Adv. Appl. Clifford Algebras (2019) 29:45

0188-7009/030001-22
published online April 30, 2019
https://doi.org/10.1007/s00006-019-0957-0

Advances in Applied Clifford Algebras

Fractals and Chaos Related to Ising-Onsager-Zhang Lattices. Quaternary Approach vs. Ternary Approach

J. Ławrynowicz, O. Suzuki, A. Niemczynowicz* and
M. Nowak-Kępczyk

In memory of Professor W. A. Rodrigues Jr.

Abstract. Continuing our idea (*Int. J. Geom. Methods Mod. Phys.*) of extending some aspects of Onsager (Phys Rev 65:117–149, 1944) crystal statistics to three dimensions taking into account binary and ternary crystal structures in connection with fractals and chaos related to Ising–Onsager–Zhang lattices, we (1) use the Galois extension structure of the nonion algebra, (2) analyze ternary and binary structures of $\mathfrak{su}(3)$ as well as (3) analyze the identification of the construction of the collection of two ternaries with the collection of three binaries, (4) observe that the approach is applicable to quarks and elementary particles including introduction of colors and, finally, (5) suggest an analysis of three quaternaries vs. four ternaries, involving duodevicenion and/or quindenion algebra.

Mathematics Subject Classification. Primary 82C44; Secondary 82D25, 81R05, 15A66.

Keywords. Crystal lattice, Ising lattice, Octonions, Quaternions, Sedenions, Nonions, Binary structures, Ternary alloy, Composition algebra.

1. Introduction and Motivation

In order to develop further our previous idea [1]; Ławrynowicz et al. [16] of identification of two ternaries and three binaries with the help of non-commutative Galois extension applied to fractals and chaos related to Ising

This article is part of the Topical Collection on Homage to Prof. W.A. Rodrigues Jr. edited by Jayme Vaz Jr..

*Corresponding author.

[2]–Onsager [3]–Zhang [4] lattices, in this considerations we start with involving the Galois extension structure of the nonion algebra and analyzing the ternary and binary structures of $\mathfrak{su}(3)$. Our approach is also motivated by unclear statements in the Zhang [4] paper and further treatise [5,6]. The main results of this paper are summarized as Theorems 1–3. Next we analyze the identification of construction of the collection of two ternaries with the collection of three binaries, and observe that the approach is applicable to quarks and elementary particles including introduction of colours. Our results announced above provide an example of a mathematical procedure applicable parallelly in the physics of condensed matter and physics of elementary particles. In this place it seems naturally to quote the suprising results due to [58–65] dealing with discovery of topological phase transition and topological phases matter [57].

Finally we suggest an analysis of three quaternaries vs. four ternaries, involving duodevicenion and/or quindenion algebra. More precisely, we suggest the construction of the collection of four quaternaries and the corresponding Dirac-like operators in connection with the *nonion* algebra. We suggest to construct then the collection of four quaternaries and the corresponding Dirac-like operators in connection with the *duodevicenion* or *quindenion* algebra [52–54] and its ternary extension. The next steps suggested are noncommutative Galois extensions and a study of their basic relations and Galois extensions of ternary Clifford type [67], quaternary Galois extensions and Galois extensions of quaternary Clifford type [9], the Galois extension structure of the duodevicenion and quindenion algebras, quaternary and ternary structure of $\mathfrak{su}(3)$, ternary and quaternary Dirac-like operators of noncommutative Galois extensions [26,27,68] and, finally, identification of the constructed collection of three quaternaries with the proper collection of four ternaries.

An approach of ternary numbers, algebras, and complex analysis, coming back to geometric ideas suggested in [66].

2. The Galois Extension Structure of the Nonion Algebra

We recall the concept of nonion algebra $\mathbb{N}$ [52–54] and discuss its Galois extension structure.

Consider the following matrices

$$Q_1 = \begin{pmatrix} 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \tag{2.1}$$

$$\overline{Q}_1 = \begin{pmatrix} 0 & 0 & 1 \\ \mathbf{j}^2 & 0 & 0 \\ 0 & \mathbf{j} & 0 \end{pmatrix}, \quad \overline{Q}_2 = \begin{pmatrix} 0 & 0 & 1 \\ \mathbf{j} & 0 & 0 \\ 0 & \mathbf{j}^2 & 0 \end{pmatrix}, \quad \overline{Q}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \tag{2.2}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \end{pmatrix}. \tag{2.3}$$

where $\mathbf{j}$ is one of the roots of $z^3 - 1 = 0$ different from 1. The matrix algebra which is generated by two of the three elements (2.1) over $\mathbb{R}[\sqrt[3]{I_3}]$ is called *nonion algebra* $\mathbb{N}$.

Consider in addition the matrices of the form

$$T_1 = I_6 := \begin{pmatrix} I_3 & 0_3 \\ 0_3 & I_3 \end{pmatrix}, \quad T_2 = \begin{pmatrix} T_1^0 & 0_3 \\ 0_3 & T_1^0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} T_2^0 & 0_3 \\ 0_3 & T_2^0 \end{pmatrix}, \tag{2.4}$$

$$T_4 = \begin{pmatrix} 0_3 & I_3 \\ -I_3 & 0_3 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 0_3 & T_1^0 \\ -T_1^0 & 0_3 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 0_3 & T_2^0 \\ -T_2^0 & 0_3 \end{pmatrix}, \tag{2.5}$$

where

$$T_1^0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_2^0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.6}$$

We recall that the nine elements (2.1)–(2.3) constitute linear basis of the nonion algebra $\mathbb{N}$. The matrix algebra which is generated by two of the three elements (2.5) over the real field $\mathbb{R}$ will be denoted by $\mathbb{B}'$. The algebra generated by T_1^0 or T_2^0 is called *cubic algebra*, denoted by $\mathbb{B}$. The algebra $\widetilde{\mathbb{N}}$ generated by T_4 is the binary extension of $\mathbb{N}$,

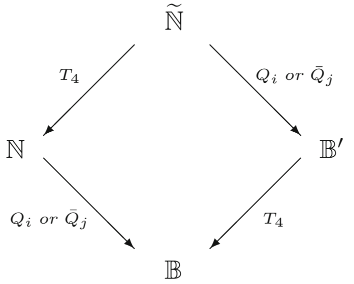
$$\widetilde{\mathbb{N}} = \mathbb{N}[\sqrt[2]{I_3}] \;:\; \widetilde{\mathbb{N}} = \{x + yT_4 \,|\, x, y \in \mathbb{N}\}.$$

The three elements (2.4) are a linear basis of $\mathbb{B}$. The six elements (2.4), (2.5) form a linear basis of $\mathbb{B}'$. Besides, $\mathbb{B}$ and $\mathbb{B}'$ are subalgebras of $\mathbb{N}$ and $\widetilde{\mathbb{N}}$ respectively. We can prove the above assertions with the help of the two enclosed product tables.

Now, owing to hints [13] and motivation given in references [7,8,13,15], following some earlier demands appearing in references [11,12,16–18], we can prove the following basic theorems on the Galois extensions.

Theorem 1. **(1)** *The nonion algebra is a ternary Galois extension of the algebra* $\mathbb{B} : \mathbb{N} = \mathbb{B}[\sqrt[3]{I_3}]$. *The extension can be realized by* $\mathbb{B}[\tau]$ $(\tau^3 = I_3)$ *with the choice of* $\tau = Q_i, \bar{Q}_i$ $(i = 1, 2, 3)$. **(2)** $\widetilde{\mathbb{N}}$ *is a binary extension of* $\mathbb{B}' : \widetilde{\mathbb{N}} = \mathbb{B}'[\sqrt[2]{I_3}]$. *Hence we have the following commutative diagram:*

(2.7)



$(i, j = 1, 2, 3)$

Proof will be given in the next section.

