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RESEARCH

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A probabilistic approach for the valuation of variance swaps under stochastic volatility with jump clustering and regime switching

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Abstract

The effects of stochastic volatility, jump clustering, and regime switching are considered when pricing variance swaps. This study established a two-stage procedure that simplifies the derivation by first isolating the regime switching from other stochastic sources. Based on this, a novel probabilistic approach was employed, leading to pricing formulas with time-dependent and regime-switching parameters. The formulated solutions were easy to implement and differed from most existing results of variance swap pricing, where Fourier inversion or fast Fourier transform must be performed to obtain the final results, since they are completely analytical without involving integrations. The numerical results indicate that jump clustering and regime switching have a significant influence on variance swap prices.

Keywords: Stochastic volatility, Jump clustering, Regime switching, Variance swaps, Probabilistic approach, Closed-form solution

Introduction

The effective management of financial risk, which is vital for market stability, is an ongoing topic in finance practice. Volatility derivatives can efficiently provide volatility exposure without having to invest in target assets directly. Such favorable properties have made these derivatives, among which variance swaps are representative, especially attractive.

To affect the risk-management process, the prices of variance swaps need to be determined accurately. A key determinant of the variance swap prices is the sampling approach, which can be continuous or discrete. Different results have been obtained when continuous sampling is adopted under various stochastic volatility models (Swishchuk 2006; Carr and Lee 2007). Despite the simplicity of this framework and the convenience of its practical implementation, it does not match existing practices in financial markets, where discrete sampling is selected. These inconsistencies have prompted research in the direction of discrete sampling.

The selection of an appropriate model that can reflect market characteristics is significant when the sampling of realized variance is discrete and the obtained solutions can greatly vary according to the chosen underlying model. For example, an analytical solution was presented by Zhu and Lian (2011) when the Cox-Ingersoll-Ross (CIR) process was used to model stochastic volatility (Heston 1993), whereas analytical and asymptotic solutions were obtained Bernard and Cui (2014) using three different stochastic volatility models. Multifactor stochastic volatility has gained attention because of its ability to produce implied volatility closer to market data (Rouah 2013; Lin et al. 2024; He and Lin 2024), leading to the consideration of variance swap valuation under this framework (Kim and Kim 2019; Issaka 2020). The stochastic interest rate is incorporated when valuing variance swaps (Cao et al. 2020; Wu et al. 2022; Badescu et al. 2019).

The abovementioned works neglect the possibility of price jumps in underlying stocks, which violates statistical/empirical observations (Bates 1996; Eraker 2004; Hu et al. 2024). This leads to an investigation of pricing variance swaps with Poisson jump-diffusion or Levy jumps (Broadie and Jain 2008; Hong and Jin 2023; Pun et al. 2015; Carr et al. 2012). However, independent increments assumed in these jump models prevented the incorporation of jump clustering, which often occurs in practice (Aït-Sahalia et al. 2015). Therefore, various research interests have guided the adoption of Hawkes jump-diffusion models for valuing financial derivatives, including vulnerable options (Ma et al. 2017), power exchange options (Pasricha and Goel 2020), and Volatility Index (VIX) options (Jing et al. 2021). Stochastic volatility and jump clustering are combined by Liu and Zhu (2019), who present an analytical valuation of variance swaps.

Another strand of research requires the consideration of the variable economic status that can greatly influence asset prices (Hamilton 1990). Consequently, regime switching has been incorporated into different dynamics in the process of pricing derivatives (He and Chen 2022; Siu and Elliott 2022) and has also been applied to determine volatility derivative prices. For instance, regime-switching stochastic volatility was considered by Elliott and Lian (2013), Lin and He (2023) when pricing variance swaps, which were extended by Cao et al. (2018) to incorporate the regime-switching stochastic interest rate further. Jump diffusion was added to a similar framework for pricing variance and volatility swaps (Yang et al. 2021).

This study attempted to solve the pricing problem of variance swaps when stochastic volatility, jump clustering, and regime switching coexist. To the best of our knowledge, this has not been studied in previous literature. A novel probability approach was employed in the valuation yielding a closed-form formula for variance swap prices. In contrast with existing literature, the formula is simple and no longer requires Fourier inversion, fast Fourier transform, or numerical integration. This unique property renders the formula extremely flexible when adopted by market participants.

A brief summary of the content is presented in the following sections. The combination of regime switching stochastic volatility and Hawkes jump diffusion is illustrated in Sect. 2. Section 3 discusses the probability approach to the analytical formulation of variance swap prices. Before concluding the paper, an implementation of this formula is presented in Sect. 4.

The model

The risk-neutral measure $\mathbb{Q}$ was emphasized as a part of the complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. A continuous-time Markov chain,[1] denoted by $\{X_t\}_{t \geq 0}$, is introduced to model the variation in economic conditions, such that its value is equal to that of the two-dimensional unit vectors. When the generator of the Markov chain is $\Lambda(t) = [\lambda_{ij}(t)]_{i,j=1,2}$ and the Martingale increment process conditional on the filtration generated by the Markov chain is $\{V_t\}_{t \geq 0}$, one can have

$$X_t = X_0 + \int_0^t \Lambda(s) X_s ds + V_t, \tag{1}$$

resulting from the semimartingale representation theorem (Elliott et al. 2008; He and Lin 2023a, b).

In the considered risk-neutral world where the risk-free interest rate is denoted by r, Y_t, ν_t, and γ_t are respectively used to denote the price of the stock, its variance, and the intensity of the Hawkes-type jump. Their dynamics in the formulated risk-neutral world are

$$\begin{aligned} \frac{dY_t}{Y_{t-}} &= (r - \omega\gamma_t)dt + \sqrt{\nu_t} dW_{1,t} + J_t dH_t, \\ d\nu_t &= p(q_{X_t} - \nu_t)dt + \zeta \sqrt{\nu_t} dW_{2,t}, \\ d\gamma_t &= \eta(\gamma_\infty - \gamma_t)dt + \delta dH_t. \end{aligned} \tag{2}$$

$\{W_{i,t}\}_{t \geq 0}, i = 1, 2$ are two Wiener processes, with $dW_{1,t} dW_{2,t} = \rho dt$ representing the connection between the stock price and its variance. ν_t is mean-reverting, the speed of which is p. This process is affected by the volatility, ζ. The long-term variance level to be approached is assumed to be influenced by economic conditions, and regime switching is performed to reflect such effects, thus one can compute q_{X_t} using the inner product between the Markov chain X_t and the vector $\hat{q} = (q_1, q_2)$, that is, $q_{X_t} =< \hat{q}, X_t >$ (He and Lin 2023c). The jump sizes with support for $(-1, \infty)$ have identical and independent distributions, with $E(J_t) = \omega$. H_t is the Hawkes process, which is defined as:

$$H_t = \sum_{i=1}^{\infty} I(t_i \leq t), \tag{3}$$

with jump time moments denoted by $\{t_i\} > 0$. The expression of jump intensity γ_t considers the jump clustering effect, indicating that the occurrence of a jump contributes to a greater intensity at the rate of δ, and the increment then experiences exponential decay at the rate of η.

A natural transform applied to stock price dynamics is $Z_t = \ln(Y_t)$, which, when combined with Ito's lemma, yields the dynamics of Z_t as

$$dZ_t = \left(r - \frac{1}{2}\nu_t - \omega\gamma_t \right) dt + \sqrt{\nu_t} dW_{1,t} + \tilde{J}_t dH_t, \tag{4}$$

[1] It is assumed to be of two states for the ease of discussion, but the results with $N > 2$ states can easily be obtained in a similar manner.

where the log stock price jumps to $\tilde{J}_t = \ln(1 + J_t)$. $\tilde{J}_t$ is assumed to be identically and independently distributed following a normal distribution for any given t with μ and σ^2 as its mean and variance, respectively. In this setting, $\omega = E\left(e^{\tilde{J}_t} - 1\right) = e^{\mu + \frac{1}{2}\sigma^2} - 1$.

Variance swap pricing

The model established in the previous section considers the effects of stochastic volatility, jump clustering, and varying economic conditions when modelling stock prices. Although appealing, the complex structure of the model dynamics can be an obstacle to efficiently evaluating financial derivatives, and it is certainly of interest to investigate the analytical valuation of variance swaps under this framework. The related details are presented below.

Unlike other types of financial derivatives, the prices of the variance swaps to be computed are referred to as delivery prices K agreed upon by both parties listed in the contract. The target delivery prices rely on how the realized variance, denoted by RV, is defined. This study adopts the following model

$$RV = \frac{100^2}{T} \sum_{i=1}^{N} \left[\ln \left(\frac{Y_{t_i}}{Y_{t_{i-1}}} \right) \right]^2,$$

where the time period between the current time $t = 0$ and the expiry of the contract $t = T$ is split into N sub-intervals uniformly. This yields the swap payoff $(RV - K)Z$, where Z denotes a constant notional value. This specific definition of realized variance was selected not only for its ability to derive an analytical solution but also due to its extensive adoption in the existing literature (Pun et al. 2015; Badescu et al. 2019; Zheng and Kwok 2014; Lin and He 2023).

Owing to the nature of the swap contract's initial zero value, we must obtain

$$K = E\left(RV | \mathcal{F}_0^{W_1} \vee \mathcal{F}_0^{W_2} \vee \mathcal{F}_0^{H} \vee \mathcal{F}_0^{X}\right) = \frac{100^2}{T} \sum_{i=1}^{N} g_1(Y_0, v_0, \gamma_0, X_0, t_i, t_{i-1}), \tag{5}$$

where

$$g_1(Y_0, v_0, \gamma_0, X_0, s, t) = E\left\{ \left[\ln \left(\frac{Y_t}{Y_s} \right) \right]^2 | \mathcal{F}_0^{W_1} \vee \mathcal{F}_0^{W_2} \vee \mathcal{F}_0^{H} \vee \mathcal{F}_0^{X} \right\},\ 0 \le s < t \le T, \tag{6}$$

which requires the computation of the unknown function g_1. However, the multiple stochastic sources involved complicate analytically computing expectations, and the separation of certain stochastic variables is first performed such that the number of stochastic sources to be dealt with at one time can be reduced.

