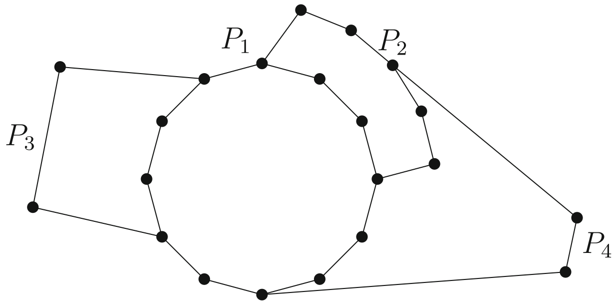


FIGURE 1. An exemplary ear decomposition $ED = \{P_1, P_2, P_3, P_4\}$ of some graph with cycles of length divisible by 3 (P_1 is a cycle with 12 edges, P_2 is a path with 6 edges, P_3 and P_4 are paths with 3 edges)

D is strongly connected (there is a directed path between any ordered pair of vertices in D), so our coloring will give us even more than the required edge coloring of G.

Let C be a 3-coloring of the edges of D by colors $\{1, 2, 3\}$, and let P be a directed path in D. We say that P has a *canonical pattern* if the sequence of colors of P forms a *block* (subsequences of consecutive terms) in the infinite periodic sequence $(1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots)$. Note that a path P has a canonical pattern if and only if P is strongly proper.

Claim 2. There exists a 3-coloring C of the edges of D, such that for every restriction C_i of C to the subgraph D_i, $1 \leq i \leq h$, the following properties hold:

(i) For every $v \in V(D_i)$, all edges going out of v have the same color.
(ii) For every $v \in V(D_i)$, all edges going into v have the same color.
(iii) Every path in D_i has a canonical pattern.
(iv) Every digraph D_i is strongly connected.

Proof. We will construct the coloring C by inductively defining C_i, for $i = 1, 2, \ldots, h$. The base case is when D_1 is a directed cycle of length divisible by 3. We color the edges of D_1 consecutively with colors $1, 2, 3$, obtaining thereby the coloring C_1. The properties (i)–(iv) are obviously satisfied.

Let $2 \leq i \leq h$ be fixed and suppose that we have the coloring C_{i-1} of the edges of D_{i-1} satisfying properties (i)–(iv). Recall that D_i is obtained from D_{i-1} by adding the path P_i from s_i to t_i. We will construct the coloring C_i from C_{i-1} by coloring the edges of P_i as follows (see Fig. 2):

(a) the first edge of P_i has the same color as the edges going out of s_i in D_{i-1},
(b) the remaining edges of P_i are colored in such a way that P_i has a canonical pattern (for example, if the first edge has color 2, then the second one has color 3, the third one 1, the fourth one 2, and so on). We will prove that C_i satisfies properties (i)–(iv).

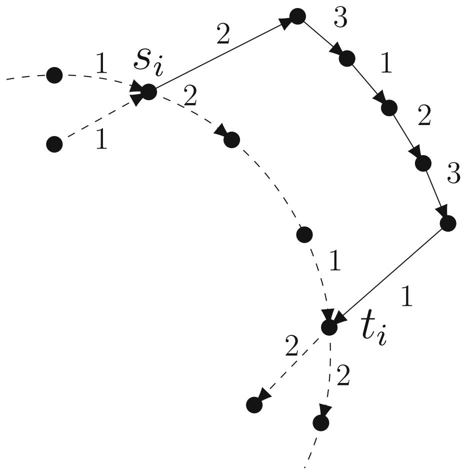


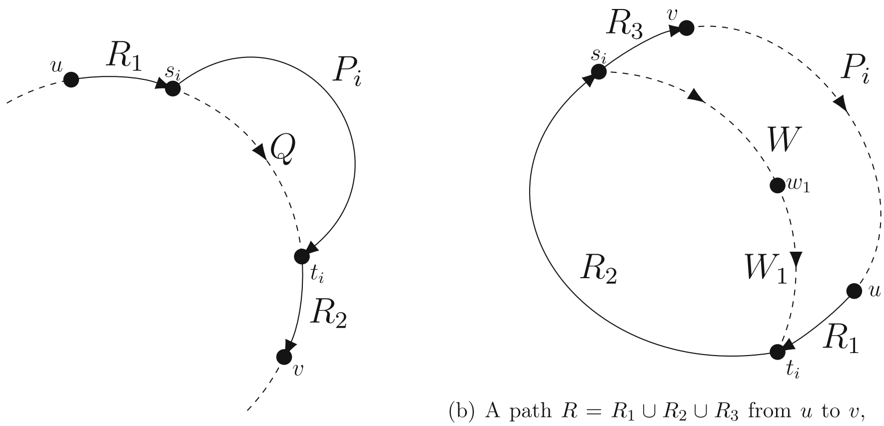
FIGURE 2. A coloring of P_i (a fragment of the graph D_{i-1} is dashed, the path P_i is drawn by normal lines, and the numbers $1, 2, 3$ mean colors of edges)

Properties (i) and (ii) remain obviously satisfied for vertices from $V(D_i) \setminus V(P_i)$. They are also satisfied for all *internal* vertices of P_i (all vertices of the path excluding the two end vertices), as they have both indegree and outdegree equal to 1 in D_i, and for s_i by (a). For t_i, property (i) is of course still satisfied, but to see that property (ii) holds, let us take a closer look at paths from s_i to t_i. We know that all edges going out of s_i in D_i have the same color [by (i)], that all paths from s_i to t_i in D_i have a canonical pattern [for paths from D_{i-1} from (iii), for P_i from (b)] and that the lengths of P_i and of every path from s_i to t_i in D_{i-1} are divisible by 3 (from Claim 1). Therefore, both P_i and all other paths from s_i to t_i in D_i end with the same color, so all edges going into t_i have the same color and property (ii) is satisfied also for t_i.