Theorem 2. **(3)** *We have the following Galois extensions:*

$$\begin{cases} \mathbb{A}[R_1, R_2, R_3] = \{xR_1 + yR_2 + zR_3 \,|\, x, y, z \in \mathbb{R}[\mathbf{j}]\}, \\ \mathbb{A}[R_1, Q_i, \bar{Q}_i] = \{xR_1 + yQ_i + z\bar{Q}_i \,|\, x, y, z \in \mathbb{R}[\mathbf{j}]\}, \quad (i = 1, 2, 3), \\ \mathbb{A}[R_1, \bar{Q}_i, Q_i] = \{xR_1 + y\bar{Q}_i + zQ_i \,|\, x, y, z \in \mathbb{R}[\mathbf{j}]\}, \quad (i = 1, 2, 3). \end{cases} \tag{2.8}$$

The extension does not depend on the choice of τ *with* $\mathbb{B}[\tau]$ $(\tau^3 = I_3)$*: we have*

$$\mathbb{N} = \mathbb{A}[R_1, Q_1, \bar{Q}_1] = \mathbb{A}[R_1, Q_2, \bar{Q}_2] = \mathbb{A}[R_1, Q_3, \bar{Q}_3]. \tag{2.9}$$

(4) Q_i, $\bar{Q}_j$ $(i, j = 1, 2, 3)$ *give a part of generators of the Galois group of* $\mathbb{N} : \mathbb{N} = \mathbb{B}[\sqrt[3]{I_3}]$. *Namely putting*

$$\begin{gathered} \mathbb{A}_U[R_1, R_2, R_3] = \{xR_1 + yUR_2 + z\bar{U}R_3 \,|\, x, y, z \in \mathbb{R}[\mathbf{j}]\}, \\ \text{where } U = Q_i, \bar{Q}_i, \ (i = 1, 2, 3), \end{gathered} \tag{2.10}$$

we can obtain new Galois extensions

$$\begin{cases} \mathbb{A}_{Q_1}[R_1, R_2, R_3] = \mathbb{A}[R_1, Q_2, \bar{Q}_2], \ \ \mathbb{A}_{Q_2}[R_1, R_2, R_3] = \mathbb{A}[R_1, Q_3, \bar{Q}_3], \\ \mathbb{A}_{\bar{Q}_1}[R_1, R_2, R_3] = \mathbb{A}[R_1, Q_3, \bar{Q}_3], \ \ \mathbb{A}_{\bar{Q}_2}[R_1, R_2, R_3] = \mathbb{A}[R_1, Q_1, \bar{Q}_1], \\ \mathbb{A}_{R_2}[R_1, Q_1, \bar{Q}_1] = \mathbb{A}[R_1, Q_2, \bar{Q}_2], \ \ \mathbb{A}_{R_2}[R_1, Q_2, \bar{Q}_2] = \mathbb{A}[R_1, Q_3, \bar{Q}_3], \\ \mathbb{A}_{R_2}[R_1, Q_3, \bar{Q}_3] = \mathbb{A}[R_1, Q_1, \bar{Q}_1], \ \ \mathbb{A}_{\bar{Q}_1}[R_1, R_2, R_3] = \mathbb{A}[R_1, Q_3, \bar{Q}_3], \\ \mathbb{A}_{\bar{Q}_2}[R_1, R_2, R_3] = \mathbb{A}[R_1, Q_1, \bar{Q}_1], \ \ \mathbb{A}_{\bar{Q}_3}[R_1, R_2, R_3] = \mathbb{A}[R_1, Q_2, \bar{Q}_2]. \end{cases} \tag{2.11}$$

(5) *We have the following results for the adjoint operations:*

$$\begin{cases} \mathrm{Ad}_{Q_i} R_1 = R_1, \ \mathrm{Ad}_{Q_i} R_2 = \mathbf{j} R_2, \ \ \mathrm{Ad}_{Q_{i1}} R_3 = \mathbf{j}^2 R_3, \ \ (i = 1, 2, 3), \\ \mathrm{Ad}_{Q_i} Q_1 = Q_1, \ \mathrm{Ad}_{Q_i} Q_2 = \mathbf{j} Q_2, \ \ \mathrm{Ad}_{Q_{i1}} Q_3 = \mathbf{j}^2 Q_3, \ \ (i = 1, 2, 3), \\ \mathrm{Ad}_{Q_i} \bar{Q}_1 = \bar{Q}_1, \ \mathrm{Ad}_{Q_i} \bar{Q}_2 = \mathbf{j}^2 \bar{Q}_2, \ \mathrm{Ad}_{Q_{i1}} \bar{Q}_3 = \mathbf{j} \bar{Q}_3, \ \ (i = 1, 2, 3), \end{cases} \tag{2.12}$$

where $\mathbf{j} \neq 1$, $\mathbf{j}^3 = 1$.

The proof will be given in the next section.

3. Proofs of Basic Theorems 1 and 2 [Assertions (1)–(5)]

Proof. Ad **(1)**. We notice that $\mathbb{B}$ is the commutative Galois extension: $\mathbb{B} = \mathbb{R}[\sqrt[3]{I_3}]$.

Then we observe that, for a Clifford algebra $\mathcal{A}$ with generators $T_1, T_2, \ldots, T_n$, there is a sequence of noncommutative binary Galois extensions of $\mathbb{R}$ which realizes the given Clifford algebra $\mathcal{A}$ [19].

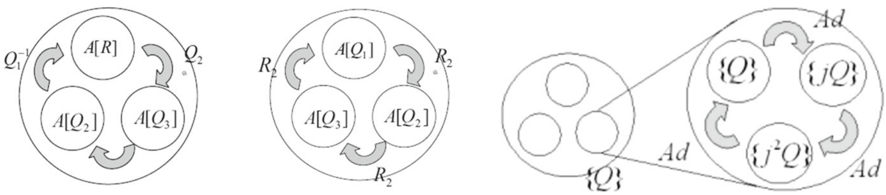


FIGURE 1. Schemes illustrating reasoning leading to Assertions **(1)**–**(5)**

Indeed, let us prove the above statement by induction with respect to m, where $\mathcal{A}_0 = \mathbb{R}$, $\mathcal{A}_m = \mathcal{A}$, and

$$T_iT_j + T_jT_i = -\delta_{ij}I_n \;\Rightarrow\; \mathcal{A}_k = \mathcal{A}_{k-1}[\sqrt[2]{-I_n}], \quad k = 1, 2, \ldots, m.$$

Complex numbers can be obtained by commutative extensions of real numbers. Now, setting

$$\hat{T}_i = \begin{pmatrix} T_i & 0 \\ 0 & -T_i \end{pmatrix}, \quad i = 1, 2, \ldots, m; \quad \hat{H}_{n+1} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

we get Clifford algebra which is generated by $(\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_n, \hat{H}_{n+1})$ on one hand, and the right (or left) module binary extension of $\mathcal{A}_n$ by $\hat{H}_{n+1}$ on the other hand.

If we choose in **(1)** $\tau = Q_i\overline{Q}_i$, $i = 1, 2, 3$, we make the Galois extension $\mathbb{B}[\sqrt[3]{I_3}]$. Then we can see that it is identical with $\mathbb{N}$.

Ad **(2)**. We notice that $\mathbb{B}'$ is the noncommutative Galois extension of $\mathbb{B} : \mathbb{B}' = \mathbb{B}[\sqrt[2]{I_3}]$, where $\sqrt[2]{I_3} = T_4$. Choosing $\tau = T_4$, we make the Galois extension. Then we can see that it is identical with $\widetilde{\mathbb{N}} : \widetilde{\mathbb{N}} = \mathbb{B}'[\sqrt[3]{I_3}]$.

Ad **(3)**–**(5)**. Clearly, for $i = 1$ we have

$$\mathbb{R}[\mathbf{j}, Q_1] = \{xR_1 + yQ_1 + zQ_1^2 | x, y, z \in \mathbb{R}[\mathbf{j}]\}.$$

From $Q_1^2 = \bar{Q}_1$ it follows that

$$\mathbb{R}[\mathbf{j}, Q_1] = \{xR_1 + yQ_1 + z\bar{Q}_1 | x, y, z \in \mathbb{R}[\mathbf{j}]\}.$$

Hence $\mathbb{R}[\mathbf{j}, Q_1]$ is a ternary Galois extension with the corresponding Galois group $1, Q_1, Q_1^2$. Analogously we perform the reasoning for $\mathbb{R}[\mathbf{j}, Q_2]$ and $\mathbb{R}[\mathbf{j}, Q_3]$.

In general, we follow the enclosed schemes (Fig. 1) which indicate how to use in the clockwise way Tables 1 and 2. The scheme is clarified by a detailed calculation in the case expressed in the second line of the formulae (2.10).

