High-order Mesh-free Numerical Quadrature for Trimmed Curved Parametric Domains

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Abstract

This work presents a high-order, mesh-free, generalized Stokes' theorem-based numerical quadrature scheme for integrating arbitrary functions over domains in $\mathbb{R}^3$ bounded by trimmed curved parametric surfaces. The algorithm proceeds in three steps: (1) surface-surface intersection is performed to find a high-order approximation to each edge curve of the trimmed region, (2) Stokes' theorem is applied to the integrand to transform volumetric integrals into surface integrals over each boundary parametric surface, and (3) Green's theorem is applied over each boundary parametric surface to transform surface integrals into line integrals along each approximated edge curve. Preliminary results indicate that this approach can achieve high efficiency due to its ability to attain high-order convergence without meshing of the trimmed geometry. Applications include moment-fitting methods, remapping between high-order meshes, fictitious domain methods, among others.

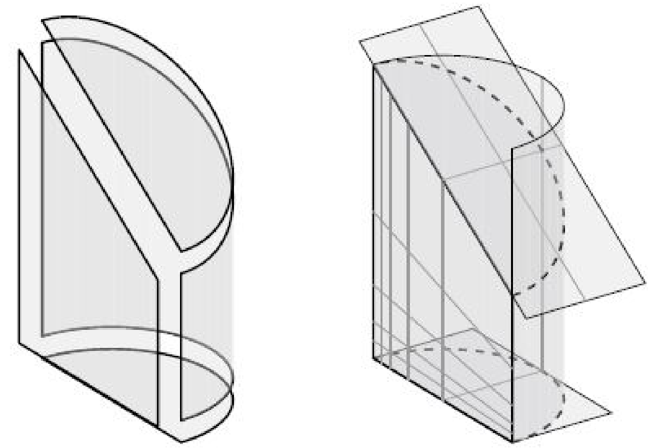

Figure 1: *An example of a region formed by trimmed surface patches. On the left, the region is bounded by four surfaces. On the right, the original untrimmed parametric patches are shown, with trimming curves in physical space shown as dashed lines. The region's boundaries are difficult to approximate using only tensor-product patches. Image courtesy of [3].*

1 Introduction

Quadrature over regions in $\mathbb{R}^3$ bounded by trimmed parametric surfaces is important in a variety of applications. For example, in computer-aided design (CAD) three-dimensional geometric objects are often given solely in terms of boundary representations (BREPs), often isoparametrically in terms of tensor-product non-uniform rational B-spline (NURBS) surfaces. In many CAD programs, trimmed NURBS patches – formed by discarding all parts of a NURBS patch that are outside another "trimming" patch – are also allowed as boundary surfaces (c.f. Figure 1). The edge curves of the resulting object are called trimming curves and can be expressed in either $\mathbb{R}^3$ or in the parametric space of either parametric patch. Efficient and accurate integration of material properties, such as density, over regions bounded by these trimmed patches is an important operation in the design process.

In addition to its application to design, integration over regions bounded by trimmed parametric surfaces has become increasingly important in the context of computer-aided analysis, where it has become more common to use high-order polynomials and/or rational functions as a basis for analysis in the finite element method (FEM). In analysis paradigms such as moment-fitting methods, fictitious domain methods, and Lagrangian remap methods, integration over intersections (i.e. trimmings) of typical hexahedral or tetrahedral elements is a fundamental operation. See Figure 2 for an example in the Lagrangian remap context.

Integration over regions bounded by trimmed surfaces has been studied recently from a variety of standpoints, two of which are (1) volumetric domain decomposition methods, such as octree decomposition or meshing of the trimmed geometry and (2) geometric correction-based methods, in which the region's boundary is approximated by a linear polyhedron, then a geometric correction term is applied to any computed integrals. For a relatively complete short survey, see [6].

Volumetric domain decomposition methods generate an approximate decomposition of the region into simpler subregions and integrate over the subregions. As shown in Figure 3, adaptive quadtree/octree integration decomposes the domain into variable-sized grid-aligned rectangular prisms, then high-order tensor-product quadrature rules are used to integrate over each rectangular prism. Because the boundary is approximated by a staircase profile, adaptive quadtree can converge at most linearly. Higher-order domain decomposition (e.g. into a curved high-order mesh) can attain higher-order convergence; however, the initial meshing step can be expensive and difficult.

