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Results in Mathematics

Exponential Almost-Riordan Arrays

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Abstract. The group of the almost-Riordan arrays with exponential generating functions is defined. The subgroups of the exponential almost-Riordan group are presented. Also, some isomorphisms between the exponential almost-Riordan group and the exponential Riordan group are considered. Then, the production matrix for the exponential almost-Riordan array is obtained.

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1. Introduction

Generating functions are a very useful tool in combinatorics, analysis, and other areas of mathematics. By using the properties of generating functions, some combinatorial identities can be obtained more easily.

For the integer sequence $\{a_n\}_{n\geq 0}$, the formal power series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots.$$

represents an ordinary generating function. The coefficient of x^n in the formal power series $f(x)$ is

$$a_n = [x^n]f(x) \tag{1.1}$$

in [12]. Additionally, the following identities are valid:

$$[x^n](x^k f(x)) = [x^{n-k}]f(x) \tag{1.2}$$

and

$$[x^n]f'(x) = (n+1)[x^{n+1}]f(x) \tag{1.3}$$

in [18]. The detailed information regarding the generating functions and coefficient extraction can be found in [6,12,17,18,23,27]. Also, the following power series

$$g(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots$$

represents the ordinary generating function of the sequence $\{\frac{a_n}{n!}\}_{n\geq 0}$. This generating function is known as the exponential generating function of the sequence $\{a_n\}_{n\geq 0}$. Then, the coefficient a_n of x^n in $g(x)$ is denoted by

$$a_n = n![x^n]g(x) \tag{1.4}$$

in [6]. Sometimes, the properties of the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$ can be complex. In such cases, we can consider the generating function of the sequence $\{\frac{a_n}{n!}\}_{n\geq 0}$. For instance, the ordinary generating function of factorial numbers sequence is as follows:

$$\sum_{n=0}^{\infty} n! x^n = \int_{n=0}^{\infty} \frac{e^{-t}}{1-tx} dt.$$

The exponential generating function of factorial numbers sequence is given as

$$\sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \frac{1}{1-x}.$$

Obviously, the exponential generating function of factorial numbers sequence is simpler than the ordinary generating function for factorial numbers sequence. Similarly, the ordinary generating function for the first kind of Stirling numbers sequence is complicated and divergent. However, the exponential generating function of the same sequence is more useful than the ordinary generating function.

Riordan arrays play a crucial role in deriving combinatorial identities and solving combinatorial sums. Renzo Sprugnoli has contributed significantly to the utilization of Riordan arrays in computing combinatorial sums [21,22]. Also, Sprugnoli has investigated sums involving binomial coefficients and Stirling numbers by using the fundamental theorem of Riordan arrays in [20]. Riordan arrays are infinite lower triangular matrices defined by ordinary generating functions. There are many studies about Riordan arrays in literature, and some of these can be found in [6,8,11,13–16,19,23,26].

We consider the following formal power series:

$$g(x) = g_0 + g_1 x + g_2 \frac{x^2}{2!} + g_3 \frac{x^3}{3!} + \dots$$

and

$$f(x) = f_0 + f_1 x + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + \dots$$

with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. The functions $g(x)$ and $f(x)$ represent the exponential generating functions of the sequences $\{g_n\}_{n\geq 0}$ and $\{f_n\}_{n\geq 0}$, respectively. The exponential generating function of the kth column of the exponential Riordan arrays is defined as follows:

$$g(x)\frac{f(x)^k}{k!}, \quad k = 0, 1, 2, \dots .$$

Additionally, the exponential Riordan arrays are denoted as pairs of exponential generating functions, represented as $[g(x), f(x)]$. The multiplication operation between two exponential Riordan arrays is defined as follows:

$$[g(x), f(x)][h(x), l(x)] = [g(x)h(f(x)), l(f(x))]. \tag{1.5}$$

The set of exponential Riordan arrays is a group with the multiplication operation defined in (1.5). This group is known as the exponential Riordan group. The identity element of the exponential Riordan group is defined as

$$I = [1, x] \tag{1.6}$$

and the inverse of $[g(x), f(x)]$ is given by

$$[g(x), f(x)]^{-1} = \left[\frac{1}{g(\overline{f}(x))}, \overline{f}(x)\right] \tag{1.7}$$

where $\overline{f}(x)$ is the compositional inverse of $f(x)$ in [10].

Let R be the exponential Riordan matrix. Then, we have

$$P = R^{-1}\overline{R} \tag{1.8}$$

which P, R^{-1} and $\overline{R}$ represent the production matrix of the matrix R, the inverse of the matrix R and the version of the matrix R with the 0th row removed, respectively in [10]. The characterization and production matrices of exponential Riordan arrays are provided in [6,10,11].

Proposition 1.1 [11]. *Let $D = (d_{n,k})_{n,k\geq 0} = [g(x), f(x)]$ be an exponential Riordan matrix with $g_0 \neq 0, f_0 \neq 0$. Let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \dots, \;\; r(y) = r_0 + r_1 y + r_2 y^2 + \dots \tag{1.9}$$

be two formal power series such that

$$\begin{aligned} r(xf(x)) &= (xf(x))', \\ c(xf(x)) &= \frac{g'(x)}{g(x)}. \end{aligned} \tag{1.10}$$

Then

$$d_{n+1,0} = \sum_i i! c_i d_{n,i}, \tag{1.11}$$

$$d_{n+1,k} = r_0 d_{n,k-1} + \frac{1}{k!}\sum_{i\geq k} i!(c_{i-k} + kr_{i-k+1})d_{n,i} \tag{1.12}$$

or, defining $c_{-1} = 0$,

$$d_{n+1,k} = \frac{1}{k!} \sum_{i \geq k-1} i!(c_{i-k} + kr_{i-k+1})d_{n,i}. \tag{1.13}$$

Conversely, starting from the sequences defined by (1.9), the infinite array $(d_{n,k})_{n,k \geq 0}$ *defined by (1.13) is an exponential Riordan matrix.*

In [11], the production matrix P of the exponential Riordan matrix D is given as follows:

$$P = \begin{pmatrix} c_0 & r_0 & 0 & 0 & 0 & \dots \\ 1!c_1 & \frac{1!}{1!}(c_0 + r_1) & r_0 & 0 & 0 & \dots \\ 2!c_2 & \frac{2!}{1!}(c_1 + r_2) & \frac{2!}{2!}(c_0 + 2r_1) & r_0 & 0 & \dots \\ 3!c_3 & \frac{3!}{1!}(c_2 + r_3) & \frac{3!}{2!}(c_1 + 2r_2) & \frac{3!}{3!}(c_0 + 3r_1) & r_0 & \dots \\ 4!c_4 & \frac{4!}{1!}(c_3 + r_4) & \frac{4!}{2!}(c_2 + 2r_3) & \frac{4!}{3!}(c_1 + 3r_2) & \frac{4!}{4!}(c_0 + 4r_1) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.14}$$

The detailed information about the exponential Riordan arrays can be found in [2–4,6,9].