4. Ternary and Binary Galois Extension Structures of $\mathfrak{su}(3)$

In addition to the desires followed from our programme formulated in [1], the present section meets also the demands appearing in references [20–25]. We

Table 1. The (Q, $\bar{\text{Q}}$, R)-matrices product table

	Q_1	Q_2	Q_3	$\bar{Q}_1$	$\bar{Q}_2$	$\bar{Q}_3$	R_1	R_2	R_3
Q_1	$\bar{Q}_1$	$\mathbf{j}^2\bar{Q}_3$	$\mathbf{j}\bar{Q}_2$	R_1	$\mathbf{j}^2R_3$	$\mathbf{j}R_2$	Q_1	Q_2	Q_3
Q_2	$\mathbf{j}\bar{Q}_3$	$\bar{Q}_2$	$\mathbf{j}^2\bar{Q}_1$	$\mathbf{j}R_2$	R_1	$\mathbf{j}^2R_3$	Q_2	Q_3	Q_1
Q_3	$\mathbf{j}^2\bar{Q}_2$	$\mathbf{j}\bar{Q}_1$	$\bar{Q}_3$	$\mathbf{j}^2R_3$	$\mathbf{j}R_2$	R_1	Q_3	Q_1	Q_2
$\bar{Q}_1$	R_1	R_2	R_3	Q_1	$\mathbf{j}^2Q_3$	$\mathbf{j}Q_2$	$\bar{Q}_1$	$\mathbf{j}^2\bar{Q}_3$	$\mathbf{j}\bar{Q}_2$
$\bar{Q}_2$	R_3	R_1	R_2	$\mathbf{j}Q_3$	Q_2	$\mathbf{j}^2Q_1$	$\bar{Q}_2$	$\mathbf{j}^2\bar{Q}_1$	$\mathbf{j}\bar{Q}_3$
$\bar{Q}_3$	R_2	R_3	R_1	$\mathbf{j}^2Q_2$	$\mathbf{j}Q_1$	Q_3	$\bar{Q}_3$	$\mathbf{j}^2\bar{Q}_2$	$\mathbf{j}\bar{Q}_1$
R_1	Q_1	Q_2	Q_3	$\bar{Q}_1$	$\bar{Q}_2$	$\bar{Q}_3$	R_1	R_2	R_3
R_2	$\mathbf{j}^2Q_2$	$\mathbf{j}^2Q_3$	$\mathbf{j}^2Q_1$	$\bar{Q}_3$	$\bar{Q}_1$	$\bar{Q}_2$	R_2	R_3	R_1
R_3	$\mathbf{j}Q_3$	$\mathbf{j}Q_1$	$\mathbf{j}Q_2$	$\bar{Q}_2$	$\bar{Q}_3$	$\bar{Q}_1$	R_3	R_1	R_2

Table 2. The T-matrices product table

	T_1	T_2	T_3	T_4	T_5	T_6
T_1	T_1	T_2	T_3	T_4	T_5	T_6
T_2	T_2	T_3	T_1	T_5	T_6	T_4
T_3	T_3	T_1	T_2	T_6	T_4	T_5
T_4	T_4	T_6	T_5	T_1	T_3	T_2
T_5	T_5	T_4	T_6	T_2	T_1	T_3
T_6	T_6	T_5	T_4	T_3	T_2	T_1

are going to discuss the structure of the Galois extension in the context of $\mathfrak{su}(3)$.

We recall base of $\mathfrak{su}(3)$ (where selected generators are proportional to the well known Gell-Mann matrices [14]):

$$f_1=\begin{pmatrix}0&i&0\\i&0&0\\0&0&0\end{pmatrix},\quad f_2=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix},\quad f_3=\begin{pmatrix}i&0&0\\0&-i&0\\0&0&0\end{pmatrix},$$

$$f_4=\begin{pmatrix}0&0&i\\0&0&0\\i&0&0\end{pmatrix},$$

$$f_5=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},\quad f_6=\begin{pmatrix}0&0&0\\0&0&i\\0&i&0\end{pmatrix},\quad f_7=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},$$

$$f_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},$$

and construct the three linear subspaces:

$$\begin{cases} L_1: \ \ e_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ L_2: \ \ e_1' = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ e_2' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ e_3' = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \\ L_3: \ \ e_1'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \ e_2'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ e_3'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}. \end{cases} \tag{4.1}$$

We notice the relation

$$f_8 = \frac{-1}{i\sqrt{3}}(e_3' + e_3''),$$

while $\{e_3, e_3^{'}, e_3^{''}\}$ is linearly dependent. Hence we can see that $(e_1, e_2, \ldots, e_3'')$ constitute the basis omitting one of e_3, e_3', e_3''.

We can prove the following basic theorem on both *binary* and *ternary* Galois extension structures on $\mathfrak{su}(3)$, extremely important from the point of view of the $\blacksquare_4$ of our research programme formulated in Section 4 of [69]: *identification of two ternaries with three binaries with the help of noncommutative Galois extensions*; cf. [26,27]:

Theorem 3. *We have the binary and ternary extension structures on* $\mathfrak{su}(3)$*:*

(**6**) *We have the following adjoint representation on* L_i $(i = 1, 2, 3)$*:*

$$\begin{cases} He_1H^{-1} = -e_2, & He_2H^{-1} = e_1, & He_3H^{-1} = e_3, \\ H'e_1'H'^{-1} = -e_2', & H'e_2'H'^{-1} = -e_1', & H'e_3'H'^{-1} = e_3', \\ H'e_1''H'^{-1} = e_2'', & H'e_2''H'^{-1} = e_1'', & H'e_3''H'^{-1} = e_3'', \end{cases} \tag{4.2}$$

where

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ H' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}. \tag{4.3}$$

(**7**) *We can obtain the following commutation relations:*

$$\begin{cases} e_1^2 = e_2^2 = e_3^2 = -1 \\ e_1e_2 = -e_2e_1 = e_3, \ e_2e_3 = -e_3e_2 = e_1, \ e_3e_1 = -e_1e_3 = e_2. \end{cases} \tag{4.4}$$

After the central extension, we have the Clifford algebra which is isomorphic to the quaternion algebra. For the case of e_i' *and* e_i'' $(i = 1, 2, 3)$*, we have the same assertions on* L_i $(i = 1, 2, 3)$*. Hence we obtain the Dirac-like operators desired.*

(**8**) *We have*

$$\begin{aligned} &G_1e_kG_1^{-1} = e_k' \ (k = 1, 2, 3), \ G_1e_k'G_1^{-1} = e_k''(k = 1, 2), \ G_1e_3'G_1^{-1} = -e_3'' \\ &G_1e_k''G_1^{-1} = e_k \ (k = 1, 3), \ G_1e_2''G_1^{-1} = -e_2, \end{aligned} \tag{4.5}$$

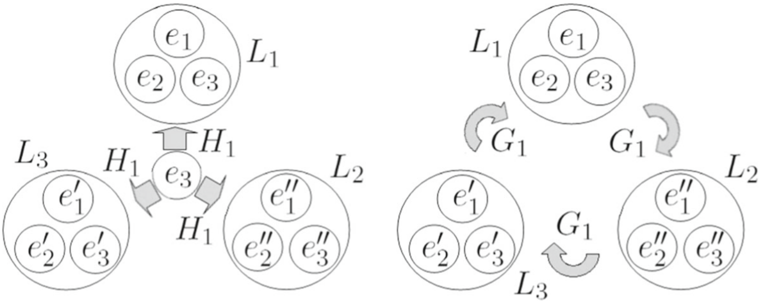


FIGURE 2. Schematic comparison of the commutative Galois extension structures on $\mathfrak{su}(3)$

where

$$G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

By this result, we can find a ternary commutative Galois extension. Hence we can introduce the ternary Dirac-like operators.

Proof. Basic elements of our considerations are summarized as properties (1) of $su(3)$, (2) of su(2), and (3) of $\mathfrak{su}(3)$; as schematic comparison of the commutative Galois extension structures on $\mathfrak{su}(3)$ (Fig. 2), and as the diagram (4.7):

$$\begin{array}{ll} (1)\ su(3) = L_1 \cup L_2 \cup L_3 & su(3) \\ & \downarrow \sqrt[3]{I_3} \\ (2)\ L_i (i = 1, 2, 3) \text{ is isomorphic to su(2) and it is a binary} & \text{su}(2) \\ \quad \text{Galois extension } L_i = B_0[\sqrt[2]{I_3}] \text{ over } B_0 = R[e_3] & \downarrow \sqrt[2]{I_3} \\ (3)\ \mathfrak{su}(3) \textit{ is a ternary Galois extension } B'[\sqrt[3]{I_3}] \text{ over } \mathbb{B}' = \text{su}(2) & \mathfrak{su}(3) \end{array} \tag{4.7}$$

All the formulae (4.2), (4.4) and (4.5) can be checked directly from the definitions concerned. A comparison of the commutative Galois extension structures on $\mathfrak{su}(3)$ is visualized in Fig. 2. As we can see, G_1 is isomorphism of (sub)algebras induced by the fact that they are isomorphic to $\mathfrak{su}(2)$ and H_1 is binary extension different for each (sub)algebra by the different choice of element τ, that $\tau^2 = I_3$.