Now, consider the property (iii). It remains clearly satisfied for paths which do not contain edges from P_i.

Now, consider some path R from u to v, where $u, v \in V(D_i)$, which contain all edges from P_i (see Fig. 3a). It means that R consists of some path R_1 from u to s_i (maybe empty), the path P_i and some path R_2 from t_i to v (maybe empty). Let Q be a path from s_i to t_i in the graph D_{i-1}. From (iii), we have that the path: $R_1 \cup Q \cup R_2$ has a canonical pattern, from (i) that the first edge of P_i has the same color as the first edge of Q and from Claim 1 that the lengths of P_i and Q are divisible by 3. Therefore, R has a canonical pattern and property (iii) remains satisfied in this case.

Next, let us consider a path R from u to v, where both u and v are internal vertices of P_i (see Fig. 3b). First, assume that R is contained in P_i (that is, the order of the vertices on the path P_i is as follows: s_i, u, v, t_i). Then, R has a canonical pattern from (b). Therefore, let us assume the opposite; it means that the order of the vertices on the path P_i is: s_i, v, u, t_i. Let us call



(a) A path $R = R_1 \cup P_i \cup R_2$ from u to v.

(b) A path $R = R_1 \cup R_2 \cup R_3$ from u to v, where u and v are internal vertices of P_i.

FIGURE 3. The property (iii)—every path R from D_i has a canonical pattern

the parts of R as follows: let R_1 be a part of R from u to t_i, let R_2 be a part of R from t_i to s_i, and let R_3 be a part of R from s_i to v. Note that R_1 and R_3 are parts of P_i and R_2 is a path from t_i to s_i in D_{i-1}. Let W be some path from s_i to t_i in D_{i-1}. Let w_1 be a vertex from W, such that the distance from w_1 to t_i on W is congruent to the same value modulo 3 as the distance from u to t_i on R_i. Let W_1 be a part of W from w_1 to t_i. Because P_i has a canonical pattern [from (b)], and also, W and $W_1 \cup R_2$ have canonical patterns [from (iii)], and R_1 ends with the same color as W_1 [from (ii)], we have that the path $R_1 \cup R_2$ has a canonical pattern. Similarly, considering w_2 as any vertex from $V(D_{i-1})$ such that w_2 belongs to some path from s_i to t_i and the distance from s_i to w_2 is congruent to the same value modulo 3 as the distance from s_i to v, we get that the path $R_2 \cup R_3$ has a canonical pattern. Therefore, the whole path R has a canonical pattern and property (iii) remains satisfied also in this case.

The last case is when one endpoint of a path is from $V(D_{i-1})$ and the second one is an internal vertex of P. Carrying out analogous considerations as in the previous case, we get that the property (iii) is fulfilled here as well, which completes the proof of (iii).

For property (iv), it suffices to notice that adding a directed ear to a strongly connected digraph D_{i-1} preserves strong connectivity for D_i.

Therefore, the proof of the claim is complete by induction. □

We constructed the coloring C of the edges of D, such that every path in D has a canonical pattern. It means that every path of D is strongly proper. Therefore, G with coloring C is strongly proper connected. The proof of Theorem 3 is complete. □

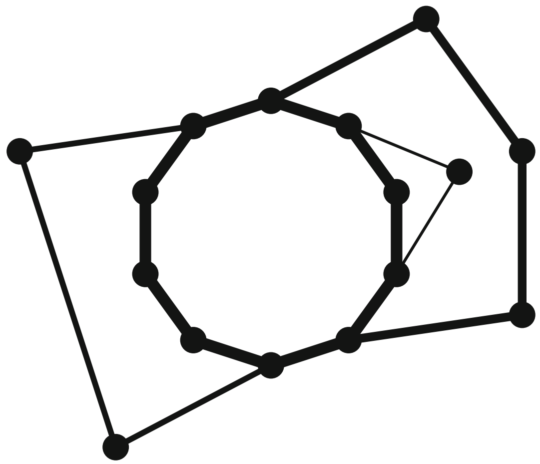


FIGURE 4. An exemplary ear decomposition $ED = \{P_1, P_2, P_3, P_4\}$ of some graph, where the thickest lines depict P_1 and the thinnest lines depict P_4 (P_1 is a cycle with 10 edges, P_2 is a path with 4 edges, P_3 is a path with 3 edges, and P_4 is a path with 2 edges)

2.2. 2-Connected Graphs Satisfy $\mathrm{spc}(G) \leq 5$

Proof of Theorem 1. Recall that a graph H is called *minimally 2-connected* if it is 2-connected, but it looses this property if we remove any single edge. Let H be a minimally 2-connected spanning subgraph of G. We will construct an edge coloring of H with at most 5 colors that makes H strongly proper connected. Note that it suffices to get the assertion of the theorem, as the remaining edges from $E(G)\backslash E(H)$ can be colored arbitrarily without affecting validity of the coloring.

From Theorem 4, we know that H has an open ear decomposition. Similarly to the proof of Theorem 3, we will color the graph while adding ears, but, this time, the order of adding ears will be important. Let $ED = (P_1, \ldots, P_h)$ be an open ear decomposition of H in which, in every step, we add the longest possible ear (see Fig. 4). Let H_i be the subgraph of H consisting of the first i ears of ED, that is, $H_i = P_1 \cup \cdots \cup P_i$. For an ear P_i, let s_i and t_i be the endpoints of P_i.

First, we present some claims concerning the properties of the ears from ED. Claim 3 shows that we will process ears from the longest to the shortest.