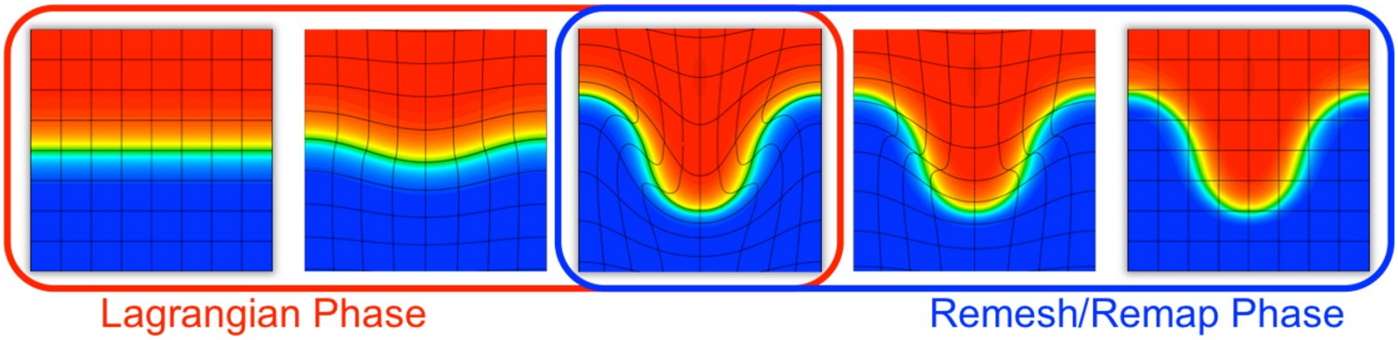

Figure 2: *In Lagrangian-remap methods, the FEM mesh is allowed to move with the fluid flow during the Lagrangian phase. When the mesh deforms so much that it is unsuitable for analysis, field variables such as density are remapped from the low-quality source mesh to a higher-quality target mesh. The field variables can be conservatively remapped by integration over the intersections between source and target mesh elements, i.e. regions bounded by trimmed parametric surfaces. Image courtesy of BLAST [2].*

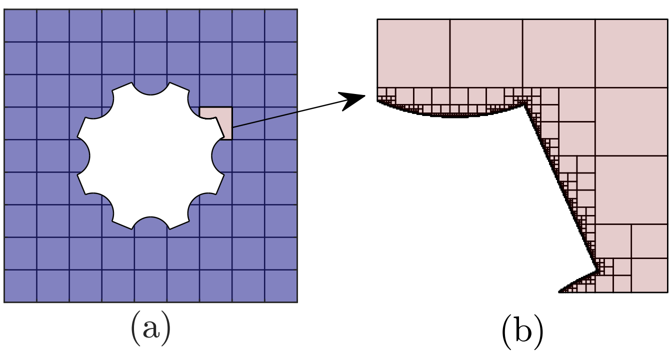

Figure 3: (a) *In immersed methods, special integration must be used over cells which intersect the immersed boundary.* (b) *The most common method currently used for such regions is adaptive quadtree integration, in which the domain is adaptively approximated by smaller and smaller square subregions. Only linear convergence can be attained, because the geometry is approximated by a staircase profile.*

In correction-based methods, the geometry is approximated by a straight-edged polyhedron, then a geometric correction term is calculated from the Taylor expansion of a linear approximation of the deformation of the true geometry from the approximate linear geometry. These methods have been shown to attain cubic convergence on trimmed regions bounded by implicitly-defined trimming surfaces, without the need for a complicated meshing of the domain [5].

However, for the common case of regions bounded by parametric trimming surfaces, neither domain decomposition nor correction-based methods take advantage of the additional information available in the form of the parametric surface description. We propose that this extra information can be effectively incorporated using a Stokes' theorem-based approach. A similar approach has been used extensively in the past for improving the efficiency of integration over linear polyhedra [6], polynomial parametric surfaces, and implicitly-defined regions [4], especially for integrands with symbolic antiderivatives.

