HISTORY AND OVERVIEW OF THE POLYNOMIAL $\mathbf{P}_B^M(X)$

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Abstract. The polynomial $\mathbf{P}_b^m(x)$ is a $2m+1$ degree polynomial in $(x,b) \in \mathbb{R}$ defined by an identity for odd-powers. The odd-power identity is derived applying certain interpolation approaches including systems of linear equations and recurrence relations. This manuscript provides a comprehensive historical survey of the milestones and evolution of the polynomial $\mathbf{P}_b^m(x)$ continuing with related works based on it. Notable results inside related works include the relation between ordinary and partial derivatives for odd-powers, finding the derivative of polynomials via double limit etc. Finally, the manuscript concludes with future research directions and activities.

Contents

1. History and evolution of the polynomial $\mathbf{P}_b^m(x)$

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Sources: https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP

Back than, in 2016 being a student of faculty of mechanical engineering, I remember myself playing with finite differences of the polynomial n^3 over the domain of natural numbers $n \in \mathbb{N}$ having at most $0 \leq n \leq 20$ values. Looking to the values in my finite difference tables, the first and very naive question that came to my mind was

Is it possible to re-assemble the value of the polynomial n^3 backwards having its finite differences?

The answer to this question is definitely *Yes*, utilizing the interpolation principles. Interpolation is a process of finding new data points based on the range of a discrete set of known data points. Interpolation has been well-developed in between 1674–1684 by Issac Newton's fundamental works, nowadays known as foundation of classical interpolation theory [1].

That time, in 2016, I was a first-year mechanical engineering undergraduate, so that due to lack of knowledge and perspective of view I started re-inventing interpolation formula myself, fueled by purest passion and feeling of mystery. All mathematical laws and relations exist from the very beginning, but we only find and describe them, I thought. That mindset truly inspired me so that my own mathematical journey has been started. Let us begin considering the table of finite differences of the polynomial n^3

n	n^3	$\Delta(n^3)$	$\Delta^2(n^3)$	$\Delta^3(n^3)$
0	0	1	6	6
1	1	7	12	6
2	8	19	18	6
3	27	37	24	6
4	64	61	30	6
5	125	91	36	
6	216	127		
7	343			

Table 1. Table of finite differences of the polynomial n^3.

First and foremost, we can observe that finite difference $\Delta(n^3)$ of the polynomial n^3 can be expressed via summation over n, e.g

$$\begin{aligned}
\Delta(0^3) &= 1 + 6 \cdot 0 \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 \\
&\vdots
\end{aligned} \tag{1.1}$$

Finally reaching its generic form

$$\Delta(n^3) = 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n = 1 + 6 \sum_{k=0}^{n} k \tag{1.2}$$

The one experienced mathematician would immediately notice a spot to apply Faulhaber's formula [2] to expand the term $\sum_{k=0}^{n} k$ reaching expected result that matches Binomial theorem [3], so that

$$\sum_{k=0}^{n} k = \frac{1}{2}(n + n^2)$$

Then our relation (1.2) immediately turns into Binomial expansion

$$\Delta(n^3) = (n+1)^3 - n^3 = 1 + 6\left[\frac{1}{2}(n + n^2)\right] = 1 + 3n + 3n^2 = \sum_{k=0}^{2} \binom{3}{k} n^k \tag{1.3}$$

However, as it said, I was not the experienced one mathematician back than, so that I reviewed the relation (1.2) from a little bit different perspective. Not following the convenient solution (1.3), I have rearranged the first order finite differences from the table (1) using (1.1) to get the polynomial n^3

$$\begin{aligned}
n^3 = {} & [1 + 6 \cdot 0] + [1 + 6 \cdot 0 + 6 \cdot 1] + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2] + \cdots \\
& + [1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot (n-1)]
\end{aligned} \tag{1.4}$$

Then, rearranging the terms of the equation (1.4) so that it turns into summation in terms of $k(n-k)$

$$n^3 = n + [(n-0)\cdot 6\cdot 0] + [(n-1)\cdot 6\cdot 1] + [(n-2)\cdot 6\cdot 2] + \cdots$$
$$\cdots + [(n-k)\cdot 6\cdot k] + \cdots + [1\cdot 6\cdot (n-1)]$$

Gives the interpolation of the polynomial n^3

$$n^3 = \sum_{k=1}^{n} 6k(n-k) + 1 \tag{1.5}$$

It is immediately seen that (1.5) is true by observing the table of $6k(n-k)+1$ values

n/k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	7	1					
3	1	13	13	1				
4	1	19	25	19	1			
5	1	25	37	37	25	1		
6	1	31	49	55	49	31	1	
7	1	37	61	73	73	61	37	1

Table 2. Values of $6k(n-k)+1$. See the OEIS entry: A287326 [4]. Sequences such that row sums give the polynomials n^5 and n^7 are also registered in OEIS [5, 6].

Therefore, we have reached our base case by successfully interpolating the polynomial n^3. Fairly enough that the next curiosity would be

Well, if the relation (1.5) true for the polynomial n^3, then is it true that (1.5) can be generalized for higher powers, e.g. for n^4 or n^5 either?