Claim 3. The ears of ED are in the opposite order to the number of their edges.

Proof. Let us assume the opposite. Let $P_i \in ED$ be an ear with the smallest possible index, such that P_i has more edges than P_{i-1}. Note that $i > 1$. From the definition of ED, we know that both endpoints of P_i are internal vertices of some ears with smaller indices (maybe two different). We consider three cases.

Case 1: None of s_i and t_i are internal vertices of P_{i-1}. Therefore, both s_i and t_i belong to some ears with indices smaller than $i-1$. It means that we could add a longer ear P_i to the graph H_{i-2} instead of P_{i-1}, which contradicts the definition of ED.

Case 2: Both s_i and t_i are internal vertices of P_{i-1}. It means that instead of P_{i-1}, we could add to the graph H_{i-2} a longer ear composed of the part of P_{i-1} from one endpoint to s_i, the path P_i, and the part of P_{i-1} from t_i to the second endpoint of P_{i-1}, which again contradicts the definition of ED.

Case 3: Exactly, one endpoint of P_i is an internal vertex of P_{i-1}. Assume that it is s_i. Therefore, t_i belongs to some ear with index smaller than $i-1$. It means that to the graph H_{i-2}, we could add an ear composed of the part of P_{i-1} from one endpoint to s_i and the path P_i, which has more edges than P_{i-1}. It contradicts the definition of ED again.

The proof of the claim is complete. □

Let us recall that H is minimally 2-connected, so every P_i has at least one internal vertex (no P_i may be a single edge). From the definition of ED, we know that $\{s_i, t_i\} = P_i \cap H_{i-1}$. Claim 4 shows that for every ear P_i, its endpoints cannot be adjacent in the graph H_{i-1}.

Claim 4. Vertices s_i and t_i are not adjacent in the graph H_{i-1}.

Proof. Let us assume the opposite. There is an edge $e = s_i t_i$ in H_{i-1}. Therefore, there is some ear $P_j \in ED$, where $j < i$, containing e. According to the ordering of ears in ED, in every step, we added the longest possible ear to our graph. But instead of P_j, we could add a longer ear $(P_j \backslash e) \cup P_i$, which contradicts the definition of ED. □

Claims 5 and 6 show the relationship between positions of the endpoints of ears with two or three edges.

Claim 5. Let P_i and P_j be ears from ED, such that both have two edges or both have three edges, where $j < i$. Then, no endpoint of P_i can be an internal vertex of P_j.

Proof. Let us assume the opposite. From Claim 4, we know that the endpoints of any ear cannot be adjacent in H. Therefore, the only possibility here is that exactly one endpoint of P_i, say s_i, is an internal vertex of P_j. Without loosing the generality of our consideration, we can assume that s_i is adjacent to t_j in H. Let r be some vertex from H_{j-1} different than s_j. Let R be the shortest path between t_i and r in the graph H_{i-1} which contains edges only from $E(H_{i-1}) \backslash E(H_j)$ (see Fig. 5). Notice that R may be an empty path (if $t_i \in V(H_{j-1})$); otherwise, it has at least one edge. Now, consider the ear $(P_j \backslash (s_i t_j)) \cup P_i \cup R$. It has more edges than P_j and has both endpoints in $V(H_{j-1})$, so it could be added to the graph H_{j-1}. This means that in the jth step, we did not add the longest possible ear, which contradicts the definition of ED. □

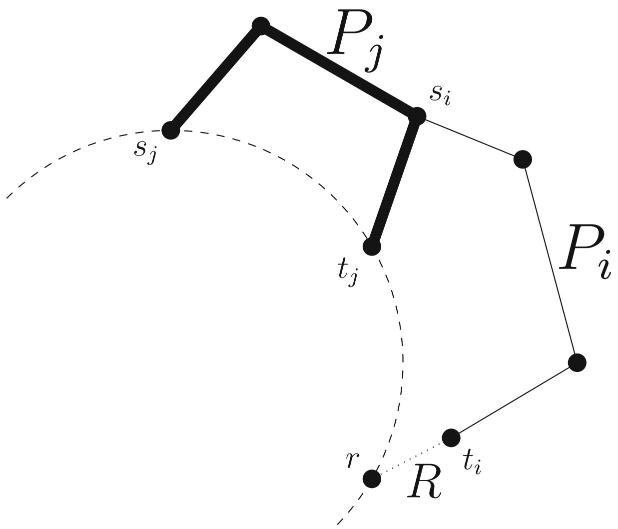


FIGURE 5. An example of an impossible case, where one endpoint of an ear P_i with 3 edges is an internal vertex of some other ear P_j with 3 edges (H_{j-1} is dashed, thick lines depict P_j, normal lines depict P_i, and dotted lines depict R)

Claim 6. Let P_i and P_j be ears from ED, such that $j < i$, P_i has two edges, P_j has three edges, and one endpoint of P_i, say s_i, is an internal vertex of P_j. Then, t_i (the second endpoint of P_i) must be an endpoint of P_j not adjacent to s_i in H_j.