Subgroups in group theory are significant topic, and there has been extensive research on the subgroups of the Riordan group. The subgroups of the Riordan group are investigated and isomorphisms among these subgroups are given by Jean-Louis and Nkwanta in [15]. Some generalization of the Riordan arrays are examined. The generalized Riordan arrays are defined using the generalized generating functions, and their properties are investigated in [26]. The generalization forms of the Riordan subgroups are defined in [8,16].

Another generalization of the Riordan arrays is the almost-Riordan arrays. Let's now present the definition and some properties of the almost-Riordan arrays, as introduced by Barry in [5].

Let's consider the following formal power series:

$$a(x) = a_0 + a_1x + a_2x^2 + \dots,$$
$$g(x) = g_0 + g_1x + g_2x^2 + \dots$$

and

$$f(x) = f_0 + f_1x + f_2x^2 + \dots$$

with $a_0 \neq 0$, $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. The notation for first order almost-Riordan arrays is $(a(x)|g(x), f(x))$. The generating function of the kth column of a first order almost-Riordan array is

$$a(x), \quad for \ \ k = 0, \tag{1.15}$$
$$xg(x)(f(x))^{k-1}, \quad for \ \ k = 1, 2, 3, \dots. \tag{1.16}$$

Additionally, the multiplication of two almost-Riordan arrays is defined as follows:

$$(a(x)|g(x), f(x))(b(x)|h(x), l(x)) = ((a(x)|g(x), f(x))b(x)\big|g(x)h(f(x)), l(f(x))) \tag{1.17}$$

where the operation $(a(x)|g(x), f(x))b(x)$ is given as

$$(a(x)|g(x), f(x))b(x) = b_0a(x) + xg(x)\frac{b(f(x)) - b_0}{f(x)}. \tag{1.18}$$

The set of the first order almost-Riordan arrays is a group with the multiplication defined in (1.17). The identity element of this group is

$$I = (1|1, x) \tag{1.19}$$

and the inverse of the almost-Riordan arrays is given as follows:

$$(a(x)|g(x), f(x))^{-1} = \left(\frac{1}{a_0}\left(1 - \frac{x}{g(\overline{f}(x))}\frac{a(\overline{f}(x)) - a_0}{\overline{f}(x)}\right)\Bigg|\frac{1}{g(\overline{f}(x))}, \overline{f}(x)\right). \tag{1.20}$$

The sequence characterizations of the almost-Riordan arrays are provided in [1]. The pseudo-involutions and involutions in the almost-Riordan arrays are studied in [7,25].

Based on the preceding studies, we introduce the exponential almost-Riordan arrays. Also, we examine the subgroups of the exponential almost-Riordan group and provide some isomorphisms among them. Furthermore, the production matrix of the exponential almost-Riordan arrays is presented in this paper.

2. Exponential almost-Riordan arrays

In this section, we define the exponential almost-Riordan arrays and give some row sums of them. Additionally, we introduce the exponential almost-Riordan group.

Definition 2.1. Let's consider the following exponential generating functions:

$$a(x) = a_0 + a_1x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \dots,$$
$$g(x) = g_0 + g_1x + g_2\frac{x^2}{2!} + g_3\frac{x^3}{3!} + \dots$$

and

$$f(x) = f_0 + f_1x + f_2\frac{x^2}{2!} + f_3\frac{x^3}{3!} + \dots$$

with $a_0 \neq 0$, $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$. The notation for the exponential almost-Riordan arrays is $[a(x)|g(x), f(x)]$. The exponential generating function of the kth column of the exponential almost-Riordan arrays is defined as follows:

$$a(x), \qquad for \;\; k=0, \tag{2.1}$$

$$\frac{1}{(k-1)!}\int_0^x g(t)f^{k-1}(t)dt, \qquad for \;\; k=1,2,3,\ldots. \tag{2.2}$$

Example 2.2. The exponential almost-Riordan array $\left[\frac{1}{1-x}|\frac{1}{1-x},\ln(\frac{1}{1-x})\right]$ is given as

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
6 & 2 & 3 & 1 & 0 & 0 & 0 & 0 & \ldots \\
24 & 6 & 11 & 6 & 1 & 0 & 0 & 0 & \ldots \\
120 & 24 & 50 & 35 & 10 & 1 & 0 & 0 & \ldots \\
720 & 120 & 274 & 225 & 85 & 15 & 1 & 0 & \ldots \\
5040 & 720 & 1764 & 1624 & 735 & 175 & 21 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

where the column 0th is composed of the factorial numbers which is the sequence A000142 in OEIS [24].

Proposition 2.3. *Let $D=(d_{n,k})_{n,k\geq 0}=[a(x)|g(x),f(x)]$ be an exponential almost-Riordan array. Then, the elements of D are as follows:*

$$d_{n,k} = n![x^n]a(x), \qquad for \;\; k=0, \tag{2.3}$$

$$d_{n,k} = \frac{(n-1)!}{(k-1)!}[x^{n-1}]g(x)f^{k-1}(x), \qquad for \;\; k\geq 1. \tag{2.4}$$

Proof. From (1.4) and (2.1), the equation (2.3) is clear. Considering (1.4) and (2.2), we have

$$d_{n,k}=\frac{n!}{(k-1)!}[x^n]F(x)$$

where $F(x)=\int_0^x g(t)f^{k-1}(t)dt$. From (1.3), we get

$$[x^n]F(x)=\frac{1}{n}[x^{n-1}]F'(x)=\frac{1}{n}[x^{n-1}]g(x)f^{k-1}(x).$$

Then, we obtain

$$d_{n,k}=\frac{(n-1)!}{(k-1)!}[x^{n-1}]g(x)f^{k-1}(x).$$

□

Example 2.4. Let's take the following exponential almost-Riordan array:

$$D=\left[\frac{\alpha e^{\alpha x}-\beta e^{\beta x}}{\alpha-\beta}\middle| e^x(1+x),x\right]$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. From (2.3), we have

$$d_{n,0} = n![x^n]\left(\frac{\alpha e^{\alpha x} - \beta e^{\beta x}}{\alpha - \beta}\right) = n![x^n]\sum_{n=0}^{\infty} F_{n+1}\frac{x^n}{n!}.$$

Using (1.4), we obtain $d_{n,0} = F_{n+1}$ that F_n is the nth Fibonacci number. By using (2.4), we get

$$d_{n,k} = \frac{(n-1)!}{(k-1)!}[x^{n-1}]e^x(1+x)x^{k-1}.$$

From (1.2), we have

$$d_{n,k} = \frac{(n-1)!}{(k-1)!}[x^{n-k}](e^x + xe^x).$$

Considering the formal power series of e^x, we obtain

$$d_{n,k} = (n-k+1)\binom{n-1}{k-1},$$

which is the sequence A093375 in OEIS [24]. The matrix D is given as follows:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 3 & 3 & 4 & 1 & 0 & 0 & 0 & 0 & \dots \\ 5 & 4 & 9 & 6 & 1 & 0 & 0 & 0 & \dots \\ 8 & 5 & 16 & 18 & 8 & 1 & 0 & 0 & \dots \\ 13 & 6 & 25 & 40 & 30 & 10 & 1 & 0 & \dots \\ 21 & 7 & 36 & 75 & 80 & 45 & 12 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proposition 2.5. *Let $[a(x)|g(x), f(x)]$ be an exponential almost-Riordan array, and let $h(x)$ be an exponential generating function of the sequence $\{h_n\}_{n\geq 0}$. Then,*

$$[a(x)|g(x), f(x)]h(x) = h_0 a(x) + \int_0^x g(t)h'(f(t))dt \tag{2.5}$$

where $h'(x)$ is the first order derivative of $h(x)$.