That was my next question, however without any expectation of the final form of generalized relation. Soon enough my idea was caught by other people. In 2018, Albert Tkaczyk has published two of his works [7, 8] showing the cases for polynomials n^5, n^7 and n^9 that were obtained similarly as (1.5). In short, it appears that relation (1.5) could be generalized for any non-negative odd power $2m+1$ solving a system of linear equations. It was proposed that the case for n^5 has explicit form

$$n^5 = \sum_{k=1}^{n} \left[Ak^2(n-k)^2 + Bk(n-k) + C\right]$$

where A, B, C are yet-unknown coefficients. Denote A, B, C as $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ to reach the form of a compact double sum

$$n^5 = \sum_{k=1}^{n} \sum_{r=0}^{2} \mathbf{A}_{2,r} k^r (n-k)^r$$

Observing the equation above, the potential form of generalized odd-power identity becomes more obvious. To evaluate the coefficients $\mathbf{A}_{2,0}, \mathbf{A}_{2,1}, \mathbf{A}_{2,2}$ it is necessary construct and solve a system of linear equations following the process

$$\begin{aligned} n^5 &= \sum_{r=0}^{2} \mathbf{A}_{2,r} \sum_{k=1}^{n} k^r (n-k)^r \\ &= \mathbf{A}_{2,0} \sum_{k=1}^{n} k^0 (n-k)^0 + \mathbf{A}_{2,1} \sum_{k=1}^{n} k^1 (n-k)^1 + \mathbf{A}_{2,2} \sum_{k=1}^{n} k^2 (n-k)^2 \end{aligned}$$

Expand the terms $\sum_{k=1}^{n} k^r (n-k)^r$ applying the Faulhaber's formula [2] to get the equation

$$\mathbf{A}_{m,0} n + \mathbf{A}_{m,1} \left[\frac{1}{6}(-n+n^3)\right] + \mathbf{A}_{m,2} \left[\frac{1}{30}(-n+n^5)\right] - n^5 = 0$$

Multiplying by 30 both right-hand side and left-hand side, we get

$$30\mathbf{A}_{2,0} n + 5\mathbf{A}_{2,1}(-n+n^3) + \mathbf{A}_{2,2}(-n+n^5) - 30n^5 = 0$$

Expanding the brackets and rearranging the terms gives

$$30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} n + 5\mathbf{A}_{2,1} n^3 - \mathbf{A}_{2,2} n + \mathbf{A}_{2,2} n^5 - 30n^5 = 0$$

Combining the common terms yields

$$n(30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2}) + 5\mathbf{A}_{2,1} n^3 + n^5(\mathbf{A}_{2,2} - 30) = 0$$

Therefore, the system of linear equations follows

$$\begin{cases} 30\mathbf{A}_{2,0} - 5\mathbf{A}_{2,1} - \mathbf{A}_{2,2} = 0 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,2} - 30 = 0 \end{cases}$$

Solving it, we get

$$\begin{cases} \mathbf{A}_{2,2} = 30 \\ \mathbf{A}_{2,1} = 0 \\ \mathbf{A}_{2,0} = 1 \end{cases}$$

So that the odd-power identity holds

$$n^5 = \sum_{k=1}^{n} 30k^2(n-k)^2 + 1$$

It is also clearly seen why the above identity is true by arranging the terms $30k^2(n-k)^2+1$ over $0 \leq k \leq n$ as tabular. See the OEIS sequence [5].

Now we can see that the relation (1.5) we got via interpolation of cubes can be generalized for all non-negative odd-powers $2m+1$ by constructing and solving a system of linear equations. Therefore, the generalized form of odd-power identity is

$$n^{2m+1} = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=1}^{n} k^r (n-k)^r \tag{1.6}$$

where $\mathbf{A}_{m,r}$ are real coefficients. In more details, the equation (1.6) is discussed separately in [9, 10].

However, constructing and solving a system of linear equations for every odd-power $2m+1$ requires a huge effort, there must be a formula that generates a set of real coefficients $\mathbf{A}_{m,r}$ for each fixed m, I thought. As it turned out, that assumption was correct. So that I reached MathOverflow community in search of answers that arrived quite shortly. In [11], Dr. Max Alekseyev has provided a complete and comprehensive formula to calculate coefficient $\mathbf{A}_{m,r}$ for each natural $m \geq 0,\ 0 \leq r \leq m$. The main idea of Alekseyev's approach was to utilize dynamic programming methods to evaluate the $\mathbf{A}_{m,r}$ recursively, taking base case $\mathbf{A}_{m,m}$

then evaluating the next coefficient $\mathbf{A}_{m,m-1}$ via backtracking, continuing similarly up to $\mathbf{A}_{m,0}$. Before we consider the derivation of the recurrent formula for coefficients $\mathbf{A}_{m,r}$, a few words must be said regarding the Faulhaber's formula [2]

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j}$$

it is important to notice that iteration step j is bounded by the value of exponent p, while the upper bound of the binomial coefficient is $p+1$. That means we cannot omit summation bounds letting j run over infinity, unless we perform the following action on the Faulhaber's formula

$$\begin{aligned}\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j} &= \left[\frac{1}{p+1}\sum_{j=0}^{p+1}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1} \\ &= \left[\frac{1}{p+1}\sum_{j}\binom{p+1}{j}B_j n^{p+1-j}\right] - B_{p+1}\end{aligned} \tag{1.7}$$