Proof. Without loosing the generality of our consideration, we can assume that s_j is adjacent to s_i in H_j. Note that if t_i is the same as t_j, then our assumptions about the order of the ears in ED are satisfied; see Fig. 6. Therefore, it remains to show that t_i cannot be different from t_j. There are two cases to consider. If t_i were a vertex of P_j different from t_j, then vertices s_i and t_i would be adjacent in the graph H_{i-1}, which contradicts Claim 4. If t_i were from $H_{i-1} \backslash P_j$, it would mean that we added the ear P_j to H_{j-1} when we could add a longer ear. To construct this ear, we could take the shortest path between t_i and some vertex from H_{j-1} different than t_j which contains edges only from $E(H_{i-1}) \backslash E(H_j)$ and append to it $(P_i \cup P_j \backslash (s_i s_j)))$ (the construction is analogous as in the proof of Claim 5). This contradicts the definition of ED. Therefore, the proof of the claim is complete. □

Let i^* be a number defined, such that P_{i^*} is the last ear in ED with at least three edges. Let us define a sequence of oriented graphs $D_0, D_1, \ldots, D_{i^*}$, such that D_0 is empty and for each $i > 0$, D_i is obtained from D_{i-1} by adding the path P_i, oriented from s_i to t_i. Take D to be the last oriented graph in this sequence, i.e., $D := D_{i^*}$.

Now, we will color the edges of D with 5 colors, such that for every two vertices u, v, there is a strongly proper *directed* path from u to v, which

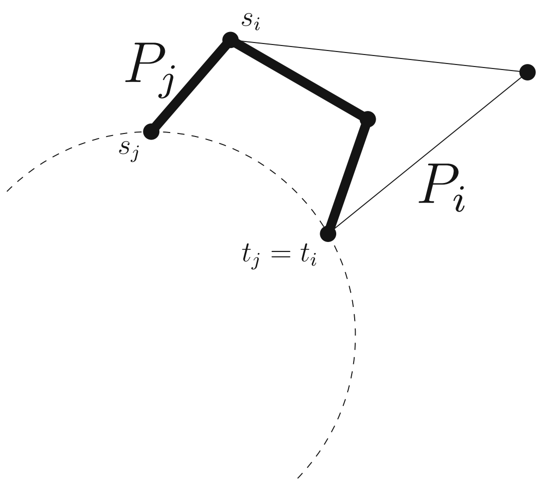


FIGURE 6. Ears $P_i, P_j \in ED$, such that P_i has 2 edges, P_j has 3 edges (H_{j-1} is dashed, thick lines depict P_j, and normal lines depict P_i)

is a strengthening of the property required from edge coloring of H. In the following, claim (iii) is the crucial part, while (i), (ii), and (iv) are invariants needed in an inductive proof.

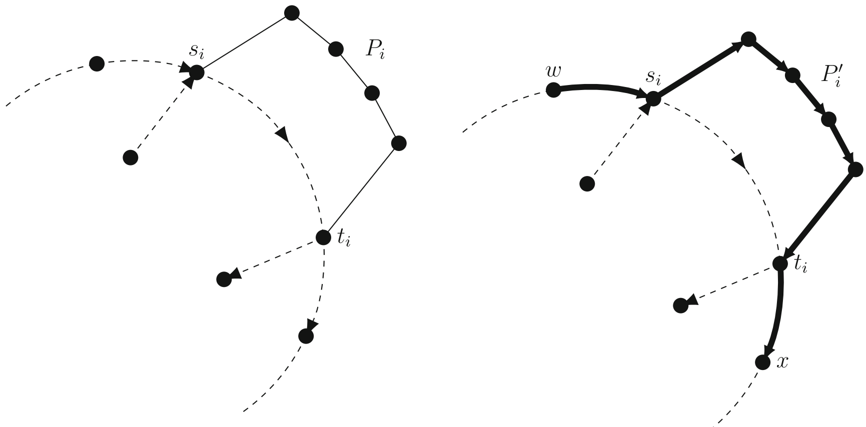
Claim 7. There exists a coloring C, such that for every $i \leq i^*$, the restriction of C_i to the edges of D_i satisfies the following properties:

(i) For every $v \in V(D_i)$, all edges going out of v have the same color.
(ii) For every $v \in V(D_i)$, all edges going into v have the same color, other than the color of edges going out of v.
(iii) For every two vertices $u, v \in V(D_i)$, there exists a strongly proper path from u to v and from v to u.
(iv) For every edge $uv \in E(D_i)$, such that u and v are not internal vertices of the same ear with 3 edges, and for every two directed edges $xu, vy \in E(D_i)$, we have $C_i(xu) \neq C_i(vy)$.

Proof. We will construct the coloring C by inductively defining C_i for $i = 1, 2, \ldots i^*$.

The base case of the induction is when D_1 is a directed cycle. It suffices to color the edges of D_1 greedily, making sure that each edge is colored differently than the two preceding and the two following edges.

Now, consider any $i \leq i^*$ and suppose that we already have the coloring C_{i-1} which satisfies properties (i)–(iv) (for $i-1$). For convenience, let us consider a directed path P'_i obtained from P_i by directing it from s_i to t_i and adding two edges ws_i and t_ix from D_{i-1}, where w is any vertex incoming to s_i and x is any vertex outgoing from t_i in the digraph D_{i-1} (such vertices must exist in D_{i-1} because of property (iii), see Fig. 7).



(a) A fragment of the graph D_{i-1} (dashed) and the path P_i (normal lines).

(b) The directed path P_i' (thick lines).

FIGURE 7. Constructing a path P_i' for a given directed graph D_{i-1} and a path P_i

Now, we will construct C_i from C_{i-1} by coloring the internal edges of P_i', such that

(a) the second edge of P_i' has the same color as the edges going out of s_i in D_{i-1},
(b) the penultimate edge of P_i' has the same color as the edges going into t_i in D_{i-1},
(c) the remaining edges of P_i' are colored with a different color than the two preceding and the two following edges on P_i' (note that it is possible, as we have five colors in use).

We will prove that C_i satisfies properties (i)–(iv).

It is clear that properties (i) and (ii) remain satisfied for vertices from $V(D_i)\backslash V(P_i)$. They are also satisfied for s_i and t_i by (a) and (b), and for the other vertices of P_i, as they have indegree and outdegree 1 in D_i.