Proof. If we consider the product of $[a(x)|g(x), f(x)]$ and $h(x)$, we obtain

$$\begin{aligned} &[a(x)|g(x), f(x)]h(x) \\ &\quad = h_0 a(x) + h_1\int_0^x g(t)dt + h_2\int_0^x g(t)f(t)dt + h_3\int_0^x g(t)\frac{f^2(t)}{2!}dt + \dots. \end{aligned}$$

Then, we have

$$\begin{aligned}[a(x)|g(x), f(x)]h(x) &= h_0 a(x) + \int_0^x g(t)\left(h_1 + h_2 f(t) + h_3 \frac{f^2(t)}{2!} + \dots\right) dt \\ &= h_0 a(x) + \int_0^x g(t) h'(f(t)) dt.\end{aligned}$$

□

For example, let's consider $[e^{2x}|e^x, x]$ and $h(x) = e^x$, we have

$$[e^{2x}|e^x, x]e^x = e^{2x} + \int_0^x e^{2t} dt = \frac{1}{2}(3e^{2x} - 1).$$

The sequence of this exponential generating function is A003945 in OEIS [24].

Proposition 2.6. *Let $D = (d_{n,k})_{n,k\geq 0} = [a(x)|g(x), f(x)]$ be an exponential almost-Riordan array, and let $h(x)$ be the exponential generating function of the sequence $\{h_n\}_{n\geq 0}$. Then,*

$$\sum_{k=0}^{n} d_{n,k} h_k = h_0 a_n + n![x^n] \int_0^x g(t) h'(f(t)) dt. \tag{2.6}$$

Specially, taking $h(x) = e^x$ in (2.6), we obtain the row sums for the exponential almost-Riordan array. Similarly, taking $h(x) = e^{-x}$ in (2.6), the alternating row sums for the exponential almost-Riordan array are obtained, as stated in the following corollary.

Corollary 2.7. *The row sums and the alternating row sums for an exponential almost-Riordan array $D = (d_{n,k})_{n,k\geq 0} = [a(x)|g(x), f(x)]$ are given by the following expressions:*

$$\sum_{k=0}^{n} d_{n,k} = a_n + n![x^n] \int_0^x g(t) e^{f(t)} dt \tag{2.7}$$

and

$$\sum_{k=0}^{n} (-1)^k d_{n,k} = a_n - n![x^n] \int_0^x g(t) e^{-f(t)} dt. \tag{2.8}$$

Example 2.8. Let's consider the exponential almost-Riordan array $D = [1|2 - e^x, x]$. The matrix D is given as

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & -3 & -3 & 1 & 0 & 0 & 0 & \dots \\ 0 & -1 & -4 & -6 & -4 & 1 & 0 & 0 & \dots \\ 0 & -1 & -5 & -10 & -10 & -5 & 1 & 0 & \dots \\ 0 & -1 & -6 & -15 & -20 & -15 & -6 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Firstly, let's find the row sums of the matrix $D = (d_{n,k})_{n,k\geq 0}$. Using (2.7), we have

$$\sum_{k=0}^{n} d_{n,k} = a_n + n![x^n] \int_0^x (2-e^t)e^t dt.$$

Then, we obtain

$$\sum_{k=0}^{n} d_{n,k} = a_n + n![x^n] \left(2\sum_{n=1}^{\infty} \frac{x^n}{n!} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^n x^n}{n!} \right).$$

Hence, we have

$$\sum_{k=0}^{n} d_{n,k} = \begin{cases} 1, & n=0 \\ 2-2^{n-1}, & n \geq 1 \end{cases}$$

where is the sequence A122958 in OEIS [24].

Now, let's find the alternating row sums of the matrix $D = (d_{n,k})_{n,k\geq 0}$. By utilizing the equation (2.8), we obtain

$$\sum_{k=0}^{n} (-1)^k d_{n,k} = a_n - n![x^n] \int_0^x (2-e^t)e^{-t} dt.$$

Considering the formal power series of e^{-x}, we find

$$\sum_{k=0}^{n} (-1)^k d_{n,k} = \begin{cases} 1, & n=0 \\ -1, & n=1 \\ 2(-1)^n, & n \geq 2 \end{cases}.$$

Specially, if we take $h(x) = e^x(x+1)$ in (2.6), we obtain the weighted row sums for the exponential almost-Riordan array. Similarly, taking $h(x) = e^{-x}(1-x)$ in (2.6), the alternating weighted row sums for the exponential almost-Riordan array are obtained as stated in the following corollary:

Corollary 2.9. *The weighted row sums and the alternating weighted row sums for an exponential almost-Riordan array* $D = (d_{n,k})_{n,k\geq 0} = [a(x)|g(x), f(x)]$ *are given as follows:*

$$\sum_{k=0}^{n} (k+1) d_{n,k} = a_n + n![x^n] \int_0^x (f(t)+2)g(t)e^{f(t)} dt \tag{2.9}$$

and

$$\sum_{k=0}^{n} (-1)^k (k+1) d_{n,k} = a_n + n![x^n] \int_0^x (f(t)-2)g(t)e^{-f(t)} dt. \tag{2.10}$$

Example 2.10. Let's consider the exponential almost-Riordan array given by $D = \left[\frac{1}{1-x}\middle| e^{x^2}, 2x\right]$. The matrix D is obtained as

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 6 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & \dots \\ 24 & 0 & 12 & 0 & 8 & 0 & 0 & 0 & \dots \\ 120 & 12 & 0 & 48 & 0 & 16 & 0 & 0 & \dots \\ 720 & 0 & 120 & 0 & 160 & 0 & 32 & 0 & \dots \\ 5040 & 120 & 0 & 720 & 0 & 480 & 0 & 64 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let's calculate the weighted row sums of the matrix D. Using (2.9), we have

$$\sum_{k=0}^{n}(k+1)d_{n,k} = a_n + n![x^n]\int_0^x (2t+2)e^{t^2+2t}dt.$$

Considering the formal power series of e^{x^2+2x}, we find the weighted row sums of the matrix D as follows:

$$\sum_{k=0}^{n}(k+1)d_{n,k} = \begin{cases} 1, & n=0 \\ n! + d_n, & n \geq 1 \end{cases}$$

where $\{d_n\}$ is the sequence A000898 in OEIS [24].

Now, we find the alternating weighted row sums of the matrix D. If we use (2.10), we get

$$\sum_{k=0}^{n}(-1)^k(k+1)d_{n,k} = a_n + n![x^n]\int_0^x (2t-2)e^{t^2-2t}dt.$$

From the formal power series of e^{x^2-2x}, we obtain the alternating weighted row sums of the matrix D as follows:

$$\sum_{k=0}^{n}(-1)^k(k+1)d_{n,k} = \begin{cases} 1, & n=0 \\ n! + (-1)^n d_n, & n \geq 1 \end{cases}$$

where $\{d_n\}$ is the sequence A000898 in OEIS [24].