At this point we are good to go through the entire derivation of the recurrent formula for coefficients $\mathbf{A}_{m,r}$. Applying both Binomial theorem and Faulhaber's formula (1.7) to the equation (1.6) we get

$$\begin{aligned}&\sum_{k=1}^{n} k^r(n-k)^r = \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\sum_{k=1}^{n}k^{t+r} \\ &= \sum_{t=0}^{r}(-1)^t\binom{r}{t}n^{r-t}\left[\frac{1}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{t+r+1-j} - B_{t+r+1}\right] \\ &= \sum_{t=0}^{r}\binom{r}{t}\left[\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - B_{t+r+1}n^{r-t}\right] \\ &= \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}\sum_{j}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t} \\ &= \sum_{j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j}B_j n^{2r+1-j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t} \\ &= \sum_{j}B_j n^{2r+1-j}\sum_{t}\binom{r}{t}\frac{(-1)^t}{t+r+1}\binom{t+r+1}{j} - \sum_{t=0}^{r}\binom{r}{t}\frac{(-1)^t}{t+r+1}B_{t+r+1}n^{r-t}\end{aligned}$$

We can notice that

$$\sum_t \binom{r}{t} \frac{(-1)^t}{r+t+1} \binom{r+t+1}{j} = \begin{cases} \frac{1}{(2r+1)\binom{2r}{r}} & \text{if } j=0 \\ \frac{(-1)^r}{j}\binom{r}{2r-j+1} & \text{if } j>0 \end{cases} \tag{1.8}$$

An elegant proof of the binomial identity (1.8) is presented in [12].

In particular, the equation (1.8) is zero for $0 < t \leq j$. So that taking $j = 0$ we have

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{j \geq 1} B_j n^{2r+1-j} \sum_t \binom{r}{t} \frac{(-1)^t}{t+r+1} \binom{t+r+1}{j} \right] - \left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]$$

Simplifying above equation via (1.8) we get

$$\sum_{k=1}^{n} k^r (n-k)^r = \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \underbrace{\left[\sum_{j \geq 1} \frac{(-1)^r}{j} \binom{r}{2r-j+1} B_j n^{2r-j+1} \right]}_{(\star)} - \underbrace{\left[\sum_{t=0}^{r} \binom{r}{t} \frac{(-1)^t}{t+r+1} B_{t+r+1} n^{r-t} \right]}_{(\diamond)}$$

Hence, introducing $\ell = 2r - j + 1$ to $(\star)$ and $\ell = r - t$ to $(\diamond)$ we collapse the common terms in the equation above so that

$$\begin{aligned} \sum_{k=1}^{n} k^r (n-k)^r &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + \left[\sum_{\ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &\quad - \left[\sum_{\ell} \binom{r}{\ell} \frac{(-1)^{r-\ell}}{2r+1-\ell} B_{2r+1-\ell} n^{\ell} \right] \\ &= \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \end{aligned}$$

Assuming that $\mathbf{A}_{m,r}$ is defined by (1.6), we obtain the following relation for polynomials in n

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2 \sum_r \mathbf{A}_{m,r} \sum_{\text{odd } \ell} \frac{(-1)^r}{2r+1-\ell} \binom{r}{\ell} B_{2r+1-\ell} n^{\ell} \equiv n^{2m+1}$$

Replacing odd ℓ by d we get

$$\sum_r \mathbf{A}_{m,r} \frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1} + 2\sum_r \mathbf{A}_{m,r} \sum_d \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1} \equiv n^{2m+1}$$

$$\sum_r \mathbf{A}_{m,r} \left[\frac{1}{(2r+1)\binom{2r}{r}} n^{2r+1}\right] + 2\sum_r \mathbf{A}_{m,r} \left[\sum_d \frac{(-1)^r}{2r-2d}\binom{r}{2d+1} B_{2r-2d} n^{2d+1}\right] - n^{2m+1} = 0 \tag{1.9}$$

Let be $r = m$, then taking the coefficient of n^{2m+1} in (1.9) we get

$$\mathbf{A}_{m,m} = (2m+1)\binom{2m}{m}$$

and taking the coefficient of n^{2d+1} for an integer d in the range $m/2 \leq d < m$, we get

$$\mathbf{A}_{m,d} = 0$$

Taking the coefficient of n^{2d+1} for d in the range $m/4 \leq d < m/2$ we get

$$\mathbf{A}_{m,d} \frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d} B_{2m-2d} = 0$$

i.e

$$\mathbf{A}_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d} B_{2m-2d}$$

Continue similarly we can compute $\mathbf{A}_{m,r}$ for each integer r in range $m/2^{s+1} \leq r < m/2^s$ (iterating consecutively $s = 1, 2, \ldots$) via previously determined values of $\mathbf{A}_{m,d}$ as follows

$$\mathbf{A}_{m,r} = (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d} \binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r} B_{2d-2r}$$

Finally, we are capable to define the coefficient $\mathbf{A}_{m,r}$ via the next recurrent relation

