Semisimple Categories

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1 Preamble: Representation Theory of Categories

A natural generalisation of traditional representation theory is to study the representation theory of categories. The idea of representation theory is to take some abstract algebraic object and study it by mapping it into a well known structure. With this mindset, a representation is simply a functor from the category we want to study to a familiar category.

For example, consider a group G as a one object category where every morphism is invertible. A representation of G is typically a group homomorphism from G into $\mathsf{End}(V)$, where V is some vector space. Notice that this is exactly a functor $G \to \mathbf{Vec}$.

Now consider a homomorphism between representations. We require such a map to commute with the actions on representations. Such a condition is exactly saying that the homomorphism is a natural transformation between the two representations.

Thus, the most general theory of representations amounts to studying functors and natural transformations. Unfortunately, in this general setting, there isn't too much we can say. To 'get some more meat on the bones', we might ask that our categories have extra structure.

For example, we might want our categories to be monoidal, linear, abelian or semisimple. If you wish, you can justify these extra structures by appealing to the fact that most categories that arise in physics automatically come with them. This report will focus on what we mean by a semisimple category.

Note that we have already studied semisimple representations and semisimple algebras (c.f. sections 2.1 and 2.5 respectively of [EtR]).

2 Introduction

There are a number of competing definitions of a semisimple category. The formal definitions are given in section 5, but informally they are:

1. Abelian semisimple: this is the usual definition for an abelian category to be semisimple (c.f. [Et]).
2. Müger semisimple: every map factors through a direct sum of simple objects.
3. Object semisimple: every object is a direct sum of simple objects.
4. Endomorphism semisimple: every endomorphism algebra is semisimple.
5. Morphism semisimple: $\bigoplus_{X,Y \in \mathscr{C}} \mathscr{C}(X \to Y)$ is a semisimple algebra.
6. Representation semisimple: every representation $F : \mathscr{C} \to \mathbf{fdVec}$ is representable.

This essay will explore the question of when definitions 1. to 4. are equivalent. Our results are summarised in the diagram below:

$$
\begin{tikzcd}
 & \text{Müger semisimple} \arrow[d, Rightarrow] \\
\text{Abelian semisimple} \arrow[r, Rightarrow] & \text{Object semisimple} \arrow[d, Rightarrow] \\
 & \text{Endomorphism semisimple} \arrow[ul, Rightarrow]
\end{tikzcd}
$$

Note however that this diagram is somewhat deceiving, since it does not capture the added assumptions necessary to prove some of these implications.

Sections 3 and 4 will introduce some background necessary for understanding and proving the results of this report. Section 5 provides the formal definitions of semisimplicity. Section 6 focuses on how abelian semisimplicity implies many of the other notions of semisimplicity. Müger semisimplicity and how it implies object semisimplicity is discussed in section 7.

This report assumes basic knowledge of abelian categories. The necessary background can be found in [Ba]. Let R be a ring and k a field throughout.

3 Some Useful Background

For this report, we need some background regarding semisimple algebras. Most importantly, we need the different characterisations of semisimple finite dimensional algebras and the Artin-Wedderburn theorem (Theorem 9).

Definition 1. An R-module is *simple* if it is non-zero and has no proper R-submodules. An R-module is *semisimple* if it is the direct sum of a family of simple submodules.

Definition 2. A ring R is *semisimple* if $1 \neq 0$ and R is semisimple as a left module over itself.

Note that there are also definitions of a simple ring and, confusingly, a simple ring is not always semisimple.

Definition 3. An algebra is *semisimple* if it is semisimple as a ring.

Definition 4. The *Jacobson radical* of a ring R is the left ideal N which is the intersection of all maximal left ideals of R.

Theorem 5 (Theorem 6.1 of [La])**.** *A finite dimensional algebra is semisimple if and only if its Jacobson radical is trivial.*

Definition 6. Let R be a ring. An R-module E is *Artinian* if any sequence

$$E_1 \supset E_2 \supset E_3 \supset ...$$

of submodules of E stabilises: there exists an integer N such that $E_n = E_{n+1}$ for $n \geq N$.

A ring R is *left* (resp. *right*) *Artinian* if it is Artinian when considered as a left (resp. right) R-module. A ring R is *Artinian* if it is both left and right Artinian.

Any finite dimensional k-algebra is Artinian.

Recall that an ideal I of a ring R is nilpotent if $I^n = 0$ for some integer $n \geq 1$. That is, every n-fold product of elements from I is zero.