To the best of our knowledge, the use of a Stokes' theorem-based method for integrating arbitrary functions over regions bounded by trimmed parametric surfaces has not been described in the literature before. In this poster we describe ongoing work on the development, implementation, and testing of a generalized Stokes' theorem-based strategy for this situation. The algorithm proceeeds in three steps:

(1) Surface-surface intersection (SSI) is used to find a high-order polynomial approximation of each trimming curve of the region within the parametric spaces of the corresponding trimmed boundary surfaces.
(2) The parametric version of generalized Stokes' theorem is numerically applied to the integrand to transform volumetric integrals into surface integrals in the parametric space of each boundary surface.
(3) Green's theorem is numerically applied to each 2D parametric integral to transform it into line integrals along each approximated trimming curve.

At this point in the research project, we have successfully implemented steps (2) and (3) of the algorithm. Preliminary results indicated that spectral convergence can be attained, given an exact representation of all the trimming curves. SSI algorithms in the literature approximate intersection curves with at most algebraic orders of accuracy (i.e. using polynomial bases). Therefore, with a high-order implementation of step one in the algorithm, we expect our quadrature schemes to achieve high orders of algebraic convergence to the correct value.

2 Quadrature for Domains Bounded by Trimmed Parametric Surfaces

2.1 Problem Definition

Formally, we consider the problem of integrating a function $f(x,y,z)$ over a region $D \in \mathbb{R}^3$ defined as the volume enclosed by n bounding parametric surfaces, $\partial D = \cup_{i=1}^{n} S_i'$. Each bounding surface S_i' is a subset of a corresponding parametric surface S_i:

$$S_i = \begin{cases} x_i(u,v) \\ y_i(u,v) \\ z_i(u,v), \end{cases} \quad 0 \le u,v \le 1, \tag{2.1}$$

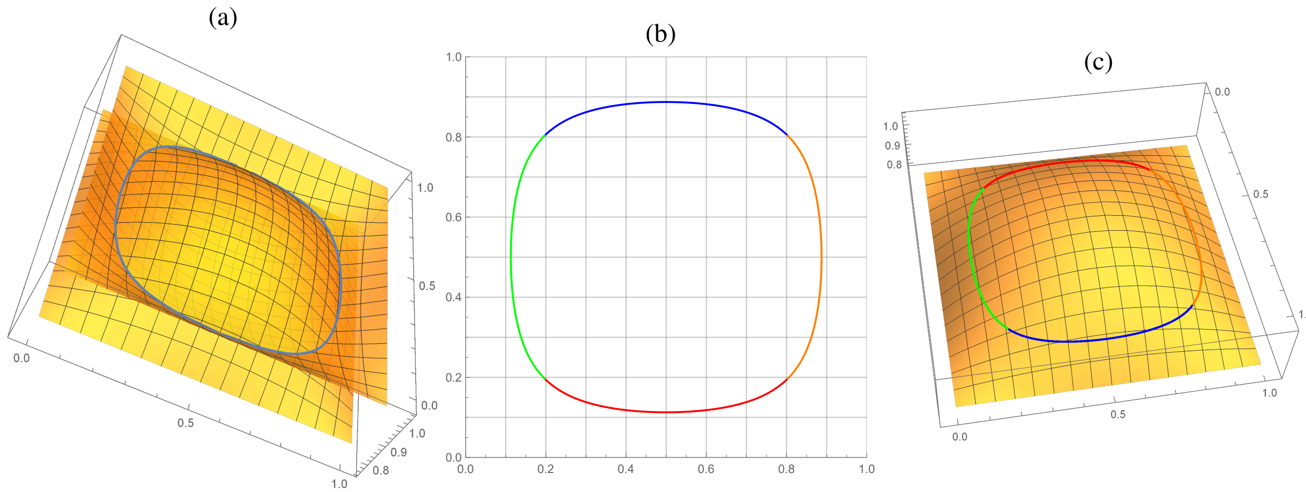

Figure 4: *An example illustration of a region bounded by trimmed parametric surfaces.* (a) *The true physical space edge curve is formed by the intersection of the two biquadratic faces, S_1 and S_2.* (b) *The edge curves are approximated in the parametric space of S_1 by some set of approximated trimming curves $\{c'_{1,j}\}_{j=1}^{4}$, typically B-splines.* (c) *The approximated trimming curves $\{c'_{1,j}\}_{j=1}^{4}$ map back through $S_1(u,v)$ approximately onto the true edge curve.*