In particular, we apply the tower rule of expectation, which leads to

$$g_1(Y_0, v_0, \gamma_0, X_0, s, t) = E\left[g_2(Y_0, v_0, \gamma_0, s, t) | \mathcal{F}_0^{X}\right], \tag{7}$$

with

$$g_2(Y_0, \nu_0, \gamma_0, s, t) = E\left\{ \left[\ln\left(\frac{Y_t}{Y_s}\right)\right]^2 | \mathcal{F}_0^{W_1} \vee \mathcal{F}_0^{W_2} \vee \mathcal{F}_0^{H} \vee \mathcal{F}_t^{X} \right\}. \tag{8}$$

Obviously, the calculation of $g_2(Y_0, \nu_0, \gamma_0, s, t)$ is an a priori step in determining $g_1(Y_0, \nu_0, \gamma_0, X_0, s, t)$ as well as the variance swap delivery prices. This first step assumes a given Markov chain, reducing the problem of finding the variance swap-pricing formula with the long-term variance q_t being no longer stochastic but time-dependent. First, we present a solution to this simplified problem.

The formula with time-dependent long-term variance

Before formally dealing with the problem of $g_2(Y_0, \nu_0, \gamma_0, s, t)$, we first present some useful results associated with ν_t and γ_t that are needed in the derivation.

Proposition 1 *If the stock variance ν_t and jump intensity γ_t are governed by the stochastic differential equations presented in Eq.* (2), *with the stochastic variable q_{X_t} replaced by the time-dependent variable q_t, we must have*

$$\begin{aligned}
h_1(\nu_s, s, t) &= E(\nu_t | \mathcal{F}_s^{W_2}) = e^{-p(t-s)}\nu_s + p\int_s^t q_\xi e^{-p(t-\xi)} d\xi,\\
h_2(\gamma_s, s, t) &= E(\gamma_t | \mathcal{F}_s^{H}) = a_2 + (\gamma_s - a_2)e^{-a_1(t-s)},\\
h_3(\nu_s, s, t) &= var(\nu_t | \mathcal{F}_s^{W_2}) = \frac{\zeta^2}{p}\left[e^{-p(t-s)} - e^{-2p(t-s)}\right]\nu_s + \zeta^2 \int_s^t q_\xi \left[e^{-p(t-\xi)} - e^{-2p(t-\xi)}\right] d\xi,\\
h_4(\gamma_s, s, t) &= var(\gamma_t | \mathcal{F}_s^{H}) = \delta^2 \left\{ \frac{a_2}{2a_1}\left[1 - e^{-2a_1(t-s)}\right] + \frac{\gamma_s - a_2}{a_1}\left[e^{-a_1(t-s)} - e^{-2a_1(t-s)}\right] \right\},
\end{aligned} \tag{9}$$

where $a_1 = \eta - \delta$ and $a_2 = \frac{\eta\gamma_\infty}{a_1}$.

Appendix 1 provides the proof of Proposition 1.

With the expectations and variances of ν_t and γ_t presented, we are now ready to compute $g_2(Y_0, \nu_0, \gamma_0, s, t)$. This requires us to calculate the conditional expectation of $\ln\left(\frac{Y_t}{Y_s}\right)$, which contains two random variables when $s > 0$: To simplify the computation, the tower rule of expectation is used, such that

$$g_2(Y_0, \nu_0, \gamma_0, s, t) = E\left[g_3(Y_s, \nu_s, \gamma_s, s, t) | \mathcal{F}_0^{W_1} \vee \mathcal{F}_0^{W_2} \vee \mathcal{F}_0^{H} \vee \mathcal{F}_t^{X} \right], \tag{10}$$

where

$$g_3(Y_s, \nu_s, \gamma_s, s, t) = E\left\{ \left[\ln\left(\frac{Y_t}{Y_s}\right)\right]^2 | \mathcal{F}_s^{W_1} \vee \mathcal{F}_s^{W_2} \vee \mathcal{F}_s^{H} \vee \mathcal{F}_t^{X} \right\}, \; 0 \le s < t \le T. \tag{11}$$

Clearly, the target $g_2(Y_u, \nu_u, \gamma_u, u, s, t)$ is the expectation of $g_3(Y_s, \nu_s, \gamma_s, s, t)$, which we can obtain if

$$\begin{aligned}
g_{3,1} &= E\left[\ln\left(\frac{Y_t}{Y_s}\right) | \mathcal{F}_s^{W_1} \vee \mathcal{F}_s^{W_2} \vee \mathcal{F}_s^{H} \vee \mathcal{F}_t^{X} \right],\\
g_{3,2} &= var\left[\ln\left(\frac{Y_t}{Y_s}\right) | \mathcal{F}_s^{W_1} \vee \mathcal{F}_s^{W_2} \vee \mathcal{F}_s^{H} \vee \mathcal{F}_t^{X} \right],
\end{aligned} \tag{12}$$

are both known. Thus, we present the corresponding results in the following proposition:

Proposition 2 *If the dynamics of the stock price Y_t are given in Eq. (2), then we must have*

$$g_{3,1} = (r - a_3 a_2)\tau - \frac{1}{2p}(1 - e^{-p\tau})\nu_s - \frac{1}{2}\int_s^t q_\xi[1 - e^{-p(t-\xi)}]d\xi - \frac{a_3}{a_1}(1 - e^{-a_1\tau})(\gamma_s - a_2), \tag{13}$$

and

$$\begin{aligned} g_{3,2} =& A_1 + A_2(\gamma_s - a_2) + \frac{\zeta^2}{4p^2}\int_s^t [e^{-p(t-\xi)} - e^{-2p(t-\xi)}]q_\xi d\xi \\ &+ \left\{\frac{1}{p}(1 - e^{-p\tau}) + \frac{\zeta^2}{4p^2}\left[\frac{1}{p}(1 - e^{-2p\tau}) - 2e^{-p\tau}\tau\right] - \frac{\rho\zeta}{p}\left[\frac{1}{p}(1 - e^{-p\tau}) - e^{-p\tau}\tau\right]\right\}\nu_s \\ &+ \left(1 + \frac{\zeta^2}{4p^2} - \frac{\rho\zeta}{p}\right)\int_s^t q_\xi[1 - e^{-p(t-\xi)}]d\xi + \left(\rho\zeta - \frac{\zeta^2}{2p}\right)\int_s^t e^{-p(t-\xi)}(t - \xi)q_\xi d\xi, \end{aligned} \tag{14}$$

where

$$A_1 = \left[a_4 a_2 \tau + \frac{2\delta a_2 a_3}{a_1^2}\left(\mu - \frac{\delta a_3}{\beta - \alpha}\right)(1 - e^{-a_1\tau}) + \frac{\delta^2 a_2 a_3^2}{2a_1^3}(1 - e^{-2a_1\tau})\right],$$

and

$$A_2 = \left\{\frac{2\delta a_3}{a_1}\left(\mu - \frac{\delta a_3}{a_1}\right)\tau e^{-a_1\tau} + \left(\frac{a_4}{a_1} - \frac{\delta^2 a_3^2}{a_1^3}\right)(1 - e^{-a_1\tau}) + \frac{\delta^2 a_3^2}{a_1^3}(1 - e^{-2a_1\tau})\right\}.$$

with $a_3 = \omega - \mu$, $a_4 = \sigma^2 + \left(\mu - \frac{\delta a_3}{a_1}\right)^2$, *and* $\tau = t - s$.

We prove Proposition 2 in Appendix 2.

Now, we obtain $g_3(Y_s, \nu_s, \gamma_s, s, t)$ in Eq. (11) by using $g_3 = (g_{3,1})^2 + g_{3,2}$, providing

$$\begin{aligned} g_3 =& A_1 + (r - a_3 a_2)^2\tau^2 + \frac{1}{4}\left\{\int_s^t q_\xi[1 - e^{-p(t-\xi)}]d\xi\right\}^2 + \left(\rho\zeta - \frac{\zeta^2}{2p}\right)\int_s^t e^{-p(t-\xi)}(t - \xi)q_\xi d\xi \\ &+ \left[1 + \frac{\zeta^2}{4p^2} - \frac{\rho\zeta}{p} - (r - a_3 a_2)\tau\right]\int_s^t q_\xi[1 - e^{-p(t-\xi)}]d\xi + \frac{\zeta^2}{4p^2}\int_s^t [e^{-p(t-\xi)} - e^{-2p(t-\xi)}]q_\xi d\xi \\ &+ \frac{1}{4p^2}(1 - e^{-p\tau})^2\nu_s^2 + \frac{a_3^2}{a_1^2}(1 - e^{-a_1\tau})^2(\gamma_s - a_2)^2 \\ &+ \left[\frac{1}{2p}(1 - e^{-p\tau})\nu_s + \frac{a_3}{a_1}(1 - e^{-a_1\tau})(\gamma_s - a_2)\right]\int_s^t q_\xi[1 - e^{-p(t-\xi)}]d\xi + A_3\nu_s \\ &+ \left[A_2 - \frac{2a_3}{a_1}(1 - e^{-a_1\tau})(r - a_3 a_2)\tau\right](\gamma_s - a_2) + A_4(\gamma_s - a_2)\nu_s, \end{aligned} \tag{15}$$

where

$$
\begin{aligned}
A_3 =&\frac{1}{p}(1-e^{-p\tau}) + \frac{\zeta^2}{4p^2}\left[\frac{1}{p}(1-e^{-2p\tau}) - 2e^{-p\tau}\tau\right] - \frac{\rho\zeta}{p}\left[\frac{1}{p}(1-e^{-p\tau}) - e^{-p\tau}\tau\right] \\
&-\frac{1}{p}(1-e^{-p\tau})(r-a_3a_2)\tau, \\
A_4 =&\frac{a_3}{a_1p}(1-e^{-p\tau})(1-e^{-a_1\tau}).
\end{aligned}
\tag{16}
$$