Now, consider the property (iii). It remains satisfied if $u, v \in V(D_{i-1})$. If both u and v are internal vertices of P_i, then (iii) holds by (c).

Now, consider the case when $u \in V(D_{i-1})$ and v is an internal vertex of P_i. The required $u - v$ path is obtained by composing a strongly proper path Q from u to s_i in D_{i-1} [note that Q exists by (iii) applied for $i-1$] with a fragment of P_i from s_i to v. We will show that this path is strongly proper. Since both Q and a fragment of P_i are strongly proper, we only need to exclude color conflicts between four[2] edges around s_i. First edge of P_i has different color than the last edge of Q by (a) and (ii). Second edge of P_i is colored differently than the last edge of Q by (c). For the remaining pair note that from Claim 5, we know that s_i is not an internal vertex of any ear P_j

[2]This statement is technically incorrect if Q has one or zero edges. However, we ignore this case as it is much easier.

with 3 edges, where $j < i$. It follows that (iv) holds for the last edge of Q in the coloring C_{i-1}. It follows that last but one edge of Q is colored differently than edges going out of s_i in D_{i-1}, and hence, by (a), there is no conflict with the first edge of P_i.

In the remaining case, when u is an internal vertex of P_i and $v \in V(D_{i-1})$, the argument is analogous (the required path is obtained by composing a fragment of P_i from u to t_i and a strongly proper path from t_i to v). Therefore, the proof of (iii) is complete.

Now, consider the property (iv) that involves three edges xu, uv, and vy. It remains satisfied if both u and v are in $V(D_i)\backslash V(P_i)$. If both u and v are internal vertices of P_i, we only need to consider the case when P_i has at least 4 edges; in this case, (iv) follows directly from (c). If $u = s_i$ or $v = t_i$, the property (iv) follows directly from (c) again. Now, consider the case $v = s_i$ or $u = t_i$. Let y' be an out-neighbor of v in D_{i-1} and x' be an in-neighbor of u in D_{i-1} (note an easy case when $x' = x$ and $y' = y$). By the induction assumption (iv), it holds that $C_{i-1}(x'u) \neq C_{i-1}(vy')$, and by (a) and (b), it follows that $C_i(vy) = C_{i-1}(vy')$ and $C_i(xu) = C_{i-1}(x'u)$, which completes the proof of (iv).

Therefore, the proof of the claim is complete by induction. □

An *internal* edge of a path is any edge whose endpoints are internal vertices of the path. Before coloring the edges of H, we need to orient ears with two edges. Let P_i be an ear with two edges. We pick s'_i and t'_i from $\{s_i, t_i\}$, such that no edge going into s'_i is an internal edge of an ear with three edges and no edge going out of t'_i is an internal edge of an ear with three edges. Such a choice is possible, because by Claim 6 if one of the vertices from $\{s_i, t_i\}$ is an internal vertex of some ear P_j with 3 edges, then the other one is an endpoint of P_j, and hence, by Claim 5, it is not an internal vertex of any other ear with 3 edges. We will think of the ear P_i as oriented from s'_i to t'_i.

Now, let C be a 5-coloring of edges of D given by Claim 7. Define a 5-coloring C' of edges of H, such that $C'(uv) = C(uv)$ for every edge $uv \in E(D)$ and for every vertex v, such that v is an internal vertex of an ear P_i with two edges, let $C'(s'_i v)$ be the color of edges going out of s'_i in C and $C'(vt'_i)$ be the color of edges going into t'_i in C. Note that this definition is correct, since, by Claim 5, s'_i and t'_i are vertices of D, and it is unambiguous by Claim 7 (i) and (ii). We will show that C' is the desired coloring of H.

Consider any two vertices $u, v \in V(H)$. We need to find a strongly proper path between u and v. If both u and v are in $V(D)$, such a path exists by Claim 7 (iii). If $u, v \notin V(D)$, pick i and j, such that u is an internal vertex of the ear P_i and v is an internal vertex of P_j (where P_i and P_j have two edges). In this case, the desired path between u and v is obtained by taking a strongly proper path from t'_i to s'_j in D (which exists by Claim 7 (iii)) and appending to it the edge ut'_i at the start and the edge $s'_j v$ at the end; let us denote the constructed path by P. We will show that P is strongly proper. If P has less than three edges, it follows directly from Claim 7 (ii), so we assume otherwise. By Claim 7 (ii) the first and second edge have different colors, and last and last but one edge on P also have different colors. By the choice of t'_i, the second edge of P