5. Identification of Construction of the Collection of Two Ternaries with the Collection of Three Binaries

Starting with definitions [1]

$$x_1 + x_2 i = x_1 + x_2\sqrt{-1} \Leftrightarrow \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix},$$

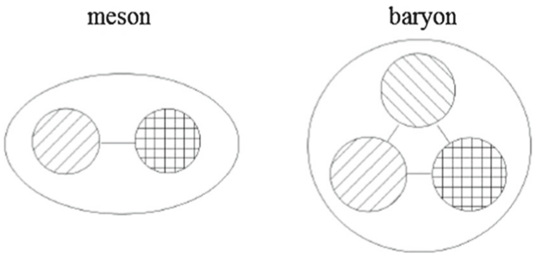


FIGURE 3. Examples of binary and ternary particles

$$y_1 + \mathbf{j}y_2 + \mathbf{j}^2 y_3 \Leftrightarrow \begin{pmatrix} y_1 & y_2 & y_3 \\ y_3 & y_1 & y_2 \\ y_2 & y_3 & y_1 \end{pmatrix} \ (= Y_1) \tag{5.1}$$

and

$$Y_2 = \begin{pmatrix} y_4 & y_5 & y_6 \\ y_6 & y_4 & y_5 \\ y_5 & y_6 & y_4 \end{pmatrix} \ (\Leftrightarrow y_4 + \mathbf{j}y_5 + \mathbf{j}^2 y_6)$$

with $x_1, x_2, y_1, \ldots, y_6 \in \mathbb{R}$,

we observe that

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_3 & X_1 & X_2 \\ X_2 & X_3 & X_1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} Y_1 & Y_2 \\ -Y_2 & Y_1 \end{pmatrix}$$

follows from the definitions of X_1, X_2, X_3 and Y_1, Y_2.

The identification includes a correspondence of the related ternary and binary Galois structures in the sense precised in assertions of Theorem 1–3 regarding the Galois extension structures on the nonion algebra (enlightening, in particular, the passage from cubic algebra $\mathbb{B}$ to the nonion algebra $\mathbb{N}$) and the analogous structure on su(3).

6. An Analogue for Sects. 2–5: Two Ternaries vs. Three Binaries for Quarks and Elementary Particles

At the beginning of this and next section we shortly present the idea which is essentially contained in [11,27] and related papers. We present it here in a slightly altered form and order for the seek of completeness.

In the procedure (4.1)–(4.4) and Fig. 6 of [1] we may replace crystallographic lattices by quarks and elementary particles with the minimal requirements for their definition at the initial stage of model [7,8,11,19–21]. In particular, we may start with mesons and baryons as examples of binary and ternary particles (Fig. 3).

We know that each meson constitutes a quark and an anti-quark, and that each baryon constitutes only three quarks or anti-quarks. This generates the duality related with the ternary Pauli exclusion principles generalized by Kerner [11] and specifying in [27] by Th. 4 (Fig. 4).

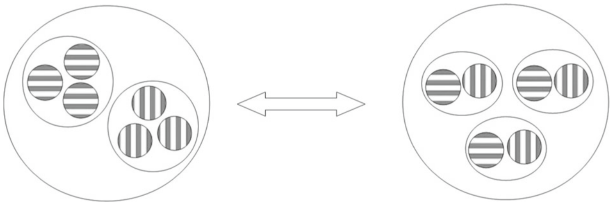


Figure 4. Duality related with the ternary Pauli exclusion principle

In turn we may study the quark realization of mesons and baryons by use of the binary Galois extension on $\mathfrak{su}(3)$ [27], Figs. 3 and 4. We come to the quark realization of mesons and baryons by use of the binary resp. ternary Galois extension of $\mathfrak{su}(3)$ refering to the Kobayashi–Masukawa theory; [27,28], Figs. 8, 9 and 10.

Following the Kobayashi–Masukawa model, we consider Galois extension of $\mathfrak{su}(3) : R\left[\sqrt{-1}, \sqrt{-1}, \sqrt[3]{1}, \sqrt{-1}\right]$. In this way we obtain six kinds of quarks. We remark that, usually, the Kobayashi–Masukawa theory is attributed to their study of two kinds of quarks.

Introducing the extension by $e_0\,(= diag[1,1,0])$ we may identify the quarks as up-quark, down-quark, strange-quark, as follows

$$\{e_0, e_1, e_2, e_3\} \Rightarrow u, \;\; \{e_0', e_1', e_2', e_3'\} \Rightarrow d, \;\; \{e_0'', e_1'', e_2'', e_3''\} \Rightarrow s. \tag{6.1}$$

Further, using the conjugate elements of (6.1)

$$\{\bar{e}_0, \bar{e}_1, \bar{e}_2, e_3\} \Rightarrow \bar{u}, \;\; \{\bar{e}_0', \bar{e}_1', \bar{e}_2', \bar{e}_3'\} \Rightarrow \bar{d}, \;\; \{\bar{e}_0'', \bar{e}_1'', \bar{e}_2'', \bar{e}_3''\} \Rightarrow \bar{s}. \tag{6.2}$$

we realize mesons and baryons in Gell-Mann model by use of the Golois extension. Similarly, in the Kobayashi–Masukawa model—by use of the binary Galois extension structure of $\mathfrak{su}(3)$ [27].

Noticing the duality for the 3-generation structures of quarks responsible for flavour; cf. [29]:

the binary structure of u, c, b $\Longleftrightarrow$ the ternary structure of u, c, b
the binary structure of d, s, t $\Longleftrightarrow$ the ternary structure of d, s, t

we arrive at a cumulative scheme for elementary particles including quarks, leptons and bozons, and generation of Matter (Fig. 5) as well as the corresponding 3-generations, more precisely: 3 objects consisting of 2 collections of 2 particles, constructed from 2 *proper* generations, more precisely from:

- 2 objects consisting of 3 collections of 2 particles (in each case),
- 2 objects consisting of 2 collections of 3 particles (in either case),
- 1 object consisting of 3 collections of 2 particles (in each case) and 1 object consisting of 2 collections of 3 particles (in either case).

It is natural to give an example of a corresponding 3-generation (Fig. 6).

For introducing colours in terms of noncommutative Galois extensions we use the nonion extension of $\mathfrak{su}(3)$; cf. the next section.

Summing up, we replace the programme $\blacksquare_{13}$–$\blacksquare_{23}$ by the following

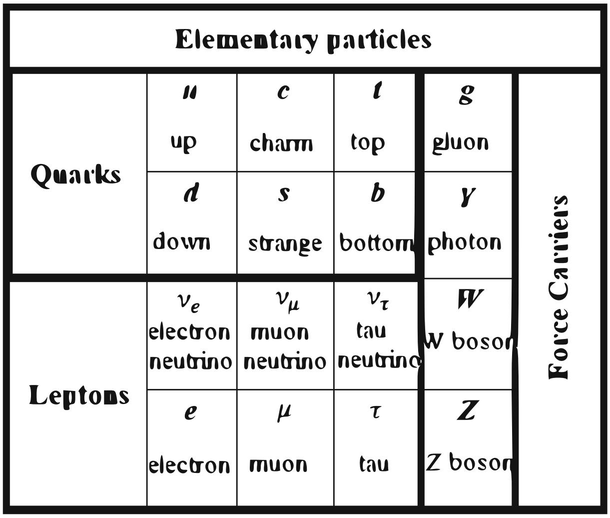


FIGURE 5. A cumulative scheme for elementary particles including quarks, leptons and bozons, and generations of matter

- $\square_{13}$ binary and ternary elementary particles
- $\square_{14}$ successive extensions of the related binary and ternary extensions on $\mathfrak{su}(3)$
- $\square_{15}$ relationships with the Gell-Mann model [13,14] and Kobayashi–Masukawa model [28]
- $\square_{16}$ binary and ternary Pauli exclusion principles
- $\square_{17}$ quark realization of mezons by the use of the binary Galois extension on $\mathfrak{su}(3)$
- $\square_{18}$ construction of quark models of mesons
- $\square_{19}$ construction of quark models for baryons
- $\square_{20}$ the 3-generations of quarks
- $\square_{21}$ adjoint representations
- $\square_{22}$ duality between the collections of corresponding binary and ternary Dirac operators
- $\square_{23}$ generation of colours and the identification problem

The steps $\square_{13}$–$\square_{23}$ may be composed in the scheme shown on Fig. 7.