The problem that remains to be solved involves the computation of $g_2(Y_0, \nu_0, \gamma_0, s, t)$ using Eq. (10). With the solution to $g_3(Y_s, \nu_s, \gamma_s, s, t)$ given in Eq. (15), the target expectation requires the computation of $E(\gamma_s|\mathcal{F}_0^H)$, $E\left[(\gamma_s - a_2)^2|\mathcal{F}_0^H\right]$ and $E(\nu_s|\mathcal{F}_0^{W_2})$. Because the required solutions have already been presented in Proposition 1, the results of $g_2(Y_0, \nu_0, \gamma_0, s, t)$ are illustrated directly without requiring tedious calculations and rearrangements.

$$
\begin{aligned}
g_2 =&A_1 + (r-a_3a_2)^2\tau^2 + \frac{a_3^2}{a_1^2}(1-e^{-a_1\tau})^2\left[A_5^2 + h_4(\gamma_0, 0, s)\right] + (A_3 + A_4A_5)e^{-ps}\nu_0 \\
&+\left[A_2 - \frac{2a_3}{a_1}(1-e^{-a_1\tau})(r-a_3a_2)\tau\right]A_5 + \frac{1}{4p^2}(1-e^{-p\tau})^2\left[\frac{\zeta^2}{p}(e^{-ps} - e^{-2ps})\nu_0 + e^{-2ps}\nu_0^2\right] \\
&+\frac{1}{2}(1-e^{-p\tau})U_1 + \frac{1}{4}U_2 + \left(\rho\zeta - \frac{\zeta^2}{2p}\right)U_4 + \left[\frac{\zeta^2}{4p^2} + \frac{\zeta^2}{4p^2}(1-e^{-p\tau})^2\right]U_5 \\
&+\left[1 + \frac{\zeta^2}{4p^2} - \frac{\rho\zeta}{p} - (r-a_3a_2)\tau + \frac{a_3}{a_1}(1-e^{-a_1\tau})A_5 + \frac{1}{2p}(1-e^{-p\tau})e^{-ps}\nu_0\right]U_3 \\
&+\left[\frac{1}{2p}(1-e^{-p\tau})^2e^{-ps}\nu_0 + p(A_3 + A_4A_5)\right]U_6 + \frac{1}{4}(1-e^{-p\tau})^2U_7,
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
U_1 &= \int_0^s q_\xi e^{-k(s-\xi)}d\xi \cdot \int_s^t q_\xi[1-e^{-p(t-\xi)}]d\xi, \\
U_2 &=\left\{\int_s^t q_\xi[1-e^{-p(t-\xi)}]d\xi\right\}^2, \\
U_3 &= \int_s^t q_\xi[1-e^{-p(t-\xi)}]d\xi, \\
U_4 &= \int_s^t e^{-p(t-\xi)}(t-\xi)q_\xi d\xi, \\
U_5 &= \int_s^t [e^{-p(t-\xi)} - e^{-2p(t-\xi)}]q_\xi d\xi, \\
U_6 &= \int_0^s q_\xi e^{-p(s-\xi)}d\xi, \\
U_7 &=\left[\int_0^s q_\xi e^{-p(s-\xi)}d\xi\right]^2,
\end{aligned}
$$

with $A_5 = h_2(\gamma_0, 0, s) - a_2 = (\gamma_0 - a_2)e^{-a_1s}$.

Once this stage is reached, one should realize that when the long-term stock variance, q_ξ, $\xi \in [0, T]$, is time dependent, Eq. (5) should be revised such that the corresponding delivery price is a result of

$$K_c = E\left(\sigma^2 | \mathcal{F}_0^{W_1} \vee \mathcal{F}_0^{W_2} \vee \mathcal{F}_0^{H} \vee \mathcal{F}_t^{X}\right) = \frac{100^2}{T} \sum_{i=1}^{N} g_2(Y_0, \nu_0, \gamma_0, t_i, t_{i-1}). \tag{18}$$

This formula can be used with the formulation of $g_2(Y_0, \nu_0, \gamma_0, s, t)$ as shown in Eq. (17). The results associated with the time dependence of q_ξ, $\xi \in [0, T]$ are not the target values, and their regime-switching nature should be re-introduced to capture the varying economic states.

The formula with regime switching long-term variance

By removing the assumption that the Markov chain is foreseeable, the variance swap delivery prices should be computed using Eq. (5), implying that the calculation of $g_1(Y_0, \nu_0, \gamma_0, X_0, s, t)$ using Eq. (7) is required. The Markov chain is the only stochastic source when deriving $g_1(Y_0, \nu_0, \gamma_0, X_0, s, t)$ using the expression $g_2(Y_0, \nu_0, \gamma_0, s, t)$. Therefore, the terms that one needs to find are $U_i = E(M_i | \mathcal{F}_0^X)$, $i = 1, 2, ..., 7$, and we now solve them individually. Before presenting these results, we first provide some frequently used notations. We define three general functions asfollows:

$$\begin{aligned} k_1(s,t) =& \tau - \frac{1}{p}(1 - e^{-p\tau}), \\ k_2(\lambda, s, t) =& \frac{1}{\lambda}(e^{\lambda t} - e^{\lambda s}), \\ .k_3(\lambda, s, t) =& t e^{\lambda t} - s e^{\lambda s}, \end{aligned} \tag{19}$$

and also define $d_1 = (q_1^2 - q_1 q_2)\frac{\lambda_{21}}{\lambda} + q_1 q_2$, $d_2 = \frac{\lambda_{12}}{\lambda}(q_1^2 - q_1 q_2)$, $d_3 = (q_2^2 - q_1 q_2)\frac{\lambda_{12}}{\lambda} + q_1 q_2$, $d_4 = \frac{\lambda_{21}}{\lambda}(q_2^2 - q_1 q_2)$, $d_5 = \frac{d_1 - d_3}{\lambda}$, $d_6 = \frac{d_2 - d_4}{\lambda}$.

Thus, the solution to U_1 is presented in Proposition 3.

Proposition 3 *With $q_{X_\xi}, \xi \in [0, t]$ being a regime-switching parameter controlled by Markov chain X_t, we have*

$$U_1 = < \bar{U}_1, X_0 >, \ \ \bar{U}_1 = (< \bar{c}_1^1, \bar{D}_1 >, < \bar{c}_2^1, \bar{D}_1 >)^T, \tag{20}$$

where

$$\bar{D}_1 = \begin{pmatrix} \frac{1}{p}(1 - e^{-ps}) \\ \frac{1}{p+\lambda}(e^{\lambda s} - e^{-ps}) \\ \frac{1}{p-\lambda}(e^{-\lambda s} - e^{-ps}) \end{pmatrix},$$

and

$$\bar{c}_1^1 = \begin{pmatrix} (d_3 + \lambda_{21} d_5) k_1(s,t) + \lambda_{12} d_6 \left[k_2(-\lambda, s, t) - e^{-pt} k_2(p - \lambda, s, t)\right] \\ (d_4 + \lambda_{21} d_6)\left[k_2(-\lambda, s, t) - e^{-pt} k_2(p - \lambda, s, t)\right] \\ \lambda_{12} d_5 k_1(s,t) \end{pmatrix},$$

$$\bar{c}_2^1 = \begin{pmatrix} (d_1 - \lambda_{12} d_5) k_1(s,t) - \lambda_{21} d_6 \left[k_2(-\lambda, s, t) - e^{-pt} k_2(p - \lambda, s, t)\right] \\ (d_4 - \lambda_{12} d_6)\left[k_2(-\lambda, s, t) - e^{-pt} k_2(p - \lambda, s, t)\right] \\ -\lambda_{21} d_5 k_1(s,t) \end{pmatrix}.$$

Proposition 3 will be validated in Appendix 3.

Because the derivation of U_2 is similar to that of U_7, we provide their solutions in Proposition 4.

Proposition 4 *With $q_{X_\xi}, \xi \in [0,t]$ being a regime-switching parameter controlled by Markov chain X_t, we have*

$$U_i =< \bar{U}_i, X_0 >, \;\; \bar{U}_i = (< \bar{c}_1^i, \bar{D}_i >, < \bar{c}_2^i, \bar{D}_i >)^T, i = 2, 7, \tag{21}$$

where

$$\bar{D}_2 = \begin{pmatrix} k_1(s,t) \\ k_2(-\lambda,s,t) - e^{-pt}k_2(p-\lambda,s,t) \\ \frac{1}{\lambda}[k_3(-\lambda,s,t) - k_2(\lambda,s,t)] + \frac{1}{p-\lambda}e^{-pt}[k_3(p-\lambda,s,t) - k_2(p-\lambda,s,t)] \\ k_2(p-\lambda,s,t) - e^{-pt}k_2(2p-\lambda,s,t) \\ k_2(\lambda,s,t) - e^{-pt}k_2(p+\lambda,s,t) \\ k_2(p,s,t) - e^{-pt}k_2(2p,s,t) \end{pmatrix},$$

$$\bar{D}_7 = \begin{pmatrix} k_1(s,t) \\ k_2(-\lambda,s,t) - e^{-pt}k_2(p-\lambda,s,t) \\ k_2(p-\lambda,s,t) - e^{-pt}k_2(2p-\lambda,s,t) \\ k_2(\lambda,s,t) - e^{-pt}k_2(p+\lambda,s,t) \\ k_2(p,s,t) - e^{-pt}k_2(2p,s,t) \end{pmatrix},$$

and

$$\bar{c}_1^2 = \begin{pmatrix} (d_3+\lambda_{21}d_5)k_1(s,t) + \frac{2(d_4+\lambda_{21}d_6)}{\lambda} + \frac{\lambda_{12}}{\lambda}(d_5e^{-\lambda s} - d_6e^{-\lambda t}) + \frac{\lambda_{12}}{p-\lambda}[d_5e^{(p-\lambda)s-pt} - d_6e^{-\lambda t}] \\ \lambda_{12}(d_5t - d_6s) - \frac{\lambda_{12}}{p}(d_5 - d_6e^{-p\tau}) + (d_4+\lambda_{21}d_6)[\frac{1}{p+\lambda}e^{(p+\lambda)s-pt} - \frac{1}{\lambda}e^{\lambda s}] - \frac{\lambda_{12}}{\lambda}(d_5-d_6) \\ -\lambda_{12}(d_5-d_6) \\ (\frac{\lambda_{12}}{p} - \frac{\lambda_{12}}{p-\lambda})(d_5-d_6)e^{-pt} \\ -(d_4+\lambda_{21}d_6)(\frac{1}{\lambda} + \frac{1}{p-\lambda})e^{-\lambda t} \\ (d_4+\lambda_{21}d_6)(\frac{1}{p-\lambda} - \frac{1}{p+\lambda})e^{-pt} \end{pmatrix},$$