Definition 1.1. *(Definition of the real coefficients $\mathbf{A}_{m,r}$.)*

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r} & \text{if } r = m \\ (2r+1)\binom{2r}{r} \sum_{d \geq 2r+1}^{m} \mathbf{A}_{m,d}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r} B_{2d-2r} & \text{if } 0 \leq r < m \\ 0 & \text{if } r < 0 \text{ or } r > m \end{cases} \tag{1.10}$$

where B_t are Bernoulli numbers [13]. It is assumed that $B_1 = \frac{1}{2}$. For example,

m/r	0	1	2	3	4	5	6	7
0	1							
1	1	6						
2	1	0	30					
3	1	-14	0	140				
4	1	-120	0	0	630			
5	1	-1386	660	0	0	2772		
6	1	-21840	18018	0	0	0	12012	
7	1	-450054	491400	-60060	0	0	0	51480

Table 3. Coefficients $\mathbf{A}_{m,r}$.

The nominators and denominators of the coefficients $\mathbf{A}_{m,r}$ are also registered as sequences in OEIS [14, 15]. It is as well interesting to notice that row sums of the $\mathbf{A}_{m,r}$ give powers of 2

$$\sum_{r=0}^{m} \mathbf{A}_{m,r} = 2^{2m+1} - 1$$

Let be a theorem

Theorem 1.2. *For every $n \geq 1,\ n, m \in \mathbb{N}$ there are $\mathbf{A}_{m,0}, \mathbf{A}_{m,1}, \dots, \mathbf{A}_{m,m}$, such that*

$$n^{2m+1} = \sum_{k=1}^{n} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

where $\mathbf{A}_{m,r}$ is a real coefficient defined recursively by (1.10).

Finally, we got our road to the main definition of the polynomial $\mathbf{P}_b^m(x)$. Introducing the parameter b to the upper summation bound of the equation (1.2), we have the definition

Definition 1.3. *(Polynomial $\mathbf{P}_b^m(x)$of degree $2m+1$.)*

$$\mathbf{P}_b^m(x) = \sum_{k=0}^{b-1} \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (x-k)^r \tag{1.11}$$

where $\mathbf{A}_{m,r}$ are real coefficients (1.10). A comprehensive discussion on the polynomial $\mathbf{P}_b^m(x)$ as well as its properties can be found at [16]. In 2023, Albert Tkaczyk yet again extended the theorem (1.2) to the so-called three dimension case so that it gives polynomials of the form n^{3l+2} at [17].

2. Related works

In this section let's give a short overview of related works that are based onto definition of polynomials $\mathbf{P}_b^m(x)$. In [18] is given a relation in terms of partial differential differential equations such that ordinary derivative of odd-power $2m+1$ can be reached in terms of partial derivatives of $\mathbf{P}_b^m(x)$. Let be a fixed point $v \in \mathbb{N}$, then ordinary derivative $\frac{d}{dx} g_v(u)$ of the odd-power function $g_v(x) = x^{2v+1}$ evaluate in point $u \in \mathbb{R}$ equals to partial derivative $(f_v)_x'(u,u)$ evaluate in point (u,u) plus partial derivative $(f_v)_z'(u,u)$ evaluate in point (u,u)

$$\frac{d}{dx} g_v(u) = (f_v)_x'(u,u) + (f_v)_z'(u,u) \tag{2.1}$$

where $f_y(x,z) = \sum_{k=1}^{z} \sum_{r=0}^{y} \mathbf{A}_{y,r} k^r (x-k)^r = \mathbf{P}_z^y(x)$. Afterward, the equation (2.1) is generalized over the timescales $\mathbb{T} \times \mathbb{T}$ providing its dynamic equation analog in [19].

Second article [20] gives another perspective of ordinary derivatives of polynomials expressing them via double limit as

$$\lim_{h \to 0} \mathbf{P}_{x+h}^m(x) = x^{2m+1}$$

that opens such opportunity.

In [21] based on (1.9), the authors give a new identity involving Bernoulli polynomials and combinatorial numbers that provides, in particular, the Faulhaber-like formula for sums of the form $1^m(n-1)^m + 2^m(n-2)^m + \cdots + (n-1)^m 1^m$ for positive integers m and n.

Three sequences were contributed to OEIS [22, 23, 24] showing the coefficients of the polynomial $\mathbf{P}_b^m(x)$ having fixed points m, b while $x \in \mathbb{R}$.

3. Future research and activities

- Differential equation (2.1) can also be expressed in terms of backward and central differential operators, including derivatives on time-scales so that results of [19] could be generalized further.
- Theorem (1.2) gives an opportunity to express odd-power identity in terms of multiplication of certain matrices.
- There are Taylor series and Maclaurin series versions in terms of $\mathbf{P}_b^m(x)$.
- The summation bounds of definition (1.11) can be altered so that k runs over $1 \leq k \leq b$, by symmetry.
- Prove that $\mathbf{P}_b^m(x)$ is an integer valued polynomial in (x, b).
- Definition (1.11) is closely related to discrete convolution because

$$\mathbf{P}_b^m(x) = \sum_{r=0}^{m} \mathbf{A}_{m,r} \sum_{k=0}^{b-1} k^r (x-k)^r$$

where $\sum_{k=0}^{b-1} k^r (x-k)^r$ is the discrete convolution of x^r. It is worth to get a closer look at it so that new relations in terms of discrete convolution may be found.