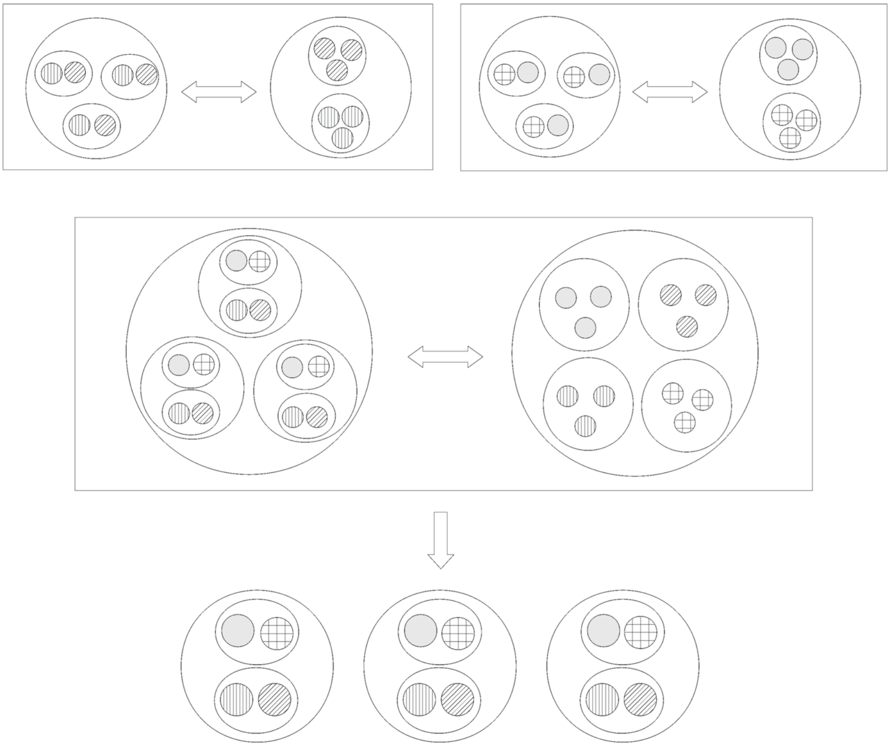


FIGURE 6. An example of a corresponding 3-generation constructed from 2 proper generations (*proper* in the sense described above)

7. Introduction of Colours

If we wish to introduce the concept of colours, we may consider the succesive ternary extension

$$\mathbb{C}[\sqrt[2]{-I_2}, \sqrt[3]{I_3}, \sqrt[3]{I_3}] \simeq \mathrm{su}(3)[\sqrt[3]{I_3}].$$

Indeed, $\mathbb{C}[\sqrt[2]{-I_2}]$ is the quaternionic algebra, so making its extension by T_2 as in (8) we get $\mathbb{C}[\sqrt[2]{-I_2}, \sqrt[3]{I_3}]$. Next, making the extension by the nonion algebra we arrive at $\mathbb{C}[\sqrt[2]{-I_2}, \sqrt[3]{I_3}, \sqrt[3]{I_3}]$, as desired. This means that we apply the *nonion extension of* $\mathfrak{su}(3)$; cf. Sects. 1, 2, and 3:

$$\tilde{\sigma}_i e_k \tilde{\sigma}_i^{-1} = e_k'', \quad i, k = 1, 2, 3 \tag{7.1}$$

with e_k, e_k'', $k = 1, 2, 3$, as in (17) and, in analogy to (21) and (22):

$$\tilde{\sigma}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tilde{\sigma}_2 = \begin{pmatrix} 0 & \mathbf{j}^2 & 0 \\ 0 & 0 & \mathbf{j} \\ 1 & 0 & 0 \end{pmatrix}, \quad \tilde{\sigma}_3 = \begin{pmatrix} 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{j}^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{j}^3 = 1, \ \ \mathbf{j} \neq 1.$$

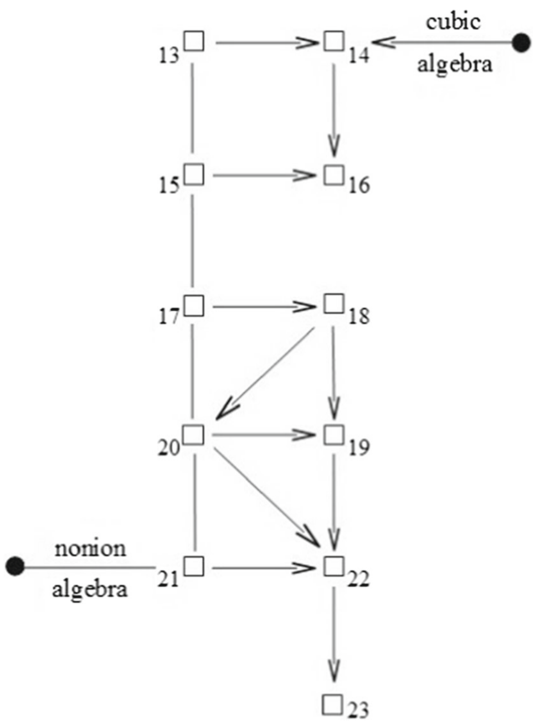


FIGURE 7. Ternary approach related to the *nonion* algebra for quarks and elementary particles

For $\tilde{\sigma}_1 e_k \tilde{\sigma}_1^{-1}$ we have e_k and e''_k, $k = 1, 2, 3$, as in (15) and

$$\begin{cases} \tilde{\sigma}_1 e_1 \tilde{\sigma}_1^{-1} = e'_1, & \tilde{\sigma}_1 e_2 \tilde{\sigma}_1^{-1} = e'_1, \quad \tilde{\sigma}_1 e_3 \tilde{\sigma}_1^{-1} = e'_3, \\ \tilde{\sigma}_1 e'_1 \tilde{\sigma}_1^{-1} = e''_1, & \tilde{\sigma}_1 e'_2 \tilde{\sigma}_1^{-1} = e''_2, \quad \tilde{\sigma}_1 e'_3 \tilde{\sigma}_1^{-1} = e''_3, \\ \tilde{\sigma}_1 e''_1 \tilde{\sigma}_1^{-1} = e_1, & \tilde{\sigma}_1 e''_2 \tilde{\sigma}_1^{-1} = -e_2, \quad \tilde{\sigma}_1 e''_3 \tilde{\sigma}_1^{-1} = e_3. \end{cases} \tag{7.2}$$

For $\tilde{\sigma}_2 e_k \tilde{\sigma}_2^{-1}$, $k = 1, 2, 3$, we have

$$\tilde{e}_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{e}_2 = i \begin{pmatrix} 0 & \mathbf{j} & 0 \\ \mathbf{j}^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{e}_3 = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} = e'_3,$$

$$\tilde{\tilde{e}}_1 = \begin{pmatrix} 0 & 0 & \mathbf{j} \\ 0 & 0 & 0 \\ \mathbf{j}^2 & 0 & 0 \end{pmatrix}, \quad \tilde{\tilde{e}}_2 = i \begin{pmatrix} 0 & 0 & \mathbf{j} \\ 0 & 0 & 0 \\ \mathbf{j}^2 & 0 & 0 \end{pmatrix}, \quad \tilde{\tilde{e}}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = ie''_3,$$

and

$$\begin{cases} \tilde{\sigma}_2 e_1 \tilde{\sigma}_2^{-1} = \tilde{e}'_1, & \tilde{\sigma}_2 e_2 \tilde{\sigma}_2^{-1} = \tilde{e}_2, \quad \tilde{\sigma}_2 e_3 \tilde{\sigma}_2^{-1} = \tilde{e}_3, \\ \tilde{\sigma}_2 \tilde{e}_1 \tilde{\sigma}_2^{-1} = \tilde{\tilde{e}}_1, & \tilde{\sigma}_2 \tilde{e}_2 \tilde{\sigma}_2^{-1} = \tilde{\tilde{e}}_2, \quad \tilde{\sigma}_2 \tilde{e}_3 \tilde{\sigma}_2^{-1} = \tilde{\tilde{e}}_3 = -ie''_3, \\ \tilde{\sigma}_2 \tilde{\tilde{e}}_1 \tilde{\sigma}_2^{-1} = e_1, & \tilde{\sigma}_2 \tilde{\tilde{e}}_2 \tilde{\sigma}_2^{-1} = e_2, \quad \tilde{\sigma}_2 \tilde{\tilde{e}}_3 \tilde{\sigma}_2^{-1} = ie_3. \end{cases} \tag{7.3}$$