The multiplication operation of two exponential almost-Riordan arrays is defined as follows:

$$\left[h_0 a(x) + \int_0^x g(t)h'(f(t))dt \middle| g(x)h(f(x)), l(f(x))\right]. \tag{2.11}$$

Example 2.11. Let D_1 and D_2 be the exponential almost-Riordan arrays as follows:

$$D_1 = \left[\frac{1}{1-x}\middle| 2-e^x, x\right] \quad \textit{and} \quad D_2 = \left[1 \middle| 1-x, x\left(1-\frac{x}{2}\right)\right].$$

Then, we have

$$D_1D_2 = \left[\frac{1}{1-x}\middle|(2-e^x)(1-x), x\left(1-\frac{x}{2}\right)\right].$$

Namely, the matrix

$$D_1D_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 6 & 1 & -5 & 1 & 0 & 0 & 0 & 0 & \dots \\ 24 & 2 & 9 & -9 & 1 & 0 & 0 & 0 & \dots \\ 120 & 3 & 2 & 33 & -14 & 1 & 0 & 0 & \dots \\ 720 & 4 & -5 & -40 & 85 & -20 & 1 & 0 & \dots \\ 5040 & 5 & -21 & -30 & -245 & 180 & -27 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is equal to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 6 & -1 & -2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 24 & -1 & -3 & -3 & 1 & 0 & 0 & 0 & \dots \\ 120 & -1 & -4 & -6 & -4 & 1 & 0 & 0 & \dots \\ 720 & -1 & -5 & -10 & -10 & -5 & 1 & 0 & \dots \\ 5040 & -1 & -6 & -15 & -20 & -15 & -6 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & -6 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 15 & -10 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -15 & 45 & -15 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -105 & 105 & -21 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 2.12. *The set of the exponential almost-Riordan arrays is a group with the multiplication defined in (2.11), and denoted by $\mathcal{R}_e^a$.*

Proof. The set $\mathcal{R}_e^a$ is closed and associative for multiplication in (2.11). The identity element of this set is $[1|1,x]$. Additionally, the inverse of the exponential almost-Riordan arrays is defined as follows:

$$[a(x)|g(x),f(x)]^{-1}=\left[\frac{1}{a_0}\left(1-\int_0^x\frac{a'(\overline{f}(t))}{g(\overline{f}(t))}dt\right)\middle|\frac{1}{g(\overline{f}(x))},\overline{f}(x)\right] \tag{2.12}$$

where $\overline{f}(x)$ is the compositional inverse of $f(x)$. □

Example 2.13. Let's consider an exponential almost-Riordan array $D=\left[1\middle|1-x,x\left(1-\frac{x}{2}\right)\right]$. The matrix D is given as

$$D=\begin{pmatrix}1&0&0&0&0&0&0&0&\cdots\\0&1&0&0&0&0&0&0&\cdots\\0&-1&1&0&0&0&0&0&\cdots\\0&0&-3&1&0&0&0&0&\cdots\\0&0&3&-6&1&0&0&0&\cdots\\0&0&0&15&-10&1&0&0&\cdots\\0&0&0&-15&45&-15&1&0&\cdots\\0&0&0&0&-105&105&-21&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Using (2.12), we have $D^{-1}=\left[1\middle|\frac{1}{\sqrt{1-2x}},1-\sqrt{1-2x}\right]$. The matrix D^{-1} is as follows:

$$D^{-1}=\begin{pmatrix}1&0&0&0&0&0&0&0&\cdots\\0&1&0&0&0&0&0&0&\cdots\\0&1&1&0&0&0&0&0&\cdots\\0&3&3&1&0&0&0&0&\cdots\\0&15&15&6&1&0&0&0&\cdots\\0&105&105&45&10&1&0&0&\cdots\\0&945&945&420&105&15&1&0&\cdots\\0&10395&10395&4725&1260&210&21&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

3. Subgroups and isomorphisms

In this section, we consider the subgroups of the exponential almost-Riordan group $\mathcal{R}_e^a$ and provide the isomorphisms between these subgroups.

Proposition 3.1. *The set of the elements in the form $[a(x)|g(x),x]$ is a subgroup of $\mathcal{R}_e^a$.*

Proof. Let $[a(x)|g(x),x]$ and $[b(x)|h(x),x]$ be the elements in the set. Using (2.11) and (2.12), we have

$$[a(x)|g(x),x][b(x)|h(x),x]=\left[b_0a(x)+\int_0^x g(t)b'(t)dt\middle|g(x)h(x),x\right]$$

and

$$[a(x)|g(x),x]^{-1}=\left[\frac{1}{a_0}\left(1-\int_0^x\frac{a'(t)}{g(t)}dt\right)\middle|\frac{1}{g(x)},x\right].$$

Therefore, the set of the exponential almost-Riordan arrays in the form $[a(x)|g(x),x]$ is a subgroup of $\mathcal{R}_e^a$. □

For example, an exponential almost-Riordan array belonging to the subgroup, denoted as $\left[e^{\frac{x}{1-x}}\middle|\frac{1}{1-x},x\right]$, is given as follows:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\\
3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots\\
13 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots\\
73 & 6 & 6 & 3 & 1 & 0 & 0 & 0 & \cdots\\
501 & 24 & 24 & 12 & 4 & 1 & 0 & 0 & \cdots\\
4051 & 120 & 120 & 60 & 20 & 5 & 1 & 0 & \cdots\\
37633 & 720 & 720 & 360 & 120 & 30 & 6 & 1 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

where the column 0th consists of the elements of the sequence A000262 in OEIS [24].

Proposition 3.2. *The set of the elements in the form $[a(x)|1,f(x)]$ is a subgroup of $\mathcal{R}_e^a$.*

Proof. Let $[a(x)|1,f(x)]$ and $[b(x)|1,l(x)]$ be the elements in the set. Using (2.11) and (2.12), we obtain

$$[a(x)|1,f(x)][b(x)|1,l(x)]=\left[b_0a(x)+\int_0^x b'(f(t))dt\middle|1,l(f(x))\right].$$

and

$$[a(x)|1,f(x)]^{-1}=\left[\frac{1}{a_0}\left(1-\int_0^x a'(\overline{f}(t))dt\right)\middle|1,\overline{f}(x)\right].$$

Namely, the set constitutes a subgroup of the exponential almost-Riordan group. □

For example, an exponential almost-Riordan array belonging to the subgroup, denoted as $\left[e^x \middle| 1, ln(\frac{1}{1-x})\right]$, is provided as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 2 & 3 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 6 & 11 & 6 & 1 & 0 & 0 & \dots \\ 1 & 0 & 24 & 50 & 35 & 10 & 1 & 0 & \dots \\ 1 & 0 & 120 & 274 & 225 & 85 & 15 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proposition 3.3. *The set of the elements in the form* $[a(x)|f'(x), f(x)]$ *is a subgroup of* $\mathcal{R}_e^a$.