$$\bar{c}_2^2 = \begin{pmatrix} (d_1-\lambda_{12}d_5)k_1(s,t) + \frac{2(d_2-\lambda_{12}d_6)}{\lambda} - \frac{\lambda_{21}}{\lambda}(d_5e^{-\lambda s} - d_6e^{-\lambda t}) - \frac{\lambda_{21}}{p-\lambda}[d_5e^{(p-\lambda)s-pt} - d_6e^{-\lambda t}] \\ -\lambda_{21}(d_5t - d_6s) + \frac{\lambda_{21}}{p}(d_5 - d_6e^{-p\tau}) + (d_2-\lambda_{12}d_6)[\frac{1}{p+\lambda}e^{(p+\lambda)s-pt} - \frac{1}{\lambda}e^{\lambda s}] + \frac{\lambda_{21}}{\lambda}(d_5-d_6) \\ \lambda_{21}(d_5-d_6) \\ -(\frac{\lambda_{21}}{p} - \frac{\lambda_{21}}{p-\lambda})(d_5-d_6)e^{-pt} \\ -(d_2-\lambda_{12}d_6)(\frac{1}{\lambda} + \frac{1}{p-\lambda})e^{-\lambda t} \\ (d_2-\lambda_{12}d_6)(\frac{1}{p-\lambda} - \frac{1}{p+\lambda})e^{-pt} \end{pmatrix},$$

$$\bar{c}_1^7 = \begin{pmatrix} (d_3+\lambda_{21}d_5)k_2(-p,\tau,0) + \frac{\lambda_{12}}{p-\lambda}[d_5e^{(p-\lambda)s-pt} - d_6e^{-\lambda t}] \\ -\frac{\lambda_{12}}{p}(d_5 - d_6e^{-p\tau}) + (d_4+\lambda_{21}d_6)\frac{1}{p+\lambda}e^{(p+\lambda)s-pt} \\ (\frac{\lambda_{12}}{p} - \frac{\lambda_{12}}{p-\lambda})(d_5-d_6)e^{-pt} \\ -(d_4+\lambda_{21}d_6)\frac{1}{p-\lambda}e^{-\lambda t} \\ (d_4+\lambda_{21}d_6)(\frac{1}{p-\lambda} - \frac{1}{p+\lambda})e^{-pt} \end{pmatrix},$$

$$\bar{c}_2^7 = \begin{pmatrix} (d_1-\lambda_{12}d_5)k_2(-p,\tau,0) - \frac{\lambda_{21}}{p-\lambda}[d_5e^{(p-\lambda)s-pt} - d_6e^{-\lambda t}] \\ \frac{\lambda_{21}}{p}(d_5 - d_6e^{-p\tau}) + (d_2-\lambda_{12}d_6)\frac{1}{p+\lambda}e^{(p+\lambda)s-pt} \\ -(\frac{\lambda_{21}}{p} - \frac{\lambda_{21}}{p-\lambda})(d_5-d_6)e^{-pt} \\ -(d_2-\lambda_{12}d_6)\frac{1}{p-\lambda}e^{-\lambda t} \\ (d_2-\lambda_{12}d_6)(\frac{1}{p-\lambda} - \frac{1}{p+\lambda})e^{-pt} \end{pmatrix}.$$

Appendix 4 provides the proof of Proposition 4.

The remaining unknown value $U_i, i = 3, 4, 5, 6$ can be determined similarly. Their formulae are presented together in Proposition 5.

Proposition 5 *With $q_{X_\xi}, \xi \in [0,t]$ being a regime-switching parameter controlled by Markov chain X_t, we have*

$$U_i =< \bar{U}_i, X_0 >, \ \ \bar{U}_i = (< \bar{c}_1^i, \bar{D}_i >, < \bar{c}_2^i, \bar{D}_i >)^T, i = 3,4,5,6, \tag{22}$$

where

$$\bar{D}_3 = \begin{pmatrix} k_1(s,t) \\ k_2(-\lambda,s,t) - e^{-pt}k_2(p-\lambda,s,t) \end{pmatrix}, \ \ \bar{D}_4 = \begin{pmatrix} -\tau e^{-p\tau} + \frac{1}{p}(1-e^{-p\tau}) \\ e^{-pt}[-\tau e^{(p-\lambda)s} + k_2(p-\lambda,s,t)] \end{pmatrix},$$
$$\bar{D}_5 = \begin{pmatrix} k_2(p,-\tau,0) - k_2(2p,-\tau,0) \\ e^{-pt}k_2(p-\lambda,s,t) - e^{-2pt}k_2(2p-\lambda,s,t) \end{pmatrix}, \ \ \bar{D}_6 = \begin{pmatrix} k_2(p,-\tau,0) \\ e^{-pt}k_2(p-\lambda,s,t) \end{pmatrix},$$

and

$$\bar{c}_1^3 = \bar{c}_1^5 = \bar{c}_1^6 = \begin{pmatrix} q_2 + \frac{\lambda_{21}}{\lambda}(q_1-q_2) \\ \frac{\lambda_{12}}{\lambda}(q_1-q_2) \end{pmatrix}, \ \ \bar{c}_2^3 = \bar{c}_2^5 = \bar{c}_2^6 = \begin{pmatrix} q_1 - \frac{\lambda_{12}}{\lambda}(q_1-q_2) \\ -\frac{\lambda_{21}}{\lambda}(q_1-q_2) \end{pmatrix},$$
$$\bar{c}_1^4 = \begin{pmatrix} \frac{1}{p}[q_2 + \frac{\lambda_{21}}{\lambda}(q_1-q_2)] \\ \frac{\lambda_{12}}{\lambda(p-\lambda)}(q_1-q_2) \end{pmatrix}, \ \ \bar{c}_2^4 = \begin{pmatrix} \frac{1}{p}[q_1 - \frac{\lambda_{12}}{\lambda}(q_1-q_2)] \\ -\frac{\lambda_{21}}{\lambda(p-\lambda)}(q_1-q_2) \end{pmatrix}.$$

Appendix validation proposition 5.

Once we have determined $U_i, i = 1,2,...,7$, the results of target $g_1(Y_0, \nu_0, \gamma_0, X_0, s, t)$ are summarized in the following theorem.

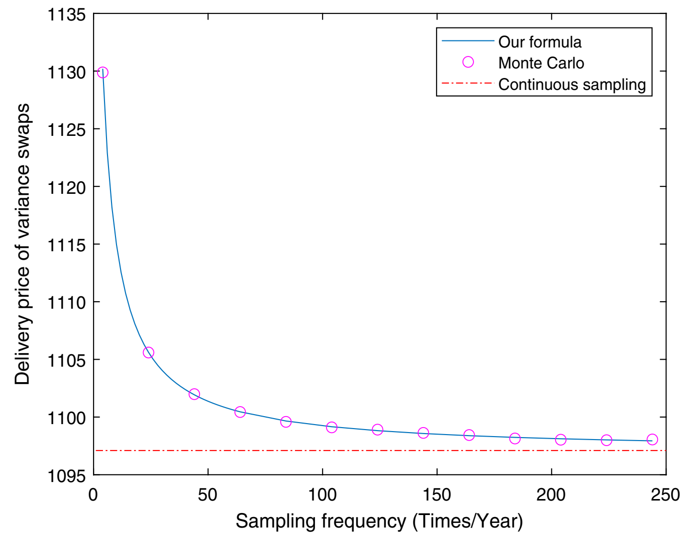
Theorem 6 *With the formulae of $U_i, i = 1,2,...,7$ being provided in Propositions* 3.3–3.5, *the function $g_1(Y_0, \nu_0, \gamma_0, X_0, s, t)$, when Y_t follows Eq.* (2), *has a solution of*

$$\begin{aligned} g_1 =& A_1 + (r - a_3a_2)^2\tau^2 + \frac{a_3^2}{a_1^2}(1-e^{-a_1\tau})^2\left[A_5^2 + h_4(\gamma_0,0,s)\right] + (A_3 + A_4A_5)e^{-ps}\nu_0 \\ &+ \left[A_2 - \frac{2a_3}{a_1}(1-e^{-a_1\tau})(r-a_3a_2)\tau\right]A_5 + \frac{1}{4p^2}(1-e^{-p\tau})^2\left[\frac{\zeta^2}{p}(e^{-ps}-e^{-2ps})\nu_0 + e^{-2ps}\nu_0^2\right] \\ &+ \frac{1}{2}(1-e^{-p\tau})U_1 + \frac{1}{4}U_2 + \left(\rho\zeta - \frac{\zeta^2}{2p}\right)U_4 + \left[\frac{\zeta^2}{4p^2} + \frac{\zeta^2}{4p^2}(1-e^{-p\tau})^2\right]U_5 \\ &+ \left[1 + \frac{\zeta^2}{4p^2} - \frac{\rho\zeta}{p} - (r-a_3a_2)\tau + \frac{a_3}{a_1}(1-e^{-a_1\tau})A_5 + \frac{1}{2p}(1-e^{-p\tau})e^{-ps}\nu_0\right]U_3 \\ &+ \left[\frac{1}{2p}(1-e^{-p\tau})^2e^{-ps}\nu_0 + p(A_3+A_4A_5)\right]U_6 + \frac{1}{4}(1-e^{-p\tau})^2U_7. \end{aligned} \tag{23}$$

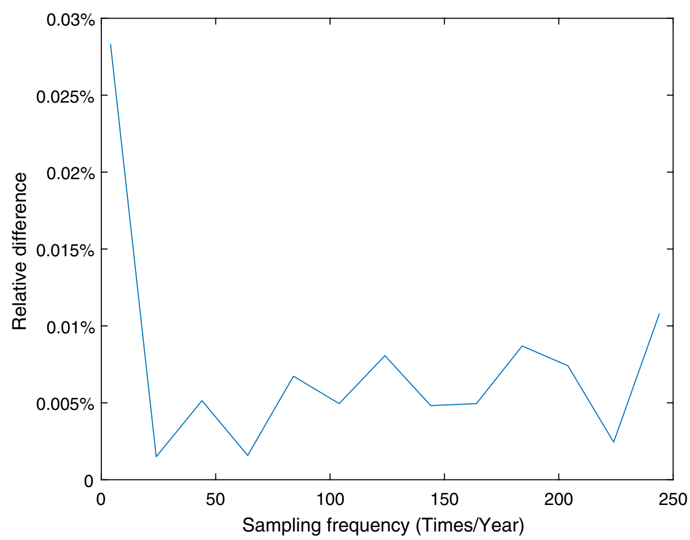
Proof The proof of this theorem is straightforward after considering the expectations of $g_2(Y_0, \nu_0, \gamma_0, s, t)$. □

The strike prices of the variance swaps can now be analytically computed using Eq. (5) when the effects of jump clustering, stochastic volatility, and varying economic conditions are incorporated. Note that this formula does not involve Fourier inversion or any other integration, which can significantly improve efficiency in practice.