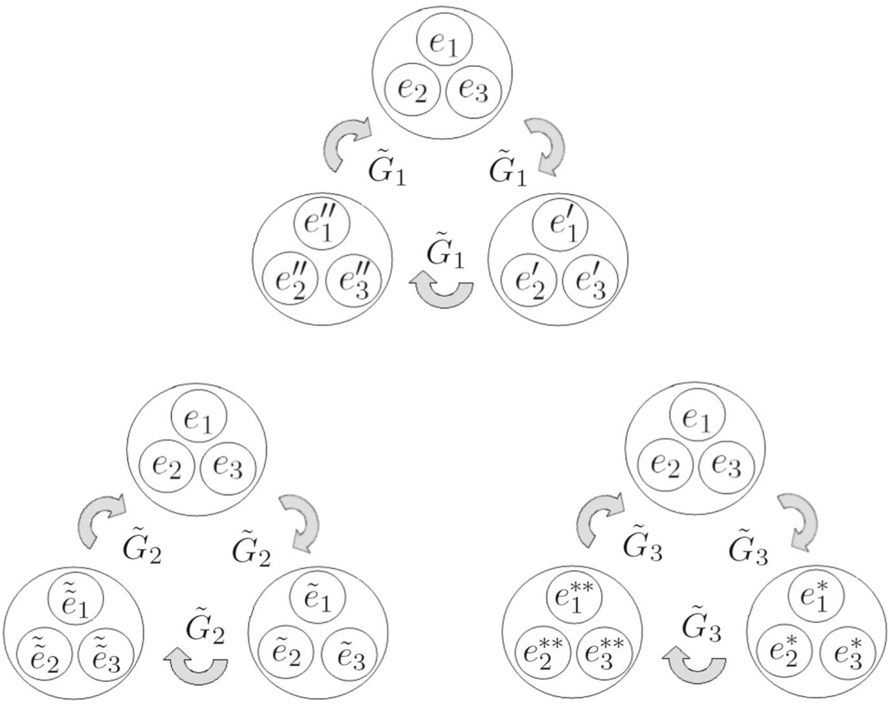


FIGURE 8. Generation of transformations $\tilde\sigma_1$, $\tilde\sigma_2$, $\tilde\sigma_3$ by the nonion extension of $\mathfrak{su}(3)$

For $\tilde\sigma_3 e_k \tilde\sigma_3^{-1}$, $k = 1, 2, 3$, we have

$$e_1^* = i\begin{pmatrix} 0 & 0 & \mathbf{j} \\ 0 & 0 & 0 \\ \mathbf{j}^2 & 0 & 0 \end{pmatrix},\quad e_2^* = i\begin{pmatrix} 0 & 0 & \mathbf{j} \\ 0 & 0 & 0 \\ -\mathbf{j}^2 & 0 & 0 \end{pmatrix},\quad e_3^* = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} = e_3',$$
$$e_1^{**} = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{j} \\ 0 & \mathbf{j}^2 & 0 \end{pmatrix},\quad e_2^{**} = i\begin{pmatrix} 0 & 0 & \mathbf{j} \\ 0 & 0 & 0 \\ \mathbf{j}^2 & 0 & 0 \end{pmatrix},\quad e_3^{**} = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} = ie_3,$$

and

$$\begin{cases} \tilde\sigma_3 e_1 \tilde\sigma_3^{-1} = e_1^*, \quad \tilde\sigma_3 e_2 \tilde\sigma_3^{-1} = e_2^*, \quad \tilde\sigma_3 e_3 \tilde\sigma_3^{-1} = e_3^*, \\ \tilde\sigma_3 e_1^* \tilde\sigma_3^{-1} = e_1^{**}, \quad \tilde\sigma_3 e_2^* \tilde\sigma_3^{-1} = e_2^{**}, \quad \tilde\sigma_3 e_3^* \tilde\sigma_3^{-1} = e_3^{**} = e_3, \\ \tilde\sigma_3 e_1^{**} \tilde\sigma_3^{-1} = e_1, \quad \tilde\sigma_3 e_2^{**} \tilde\sigma_3^{-1} = e_2, \quad \tilde\sigma_3 e_3^{**} \tilde\sigma_3^{-1} = e_3. \end{cases} \tag{7.4}$$

the rules (7.1) or (7.2)–(7.4) for the transformations $\tilde\sigma_1$, $\tilde\sigma_2$, $\tilde\sigma_3$ are illustrated by the schemes in Fig. 8.

The Dirac-like operators for

$$(e_0', e_1', e_2', e_3'),\quad (e_0'', e_1'', e_2'', e_3''),\quad (\tilde e_0, \tilde e_1, \tilde e_2, \tilde e_3),$$
$$(\tilde{\tilde e}_0, \tilde{\tilde e}_1, \tilde{\tilde e}_2, \tilde{\tilde e}_3),\quad (e_0^*, e_1^*, e_2^*, e_3^*),\quad (e_0^{**}, e_1^{**}, e_2^{**}, e_3^{**})$$

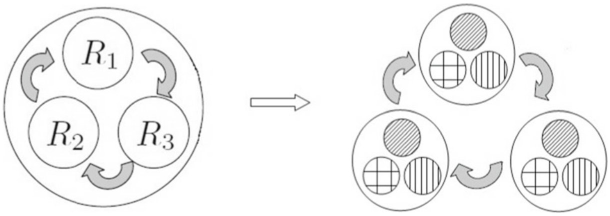


FIGURE 9. The concept of introducing colours for elementary particles via the transformations $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$ governing the nonion extension of $\mathfrak{su}(3)$

can be introduced in an analogous way as shown in Fig. 9. The proposed approach can be naturally modified for more general quark models what is planned for a future paper.

It is worth-while to notice several other important applications of Galois extensions and noncommutative Galois theory in contemporary physics [30–37]. Also, because of an experimental confirmation of the existence of the Higgs boson [38–41], it is natural to extend our approach combining the Kobayashi–Masukawa model with the BEH-mechanism (here B stands for Robert Brout, E for Franois Englert, Nobel Prize 2013, and H for Peter W. Higgs, Nobel Prize 2013) [42–47]. In this context we can see an elegant interpretation of considering our approach parallelly in the theories of condensed matter and elementary particles—the appearance of the Nambu [48–50]–Goldstone [51] boson and the Meissner effect in superconductivity within the Ginzburg–Landau model: temperature-dependent vector potential and photons getting the mass. Around 1960 Yôichirô Nambu (Nobel Prize 2008) extended some ideas from superconductivity to particle physics. Jeffrey Goldstone introduced a complex massive scalar field and gave a theorem on appearance of a new massless scalar (Nambu–Goldstone) boson. Note that Murray Gell-Mann (Nobel Prize 1969) and Yuval Ne'eman in connection with the symmetry group SU(3) introduced the new quantum number called flavour [13].

8. Conclusions

For conclusions, we combine the step $\bullet_6$ of Sect. 2 in our previous paper [1] and the steps $\bullet_{13}$–$\bullet_{23}$ formulated there in Sect. 6 with the steps $\blacksquare_{13}$–$\blacksquare_{23}$:

$\downarrow$ $\bullet_{13}$ binaries and ternaries; construction of the collection of two ternaries

$\blacksquare_{13}$ ternaries and quaternaries; construction of the collection of three quaternaries

↓ $\bullet_{14}$ the corresponding Dirac-like operators in connection with the *cubic* algebra

$\blacksquare_{14}$ the corresponding Dirac-like operators in connection with the *nonion* algebra

↑ $\bullet_{16}$ the Dirac-like operators corresponding to the collection of three binaries in connection with the *nonion* algebra and its binary extension

↓ $\bullet_{15}$ construction of the collection of three binaries

$\blacksquare_{15}$ construction of the collection four ternaries

↓ $\bullet_{6}$ Jordan–von Neumann–Wigner elements, complete elements, perfect elements; an example of perfect 15-element system

$\blacksquare_{16}$ the corresponding Dirac-like operators in connection with the *duodevicenion* or *quindenion* algebra [52, 53] and its ternary extension

↓ $\bullet_{16}$ noncommutative Galois extensions for $\bullet_{16}$ and their basic relations

$\blacksquare_{17}$ noncommutative Galois extensions for $\blacksquare_{16}$ and their basic relations

↓ $\bullet_{18}$ binary Galois extensions and Galois extensions of binary Clifford type for $\bullet_{16}$

$\blacksquare_{18}$ ternary Galois extensions and Galois extensions of ternary Clifford type for $\blacksquare_{16}$

↓ $\bullet_{19}$ ternary Galois extensions and Galois extensions of ternary Clifford type for $\bullet_{16}$

$\blacksquare_{19}$ quaternary Galois extensions and Galois extensions of quaternary Clifford type for $\blacksquare_{16}$

↓ $\bullet_{6}$ Jordan–von Neumann–Wigner elements, complete elements, perfect elements; an example

$\blacksquare_{20}$ the Galois extension structure of the *duodevicenion* and *quindenion* algebras

↑ $\bullet_{20}$ the Galois extension structure of the *nonion* algebra

↓ $\bullet_{21}$ ternary and binary Galois extension structure for $\mathfrak{su}(3)$

$\blacksquare_{21}$ quaternary and ternary Galois extension structure for $\mathfrak{su}(3)$

↓ $\bullet_{22}$ binary and ternary Dirac-like operators of noncommutative Galois extensions

$\blacksquare_{22}$ ternary and quaternary Dirac-like operators of noncommutative Galois extensions

↓ $\bullet_{23}$ identification of the constructed collection of two ternaries with the proper collection of three binaries

$\blacksquare_{23}$ identification of the constructed collection of three quaternaries with the proper collections of four ternaries

The steps $\bullet_{6}$, $\bullet_{13}$–$\bullet_{23}$, $\blacksquare_{13}$–$\blacksquare_{23}$ may be composed in the scheme shown on Fig. 10.