Proof. Let $[a(x)|f'(x), f(x)]$ and $[b(x)|l'(x), l(x)]$ be the elements of the set. Using (2.11) and (2.12), we get

$$[a(x)|f'(x), f(x)][b(x)|l'(x), l(x)] = \left[b(f(x)) + b_0a(x) - b_0 \middle| f'(x)l'(f(x)), l(f(x))\right]$$

and

$$\begin{aligned}&[a(x)|f'(x), f(x)]^{-1}\\ &\quad = \left[\frac{1}{a_0}\left(1 - \int_0^x \frac{a'(\overline{f}(t))}{f'(\overline{f}(t))}dt\right) \middle| \frac{1}{f'(\overline{f}(x))}, \overline{f}(x)\right].\end{aligned}$$

Therefore, the set is a subgroup of the exponential almost-Riordan group $\mathcal{R}_e^a$. □

For example, the exponential almost-Riordan array belonging to the subgroup, denoted as $\left[e^{e^x-1} \middle| e^x, e^x - 1\right]$, is given as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 5 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & \dots \\ 15 & 1 & 7 & 6 & 1 & 0 & 0 & 0 & \dots \\ 52 & 1 & 15 & 25 & 10 & 1 & 0 & 0 & \dots \\ 203 & 1 & 31 & 90 & 65 & 15 & 1 & 0 & \dots \\ 877 & 1 & 63 & 301 & 350 & 140 & 21 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

where the column 0th is composed of the sequence A000110 in OEIS [24].

Theorem 3.4. *The set of the elements in the form* $[1|f'(x), f(x)]$ *is a subgroup of* $\mathcal{R}_e^a$*, and it's isomorphic to the associated subgroup of the exponential Riordan group.*

Proof. It follows from Proposition 3.3 that the set of the exponential almost-Riordan arrays in the form $[1|f'(x), f(x)]$ constitutes a subgroup of $\mathcal{R}_e^a$. We consider the map φ such that

$$\varphi\left([1|f'(x), f(x)]\right) = [1, f(x)].$$

Let's show that φ is a homomorphism. We get

$$\begin{aligned}\varphi\left([1|f'(x), f(x)][1|l'(x), l(x)]\right) =&\varphi([1|f'(x)l'(f(x)), l(f(x))])\\ =&[1, l(f(x))]\\ =&[1, f(x)][1, l(x)]\\ =&\varphi([1|f'(x), f(x)])\varphi([1|l'(x), l(x)]).\end{aligned}$$

Because φ is one to one and onto, φ is an isomorphism. $\square$

Theorem 3.5. *The set of the elements in the form $[1|1, f(x)]$ is a subgroup of $\mathcal{R}_e^a$ and it's isomorphic to the subgroup in the form $[1|f'(x), f(x)]$ of $\mathcal{R}_e^a$.*

Proof. It follows from the Proposition 3.2 that the set of the exponential almost-Riordan arrays in the form $[1|1, f(x)]$ constitutes a subgroup. Let's consider the map φ such that

$$\varphi([1|f'(x), f(x)]) = [1|1, f(x)].$$

Let's show that φ is a homomorphism. We have

$$\begin{aligned}\varphi\left([1|f'(x), f(x)][1|l'(x), l(x)]\right) =&\varphi([1|f'(x)l'(f(x)), l(f(x))])\\ =&[1|1, l(f(x))]\\ =&[1|1, f(x)][1|1, l(x)]\\ =&\varphi([1|f'(x), f(x)])\varphi([1|l'(x), l(x)]).\end{aligned}$$

For φ is one to one and onto, φ is an isomorphism. $\square$

Proposition 3.6. $D = [1|f'(x), f(x)]$ *is an involution if and only if* $f(x) = \overline{f}(x)$.

Proof. Let D be an involution. Then

$$D^2 = [1|f'(x), f(x)][1|f'(x), f(x)] = [1|1, x].$$

Using (2.11), we obtain

$$[1|f'(x)f'(f(x)), f(f(x))] = [1|1, x].$$

Namely, $f(x) = \overline{f}(x)$. Conversely, let's take $f(x) = \overline{f}(x)$. Then, we find

$$D^2 = [1|f'(x)f'(f(x)), f(f(x))] = [1|1, x].$$

$\square$

Proposition 3.7. *The set of the elements in the form* $\left[a(x)\middle|\frac{xf'(x)}{f(x)}, f(x)\right]$ *is a subgroup of* $\mathcal{R}_e^a$.

Proof. Let's take elements $\left[a(x)\middle|\frac{xf'(x)}{f(x)}, f(x)\right]$ and $\left[b(x)\middle|\frac{xl'(x)}{l(x)}, l(x)\right]$ from the set. Using (2.11) and (2.12), we have

$$\begin{aligned}&\left[a(x)\middle|\frac{xf'(x)}{f(x)}, f(x)\right]\left[b(x)\middle|\frac{xl'(x)}{l(x)}, l(x)\right]\\&\quad=\left[b_0a(x)+\int_0^x\frac{tf'(t)b'(f(t))}{f(t)}dt\middle|\frac{xf'(x)l'(f(x))}{l(f(x))}, l(f(x))\right]\end{aligned}$$

and

$$\left[a(x)\middle|\frac{xf'(x)}{f(x)}, f(x)\right]^{-1}=\left[\frac{1}{a_0}\left(1-\int_0^x\frac{ta'(\overline{f}(t))}{\overline{f}(t)f'(\overline{f}(t))}dt\right)\middle|\frac{x(\overline{f}(x))'}{\overline{f}(x)}, \overline{f}(x)\right].$$

Therefore, the set is a subgroup of $\mathcal{R}_e^a$. □

For example, an exponential almost-Riordan array belonging to the subgroup, denoted as $\left[e^{-x}\middle|1+\frac{x}{2+x}, x(1+\frac{x}{2})\right]$, is given as follows:

$$\begin{pmatrix}1&0&0&0&0&0&0&0&\cdots\\-1&1&0&0&0&0&0&0&\cdots\\1&\frac{1}{2}&1&0&0&0&0&0&\cdots\\-1&-\frac{1}{2}&2&1&0&0&0&0&\cdots\\1&\frac{3}{4}&0&\frac{9}{2}&1&0&0&0&\cdots\\-1&-\frac{3}{2}&0&6&8&1&0&0&\cdots\\1&\frac{15}{4}&0&0&25&\frac{25}{2}&1&0&\cdots\\-1&-\frac{45}{4}&0&0&30&\frac{135}{2}&18&1&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Theorem 3.8. *The set of the elements in the form* $\left[1\middle|\frac{xf'(x)}{f(x)}, f(x)\right]$ *is a subgroup of* $\mathcal{R}_e^a$ *and it's isomorphic to the subgroup in the form* $[1|f'(x), f(x)]$ *of* $\mathcal{R}_e^a$.