It would also be interesting to investigate how variance swaps with continuous sampling behave under the considered model, which can provide an additional check on the validity of the proposed formula. By defining



(a) Strike prices



(b) Relative errors

Fig. 1 Verification of the correctness

$$RV^c = \frac{100^2}{T}\left(\int_0^T v_t dt + \int_0^T \tilde{J}_t dH_t\right),$$

Based on Chebyshev's inequality, one can illustrate that $RV - RV^c$ converges to zero as the probability approaches one, following arguments similar to those presented in Broadie and Jain (2008), Liu and Zhu (2019): Therefore, variance swaps with continuous sampling can be expressed as:

$$K^c = E\left(RV^c | \mathcal{F}_0^{W_1} \vee \mathcal{F}_0^{W_2} \vee \mathcal{F}_0^{H} \vee \mathcal{F}_0^{X}\right),$$

yielding

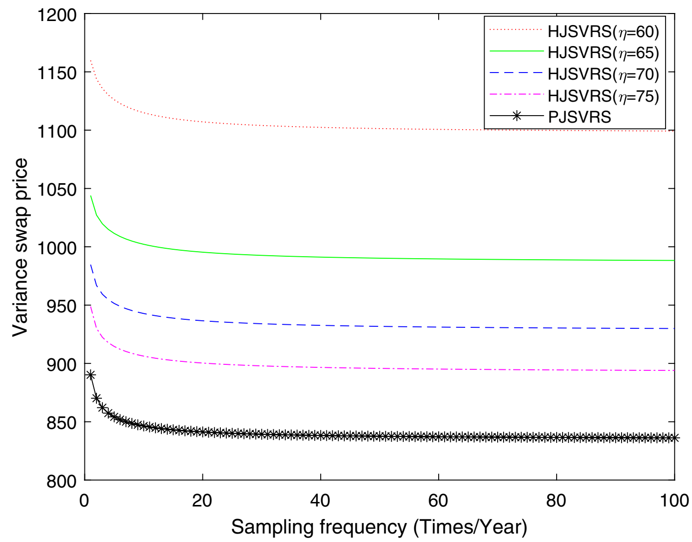


Fig. 2 The effects of η

$$K^c = \frac{100^2}{T}\left\{\int_0^T < \begin{pmatrix} (q_1 - q_2)\left(\frac{\lambda_{21}}{\lambda} + \frac{\lambda_{12}}{\lambda}e^{-\lambda u}\right) + q_2 \\ (q_2 - q_1)\left(\frac{\lambda_{12}}{\lambda} + \frac{\lambda_{21}}{\lambda}e^{-\lambda u}\right) + q_1 \end{pmatrix}, X_0 > \cdot \left[1 - e^{-p(T-u)}\right] du \right.$$
$$\left. + \frac{1}{p}\left(1 - e^{-pT}\right)v_0 + (\sigma^2 + \mu^2)\left[a_2 T + (\lambda_0 - a_2)\frac{1 - e^{-a_1 T}}{a_1}\right]\right\}, \tag{24}$$

from the results of propositions 1 and 5.

Formulae with discrete and continuous sampling are numerically implemented in the next section to demonstrate the impact of the three stochastic factors.

Numerical analysis

We first discuss the reliability of the formula (5) presented in the previous section by comparing its produced prices with the Monte Carlo results. We also analyze the sensitivity of strike prices to changes in the different model parameters. Here is a list of parameters that produce the plots in this section.

$$p = 12;\ \hat{q} = (0.02, 0.2);\ \zeta = 0.6;\ \eta = 60;\ \delta = 50;\ \gamma_\infty = 4.5;\ v_0 = 0.04;\ \gamma_0 = 10;$$
$$r = 0.1;\ \mu = -0.01;\ \sigma = 0.04;\ T = 1;\ X_0 = (1, 0);\ \lambda_{12} = \lambda_{21} = 1;\ N = 4.$$

Figure 1 displays the strike prices produced by Eq. (5) compared to Monte Carlo prices. The clear pattern shown in Fig. 1a indicates a very close agreement between the corresponding results from the two methods, which is further supported by Fig. 1b, where the errors of our results relative to the benchmark are less than 0.03%. Another evidence supporting the correctness of the formula with discrete sampling is its convergence with that with continuous sampling when the sampling frequency is increased.

Model performance can be assessed once the accuracy of the formula is confirmed. We first show the difference between our model considering stochastic volatility, jump

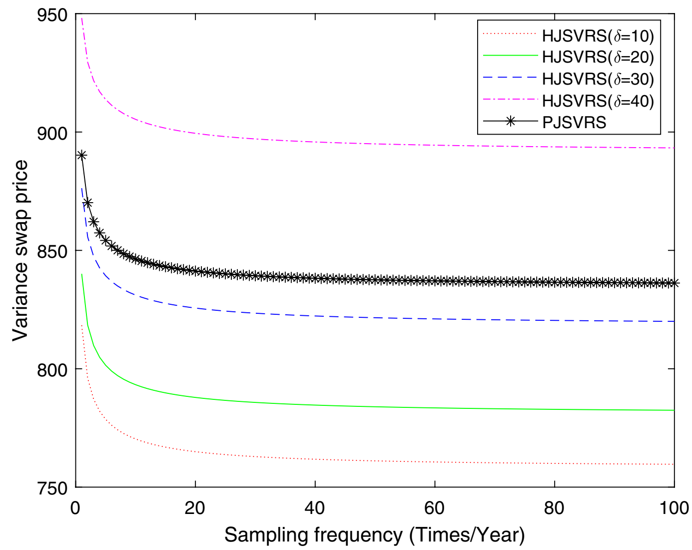


Fig. 3 The effects of δ

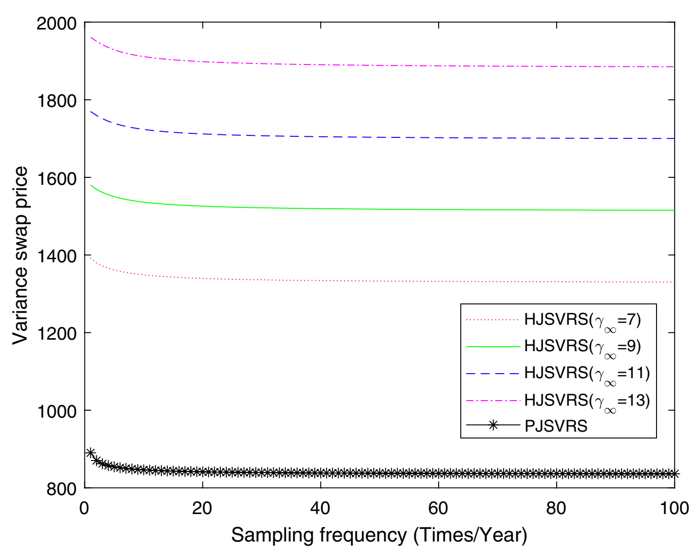


Fig. 4 The effects of γ_∞

clustering, and variations in economic status (HJSVRS) and that constructed by our model ignoring jump clustering (PJSVRS). This comparison is illustrated in Fig. 2 with the varying decay rates of jump clustering η. Some distance exists between the swap prices generated by the two models, and decreasing the value of η typically widens this gap. This phenomenon is reasonable, because a higher decay rate implies a less significant impact of jump clustering.

The opposite phenomenon is displayed in Fig. 3, where increasing the value of δ is equivalent to increasing strike prices. A larger δ implies a higher possibility for the arrival of another jump, creating more significant jump-clustering effects and leading to the corresponding strike prices. The strike prices of PJSVRS model stay in between HJSVRS

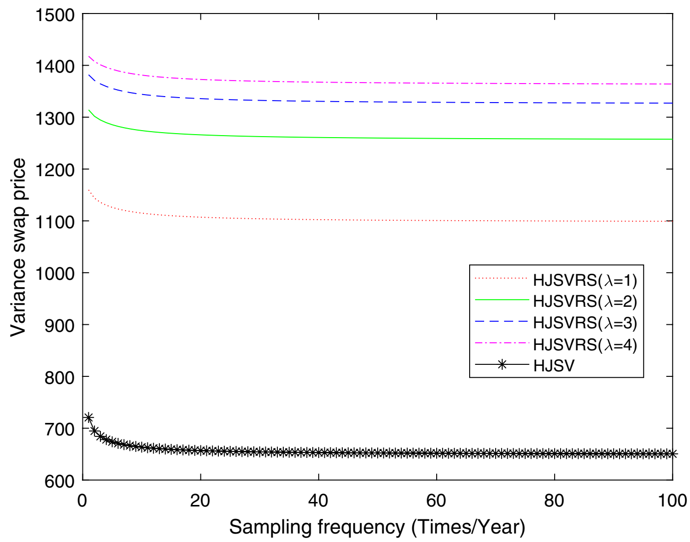


Fig. 5 The effects of λ

prices, which is probably due to the decrease in δ leads to the number of expected jumps in HJSVRS model falling below that in PJSVRS model.