Other new ideas are provided by Perk [55] and Au-Yang and Perk [56], kindly communicated to us in a private letter of Prof. Perk (21.08.,2014).

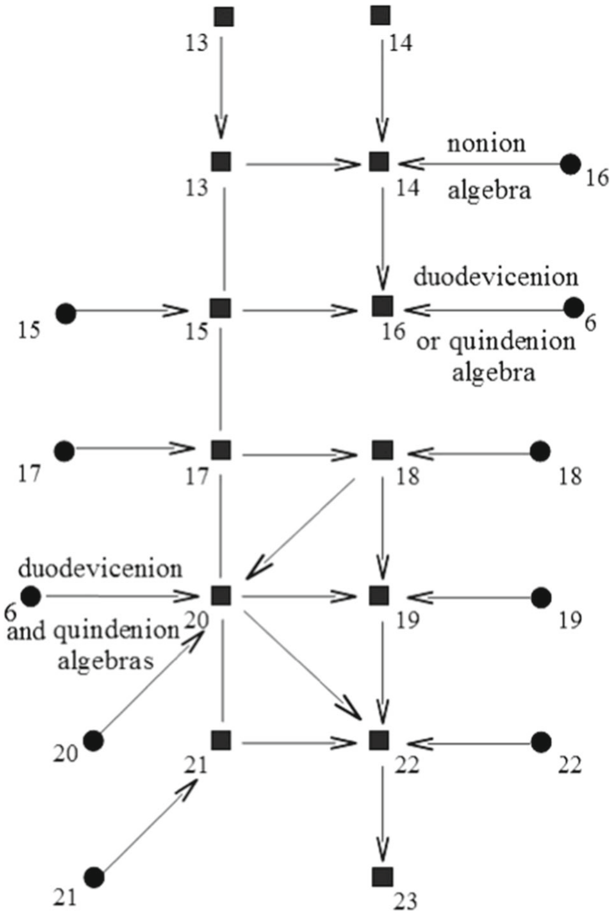


FIGURE 10. Quaternary approach related to the quindenion algebra, for crystallographic lattices

References

[1] Ławrynowicz, J., Suzuki, O., Niemczynowicz, A., Nowak-Kępczyk, M.: Fractals and chaos related to Ising-Onsager lattices. Ternary approach versus binary

approach. Int. J. Geom. Methods Mod. Phys. **15**(11), 1850187 (2018). https://doi.org/10.1142/S0219887818501876

[2] Ising, E.: Beitrag zur Theorie des Ferromagnetismus. Zschr. f. Phys. **31**, 253–258 (1925)

[3] Onsager, L.: Crystal statistics. I. A two-dimensional model with an order-disorder transition. Phys. Rev. **65**, 117–149 (1944)

[4] Zhang, Z.-D.: Conjectures on exact solution of three-dimensional (3D) simple orthorhombic Ising lattices. Phil. Mag. **87**, 5309–5419 (2007). [arXiv:0705.1045 [cond-mat] (pp.1-170)]

[5] Zhang, Z.-D.: Mathematical structure of the three-dimensional (3D) Ising model. Chinese Phys. B **22**, 030513 (2013)

[6] Zhang, Z.-D., Suzuki, O., March, N.H.: Clifford algebra approach of 3D Ising model. Adv. Appl. Clifford Algebras **29**, 12 (2019)

[7] Ławrynowicz, J., Nôno, K., Nagayama, D., and Suzuki, O., "Non-commutative Galois theory on Nonion algebra and su(3) and its application to construction of quark models", Proc. of the Annual Meeting of the Yukawa Inst. Kyoto "The Hierarchy Structure in Physics and Information Theory" Soryuusironnkennkyuu, Yukawa Institute, Kyoto, pp. 145–157 [http://www2.yukawa.kyoto-u.ac.jp] (2011)

[8] Ławrynowicz, J., Nôno, K., Nagayama, D., Suzuki, O.: A method of non-commutative Galois theory for binary and ternary Clifford Analysis. Proc. ICMPEA (Internat. Conf. on Math. Probl. in Eng. Aerospace, and Sciences) Wien, AIP (Amer. Inst. of Phys.) Conf. **1493**, 1007–1014 (2012)

[9] Ławrynowicz, J., Nowak-Kępczyk, M., Suzuki, O.: Fractals and chaos related to Ising-Onsager-Zhang lattices vs. the Jordan-von Neumann-Wigner procedures. Quaternary approach. Internat. J. of Bifurcations and Chaos **22**(1), 1230003 (19 pages) (2012)

[10] Ławrynowicz, J., Suzuki, O., Niemczynowicz, A.: Fractals and chaos related to Ising-Onsager-Zhang lattices vs. the Jordan-von Neumann-Wigner procedures. Ternary approach. Internat. J. of Nonlinear Sci. and Numer. Simul. **14**(3–4), 211–215 (2013)

[11] Kerner, R.: $\mathbb{Z}_3$- graded algebras and the cubic root of supersymmetry translations. J. Math. Phys. **33**, 403–411 (1992)

[12] Kerner, R., Suzuki, O.: Internal symmetry groups of cubic algebra. Internat. J. of Geom. Methods in Modern. Phys. **9**, 1261007 (10 pages) (2012)

[13] Gell-Mann, M., Ne'eman, Y.: The Eight-fold Way. W. A. Benjamin Inc, New York-Amsterdam (1964)

[14] Gell-Mann, M.: Symmetries of baryons and mesons. Phys. Rev. **125**(3), 1067–1084 (1962)

[15] Ławrynowicz, J., Ne'eman, Y., Rembieliński, J., Szudy, J. and Wojtczak, L. eds.: Ideas of Albert Abraham Michelson in Mathematical Physics. I-II, Bull. Soc. Sci. Lettres Łódź **52**-**53** Sér. Rech. Déform. **38**-**39**, (2002-2003), 167pp. + 165pp

[16] Ławrynowicz, J., Marchiafava, S., Nowak-Kępczyk, M.: Periodicity theorem for structure fractals in quaternionic formulation. Internat. J. of Geom. Meth. in Modern Phys. **3**, 1167–1197 (2006)

[17] Ławrynowicz, J., Suzuki, O., Castillo Alvarado, F.L.: Basic properties and applications of graded fractal bundles related to Clifford structures. An introduction. Ukrain. Mat. Zh. **60**, 603–618 (2008)

[18] Shaw, R.: Ternary composition algebras: 8 dimensions out of 4? Il nuovo Cimento **104 B**(2), 161–183 (1989)

[19] Ławrynowicz, J., Nôno, K., Nagayama, D., Suzuki, O.: Binary and ternary Clifford analysis on nonion algebra and su(3). Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform **63**(3), 33–48 (2013)

[20] Vaccaro, M.: Subspaces of a paraquaternionic Hermitian vector space. Internat. J. of Geom. Methods in Modern. Phys. **8**(7), 1487–1506 (2011)

[21] Ławrynowicz, J., Vaccaro, M.: Structure fractals in para-quaternionic geometry. Ann. Univ. Mariae Curie-Skłodowska Sect. A Math. **65**(2), 63–73 (2012)

[22] Kovacheva, R. K., Ławrynowicz, J., and Marchiafava, S. (eds.) Applied Complex and Quaternionic Approximation, Ediz. Nuova Cultura Univ. 'La Sapienza', Roma (2009) xxvi + 238pp

[23] Aubin, Th, Ławrynowicz, J., Wojtczak, L.: Nonlinear parabolic equations, relaxation and roughness. Bull. Soc. Sci. Math. (France) (2) **117**, 313–327 (2003)

[24] Ławrynowicz, J., Polatoglou, H.M.: The relaxation and stochastical relaxation problems in crystals in terms of para-quaternions. Acta Physicae Superficierum **12**, 97–107 (2012)

[25] Ławrynowicz, J., Marchiafava, S., Castillo Alvarado, F.L., Niemczynowicz, A.: (Para) quaternionic geometry, harmonic forms, and stochastical relaxation. Publ. Math. Debrecen **84**(1–2), 205–220 (2014)

[26] Ławrynowicz, J., Nôno, K., Nagayama, D., Suzuki, O.: A method for noncommutative Galois theory and constructuion of quark models (Kobayashi-Masukawa model). I. Succesive Galois extensions. Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **63**(1), 95–112 (2013)

[27] Ławrynowicz, J., Nôno, K., Nagayama, D., Suzuki, O.: A method for noncommutative Galois theory and construction of quark models (Kobayashi-Masukawa model). II. Exclusion principles, quark models and colours. Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **63**(2), 79–95 (2013)

[28] Kobayashi, M., Masukawa, T.: CP-violation in the renormalizable theory of weak interaction. Progress Theor. Physics **49**(2), 652–657 (1973)