- All kinds of derivatives e.g forward, backward and central, including the derivatives on time-scales can be expressed as double limit of $\mathbf{P}_b^m(x)$ extending the results of [20].
- Introducing the definition of coefficient $\left\llbracket {m,n \atop k} \right\rrbracket$

$$\left\llbracket {m,n \atop k} \right\rrbracket = \sum_{r=0}^{m} \mathbf{A}_{m,r} k^r (n-k)^r$$

the novel identities can be reached, for example

$$\left\llbracket {m, 2t+1 \atop 1} \right\rrbracket = \left\llbracket {m, t+2 \atop 2} \right\rrbracket$$

$$\left\llbracket {m, n \atop k} \right\rrbracket = \left\llbracket {m, n \atop n-k} \right\rrbracket$$

$$\left\llbracket {m, 2t-3r \atop r} \right\rrbracket = \left\llbracket {m, t \atop 2r} \right\rrbracket = \left\llbracket {m, 2t-3r \atop 2t-4r} \right\rrbracket$$

so that combinatorial sense of above is also a topic to research.

- Following the results of https://arxiv.org/pdf/1603.02468v15.pdf, the equation (1.11) approximates the odd-power polynomial x^{2m+1} around given points x_i as it may be observed from the following plots

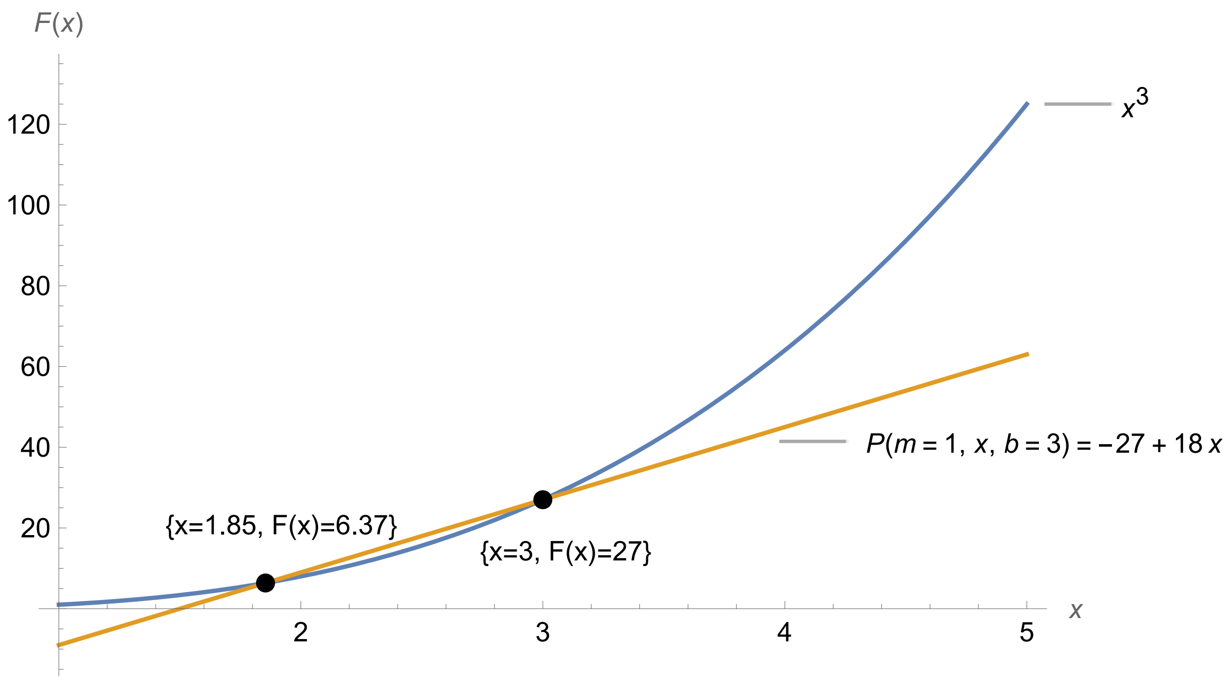

Figure 1. Approximation of x^3.