which is defined in terms of a polynomial or rational parametric tensor-product basis, such as the Bernstein-Bézier basis. Note that another basis or triangular patches could also be used, but we focus for clarity on the Bernstein-Bézier tensor-product patch case. For each S_i, there is a set of associated implicitly-defined trimming curves $\{c_{i,k}\}_{k=1}^{m_i}$ in the parametric space, $R_i = [0,1]\times[0,1]$, of S_i. This set of trimming curves $\{c_{i,k}\}_{k=1}^{m_i}$ splits R_i into an "interior" (possibly disconnected) region R'_i bounded by the set of $\{c_{i,k}\}_{k=1}^{m_i}$ and another (possibly disconnected) region $R_i \setminus R'_i$, such that each R'_i maps through the parametric mapping S_i to $S'_i \in \partial D$.

Our algorithm proceeds in three steps: (1) approximate each trimming curve $c_{i,k}$ using a surface-surface intersection (SSI) algorithm, (2) apply the parametric version of generalized Stokes' theorem to transform the volumetric integral over D into a sum of area integrals over the parametric domain of each of boundary surface S'_i, and (3) apply Green's theorem to transform each surface integral over S'_i into a sum of line integrals along each approximated boundary curve $c'_{i,k}$.

2.2 Step One: Trimming Curve Approximation

The first step is, for each S_i, to approximate the trimming curves $\{c_{i,k}\}_{k=1}^{m_i}$ using some set of p_i polynomial parametric curves in R_i:

$$c'_{i,j} = \begin{cases} u_{i,j}(s) \\ v_{i,j}(s), \end{cases} \quad 0 \le s \le 1, \tag{2.2}$$

such that $\cup_{j=1}^{p_i} c'_{i,j} \approx \cup_{i=1}^{m_i} c_{i,k}$ where the functions $u_{i,j}$ and $v_{i,j}$ are given as polynomial parametric curves in the Bernstein-Bézier basis. There is a wide body of literature on approximating trimming curves determined as the intersection of two parametric surfaces and an extensive survey can be found in [3]. We have not implemented this step yet.

2.3 Step Two: Generalized Stokes' Theorem

The second step in the algorithm consists of applying the parametric version of generalized Stokes' algorithm to transform the volumetric integral over the region D into a sum of surface integrals over the boundary surfaces S'_i:

$$\iiint_D f(x,y,z) = \sum_{i=1}^{n} \iint_{S'_i} \vec{A}_f(x,y,z)\cdot\vec{n}dS. \tag{2.3}$$

For simplicity, we choose the z-antiderivative, so that

$$\vec{A}_f(x_i,y_i,z_i) = \langle 0,0,\int_{C_z}^{z_i} f(x_i,y_i,\xi)d\xi\rangle, \tag{2.4}$$

for some appropriately chosen constant C_z. Because each boundary surface S'_i is given parametrically, the surface integrals can be evaluated in their respective parametric spaces:

$$\iint_{S'_i} \vec{A}_f(x,y,z)\cdot\vec{n}dS \tag{2.5}$$

$$= \iint_{R'_i} \left(\int_{C_z}^{z_i(u,v)} f(x_i(u,v),y_i(u,v),\xi)d\xi\right)\cdot \left(\frac{\partial x_i(u,v)}{\partial v}\frac{\partial y_i(u,v)}{\partial u} - \frac{\partial x_i(u,v)}{\partial u}\frac{\partial y_i(u,v)}{\partial v}\right)dudv. \tag{2.6}$$

Each of the two terms in the product in Equation (2.6) can be evaluated numerically: the first using high-order quadrature and the second using de Casteljau's algorithm [1].

2.4 Step Three: Green's Theorem Application

The third step in the algorithm is to use Green's theorem to convert each double integral in Equation (2.6) to a sum of line integrals over its corresponding approximated trimming curves, $\{c'_{i,j}\}_{j=1}^{p_i}$. For an arbitrary function $g(u,v)$, we have

$$\iint_{R'_i} g(u,v)dudv \approx \sum_{j=1}^{p_i}\int_{c'_{i,j}} A_g(u,v)dv, \tag{2.7}$$