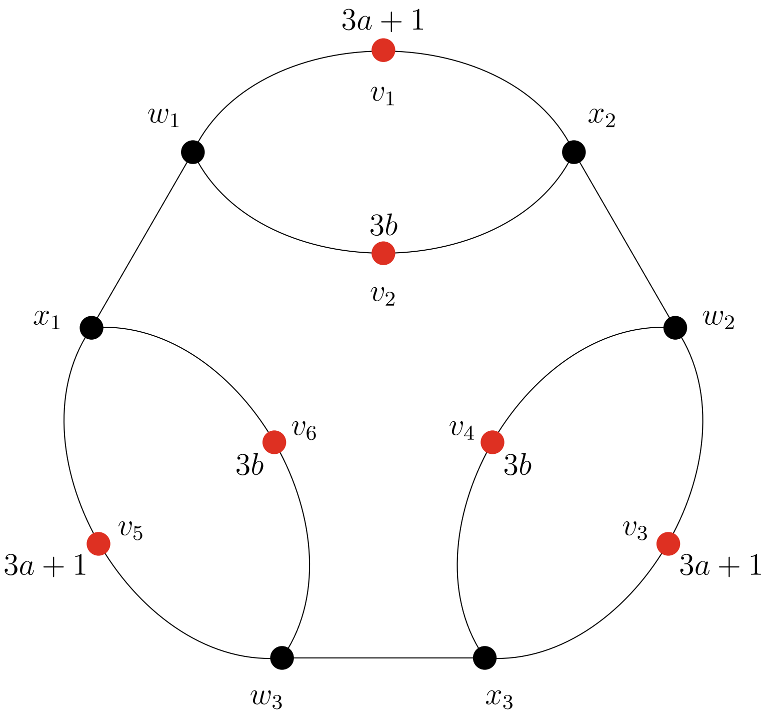


Figure 8. A graph G_d with strong proper connection number equal to 4

is not an internal edge of any ear with 3 edges, so by Claim 7 (iv) the first and third edge on P have different colors. Similarly, by the choice of s'_j, the last but one edge on the constructed path is not an internal edge of any ear with 3 edges, so P is strongly proper by Claim 7 (iv).

Note that the same argument applies when $u \notin V(D)$ and $v \in V(D)$ or $u \in V(D)$ and $v \notin V(D)$, except that only one end of P needs to be considered. This proves that H, together with the constructed coloring C', is strongly proper connected, so the proof of Theorem 1 is complete. □

2.3. Proof of the Lower Bound $\mathrm{spc}(G_d) \geq 4$

Proof of Theorem 2. Let $d \geq 3$ and let $a, b \geq \max\{3, d/3\}$ be fixed integers. Consider a graph G_d consisting of three edges, x_1w_1, x_2w_2, x_3w_3, and six paths $P_1, P_2, \ldots, P_6$, such that P_1 and P_2 go from w_1 to x_2, P_3, and P_4 go from w_2 to x_3, P_5 and P_6 go from w_3 to x_1. Moreover, P_1, P_3, P_5 have length $3a+1$ and P_2, P_4, P_6 have length $3b$ (see Fig. 8).

Suppose for the contrary that $\mathrm{spc}(G_d) \leq 3$ and fix an edge coloring of G_d with colors 1, 2 and 3 that makes it strongly proper connected. Choose vertices $v_1, v_2, \ldots, v_6$ such that for each i, v_i is a vertex from P_i at distance at least 2 from both ends of P_i, and the path with four edges closest to v_i is strongly proper. Note that such a choice is possible, since the edge-colored graph is strongly proper connected, e.g., for each i choose a vertex z_i that is at distance at least 4 from both ends of P_i and take v_i to be the third vertex on a strongly proper path from z_i to x_1.

Now define a directed graph D on vertices $v_1, \ldots, v_6$, such that there is an arc $v_i v_j$ if and only if there exists a strongly proper path from v_i to v_j in G_d with the canonical color pattern (a block of the sequence $(1, 2, 3, 1, 2, 3, \ldots)$). Note that each strongly proper path that uses three colors must exhibit this canonical pattern in one of two possible directions. Since the edge-colored graph G is strongly proper, it follows that D contains a tournament as a directed subgraph. Now, consider two cases.

Case 1: (D *is acyclic*) Note that, in this case, the vertices of D can be reordered as $u_1, u_2, \ldots, u_6$, so that for $i \in \{1, 2, \ldots, 5\}$, there exists a strongly proper path from u_i to u_{i+1} with the canonical color pattern. Let W be the walk obtained by joining all those paths (in order from u_1 to u_6). Note that no edge on W appears two times in a row, because if the last edge on the path from u_{i-1} to u_i was the same as the first edge on the path from u_i to u_{i+1}, then by the choice of v_i's the last but one and the second edge of those paths would also be the same, which contradicts the fact that both those paths have canonical color pattern; it also follows that W is strongly edge-colored. Moreover, each vertex from $\{v_1, v_2, \ldots, v_6\}$ appears on W exactly once, as, otherwise, there would be a cycle in D. This is a contradiction with the structure of G_d, i.e., there is no walk in G_d that visits each vertex from $\{v_1, v_2, \ldots, v_6\}$ exactly once and does not contain two consecutive occurrences of some edge.

Case 2: (D *contains a directed cycle* $u_1 u_2 \ldots u_k$). Consider a closed walk W obtained by joining strongly proper paths with the canonical color pattern that go from u_1 to u_2, from u_2 to u_3, and so on, up to a path from u_k to u_1. Note that, like in the previous case, W is strongly edge-colored and no edge of G_d appears on W two times in a row.

Let $S = \{x_1, w_1, x_2, w_2, x_3, w_3\}$ and consider consecutive occurrences of vertices from S on W. Note that up to natural symmetries (i.e., renaming x_i to w_i and vice versa or rotating names, so that x_i, w_i become x_{i+1} and w_{i+1}) there are two cases: either (2a) at some point w_1 is followed by x_2, followed again by w_1 or (2b) x_i is always followed by w_i, and w_1 is followed by x_2, w_2 by x_3 and w_3 by x_1.

In case (2a), note that the first occurrence of w_1 must be preceded by x_1 and the second occurrence of w_1—followed by x_1 (because otherwise P_1 and P_2 would form a cycle with the canonical color pattern, which is impossible as their total length is not divisible by 3). However, it implies that the edge $x_1 w_1$ occurs on W twice at distance exactly $3a + 3b + 2$, which is a contradiction, because colors on W must repeat every three edges.

In the remaining case (2b) a part of W from the first occurrence of the edge $x_1 w_1$ to its second occurrence must be a strongly edge-colored cycle; denote it by C. Since the length of C must be divisible by 3, it contains either vertices $\{v_1, v_3, v_5\}$ or $\{v_2, v_4, v_6\}$. Let u_1, u_2, u_3 be vertices from $\{v_1, v_2, \ldots, v_6\}$ outside C.