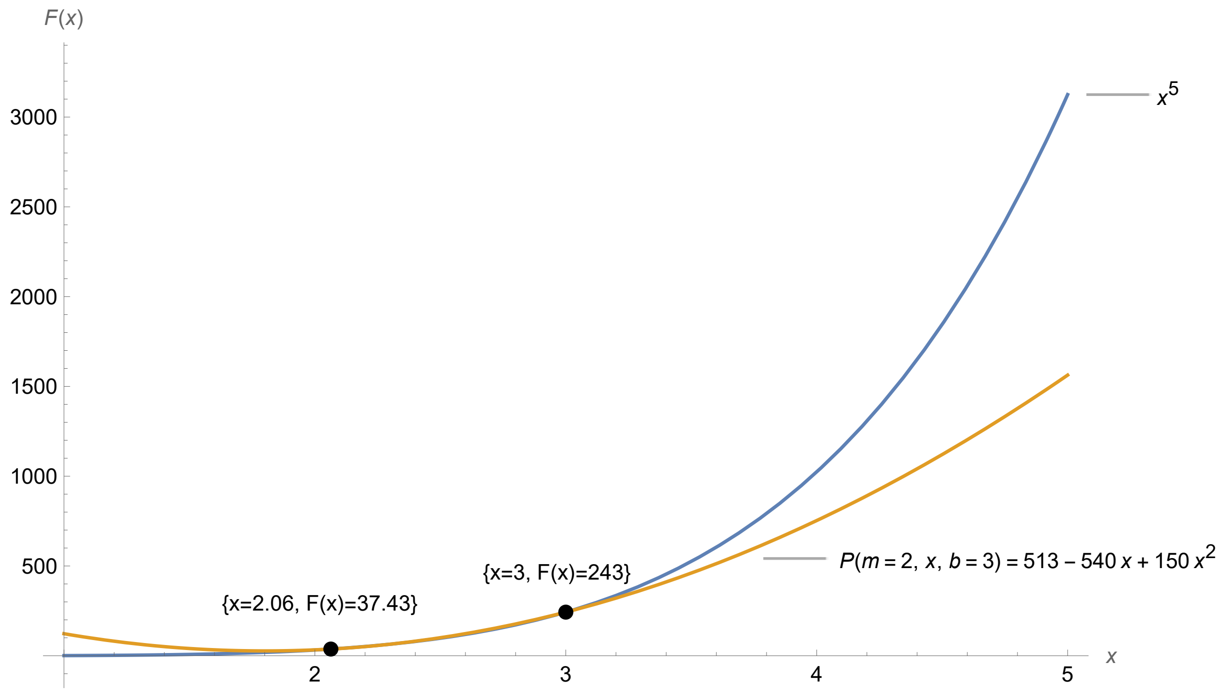

Figure 2. Approximation of x^5.

- English grammar reviews and improvements are welcome.
- Improvements and suggestions to current manuscript under open-source initiatives at https://github.com/kolosovpetro/HistoryAndOverviewOfPolynomialP

4. Conclusions

In this manuscript we have successfully provided a comprehensive historical survey of the milestones and evolution of the polynomial $\mathbf{P}_b^m(x)$ as well as related works such that based onto, for instance various polynomial identities, differential equations etc. In addition, future research directions are proposed and discussed.

References

[1] Meijering, Erik. A chronology of interpolation: from ancient astronomy to modern signal and image processing. *Proceedings of the IEEE*, 90(3):319–342, 2002. https://infoscience.epfl.ch/record/63085/files/meijering0201.pdf.

[2] Alan F. Beardon. Sums of powers of integers. *The American mathematical monthly*, 103(3):201–213, 1996. https://les-mathematiques.net/vanilla/uploads/editor/5w/3wm5emwr0i7p.pdf.

[3] Milton Abramowitz, Irene A. Stegun, and Robert H. Romer. Handbook of mathematical functions with formulas, graphs, and mathematical tables, 1988.

[4] Petro Kolosov. Numerical triangle, row sums give third power, Entry A287326 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A287326, 2017.

[5] Petro Kolosov. Numerical triangle, row sums give fifth power, Entry A300656 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A300656, 2018.

[6] Petro Kolosov. Numerical triangle, row sums give seventh power, Entry A300785 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A300785, 2018.

[7] Tkaczyk, Albert. About the problem of a triangle developing the polynomial function. Published electronically at LinkedIn, 2018.

[8] Tkaczyk, Albert. On the problem of a triangle developing the polynomial function - continuation. Published electronically at LinkedIn, 2018.

[9] Kolosov, Petro. 106.37 An unusual identity for odd-powers. *The Mathematical Gazette*, 106(567):509–513, 2022. https://doi.org/10.1017/mag.2022.129.

[10] Petro Kolosov. Polynomial identity involving Binomial Theorem and Faulhaber's formula. Published electronically at https://kolosovpetro.github.io/pdf/PolynomialIdentityInvolvingBTandFaulhaber.pdf, 2023.

[11] Alekseyev, Max. MathOverflow answer 297916/113033. Published electronically at https://mathoverflow.net/a/297916/113033, 2018.

[12] Scheuer, Markus. MathStackExchange answer 4724343/463487. Published electronically at https://math.stackexchange.com/a/4724343/463487, 2023.

[13] Harry Bateman. *Higher transcendental functions [volumes i-iii]*, volume 1. McGRAW-HILL book company, 1953.

[14] Petro Kolosov. Entry A302971 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A302971, 2018.

[15] Petro Kolosov. Entry A304042 in The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A304042, 2018.

[16] Petro Kolosov. On the link between binomial theorem and discrete convolution. *arXiv preprint arXiv:1603.02468*, 2016. https://arxiv.org/abs/1603.02468.

[17] Albert Tkaczyk. On three-dimensional expansions of the polynomial function f(n)=nm̂, September 2023. https://doi.org/10.5281/zenodo.8371454.

[18] Petro Kolosov. Another approach to get derivative of odd-power. *arXiv preprint arXiv:2310.07804*, 2023. https://arxiv.org/abs/2310.07804.

[19] Petro Kolosov. A study on partial dynamic equation on time scales involving derivatives of polynomials. *arXiv preprint arXiv:1608.00801*, 2016. https://arxiv.org/abs/1608.00801.