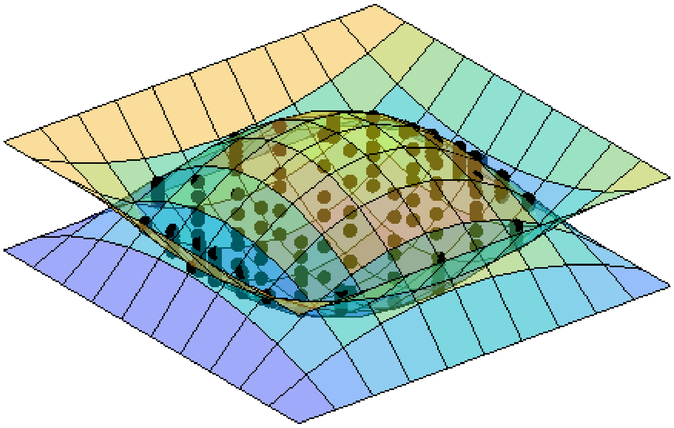

Figure 5: *Example quadrature points produced by the algorithm in Section 2 for the domain shown in Figure 4 when the trimming curves are given symbolically.*

where we take the v antiderivative,

$$A_g(u,v) = \int_{C_v}^{v} g(u,\xi)\,d\xi, \tag{2.8}$$

for some appropriately chosen constant C_v. As before, since the approximated trimming curves are given parametrically, each of the integrals in the sum on the right hand side of Equation (2.7) can be evaluated in its respective parametric space:

$$\int_{c'_{i,j}} A_g(u,v)\,dv = \int_0^1 A_g(u_{i,j}(s), v_{i,j}(s)) \frac{du_{i,j}(s)}{ds}\,ds. \tag{2.9}$$

Each of the two terms in the product in Equation (2.9) can be approximated numerically. In addition, the outer integral can be approximated using high-order numerical quadrature.

2.5 Final Quadrature Rule

Each of the one-dimensional integrals in Equations 2.4, 2.8, and 2.9 can be evaluated using high-order Gaussian quadrature. The derivatives in Equations 2.4 and 2.8 can be evaluated using de Casteljau's algorithm. The three one-dimensional quadrature rules can then be combined in a tensor-product fashion to produce a quadrature rule for D, as shown in Figure 5. Exponential convergence can be attained by increasing the order of the three 1D quadrature rules simultaneously.

3 Preliminary Results

As this project is a work in progress, we still need to implement the full algorithm and to test it on a variety of shapes and integrands. However, we have tested steps two and three of the algorithm given above on some regions, including the one in Figure 4. As a stand-in for step one of the algorithm, we computed the (non-polynomial) intersection curves $c_{i,k}$ symbolically and evaluated them to machine precision as needed. The rest of the algorithm proceeded as described. As can be seen in Figure 6, the scheme is able to attain exponential convergence, given the exact representations of the trimming curves. Because the only other approximation necessary is a polynomial form of the trimming curves, we expect the full algorithm to attain the order of convergence of the polynomial approximation curves.

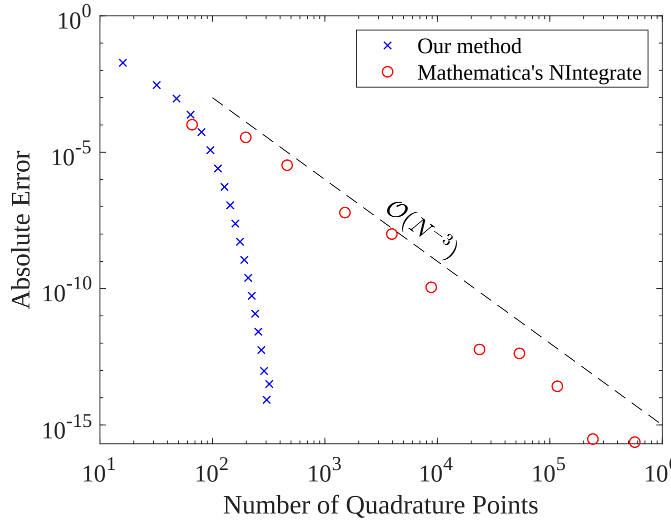

Figure 6: *The integral of $f(x,y,z) = 1$ over the domain shown in Figure 4 is approximated with exponential convergence when the trimming curves are given symbolically, greatly outperforming Mathematica's NIntegrate. We expect high algebraic orders of convergence when the trimming curves are approximated by high-order polynomial parametric curves.*

Acknowledgments

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