Proof. From Proposition 3.7, the set of $\left[1\middle|\frac{xf'(x)}{f(x)}, f(x)\right]$ is a subgroup. Let's consider the map φ such that

$$\varphi\left(\left[1\middle|\frac{xf'(x)}{f(x)}, f(x)\right]\right)=[1|f'(x), f(x)].$$

φ is one to one and onto. Then, we obtain

$$\begin{aligned}
&\varphi\left(\left[1\middle|\frac{xf'(x)}{f(x)},f(x)\right]\left[1\middle|\frac{xl'(x)}{l(x)},l(x)\right]\right)\\
&\quad=\varphi\left(\left[1\middle|\frac{xf'(x)l'(f(x))}{l(f(x))},l(f(x))\right]\right)\\
&\quad=[1|f'(x)l'(f(x)),l(f(x))]\\
&\quad=[1|f'(x),f(x)][1|l'(x),l(x)]\\
&\quad=\varphi\left(\left[1\middle|\frac{xf'(x)}{f(x)},f(x)\right]\right)\varphi\left(\left[1\middle|\frac{xl'(x)}{l(x)},l(x)\right]\right).
\end{aligned}$$

Consequently, φ is an isomorphism. □

Proposition 3.9. *The set of the elements in the forms* $[a(x)|g(x),xg(x)]$ *or* $[a(x)|\frac{f(x)}{x},f(x)]$ *is a subgroup of* $\mathcal{R}_e^a$.

For example, an exponential almost-Riordan array belonging to the subgoup, denoted as $\left[e^x\middle|\frac{1}{1-x},\frac{x}{1-x}\right]$, is given as follows:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots\\
1 & 2 & 4 & 1 & 0 & 0 & 0 & 0 & \dots\\
1 & 6 & 18 & 9 & 1 & 0 & 0 & 0 & \dots\\
1 & 24 & 96 & 72 & 16 & 1 & 0 & 0 & \dots\\
1 & 120 & 600 & 600 & 200 & 25 & 1 & 0 & \dots\\
1 & 720 & 4320 & 5400 & 2400 & 450 & 36 & 1 & \dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Theorem 3.10. *The set of the elements in the form* $\left[1\middle|\frac{f(x)}{x},f(x)\right]$ *is a subgroup of* $\mathcal{R}_e^a$ *and it's isomorphic to the subgroup of the form* $[1|1,f(x)]$ *of* $\mathcal{R}_e^a$.

Proof. Considering Proposition 3.9, the set of exponential almost-Riordan arrays in the form $\left[1\middle|\frac{f(x)}{x},f(x)\right]$ is a subgroup of $\mathcal{R}_e^a$. Let's take the map φ such that

$$\varphi([1|1,f(x)])=\left[1\middle|\frac{f(x)}{x},f(x)\right].$$

It's clear that φ is one to one and onto. Also, we have

$$\begin{aligned}\varphi\left([1|1,f(x)][1|1,l(x)]\right) =&\varphi([1|1,l(f(x))])\\ =&\left[1\middle|\frac{l(f(x))}{x},l(f(x))\right]\\ =&\left[1\middle|\frac{f(x)}{x},f(x)\right]\left[1\middle|\frac{l(x)}{x},l(x)\right]\\ =&\varphi([1|1,f(x)])\varphi([1|1,l(x)]).\end{aligned}$$

□

Now, we give the definitions of the stochastic and stabilizer subgroups of $\mathcal{R}_e^a$.

Proposition 3.11. *The following subset of the group $\mathcal{R}_e^a$ is a subgroup, known as the stochastic subgroup.*

$$\mathfrak{D}=\left\{\left[a(x)\middle|\frac{e^x-a'(x)}{e^{f(x)}},f(x)\right]:a_0=1,a_1\neq 1,f_0=0,f_1\neq 0\right\}.$$

Proof. Firstly, we show that the row sums equal to 1. Using (2.5), we get

$$\left[a(x)\middle|\frac{e^x-a'(x)}{e^{f(x)}},f(x)\right]e^x=a(x)+\int_0^x(e^t-a'(t))dt=e^x.$$

Let $\left[a(x)|\frac{e^x-a'(x)}{e^{f(x)}},f(x)\right]$ and $\left[b(x)|\frac{e^x-b'(x)}{e^{l(x)}},l(x)\right]$ be two elements of the set $\mathfrak{D}$. Using the multiplication defined in (2.11), we have

$$\left[b_0a(x)+\int_0^x\frac{e^t-a'(t)}{e^{f(t)}}b'(f(t))dt\middle|\frac{e^x-a'(x)}{e^{f(x)}}\frac{e^{f(x)}-b'(f(x))}{e^{l(f(x))}},l(f(x))\right].$$

If we consider (2.12), we find

$$\begin{aligned}&\left[a(x)\middle|\frac{e^x-a'(x)}{e^{f(x)}},f(x)\right]^{-1}\\ &\quad=\left[1-\int_0^x\frac{a'(\overline{f}(t))}{e^{\overline{f}(t)}-a'(\overline{f}(t))}e^tdt\middle|\frac{e^x}{e^{\overline{f}(x)}-a'(\overline{f}(x))},\overline{f}(x)\right].\end{aligned}$$

Thus, the set $\mathfrak{D}$ is a subgroup of $\mathcal{R}_e^a$. □

For example, an exponential almost-Riordan array belonging to the stochastic subgroup, denoted as $[e^{e^x-1-x}|1-e^{e^x-1-x}(1-e^{-x}),x]$, is given as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 4 & -4 & 3 & -3 & 1 & 0 & 0 & 0 & \dots \\ 11 & 3 & -16 & 6 & -4 & 1 & 0 & 0 & \dots \\ 41 & -21 & 15 & -40 & 10 & -5 & 1 & 0 & \dots \\ 162 & -10 & -126 & 45 & -80 & 15 & -6 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the elements of the column 0th is the elements of the sequence A000296 in OEIS [24]. Also, we can see that the row sums of this matrix equal to 1.

Theorem 3.12. *The set of the elements in the form* $[1\big|e^{x-f(x)}, f(x)]$ *is a subgroup of* $\mathcal{R}_e^a$. *The map* φ, *defined such that*

$$\varphi\left([1|e^{x-f(x)}, f(x)]\right) = [e^{x-f(x)}, f(x)],$$

is an isomorphism from the subgroup of the group $\mathcal{R}_e^a$ *to exponential Riordan group in the form* $[e^{x-f(x)}, f(x)]$.

Proof. It follows from Proposition 3.11 that, the set of the exponential almost-Riordan arrays in the form $[1|e^{x-f(x)}, f(x)]$ constitutes a subgroup. Let's show that φ is a homomorphism. We have

$$\begin{aligned} \varphi\left([1|e^{x-f(x)}, f(x)][1|e^{x-l(x)}, l(x)]\right) =& \varphi([1|e^{x-l(f(x))}, l(f(x))]) \\ =& [e^{x-l(f(x))}, l(f(x))] \\ =& [e^{x-f(x)}, f(x)][e^{x-l(x)}, l(x)] \\ =& \varphi([1|e^{x-f(x)}, f(x)])\varphi([1|e^{x-l(x)}, l(x)]). \end{aligned}$$

Also, φ is one to one and onto. □

Theorem 3.13. *The sets of the elements in the forms* $[1|f'(x), f(x)]$ *and* $[1|e^{x-f(x)}, f(x)]$ *are isomorphic subgroups.*

Proof. Let's consider the map φ such that

$$\varphi([1|f'(x), f(x)]) = [1|e^{x-f(x)}, f(x)].$$

Clearly, φ is a homomorphism, one to one and onto. □

Proposition 3.14. $D = [1|e^{x-f(x)}, f(x)]$ *is an involution if and only if* $f(x) = \overline{f}(x)$.