[20] Petro Kolosov. Finding the derivative of polynomials via double limit, January 2024. https://doi.org/10.5281/zenodo.10575485.

[21] J. Fernando Barbero G., Juan Margalef-Bentabol, and Eduardo J.S. Villaseñor. A two-sided Faulhaber-like formula involving Bernoulli polynomials. *Comptes Rendus. Mathématique*, 358(1):41–44, 2020. https://doi.org/10.5802/crmath.10.

[22] Petro Kolosov. The coefficients u(m, l, k), m = 1 defined by the polynomial identity, 2018. https://oeis.org/A320047.

[23] Petro Kolosov. The coefficients u(m, l, k), m = 2 defined by the polynomial identity, 2018. https://oeis.org/A316349.

[24] Petro Kolosov. The coefficients u(m, l, k), m = 3 defined by the polynomial identity, 2018. https://oeis.org/A316387.

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5. Addendum 1: Examples of the polynomial $\mathbf{P}_b^m(x)$

$$\mathbf{P}_b^0(x) = b$$

$$\mathbf{P}_b^1(x) = 3b^2 - 2b^3 - 3bx + 3b^2x$$

$$\mathbf{P}_b^2(x) = 10b^3 - 15b^4 + 6b^5 - 15b^2x + 30b^3x - 15b^4x + 5bx^2 - 15b^2x^2 + 10b^3x^2$$

$$\mathbf{P}_b^3(x) = -7b^2 + 28b^3 - 70b^5 + 70b^6 - 20b^7 + 7bx - 42b^2x + 175b^4x - 210b^5x + 70b^6x$$
$$+ 14bx^2 - 140b^3x^2 + 210b^4x^2 - 84b^5x^2 + 35b^2x^3 - 70b^3x^3 + 35b^4x^3$$

$$\mathbf{P}_b^4(x) = -60b^2 + 180b^3 - 294b^5 + 420b^7 - 315b^8 + 70b^9 + 60bx - 270b^2x + 735b^4x - 1470b^6x$$
$$+ 1260b^7x - 315b^8x + 90bx^2 - 630b^3x^2 + 1890b^5x^2 - 1890b^6x^2 + 540b^7x^2 + 210b^2x^3$$
$$- 1050b^4x^3 + 1260b^5x^3 - 420b^6x^3 - 21bx^4 + 210b^3x^4 - 315b^4x^4 + 126b^5x^4$$

$$\mathbf{P}_b^5(x) = -693b^2 + 2068b^3 - 330b^4 - 2640b^5 + 2772b^7 - 2310b^9 + 1386b^{10} - 252b^{11} + 693bx$$
$$- 3102b^2x + 660b^3x + 6600b^4x - 9702b^6x + 10395b^8x - 6930b^9x + 1386b^{10}x + 1034bx^2$$
$$- 330b^2x^2 - 5940b^3x^2 + 12936b^5x^2 - 18480b^7x^2 + 13860b^8x^2 - 3080b^9x^2 + 2310b^2x^3$$
$$- 8085b^4x^3 + 16170b^6x^3 - 13860b^7x^3 + 3465b^8x^3 - 330bx^4 + 2310b^3x^4 - 6930b^5x^4$$
$$+ 6930b^6x^4 - 1980b^7x^4 - 231b^2x^5 + 1155b^4x^5 - 1386b^5x^5 + 462b^6x^5$$

$$\mathbf{P}_b^6(x) = -10920b^2 + 33306b^3 - 9009b^4 - 36036b^5 + 37752b^7 - 22022b^9 + 12012b^{11} - 6006b^{12} + 924b^{13}$$
$$+ 10920bx - 49959b^2x + 18018b^3x + 90090b^4x - 132132b^6x + 99099b^8x - 66066b^{10}x + 36036b^{11}x$$
$$- 6006b^{12}x + 16653bx^2 - 9009b^2x^2 - 84084b^3x^2 + 180180b^5x^2 - 180180b^7x^2 + 150150b^9x^2$$
$$- 90090b^{10}x^2 + 16380b^{11}x^2 + 36036b^2x^3 - 120120b^4x^3 + 168168b^6x^3 - 180180b^8x^3$$
$$+ 120120b^9x^3 - 24024b^{10}x^3 - 6006bx^4 + 40040b^3x^4 - 84084b^5x^4 + 120120b^7x^4 - 90090b^8x^4$$
$$+ 20020b^9x^4 - 6006b^2x^5 + 21021b^4x^5 - 42042b^6x^5 + 36036b^7x^5 - 9009b^8x^5 + 286bx^6$$
$$- 2002b^3x^6 + 6006b^5x^6 - 6006b^6x^6 + 1716b^7x^6$$

6. Addendum 2: Derivation of the coefficients $\mathbf{A}_{m,r}$

Consider the definition (1.10) of the coefficients $\mathbf{A}_{m,r}$, it can be written as

$$\mathbf{A}_{m,r} := \begin{cases} (2r+1)\binom{2r}{r}, & \text{if } r = m; \\ \sum_{d\geq 2r+1}^{m} \mathbf{A}_{m,d} \underbrace{(2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}}_{T(d,r)}, & \text{if } 0 \leq r < m; \\ 0, & \text{if } r < 0 \text{ or } r > m, \end{cases}$$