The impact of jump clustering is also examinedwhen we assign γ_∞ different values in Fig. 4. An increasing trend in variance swap prices can be observed with the increase in γ_∞, leading to a wide difference between HJSVRS and PJSVRS model prices. The higher γ_∞ indicates a greater number of expected jumps, yielding larger risks, as well as swap delivery prices.

The impacts of varying economic statuses are crucial factors that must be studied. The HJSV model was constructed as a benchmark for this by removing the regime switching contained in HJSVRS model and plot both model prices in Fig. 5 after equating λ_{12} and λ_{21} to λ for ease of comparison. An increase in transition rates would inevitably result in an increase in strike prices, causing a greater difference between HJSVRS and HJSV model prices. The reasonableness of this observation lies in the current setting of the long-term variance, corresponding to the current state being lower than that of the other states, and the increasing transition rates actually raise the volatility level, yielding higher risks and strike prices.

Conclusion

This study solved the variance-swap pricing problem when the underlying dynamics are subject to the risks of stochastic volatility, jump clustering, and regime switching. We utilized a novel probabilistic approach and first considered a simplified case when regime-switching parameters were replaced by time-dependent ones, whose solution acts as a given condition when working on a general case. Compared with the literature requiring Fourier inversion or fast Fourier transform, the obtained solution involved no numerical integration and was written using only fundamental functions, which greatly enhances its efficiency. Variance swap prices are also numerically shown to be significantly influenced by jumps and regime switching.

Appendix 1

Here, we prove Proposition 1. To determine the first and second moments associated with ν_t, we first apply Ito's lemma to derive

$$d\left(e^{-p(t-\xi)}\nu_\xi\right) = pq_\xi e^{-p(t-\xi)}d\xi + \zeta\sqrt{\nu_\xi}dW_{2,\xi}\cdot e^{-p(t-\xi)},\ 0\le\xi<t,$$

The integration of from s to t leads to the expression ν_t asfollows:

$$\nu_t = e^{-p(t-s)}\nu_s + p\int_s^t q_\xi e^{-p(t-\xi)}d\xi + \zeta\int_s^t\sqrt{\nu_\xi}e^{-p(t-\xi)}dW_{2,\xi}. \tag{25}$$

The expectation of ν_t is straightforward, because $\int_s^t\sqrt{\nu_\xi}e^{-p(t-\xi)}dW_{2,\xi}$ is a martingale whose expectation is zero. The variance of ν_t can be computed as

$$\begin{aligned} var(\nu_t|\mathcal{F}_s^{W_2}) =& var\left(\zeta\int_s^t\sqrt{\nu_\xi}e^{-p(t-\xi)}dW_{2,\xi}|\mathcal{F}_s^{W_2}\right)\\ =&\zeta^2\int_s^t e^{-2p(t-\xi)}E\left(\nu_\xi|\mathcal{F}_s^{W_2}\right)d\xi\\ =&\zeta^2\int_s^t e^{-2p(t-\xi)}\left[e^{-p(t-s)}\nu_s + p\int_s^\xi q_x e^{-p(\xi-x)}dx\right]d\xi. \end{aligned} \tag{26}$$

With

$$\int_s^t e^{-2p(t-\xi)}\int_s^\xi q_x e^{-p(\xi-x)}dxd\xi = \int_s^t q_x\int_x^t e^{p\xi-2pt+px}d\xi dx = \frac{1}{p}\int_s^t q_\xi[e^{-p(t-\xi)} - e^{-2p(t-\xi)}]d\xi,$$

one can then obtain

$$\begin{aligned} var(\nu_t|\mathcal{F}_s^{W_2}) =&\zeta^2\int_s^t e^{-2p(t-\xi)}\cdot e^{-p(\xi-s)}d\xi\cdot\nu_s + \zeta^2 p\int_s^t e^{-2p(t-\xi)}\int_s^\xi q_x e^{-p(\xi-x)}dxd\xi,\\ =&\frac{\zeta^2}{p}\left[e^{-p(t-s)} - e^{-2p(t-s)}\right]\nu_s + \zeta^2\int_s^t q_\xi\left[e^{-p(t-\xi)} - e^{-2p(t-\xi)}\right]d\xi. \end{aligned} \tag{27}$$

We also need to derive the first and second moments associated with γ_t. We first adjust the jump process to form a martingale through $N_t = H_t - \int_0^t\gamma_\xi d\xi$ so that the expectation of dN_t is zero. As a result, we obtain

$$d\gamma_t = a_1(a_2-\gamma_t)dt + \delta dN_t.$$

One can further compute the dynamics of $d\left(e^{-a_1(t-\xi)}\gamma_\xi\right)$ with Io's Lemma, which provides a representation of γ_t as

$$\gamma_t = a_2 + (\gamma_s - a_2)e^{-a_1(t-s)} + \delta\int_s^t e^{-a_1(t-\xi)}dN_\xi.$$

This directly gives the result of $E(\gamma_t|\mathcal{F}_s^H)$ due to $\int_s^t e^{-a_1(t-\xi)}dN_\xi$ being a martingale with its mean as zero. Moreover, we can also obtain

$$\begin{aligned} var(\gamma_t|\mathcal{F}_s^H) =& var\left(\delta \int_s^t \sqrt{\nu_\xi} e^{-a_1(t-\xi)} dN_\xi |\mathcal{F}_s^H\right) \\ =&\delta^2 \int_s^t e^{-2a_1(t-\xi)} E(\gamma_\xi|\mathcal{F}_s^H) d\xi \\ =&\delta^2 \int_s^t e^{-2a_1(t-\xi)} \left[a_2 + (\gamma_s - a_2) e^{-a_1(\xi-s)}\right] d\xi, \end{aligned} \tag{28}$$

which leads to the desired explicit result after working out the integration.

Appendix 2

Here is the proof of Proposition 2.

To compute the expectation and variance of $\ln\left(\frac{Y_t}{Y_s}\right)$, a necessary step is to figure out its expression. In particular, Integrating both sides of Eq. (4) from s to t should yield

$$\ln\left(\frac{Y_t}{Y_s}\right) = \int_s^t \left[r - \frac{1}{2}\nu_\xi - (\omega - L_\xi)\gamma_\xi\right] d\xi + \int_s^t \sqrt{\nu_\xi} dW_{1,\xi} + \int_s^t L_\xi dN_\xi. \tag{29}$$

A simple treatment gives

$$\int_s^t \nu_\xi d\xi = \frac{1}{p}(1 - e^{-p\tau})\nu_s + \int_s^t q_\xi [1 - e^{-p(t-\xi)}] d\xi + \zeta \int_s^t \int_s^\xi \sqrt{\nu_x} e^{-p(\xi - x)} dW_{2,x} d\xi,$$

which directly leads to

$$\ln\left(\frac{Y_t}{Y_s}\right) = J_1(\tau) + J_2(W_{1,t}) + J_3(W_{2,t}) + J_4(N_t). \tag{30}$$

Here,

$$\begin{aligned} J_1(\tau) =& (r - \omega a_2)\tau - \frac{1}{2p}(1 - e^{-p\tau})\nu_s - \frac{1}{2}\int_s^t q_\xi [1 - e^{-p(t-\xi)}] d\xi - \frac{m(\gamma_s - a_2)}{a_1}(1 - e^{-a_1\tau}) \\ &+ \int_s^t [a_2 + (\gamma_s - a_2) e^{-a_1(\xi - s)}] L_\xi d\xi, \\ J_2(W_{1,t}) =& \int_s^t \sqrt{\nu_\xi} dW_{1,\xi}, \\ J_3(W_{2,t}) =& -\frac{1}{2}\zeta \int_s^t \int_s^\xi \sqrt{\nu_x} e^{-p(\xi - x)} dW_{2,x} d\xi = -\frac{1}{2}\zeta \int_s^t \frac{1 - e^{-p(t-\xi)}}{p} \sqrt{\nu_\xi} dW_{2,\xi}, \\ J_4(N_t) =& \int_s^t L_\xi dN_\xi = \int_s^t \left[L_\xi - \delta \int_x^t (L_\xi - \omega) e^{-a_1(\xi - x)} d\xi\right] dN_x. \end{aligned}$$

Taking the expectation on both sides of Eq. (30) and using $E(L_t) = \mu$, we obtain the desired result in Eq. (13).

However, considering the variance on both sides of Eq. (30) results in:

$$g_{3,2} = var(J_1|\mathcal{F}_s^H) + var(J_2|\mathcal{F}_s^{W_1}) + var(J_3|\mathcal{F}_s^{W_2}) + var(J_4|\mathcal{F}_s^H) + 2cov(J_2, J_3|\mathcal{F}_s^{W_1} \vee \mathcal{F}_s^{W_2}). \tag{31}$$

One can directly have

$$var(J_1|\mathcal{F}_s^H) = 0, \tag{32}$$

which is the result of J_1 being constant given $\mathcal{F}_s^H$. Based on the results of Proposition 1, we can further compute

$$\begin{aligned} var(J_2|\mathcal{F}_s^{W_1}) &= \int_s^t E(\nu_\xi|\mathcal{F}_s^{W_1})d\xi \\ &= \frac{1}{p}(1-e^{-p\tau})\nu_s + p\int_s^t \int_s^\xi q_x e^{-p(\xi-x)}dxd\xi \\ &= \frac{1}{p}(1-e^{-p\tau})\nu_s + \int_s^t q_\xi[1-e^{-p(t-\xi)}]d\xi. \end{aligned} \tag{33}$$

The third unknown term is calculated as follows:

$$\begin{aligned} var(J_3|\mathcal{F}_s^{W_2}) =& \frac{\zeta^2}{4p^2}\int_s^t [1-e^{-p(t-\xi)}]^2 E(\nu_\xi|\mathcal{F}_s^{W_2})d\xi \\ =& \frac{\zeta^2}{4p^2}\int_s^t \left[1-2e^{-p(t-\xi)}+e^{-2p(t-\xi)}\right]\left[e^{-p(\xi-s)}\nu_s + p\int_s^\xi q_x e^{-p(\xi-x)}dx\right]d\xi. \end{aligned} \tag{34}$$