Note that u_i may not be incident to both incoming and outgoing arcs in D. Indeed, if that was the case, then a part of C, together with paths from C to u_i and from u_i to C with canonical color pattern, would form a strongly 3-edge-colored cycle with length not divisible by 3, which is a contradiction. However, this implies that D does not contain an arc between two of the vertices from $\{u_1, u_2, u_3\}$ (i.e., there can be no arc between two vertices with outdegree 0), which contradicts the fact that D contains a tournament. Therefore, the proof is complete. □

2.4. Nonrepetitive Connected Coloring of Graphs

In this subsection, we prove that 4-connected graphs satisfy $\mathrm{nrc}(G) \leq 6$ and 2-connected graphs satisfy $\mathrm{nrc}(G) \leq 15$. Actually, we will derive these bounds as simple consequences of more general results.

Theorem 6. *Let G be a graph containing two edge-disjoint spanning trees. Then,* $\mathrm{nrc}(G) \leq 6$.

Proof. Let T_1 and T_2 be two spanning trees of G, such that $E(T_1) \cap E(T_2) = \emptyset$. Let r be a common root of these trees. Let $E_i(T_1)$ be the set of edges at distance i from the root r. Therefore, $E_0(T_1)$ consists of the edges of T_1 incident to r, $E_1(T_1)$ contains the edges of T_1 incident to the neighbors of r, and so on. By the theorem of Thue [18], there exists a nonrepetitive sequence $a_0a_1a_2\cdots$ of arbitrary length, such that $a_i \in \{1, 2, 3\}$. We may color the edges of the tree T_1 using this sequence, so that each edge in the set $E_i(T_1)$ gets color a_i. The same construction may be applied to the tree T_2, with similarly defined sets $E_i(T_2)$, and sufficiently long nonrepetitive sequence $b_0b_1b_2\cdots$, with $b_i \in \{4, 5, 6\}$. All other edges of G may be colored arbitrarily.

We claim that this coloring satisfies the desired property. Indeed, let u, v be any two vertices of G. Denote by $P_j(x, y)$, $j = 1, 2$, the unique path from x to y in the tree T_j. Consider the path $P_1(u, r)$. Clearly, it is nonrepetitive by the construction of the coloring. If v lies on $P_1(u, r)$, then the sub-path $P_1(u, v)$ is nonrepetitive, too, and we are done. Therefore, assume that v lies outside $P_1(u, r)$ and consider the path $P_2(r, v)$. If the only common vertex of these two paths is r, then we may glue them together into a longer path $P_1(u, r)P_2(r, v)$, which is clearly nonrepetitive, as the sets of colors on both fragments are disjoint.

Finally, suppose that the two paths, $P_1(u, r)$ and $P_2(r, v)$, have some common vertices other than the root r and let x be the one with the largest distance from r (in the tree T_1, say). Then, the two sub-paths $P_1(u, x)$ and $P_2(x, v)$ intersect in only one vertex x and, as before, we may glue them together to get the nonrepetitive path $P_1(u, x)P_2(x, v)$. This completes the proof. □

To get the second assertion of Theorem 5, it suffices to apply the following simple fact following easily from the celebrated theorem of Nash–Williams [16] (see Corollary 44 in [17]).

Theorem 7 (Nash–Williams [16]). *Every $2k$-edge-connected graph contains k edge-disjoint spanning trees.*

Indeed, it is enough to take $k = 2$ and notice that a 4-edge-connected graph is all the more 4-(vertex)-connected.

For the second bound for 2-connected graphs, we apply a similar approach with a slightly weaker property based on edge-independent trees. Recall that two spanning trees in a graph G, T_1 and T_2, having the same root r, are *edge-independent* if, for every vertex v, the unique paths $P_1(v, r)$ in T_1 and $P_2(v, r)$ in T_2 are edge disjoint.

Theorem 8. *Let G be a graph containing two edge-independent spanning trees. Then,* $\mathrm{nrc}(G) \leq 15$.

Proof. Let T_1 and T_2 be two edge-independent spanning trees of G. We will construct a similar coloring as in the proof of Theorem 6, but notice that this time the sets of edges $E(T_1)$ and $E(T_2)$ need not be disjoint. Therefore, we will color the edges of G by ordered pairs of colors whose first coordinates are controlled by an appropriate coloring of T_1, while second coordinates are determined by an analogous coloring of T_2.

Therefore, let r be a common root of trees T_1 and T_2. Let $E_i(T_1)$ be the set of edges at distance i from the root r. Therefore, $E_0(T_1)$ consists of the edges of T_1 incident to r, $E_1(T_1)$ contains the edges of T_1 incident to the neighbors of r, and so on. By Thue's theorem [18], there exist nonrepetitive sequences, $a_0a_1a_2\cdots$ and $b_0b_1b_2\cdots$, of arbitrary length, such that $a_i \in \{1, 2, 3\}$ and $b_i \in \{4, 5, 6\}$. We may color the edges of trees T_1 and T_2 using these sequences, so that each edge in the set $E_i(T_1)$ gets color a_i and each edge in $E_i(T_2)$ gets color b_i. Now, if an edge e belongs to both trees, then its final color is an ordered pair of colors (a_i, b_j). In this way, we get a partial coloring of G using at most $9 + 6 = 15$ colors. The rest of the edges of G, not belonging to trees T_i, may be colored by these colors arbitrarily.