Therefore, let be a definition of the real coefficient $T(d,r)$

Definition 6.1. *Real coefficient* $T(d,r)$

$$T(d,r) = (2r+1)\binom{2r}{r}\binom{d}{2r+1}\frac{(-1)^{d-1}}{d-r}B_{2d-2r}$$

Example 6.2. *Let be* $m = 2$ *so first we get* $\mathbf{A}_{2,2}$

$$\mathbf{A}_{2,2} = 5\binom{4}{2} = 30$$

Then $\mathbf{A}_{2,1} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $1 \leq d < 2$*. Finally, the coefficient* $\mathbf{A}_{2,0}$ *is*

$$\begin{aligned}\mathbf{A}_{2,0} = \sum_{d\geq 1}^{2} \mathbf{A}_{2,d} \cdot T(d,0) &= \mathbf{A}_{2,1} \cdot T(1,0) + \mathbf{A}_{2,2} \cdot T(2,0) \\ &= 30 \cdot \frac{1}{30} = 1\end{aligned}$$

Example 6.3. *Let be* $m = 3$ *so that first we get* $\mathbf{A}_{3,3}$

$$\mathbf{A}_{3,3} = 7\binom{6}{3} = 140$$

Then $\mathbf{A}_{3,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for* d *in* $2 \leq d < 3$*. The* $\mathbf{A}_{3,1}$ *coefficient is non-zero and calculated as*

$$\mathbf{A}_{3,1} = \sum_{d\geq 3}^{3} \mathbf{A}_{3,d} \cdot T(d,1) = \mathbf{A}_{3,3} \cdot T(3,1) = 140 \cdot \left(-\frac{1}{10}\right) = -14$$

Finally, the coefficient $\mathbf{A}_{3,0}$ *is*

$$\mathbf{A}_{3,0} = \sum_{d\geq 1}^{3} \mathbf{A}_{3,d} \cdot T(d,0) = \mathbf{A}_{3,1} \cdot T(1,0) + \mathbf{A}_{3,2} \cdot T(2,0) + \mathbf{A}_{3,3} \cdot T(3,0)$$

$$= -14 \cdot \frac{1}{6} + 140 \cdot \frac{1}{42} = 1$$

Example 6.4. *Let be* $m = 4$ *so that first we get* $\mathbf{A}_{4,4}$

$$\mathbf{A}_{4,4} = 9\binom{8}{4} = 630$$

Then $\mathbf{A}_{4,3} = 0$ *and* $\mathbf{A}_{4,2} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for d in* $2 \leq d < 4$. *The value of the coefficient* $\mathbf{A}_{4,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{4,1} = \sum_{d\geq 3}^{4} \mathbf{A}_{4,d} \cdot T(d,1) = \mathbf{A}_{4,3} \cdot T(3,1) + \mathbf{A}_{4,4} \cdot T(4,1) = 630 \cdot \left(-\frac{4}{21}\right) = -120$$

Finally, the coefficient $\mathbf{A}_{4,0}$ *is*

$$\mathbf{A}_{4,0} = \sum_{d\geq 1}^{4} \mathbf{A}_{4,d} \cdot T(d,0) = \mathbf{A}_{4,1} \cdot T(1,0) + \mathbf{A}_{4,4} \cdot T(4,0) = -120 \cdot \frac{1}{6} + 630 \cdot \frac{1}{30} = 1$$

Example 6.5. *Let be* $m = 5$ *so that first we get* $\mathbf{A}_{5,5}$

$$\mathbf{A}_{5,5} = 11\binom{10}{5} = 2772$$

Then $\mathbf{A}_{5,4} = 0$ *and* $\mathbf{A}_{5,3} = 0$ *because* $\mathbf{A}_{m,d}$ *is zero in the range* $m/2 \leq d < m$ *means that zero for d in* $3 \leq d < 5$. *The value of the coefficient* $\mathbf{A}_{5,2}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,2} = \sum_{d\geq 5}^{5} \mathbf{A}_{5,d} \cdot T(d,2) = \mathbf{A}_{5,5} \cdot T(5,2) = 2772 \cdot \frac{5}{21} = 660$$

The value of the coefficient $\mathbf{A}_{5,1}$ *is non-zero and calculated as*

$$\mathbf{A}_{5,1} = \sum_{d\geq 3}^{5} \mathbf{A}_{5,d} \cdot T(d,1) = \mathbf{A}_{5,3} \cdot T(3,1) + \mathbf{A}_{5,4} \cdot T(4,1) + \mathbf{A}_{5,5} \cdot T(5,1)$$

$$= 2772 \cdot \left(-\frac{1}{2}\right) = -1386$$

Finally, the coefficient $\mathbf{A}_{5,0}$ *is*

$$\mathbf{A}_{5,0} = \sum_{d \geq 1}^{5} \mathbf{A}_{5,d} \cdot T(d,0) = \mathbf{A}_{5,1} \cdot T(1,0) + \mathbf{A}_{5,2} \cdot T(2,0) + \mathbf{A}_{5,5} \cdot T(5,0)$$
$$= -1386 \cdot \frac{1}{6} + 660 \cdot \frac{1}{30} + 2772 \cdot \frac{5}{66} = 1$$

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