With

$$\begin{aligned} \int_s^t \left[1-2e^{-p(t-\xi)}+e^{-2p(t-\xi)}\right]e^{-p(\xi-s)}\nu_s d\xi &= \left[\frac{1}{p}(1-e^{-p\tau})-2e^{-p\tau}\tau\right]\nu_s, \\ \int_s^t p\int_s^\xi q_x e^{-p(\xi-x)}dxd\xi &= \int_s^t q_\xi[1-e^{-p(t-\xi)}]d\xi, \\ \int_s^t -2pe^{-p(t-\xi)}\int_s^\xi q_x e^{-k(\xi-x)}dxd\xi &= -2p\int_s^t e^{-p(t-\xi)}(t-\xi)q_\xi d\xi, \\ \int_s^t e^{-2p(t-\xi)}\cdot p\int_s^\xi q_x e^{-p(\xi-x)}dxd\xi &= \int_s^t [e^{-p(t-\xi)}-e^{-2p(t-\xi)}]q_\xi d\xi, \end{aligned}$$

one can arrive at

$$\begin{aligned} var(J_3|\mathcal{F}_s^{W_2}) =& \frac{\zeta^2}{4p^2}\Bigg\{\left[\frac{1}{p}(1-e^{-2p\tau})-2e^{-p\tau}\tau\right]\nu_s + \int_s^t q_\xi[1-e^{-p(t-\xi)}]d\xi \\ & -2p\int_s^t e^{-p(t-\xi)}(t-\xi)q_\xi d\xi + \int_s^t [e^{-p(t-\xi)}-e^{-2p(t-\xi)}]q_\xi d\xi\Bigg\}. \end{aligned} \tag{35}$$

From the results in Proposition 1, the fourth unknown term can be derived as

$$\begin{aligned} var(J_4|\mathcal{F}_s^H) &= \int_s^t E\left\{\left[L_\xi + \delta\int_\xi^t (L_x-\omega)e^{-a_1(x-\xi)}dx\right]^2\right\}E\left(\gamma_\xi|\mathcal{F}_s^H\right)d\xi \\ &= \int_s^t \left\{\sigma^2 + \left[\mu - \frac{\delta a_3}{a_1}\left(1-e^{-a_1(t-\xi)}\right)\right]^2\right\}\left[a_2 + (\gamma_s - a_2)e^{-a_1(\xi-s)}\right]\xi. \end{aligned} \tag{36}$$

We can further compute the fifth unknown term using the results in Proposition 1, leading to

$$
\begin{aligned}
2cov(J_2, J_3|\mathcal{F}_s^{W_1} \vee \mathcal{F}_s^{W_2}) =& -\rho\zeta \int_s^t \frac{1-e^{-p(t-\xi)}}{p} \mathbb{E}\big(\nu_\xi|\mathcal{F}_s^{W_1} \vee \mathcal{F}_s^{W_2}\big) d\xi \\
=& -\frac{\rho\zeta}{p} \int_s^t \left[1-e^{-p(t-\xi)}\right]\left[e^{-p(\xi-s)}\nu_s + p\int_s^\xi q_x e^{-p(\xi-x)}dx\right] d\xi \\
\triangleq& -\frac{\rho\zeta}{p}(\chi_1+\chi_2+\chi_3),
\end{aligned} \tag{37}
$$

where

$$
\begin{aligned}
\chi_1 =& \int_s^t \left[1-e^{-p(t-\xi)}\right] e^{-p(\xi-s)}\nu_s d\xi = \left[\frac{1}{p}(1-e^{-p\tau}) - e^{-p\tau}\tau\right]\nu_s \\
\chi_2 =& p\int_s^t \int_s^\xi q_x e^{-p(\xi-x)} dx d\xi = \int_s^t q_\xi [1-e^{-p(t-\xi)}] d\xi \\
\chi_3 =& -p\int_s^t e^{-p(t-\xi)} \int_s^\xi q_x e^{-p(\xi-x)} dx d\xi = -p\int_s^t e^{-p(t-\xi)}(t-\xi) q_\xi d\xi.
\end{aligned}
$$

The solution to $g_{3,2}$ in Eq. (31) can then be found by substituting the results in Eqs. (32)–(37), as well as some simplifications.

Appendix 3

Here, we prove Proposition 3.

We can formulate

$$
\begin{aligned}
U_1 =& \mathbb{E}\left[\int_0^s q_u e^{-k(s-u)} \cdot \int_s^t q_\xi \left(1-e^{-p(t-\xi)}\right) d\xi du|\mathcal{F}_0^X\right] \\
=& \int_0^s \mathbb{E}\left[q_u e^{-k(s-u)} \cdot \int_s^t q_\xi \left(1-e^{-p(t-\xi)}\right) d\xi|\mathcal{F}_0^X\right] du \\
=& \int_0^s \mathbb{E}\left\{q_u e^{-k(s-u)} \cdot \int_s^t \mathbb{E}\left[q_\xi \left(1-e^{-p(t-\xi)}\right)|\mathcal{F}_u^X\right] d\xi|\mathcal{F}_0^X\right\} du,
\end{aligned} \tag{38}
$$

where the final step is the result of the tower rule of expectations. This demands the solution to $I_{11} = \mathbb{E}\left\{q_u e^{-k(s-u)} \cdot \int_s^t \mathbb{E}\left[q_\xi \left(1-e^{-p(t-\xi)}\right)|\mathcal{F}_u^X\right] d\xi|\mathcal{F}_0^X\right\}$. Let

$$
\begin{aligned}
\psi_{11}(\xi,u) =& \frac{\lambda_{21}}{\lambda_{12}+\lambda_{21}} + \frac{\lambda_{12}}{\lambda_{12}+\lambda_{21}} e^{-(\lambda_{12}+\lambda_{21})(\xi-u)} = \frac{\lambda_{21}}{\lambda} + \frac{\lambda_{12}}{\lambda} e^{-\lambda(\xi-u)}, \\
\psi_{22}(\xi,u) =& \frac{\lambda_{12}}{\lambda_{12}+\lambda_{21}} + \frac{\lambda_{21}}{\lambda_{12}+\lambda_{21}} e^{-(\lambda_{12}+\lambda_{21})(\xi-u)} = \frac{\lambda_{12}}{\lambda} + \frac{\lambda_{21}}{\lambda} e^{-\lambda(\xi-u)},
\end{aligned}
$$

where $\lambda = \lambda_{12} + \lambda_{21}$. We further write

$$
\psi_{12}(\xi,u) = 1-\psi_{11}(\xi,u), \quad \psi_{21}(\xi,u) = 1-\psi_{22}(\xi,u),
$$

and denote

$$
\Psi(\xi,u) = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}(\xi,u).
$$

We can then directly obtain the inner expectation of U_{10} as

$$\mathbb{E}\Big[q_\xi\Big(1-e^{-p(t-\xi)}\Big)|\mathcal{F}_u^X\Big] = <\Psi(\xi,u)\begin{pmatrix} q_1\big(1-e^{-p(t-\xi)}\big) \\ q_2\big(1-e^{-p(t-\xi)}\big)\end{pmatrix}, X_u > \\ = <\begin{pmatrix} q_1\big(1-e^{-p(t-\xi)}\big)\psi_{11}(\xi,u)+q_2\big(1-e^{-p(t-\xi)}\big)\psi_{12}(\xi,u) \\ q_1\big(1-e^{-p(t-\xi)}\big)\psi_{21}(\xi,u)+q_2\big(1-e^{-p(t-\xi)}\big)\psi_{22}(\xi,u)\end{pmatrix}, X_u >,$$

with which we can obtain

$$\begin{aligned} I_{11} =&\mathbb{E}\Big[q_u e^{-k(s-u)} < \begin{pmatrix} q_1\iota_1+q_2\iota_2 \\ q_1\iota_3+q_2\iota_4\end{pmatrix}, X_u > |\mathcal{F}_0^X\Big] \\ =&e^{-k(s-u)}\mathbb{E}\Big[<\begin{pmatrix} q_1^2\iota_1+q_1q_2\iota_2 \\ q_1q_2\iota_3+q_2^2\iota_4\end{pmatrix}, X_u > |\mathcal{F}_0^X\Big] \\ =&e^{-k(s-u)} < \Psi(u,0)\begin{pmatrix} q_1^2\iota_1+q_1q_2\iota_2 \\ q_1q_2\iota_3+q_2^2\iota_4\end{pmatrix}, X_0 >, \end{aligned} \tag{39}$$

where

$$\iota_1 = \int_s^t \Big(1-e^{-p(t-\xi)}\Big)\psi_{11}(\xi,u)d\xi, \quad \iota_2 = \int_s^t \Big(1-e^{-p(t-\xi)}\Big)\psi_{12}(\xi,u)d\xi,$$
$$\iota_3 = \int_s^t \Big(1-e^{-p(t-\xi)}\Big)\psi_{21}(\xi,u)d\xi, \quad \iota_4 = \int_s^t \Big(1-e^{-p(t-\xi)}\Big)\psi_{22}(\xi,u)d\xi.$$

Simple manipulation yields

$$\begin{aligned} q_1^2\iota_1+q_1q_2\iota_2 =&b_1+b_2e^{\lambda u}, \\ q_1q_2\iota_3+q_2^2\iota_4 =&b_3+b_4e^{\lambda u}, \end{aligned}$$

where

$$\begin{aligned} b_1 =&d_1k_1(s,t), \quad b_2 = d_2\big[k_2(-\lambda,s,t)-e^{-pt}k_2(p-\lambda,s,t)\big], \\ b_3 =&d_3k_1(s,t), \quad b_4 = d_4\big[k_2(-\lambda,s,t)-e^{-pt}k_2(p-\lambda,s,t)\big]. \end{aligned}$$

Substituting Eq. (39) into (38) yields:

$$\begin{aligned} U_1 =& \int_0^s e^{-k(s-u)} < \Psi(u,0)\begin{pmatrix} q_1^2\iota_1+q_1q_2\iota_2 \\ q_1q_2\iota_3+q_2^2\iota_4\end{pmatrix}, X_0 > d\xi \\ =& \int_0^s e^{-p(s-u)} < \begin{pmatrix} \psi_{11}(u,0)(b_1+b_2e^{\lambda u})+\psi_{12}(u,0)(b_3+b_4e^{\lambda u}) \\ \psi_{21}(u,0)(b_1+b_2e^{\lambda u})+\psi_{22}(u,0)(b_3+b_4e^{\lambda u})\end{pmatrix}, X_0 > du. \end{aligned}$$

One can then reach The desired result can be obtained after determining the integrations involved.