We claim that this coloring satisfies the desired property. Indeed, let u, v be any two vertices of G. Denote by $P_j(x, y)$, $j = 1, 2$, the unique path from x to y in the tree T_j. Consider the path $P_1(u, r)$. Clearly, it is nonrepetitive by the construction of the coloring. If v lies on $P_1(u, r)$, then the sub-path $P_1(u, v)$ is nonrepetitive, too, and we are done. Therefore, assume that v lies outside $P_1(u, r)$ and consider the path $P_2(v, r)$. Let x_1 be the first vertex on the path $P_1(u, r)$ traversed from u to r belonging to $P_2(v, r)$. Let $e_u = yx_1$ be the last edge on the sub-path $P_1(u, x_1)$. Similarly, let x_2 be the first vertex on the path $P_2(v, r)$ traversed from v to r belonging to $P_1(u, r)$. Let $e_v = zx_2$ be the last edge on the sub-path $P_2(v, x_2)$. Now, it is not hard to verify that, by the assumption of the edge-independence of trees T_j, each of these two edges belong to only one tree, namely, e_u to T_1 and e_v to T_2. Consequently, the color of e_u, which is some a_i, cannot occur on the path $P_2(v, x_2)$. It follows that the path $P_1(u, x_2)P_2(x_2, v)$ is nonrepetitive. This completes the proof. □

It is conjectured that every k-edge-connected graph contains k edge-independent spanning trees with an arbitrarily choosen common root r (see [17]). To get the first part of Theorem 5, it suffices to invoke the results of Itai and Rodeh [12] and Khuller and Scheiber [13] confirming this conjecture for $k = 2$ (see [17]).

3. Final Remarks

Let us conclude the paper with some natural open problems. The first one is very concrete and asks for the optimum value of the strongly proper connection number $\mathrm{spc}(G)$ in the class of 2-connected graphs.

Problem 9. *Determine the least possible number k, such that every 2-connected graph G satisfies* $\mathrm{spc}(G) \leq k$.

We know that $k = 4$ or 5, but which is the correct value? It would be also nice to know what happens for graphs with higher connectivity. For instance, it is known (see [3,14]) that $\mathrm{pc}(G) = 2$ holds already for 3-connected graphs. Also, our graphs G_d from the proof of Theorem 2 are not 3-connected. This prompts us to formulate the following conjecture.

Conjecture 10. *Every 3-connected graph G satisfies* $\mathrm{spc}(G) \leq 3$.

Let us stress, however, that we do not even know if the above inequality holds for graphs with any sufficiently high connectivity.

It would be also nice to know more on nonrepetitive connected coloring and the corresponding parameter $\mathrm{nrc}(G)$.

Problem 11. *Determine the least possible number t, such that every 2-connected graph G satisfies* $\mathrm{nrc}(G) \leq t$.

By Theorem 8, we know that $t \in \{3, 4, \ldots, 15\}$. The problem seems challenging even if restricted to some classes of graphs. Consider, for instance, nonrepetitive connected coloring of *planar* graphs. Barnette [2] proved that every 3-connected planar graph contains a spanning tree of maximum degree at most three (see [17]). Using a general upper bound form [1], one gets that for such graphs, we have $\mathrm{nrc}(G) \leq 8$. Also, since 4-connected planar graphs are Hamiltonian, as proved by Tutte [19], they satisfy the best possible bound $\mathrm{nrc}(G) \leq 3$.

As mentioned in the introduction, one may consider fairly general $\mathcal{P}$-*connected colorings*, where $\mathcal{P}$ is any property of sequences. The minimum number of colors needed for such a coloring of G, with fixed property $\mathcal{P}$, is denoted by $\mathcal{P}\text{-c}(G)$. A property $\mathcal{P}$ is called *honest* if it possess the following basic features:

- (i) If a sequence S has property $\mathcal{P}$, then each nonempty block of S also satisfies $\mathcal{P}$.
- (ii) If S and T are two sequences over disjoint alphabets (color sets) satisfying $\mathcal{P}$, then their concatenation ST also satisfies $\mathcal{P}$.
- (iii) There exist arbitrarily long sequences over some finite alphabet satisfying property $\mathcal{P}$.

Let us denote by $m(\mathcal{P})$ the least possible constant in condition (iii). For instance, if $\mathcal{P}$ corresponds to strong coloring or nonrepetitive coloring, then $m(\mathcal{P}) = 3$, while if $\mathcal{P}$ stems from the usual proper coloring, then $m(\mathcal{P}) = 2$.

It is now easy to see that repeating the proof of Theorem 5 gives the following general result.

Theorem 12. *Let $\mathcal{P}$ be any honest property of sequences. If G is a graph containing two edges disjoint spanning trees, then $\mathcal{P}\text{-c}(G) \leq 2\,m(\mathcal{P})$. In particular, every 4-connected graph G satisfies $\mathcal{P}\text{-c}(G) \leq 2m(\mathcal{P})$.*

One naturally wonders if the above statement could be true for 2-connected graphs (or at least for 3-connected graphs), possibly with some larger upper bound.

Conjecture 13. *Let $\mathcal{P}$ be any honest property of sequences. Then, there exists a constant $t(\mathcal{P})$, such that every 2-connected graph G satisfies $\mathcal{P}\text{-c}(G) \leq t(\mathcal{P})$.*

One also naturally wonders if the minimum possible number of colors in a $\mathcal{P}$-connected coloring can be achieved at the expense of increasing connectivity.

Conjecture 14. *Let $\mathcal{P}$ be any honest property of sequences. Then, there exists a constant $c(\mathcal{P})$, such that every $c(\mathcal{P})$-connected graph G satisfies $\mathcal{P}\text{-c}(G) \leq m(\mathcal{P})$.*

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Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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