Proof. Let D be an involution.Then

$$D^2 = [1|e^{x-f(x)}, f(x)][1|e^{x-f(x)}, f(x)] = [1|1, x].$$

Using (2.11), we get

$$[1|e^{x-f(f(x))}, f(f(x))] = [1|1, x].$$

Thus, we have $f(x) = \overline{f}(x)$. Conversely, let's take $f(x) = \overline{f}(x)$, we find

$$D^2 = [1|e^{x-f(f(x))}, f(f(x))] = [1|1, x].$$

□

Let $h(x)$ be the exponential generating function of the sequence $\{h_n\}_{n\geq 0}$. A column vector whose elements are determined by the generating function $h(x)$ must satisfy the following condition in order to an exponential almost-Riordan array to stabilize it.

$$[a(x)|g(x), f(x)]h(x) = h(x). \tag{3.1}$$

Now, we define the stabilizer subgroup of the exponential almost-Riordan group $\mathcal{R}_e^a$.

Proposition 3.15. *The following subset of the group $\mathcal{R}_e^a$ is a subgroup, known as the stabilizer subgroup.*

$$\mathfrak{B} = \left\{ \left[a(x) \middle| \frac{h'(x) - h_0 a'(x)}{h'(f(x))}, f(x) \right] : a_0 = 1, a_1 h_0 \neq h_1, h_1 \neq 0, f_0 = 0, f_1 \neq 0 \right\}.$$

Proof. Using (2.5), we get

$$\left[a(x) \middle| \frac{h'(x) - h_0 a'(x)}{h'(f(x))}, f(x) \right] h(x) = h_0 a(x) + \int_0^x (h'(t) - h_0 a'(t))dt = h(x).$$

Let $\left[a(x) \middle| \frac{h'(x)-h_0 a'(x)}{h'(f(x))}, f(x) \right]$ and $\left[b(x) \middle| \frac{h'(x)-h_0 b'(x)}{h'(l(x))}, l(x) \right]$ be two elements of the set $\mathfrak{B}$. Using (2.11), we get

$$\left[a(x) + \int_0^x \frac{h'(t) - h_0 a'(t)}{h'(f(t))} b'(f(t))dt \middle| \frac{h'(x) - h_0 a'(x)}{h'(f(x))} \frac{h'(f(x)) - h_0 b'(f(x))}{h'(l(f(x)))}, l(f(x)) \right].$$

From (2.12), we obtain the inverse as

$$\begin{aligned} &\left[a(x) \middle| \frac{h'(x) - h_0 a'(x)}{h'(f(x))}, f(x) \right]^{-1} \\ &\quad = \left[1 - \int_0^x \frac{a'(\overline{f}(t))h'(t)}{h'(\overline{f}(t)) - h_0 a'(\overline{f}(t))} dt \middle| \frac{h'(x)}{h'(\overline{f}(x)) - h_0 a'(\overline{f}(x))}, \overline{f}(x) \right]. \end{aligned}$$

Thus, the set $\mathfrak{B}$ is a subgroup of $\mathcal{R}_e^a$. □

For example, let's take $h(x) = 2e^x + 1$ and $\left[e^x \middle| -\frac{1}{2}e^{x-e^x+1}, e^x - 1\right]$. Hence, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ 1 & \frac{1}{2} & 1 & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 & \dots \\ 1 & -1 & \frac{9}{2} & -\frac{1}{2} & -3 & -\frac{1}{2} & 0 & 0 & \dots \\ 1 & -\frac{9}{2} & \frac{9}{2} & \frac{25}{2} & -\frac{15}{2} & -5 & -\frac{1}{2} & 0 & \dots \\ 1 & -\frac{9}{2} & -25 & 52 & 10 & -25 & -\frac{15}{2} & -\frac{1}{2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It's noted that the multiplication of this matrix and $(3, 2, 2, 2, \dots)^T$ is $(3, 2, 2, 2, \dots)^T$.

4. Production matrix

In this section, we present the production matrix of the exponential almost-Riordan arrays.

Proposition 4.1. *Let $D = (d_{n,k})_{n,k\geq 0} = [a(x)|g(x), f(x)]$ be an exponential almost-Riordan array and $P = (p_{n,k})_{n,k\geq 0}$ be the production matrix of the matrix D. Then, we have*

$$d_{n+1,0} = p_{0,0}d_{n,0} + p_{1,0}d_{n,1} + p_{2,0}d_{n,2} + \dots \tag{4.1}$$

$$d_{n+1,1} = p_{0,1}d_{n,0} + p_{1,1}d_{n,1} + p_{2,1}d_{n,2} + \dots \tag{4.2}$$

$$d_{n+1,k+1} = p_{k,k+1}d_{n,k} + p_{k+1,k+1}d_{n,k+1} + p_{k+2,k+1}d_{n,k+2} + \dots. \tag{4.3}$$

Proposition 4.2. *Let the matrix P be the production matrix of $D = [a(x)|g(x), f(x)]$. The exponential generating function of the kth column of the matrix P is given as follows:*

$$P_0(x) = \frac{a_1}{a_0} + \int_0^x \frac{1}{g(\overline{f}(t))} \left(a''(\overline{f}(t)) - \frac{a_1}{a_0} a'(\overline{f}(t)) \right) dt, \tag{4.4}$$

$$P_1(x) = \frac{g_0}{a_0} + \int_0^x \frac{1}{g(\overline{f}(t))} \left(g'(\overline{f}(t)) - \frac{g_0}{a_0} a'(\overline{f}(t)) \right) dt, \tag{4.5}$$

and

$$P_{k+1}(x) = \int_0^x \frac{t^{k-1}}{(k-1)!} \left(\frac{t}{k} \frac{g'(\overline{f}(t))}{g(\overline{f}(t))} + f'(\overline{f}(t)) \right) dt \tag{4.6}$$

for $k \geq 1$.

Proof. Considering (4.1), (2.1) and (2.2), we have

$$a'(x) = p_{0,0}a(x) + p_{1,0} \int_0^x g(t)dt + p_{2,0} \int_0^x g(t)f(t)dt + \dots.$$

Then, we get

$$a''(x) = p_{0,0}a'(x) + p_{1,0}g(x) + p_{2,0}g(x)f(x) + \dots .$$

Hence, we have

$$\frac{a''(x) - p_{0,0}a'(x)}{g(x)} = P_0'(f(x)).$$

Considering (1.8), we find $p_{0,0} = \frac{a_1}{a_0}$. From the previous equation, the equation (4.4) is obtained.

Using (4.2), (2.1) and (2.2), we obtain

$$g(x) = p_{0,1}a(x) + p_{1,1}\int_0^x g(t)dt + p_{2,1}\int_0^x g(t)f(t)dt + \dots .$$

Hence,

$$g'(x) = p_{0,1}a'(x) + p_{1,1}g(x) + p_{2,1}g(x)f(x) + \dots .$$

Then, we have

$$\frac{g'(x) - p_{0,1}a'(x)}{g(x)} = P_1'(f(x)).$$

Considering (1.8), we find $p_{0,1} = \frac{g_0}{a_0}$. Therefore, the equation (4.5) is found.