[29] Huang, K.: Quarks, Leptons and Gauge Fields, 2nd edn. World Scientific, Singapore (2013)

[30] Brzeziński, T.: On modules associated to coalgebra Galois extensions. J. Algebra **(215)**(1), 290–317 (1999)

[31] Brzeziński, T., Hajac, P.M.: Coalgebra extensions and algebra coextensions of Galois type. Comm. Algebra **27**(3), 1347–1367 (1999)

[32] Dąbrowski, L., Grosse, H., Hajac, P.M.: Strong connections and Chern-Connes pairing in the Hopf-Galois theory. Comm. Math. Phys. **220**, 301–331 (2001)

[33] Brzeziński, T., Hajac, P.M.: The Chern-Galois character. C. R. Acad. Sci. Paris **338**, 113–116 (2004)

[34] Krähmer, U., Zieliński, B.: On piecewise trivial Hopf-Galois extensions. Czech. J. Phys. **56**(10/11), 1221–1226 (2006)

[35] Zieliński, B.: "Locally coalgebra-Galois extensions", [arXiv:math/0512150]

[36] Hajac, P.M., Kröhmer, U., Matthes, R., Szymański, W., and Zieliński, B.: "Topological concepts in Hopf-Galois theory", In: Quantum Symmetry in Noncommutative Geometry, Hajac, P.M., ed., EMS Publ. House, to appear

[37] Hajac, P.M., Matthes, R., Sołtan, P.M., Szymański, W., Zieliński, B.: "Hopf-Galois extensions and C*-algebras," ibid., to appear

[38] Aaltonen, T., et al.: (CDF and Do Collaborations), "Higgs boson studies at the Tevatron". Phys. Rev. D **88**, 052014 (2013). [arXiv:1303.6346]

[39] ATLAS Collaboration "Measurments of Higgs boson production and couplings in diboson final states with the ATLAS detector at the LHC", Phys. Lett. B **726**, 88–89 (2013) [arXiv:1507.1427]

[40] CMS Collaboration "Study of the mass and spin-parity of the Higgs boson Candidate via its decays to Z boson pairs", Phys. Rev. Lett. **110**, 081803 (2013); CMS Physics Analysis Summary, HIG-13-002-pas.pdf

[41] ATLAS Collaboration "Evidence for the spin-o nature of the Higgs boson using ATLAS data", *Phys. Lett. B***726**, 120–121 (2013) [arXiv:1307.1432]

[42] The Royal Swedish Academy of Sciences "Class for Physics of the Royal Swedish Academy of Sciences, Scientific background on the Nobel Prize in Physics 2013: The BEH-mechanism interactions with short range forces and scalar particles", Stockholm, 28 pp

[43] Englert, F., Higgs, P.W.: Nobel Lecture 2013 in Physics. The Royal Swedish Academy of Sciences, Stockholm (2013)

[44] Higgs, P.W.: Broken symmetries, massless particles and gauge fields. Phys. Lett. **12**, 132–133 (1964)

[45] Higgs, P.W.: Broken symmetries and the mass of the gauge bosons. Phys. Rev. Lett. **13**, 508–509 (1964)

[46] Englert, F., Brout, R.: Broken symmetry and the mass of the gauge vector mesons. Phys. Rev. Lett. **13**, 321–322 (1964)

[47] Higgs, P.W.: Spontaneous symmetry breakdown without massless boxons. Phys. Rev. **145**, 1156–1166 (1966)

[48] Nambu, Y.: Quasi-particles and gauge invariance in the theory of super- conductivity. Phys. Rev. **117**, 648–660 (1960)

[49] Nambu, Y.: "A superconductor' model of elementary particles and its consequencies", Talk given at a conference at Purdue (1960), reprinted In: Broken Symmetries. Selected Papers by Nambu, Y., ed. by Eguchi, T. and Nishijima, K., World Scientific, Singapore (1995)

[50] Han, M.Y., Nambu, Y.: Three-triplet model with double SU(3) symmetry. Phys. Rev. B **139**, 1006–1020 (1965)

[51] Goldstone, J.: Field theories with superconductor solutions. Nuovo Cim. **19**, 154–164 (1961)

[52] Nowak-Kępczyk, M.: An algebra governing reduction of quaternary structures to ternary structures I. Reductions of quaternary structures to ternary structures. Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **64**((2)), 101–109 (2014)

[53] Nowak-Kępczyk, M. [2014] "An algebra governing reduction of quaternary structures to ternary structures II. A study of the multiplication table for the resulting algebra generators", Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **64**(3), 81–90 (2014)

[54] Nowak-Kępczyk, M.: An algebra governing reduction of quaternary structures to ternary structures III. A study of generations of the resalting algebras. Bull. Soc. Sci. Lettres Łódź Sér. Rech. Déform. **66**(1), 81–90 (2016)

[55] Perk, J.H.H.: Comment on 'Mathematical structure of the three-dimensional (3D) Ising model. Chinese Phys. B **2+2**(8), 080508 (5pp.) (2013). [arXiv:1307.1753v1]

[56] Au-Yang, H., Perk, J.H.H.: Parafermions of the tau-2 model. J. Phys. A: Math. Theor. **47**, 315002 (19 pp.) (2014). [arXiv:1402.0061]

[57] Thouless, D.J., Duncan, F., Haldane, M., Kosterlitz, J.M.: Nobel Prize Lectures in Physics, Stockholm, (2016)

[58] Duncan, F., Haldane, M.: Nobel Lecture: Topological quantum matter. Rev. Mod. Phys. **89**, 040502 (2017)

[59] Kosterlitz, J.M.: Nobel Lecture: Topological defects and phase transitions. Rev. Mod. Phys. bf **89**, 040501 (2017)

[60] Duncan, F., Haldane, M.: Continuum dynamics of the 1-D Heisenberg antiferromagnet: Identification with the O(3) nonlinear sigma model. Phys. Lett. A **93**, 464–468 (1983)

[61] Duncan, F., Haldane, M.: Nonlinear Field Theory of Large-Spin Heisenberg Antiferromagnets: Semiclassically Quantized Solitons of the One-Dimensional Easy-Axis Néel State. Phys. Rev. Lett. **50**, 1153–1156 (1983)

[62] Duncan, F., Haldane, M.: Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly". Phys. Rev. Lett. **61**, 2015–2018 (1988)

[63] Kosterlitz, J.M.: The critical properties of the two-dimensional xy model. J. Phys. C: Solid State Phys. **7**, 1046–1060 (1974)

[64] Kosterlitz, J.M., Thouless, D.J.: Ordering, metastability and phase transitions in two-dimensional systems. J. Phys. C: Solid State Phys. **6**, 1181–1203 (1973)

[65] Kosterlitz, J.M., Thouless, D.J.: Long range order and metastability in two dimensional solids and superfluids. (Application of dislocation theory). J. Phys. C: Solid State Phys. **5**, L124 (1972)

[66] Lipatov, L.N., Rausch de Traubenberg, M., Volkov, G.G.: On the ternary complex analysis and its applications. J. Math. Phys. **49**, 013502 (2008)

[67] Trovon, A., Suzuki, O.: Noncommutative Galois Extensions and Ternary Clifford Analysis. Adv. Appl. Clifford Algebras **27**, 59–70 (2017). https://doi.org/10.1007/s00006-015-0565-6

[68] Connes, A.: Noncommutative geometry. Academic, New York (1994)

[69] Ławrynowicz, J., Suzuki, O., Niemczynowicz, A., Nowak-Kępczyk, M.: "Fractals and chaos related to Ising-Onsager lattices. Relation to the Onsager model" In: Current Research in Mathematical and Computer Sciences II, ed. A. Lecko 131–140 (2018)

J. Ławrynowicz
Faculty of Physics and Applied Informatics
University of Łódź
Pomorska 149/153
90-236 Lodz
Poland
e-mail: jlawryno@uni.lodz.pl

and

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
P.O.B. 2100-956 Warsaw
Poland

O. Suzuki
Department of Computer and System Analysis, College of Humanities and Sciences
Nihon University
Sakurajosui 3-25-40
Setagaya 156-8550
Japan
e-mail: osuzuki1944butterfly@gmail.com

A. Niemczynowicz
Department of Analysis and Differential Equations
University of Warmia and Mazury
Słoneczna 54
10-710 Olsztyn
Poland
e-mail: niemaga@matman.uwm.edu.pl

M. Nowak-Kępczyk
Institute of Mathematics and Computer Science
The John Paul II Catholic University of Lublin
Al. Racławickie 14
P.O. Box 12920-950 Lublin
Poland
e-mail: gosianmk@wp.pl

Received: September 24, 2018.
Accepted: March 19, 2019.