Appendix 4

Here, we prove Proposition 4. The computation of U_2 can be expressed as

$$
\begin{aligned}
U_2 =&\mathbb{E}\left\{\left[\int_s^t q_\xi(1-e^{-p(t-\xi)})d\xi\right]^2|\mathcal{F}_0^X\right\}\\
=&\mathbb{E}\left[\int_s^t\int_s^t q_u q_\xi(1-e^{-p(t-u)})(1-e^{-p(t-\xi)})d\xi du|\mathcal{F}_0^X\right]\\
=&\int_s^t(1-e^{-p(t-u)})(I_{21}+I_{22})du,
\end{aligned}
\tag{40}
$$

where

$$
I_{21}=\int_s^u\mathbb{E}(q_u q_\xi|\mathcal{F}_0^X)(1-e^{-p(t-\xi)})d\xi,\ \ I_{22}=\int_u^t\mathbb{E}(q_u q_\xi|\mathcal{F}_0^X)(1-e^{-p(t-\xi)})d\xi.
$$

This requires calculation of U_{21} and U_{22}. Specifically,

$$
\begin{aligned}
I_{21}=&\int_s^u\mathbb{E}[q_\xi\mathbb{E}(q_u|\mathcal{F}_\xi^X)|\mathcal{F}_0^X](1-e^{-p(t-\xi)})d\xi\\
=&\int_s^u\mathbb{E}\left[<\begin{pmatrix}\psi_{11}(u,\xi)q_1^2+\psi_{12}(u,\xi)q_1q_2\\ \psi_{21}(u,\xi)q_1\theta_2+\psi_{22}(u,\xi)q_2^2\end{pmatrix},X_\xi>|\mathcal{F}_0^X\right](1-e^{-p(t-\xi)})d\xi\\
=&\int_s^u<\Psi(\xi,0)\begin{pmatrix}d_1+d_2e^{-\lambda(u-\xi)}\\ d_3+d_4e^{-\lambda(u-\xi)}\end{pmatrix},X_0>(1-e^{-p(t-\xi)})d\xi,\\
=&\int_s^u<\begin{pmatrix}L_1\\ L_2\end{pmatrix},X_0>(1-e^{-p(t-\xi)})d\xi,
\end{aligned}
$$

where

$$
\begin{aligned}
L_1=&d_3+\frac{\lambda_{21}}{\lambda}(d_1-d_3)+\left[f_4+\frac{\lambda_{21}}{\lambda}(d_2-d_4)\right]e^{-\lambda(u-\xi)}+\frac{\lambda_{12}}{\lambda}(d_1-d_3)e^{-\lambda\xi}+\frac{\lambda_{12}}{\lambda}(d_2-d_4)e^{-\lambda u},\\
L_2=&d_1+\frac{\lambda_{12}}{\lambda}(d_3-d_1)+\left[d_2+\frac{\lambda_{12}}{\lambda}(d_4-d_2)\right]e^{-\lambda(u-\xi)}+\frac{\lambda_{21}}{\lambda}(d_3-d_1)e^{-\lambda\xi}+\frac{\lambda_{21}}{\lambda}(d_4-d_2)e^{-\lambda u}.
\end{aligned}
$$

Following a similar argument, one can also find that

$$
I_{22}=\int_u^t<\begin{pmatrix}L_3\\ L_4\end{pmatrix},X_0>(1-e^{-p(t-\xi)})d\xi,
\tag{41}
$$

where

$$
\begin{aligned}
L_3=&d_3+\frac{\lambda_{21}}{\lambda}(d_1-d_3)+\left[d_4+\frac{\lambda_{21}}{\lambda}(d_2-d_4)\right]e^{-\lambda(\xi-u)}+\frac{\lambda_{12}}{\lambda}(d_1-d_3)e^{-\lambda u}+\frac{\lambda_{12}}{\lambda}(d_2-d_4)e^{-\lambda\xi},\\
L_4=&d_1+\frac{\lambda_{12}}{\lambda}(d_3-d_1)+\left[d_2+\frac{\lambda_{12}}{\lambda}(d_4-d_2)\right]e^{-\lambda(\xi-u)}+\frac{\lambda_{21}}{\lambda}(d_3-d_1)e^{-\lambda u}+\frac{\lambda_{21}}{\lambda}(d_4-d_2)e^{-\lambda\xi}.
\end{aligned}
$$

Consequently, we arrive at the following hypotheses:

$$
\begin{aligned}
U_2=&\int_s^t(1-e^{-p(t-u)})<\begin{pmatrix}\int_s^u L_1(1-e^{-p(t-\xi)})d\xi+\int_u^t L_3(1-e^{-p(t-\xi)})d\xi\\ \int_s^u L_2(1-e^{-p(t-\xi)})d\xi+\int_u^t L_4(1-e^{-p(t-\xi)})d\xi\end{pmatrix},X_0>du\\
=&<\begin{pmatrix}\int_s^t(1-e^{-p(t-u)})\left[\int_s^u L_1(1-e^{-p(t-\xi)})d\xi+\int_u^t L_3(1-e^{-p(t-\xi)})d\xi\right]du\\ \int_s^t(1-e^{-p(t-u)})\left[\int_s^u L_2(1-e^{-p(t-\xi)})d\xi+\int_u^t L_4(1-e^{-p(t-\xi)})d\xi\right]du\end{pmatrix},X_0>.
\end{aligned}
\tag{42}
$$

Determining the integration contained in the above formula leads to a solution for U_2.

Similarly, we can reformulate U_7 as

$$\begin{aligned} U_7 =&\mathbb{E}\left\{\left[\int_s^t q_\xi e^{-p(t-\xi)}d\xi\right]^2 |\mathcal{F}_0^X\right\} \\ =&\mathbb{E}\left[\int_s^t\int_s^t q_u q_\xi e^{-p(t-u)}e^{-p(t-\xi)}d\xi du|\mathcal{F}_0^X\right] \\ =&\int_s^t e^{-p(t-u)}(I_{71}+I_{72})du, \end{aligned} \tag{43}$$

where

$$I_{71}=\int_s^u \mathbb{E}(q_u q_\xi|\mathcal{F}_0^X)e^{-p(t-\xi)}d\xi,\ \ I_{72}=\int_u^t \mathbb{E}(q_u q_\xi|\mathcal{F}_0^X)e^{-p(t-\xi)}d\xi.$$

Thus, we have

$$\begin{aligned} I_{71}&=\int_s^u < \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, X_0 > e^{-p(t-\xi)}d\xi, \\ I_{72}&=\int_u^t < \begin{pmatrix} L_3 \\ L_4 \end{pmatrix}, X_0 > e^{-p(t-\xi)}d\xi, \end{aligned}$$

and this further yields

$$U_7 =< \begin{pmatrix} \int_s^t e^{-p(t-u)}\left[\int_s^u L_1 e^{-p(t-\xi)}d\xi + \int_u^t L_3 e^{-p(t-\xi)}d\xi\right]du \\ \int_s^t e^{-p(t-u)}\left[\int_s^u L_2 e^{-p(t-\xi)}d\xi + \int_u^t L_4 e^{-p(t-\xi)}d\xi\right]du \end{pmatrix}, X_0 >, \tag{44}$$

to obtain the desired formulation.

Appendix 5

Here, we prove Proposition 5.

The calculation of U_3 is straightforward through

$$\begin{aligned} U_3 &= \int_s^t \mathbb{E}(q_u|\mathcal{F}_0^X)(1-e^{-p(t-u)})du \\ &= \int_s^t < \begin{pmatrix} q_1\psi_{11}(u,0)+q_2\psi_{12}(u,0) \\ q_1\psi_{21}(u,0)+q_2\psi_{22}(u,0) \end{pmatrix}, X_0 > (1-e^{-p(t-u)})du \end{aligned}$$

The corresponding solutions were then obtained.

Similarly, we compute U_4 using

$$\begin{aligned} U_4 &= \int_s^t \mathbb{E}(q_u|\mathcal{F}_0^X)(t-u)(1-e^{-p(t-u)})du \\ &= \int_s^t < \begin{pmatrix} q_1\psi_{11}(u,0)+q_2\psi_{12}(u,0) \\ q_1\psi_{21}(u,0)+q_2\psi_{22}(u,0) \end{pmatrix}, X_0 > (t-u)(1-e^{-p(t-u)})du, \end{aligned}$$

Thus, the solution is as follows:

U_5 can be calculated using

$$
\begin{aligned}
U_5 &= \int_s^t \mathbb{E}(q_u|\mathcal{F}_0^X)[e^{-p(t-\xi)} - e^{-2p(t-\xi)}]du \\
&= \int_s^t < \begin{pmatrix} q_1\psi_{11}(u,0) + q_2\psi_{12}(u,0) \\ q_1\psi_{21}(u,0) + q_2\psi_{22}(u,0) \end{pmatrix}, X_0 > [e^{-p(t-\xi)} - e^{-2p(t-\xi)}]du,
\end{aligned}
$$

yielding the final solution.

The calculation of U_6 can be performed using:

$$
\begin{aligned}
U_6 &= \int_s^t \mathbb{E}(q_u|\mathcal{F}_0^X)e^{-p(t-\xi)}du \\
&= \int_s^t < \begin{pmatrix} q_1\psi_{11}(u,0) + q_2\psi_{12}(u,0) \\ q_1\psi_{21}(u,0) + q_2\psi_{22}(u,0) \end{pmatrix}, X_0 > e^{-p(t-\xi)}du,
\end{aligned}
$$

leading to the final expression.

Abbreviations

CIR	Cox–Ingersoll–Ross
VIX	Volatility index
HJSVRS	Model with stochastic volatility, jump clustering, and regime switching
PJSVRS	Model with stochastic volatility, Poisson jumps, and regime switching
HJSV	Model with stochastic volatility and jump clustering

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Declarations

Competing interests
The authors declare that they have no competing interests.

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