Considering (4.3) and (2.2), we have

$$g(x)\frac{f^k(x)}{k!} = p_{k,k+1}\left(\int_0^x g(t)\frac{f^{k-1}(t)}{(k-1)!}dt\right) + p_{k+1,k+1}\left(\int_0^x g(t)\frac{f^k(t)}{k!}dt\right)$$
$$+ p_{k+2,k+1}\left(\int_0^x g(t)\frac{f^{k+1}(t)}{(k+1)!}dt\right) + \dots .$$

Then, we get

$$\frac{g'(x)f^k(x)}{k!} + \frac{g(x)f'(x)f^{k-1}(x)}{(k-1)!}$$
$$= g(x)\left(p_{k,k+1}\frac{f^{k-1}(x)}{(k-1)!} + p_{k+1,k+1}\frac{f^k(x)}{k!} + p_{k+2,k+1}\frac{f^{k+1}(x)}{(k+1)!} + \dots\right).$$

Thus, we have

$$\frac{f^{k-1}(x)}{(k-1)!}\left(\frac{g'(x)}{g(x)}\frac{f(x)}{k} + f'(x)\right) = P_{k+1}'(f(x)).$$

Hence, the equation (4.6) is clear. □

Corollary 4.3. *Let $P = (p_{n,k})_{n,k\geq 0}$ be the production matrix of the exponential almost-Riordan array $[a(x)|g(x), f(x)]$. For $n \geq 1$,*

$$p_{n,0} = c_{n-1} - \frac{a_1}{a_0}r_{n-1}, \tag{4.7}$$

$$p_{n,1} = z_{n-1} - \frac{g_0}{a_0}r_{n-1}, \tag{4.8}$$

and for $k \geq 2$,

$$p_{n,k} = \binom{n-1}{k-1} z_{n-k} + \binom{n-1}{k-2} s_{n-k+1} \tag{4.9}$$

where r_n, c_n, z_n and s_n are the nth elements of the following exponential generating functions, respectively.

$$r(x) = \sum_{i=0}^{\infty} r_i \frac{x^i}{i!} = \frac{a'(\overline{f}(x))}{g(\overline{f}(x))}, \tag{4.10}$$

$$c(x) = \sum_{i=0}^{\infty} c_i \frac{x^i}{i!} = \frac{a''(\overline{f}(x))}{g(\overline{f}(x))}, \tag{4.11}$$

$$Z(x) = \sum_{i=0}^{\infty} z_i \frac{x^i}{i!} = \frac{g'(\overline{f}(x))}{g(\overline{f}(x))}, \tag{4.12}$$

$$S(x) = \sum_{i=0}^{\infty} s_i \frac{x^i}{i!} = f'(\overline{f}(x)). \tag{4.13}$$

Proof. Considering (1.4) and (4.4), we get

$$p_{n,0} = n![x^n] \int_0^x \left(c(t) - \frac{a_1}{a_0} r(t) \right) dt.$$

Using (1.3), we have

$$p_{n,0} = (n-1)![x^{n-1}] \left(c(x) - \frac{a_1}{a_0} r(x) \right).$$

From (1.4), the equation (4.7) is found. By the similar way, the equations (4.8) and (4.9) are obtained. □

According to Corollary 4.3, the production matrix of the exponential almost Riordan array $[a(x)|g(x), f(x)]$ is given as follows:

$$P = \begin{pmatrix} \frac{a_1}{a_0} & \frac{g_0}{a_0} & 0 & 0 & 0 & 0 & \cdots \\ c_0 - \frac{a_1}{a_0} r_0 & z_0 - \frac{g_0}{a_0} r_0 & s_0 & 0 & 0 & 0 & \cdots \\ c_1 - \frac{a_1}{a_0} r_1 & z_1 - \frac{g_0}{a_0} r_1 & z_0 + s_1 & s_0 & 0 & 0 & \cdots \\ c_2 - \frac{a_1}{a_0} r_2 & z_2 - \frac{g_0}{a_0} r_2 & 2z_1 + s_2 & z_0 + 2s_1 & s_0 & 0 & \cdots \\ c_3 - \frac{a_1}{a_0} r_3 & z_3 - \frac{g_0}{a_0} r_3 & 3z_2 + s_3 & 3z_1 + 3s_2 & z_0 + 3s_1 & s_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.14}$$

Corollary 4.4. *Let $D = (d_{n,k})_{n,k\geq 0} = [a(x)|g(x), f(x)]$ be an exponential almost-Riordan array. Then,*

$$d_{n+1,0} = \frac{a_1}{a_0} a_n + \sum_{i=1}^{n} \left(c_{i-1} - \frac{a_1}{a_0} r_{i-1} \right) d_{n,i}, \tag{4.15}$$

$$d_{n+1,1} = \frac{g_0}{a_0} a_n + \sum_{i=1}^{n} \left(z_{i-1} - \frac{g_0}{a_0} r_{i-1} \right) d_{n,i}, \tag{4.16}$$

and

$$d_{n+1,k} = \sum_{i=k-1}^{n} \left(\binom{i-1}{k-1} z_{i-k} + \binom{i-1}{k-2} s_{i-k+1} \right) d_{n,i} \tag{4.17}$$

for $k \geq 2$.

Proof. Considering the Proposition 4.1 and Corollary 4.3, the result is clear. □

Example 4.5. Let's consider an exponential almost-Riordan array $D = \left[\frac{1}{1-x} \middle| e^{-x}, x(1+\frac{x}{2}) \right]$, the matrix D is given as

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 6 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 24 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 120 & 1 & 2 & -3 & 2 & 1 & 0 & 0 & \dots \\ 720 & -1 & -5 & 5 & -5 & 5 & 1 & 0 & \dots \\ 5040 & 1 & 9 & 0 & -5 & 0 & 9 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, we find the production matrix of the matrix D. By using (4.10), we have

$$r(x) = \frac{e^{\sqrt{2x+1}-1}}{(2-\sqrt{2x+1})^2}$$

which is exponential generating function of the sequence

$$1, 3, 8, 25, 87, 386, 1663, 11313, 39560 \dots.$$

If we use (4.11), we get

$$c(x) = \frac{2e^{\sqrt{2x+1}-1}}{(2-\sqrt{2x+1})^3}$$

which is exponential generating function of the sequence

$$2, 8, 30, 122, 548, 2802, 15638, 100760 \dots.$$

Similarly, we obtain $Z(x) = -1$. From (4.13), we find $S(x) = \sqrt{2x+1}$ which is the exponential generating function of the following sequence

$$1, 1, -1, 3, -15, 105, -945, 10395, \ldots.$$

Then, the production matrix is obtained as follows:

$$P_D = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 5 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 22 & -8 & -1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 97 & -25 & 3 & -3 & 2 & 1 & 0 & 0 & \ldots \\ 461 & -87 & -15 & 12 & -6 & 3 & 1 & 0 & \ldots \\ 2416 & -386 & 105 & -75 & 30 & -10 & 4 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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