Theorem 7 (Theorem 2.13 of [Kn])**.** *A left Artinian ring is semisimple if and only if all nilpotent two-sided ideals are trivial.*

Corollary 8. *A finite dimensional algebra is semisimple if and only if all nilpotent two-sided ideals are trivial. In particular, any matrix algebra is semisimple since it has no non-trivial two-sided ideals [Ma] and hence the product of matrix algebras is semisimple.*

Theorem 9 (Artin-Wedderburn, Theorem 2.2 of [Kn])**.** *If R is a semisimple ring, then*

$$R \cong M_{n_1}(D_1) \times ... \times M_{n_r}(D_r)$$

where $D_1, ...D_r$ are division rings and $n_1, ..., n_r$ are positive integers. This decomposition is unique up to permutation of the pairs $(D_1, n_1), ..., (D_r, n_r)$ and isomorphism of each D_i.

4 Idempotents

Idempotents turn out to play a crucial role in understanding semisimplicity, as we shall see.

Definition 10. In a category $\mathscr{C}$, an idempotent $p : X \to X$ *splits* if there are morphisms $r : X \to Y$ and $s : Y \to X$ such that $s \circ r = e$ and $r \circ s = \mathsf{id}_Y$. A category $\mathscr{C}$ is *idempotent complete* if every non-zero idempotent splits.

In linear algebra, all idempotents split. Therefore, one is often interested in idempotent complete categories. Note that the condition $r \circ s = \mathsf{id}_Y$ implies that r is epic and s monic.

Definition 11 [Kna]**.** A category $\mathscr{C}$ is *pseudo-abelian* if it is additive and every idempotent has an image.

Some authors only require a pseudo-abelian category to be pre-additive.

Every idempotent $p : X \to X$ in an additive category (or a pre-additive category with a terminal object) $\mathscr{C}$ has a cokernel $(X, \mathsf{id}_X - p)$. The proof of this is elementary. Since $\mathsf{id}_X - p$ is also an idempotent if and only if p is an idempotent, an additive category is pseudo-abelian if and only if every idempotent has a kernel.

Lemma 12. *A pseudo-abelian category is idempotent complete.*

Proof. Let $p : X \to X$ be an idempotent with image $(K, k : K \to X)$. We know k is the kernel of $\mathsf{id}_X - p$. This implies $0 = (\mathsf{id}_X - p)k$ and consequently $k = pk$. By the universal property of kernels, there exists a map $r : X \to K$ such that $p = kr$. Thus, $krk = pk = k$ and so $rk = \mathsf{id}_K$ since k is monic. □

Proposition 13. *If $p : X \to X$ is an idempotent in a pseudo-abelian category then $X \cong \mathsf{Im}\, p \oplus \mathsf{Im}(\mathsf{id}_X - p)$.*

Proof. Write $p = kr$ for maps $r : X \to \mathsf{Im}\, p$ and $k : \mathsf{Im}\, p \to X$ and similarly write $\mathsf{id}_X - p = k'r'$ for maps $r' : X \to \mathsf{Im}(\mathsf{id}_X - p)$ and $k' : \mathsf{Im}(\mathsf{id}_X - p) \to X$. Then $k'r'kr = (\mathsf{id}_X - p)p = 0$. Since k' is monic and r epic, it follows $r'k = 0$. Similarly, $rk' = 0$. Thus, the maps

$$\begin{pmatrix} r \\ r' \end{pmatrix} : X \to \mathsf{Im}\, p \oplus \mathsf{Im}(\mathsf{id}_X - p),$$

$$\begin{pmatrix} k & k' \end{pmatrix} : \mathsf{Im}\, p \oplus \mathsf{Im}(\mathsf{id}_X - p) \to X$$

are inverses. □

Note that such an isomorphism does not hold in idempotent complete categories, even when they are additive. For example consider the following category $\mathscr{C}$. It has 5 objects: M, K, C, C' and 0 and morphisms p, r, k, c, c' such that

$$\begin{tikzcd}
 & & & C \\
X \arrow[rr, "p"] \arrow[dr, "r"'] & & X \arrow[ur, "c"] \arrow[dr, "c'"] & \\
 & K \arrow[ur, "k"'] & & C'
\end{tikzcd}$$

commutes; p is a non-trivial idempotent; (r, k) is a splitting of p; and c and c' are non-zero morphisms such that $cp = c'p = 0$. (We also need to include all of the additional morphisms whose existence follows from $\mathscr{C}$ being an additive category.) We claim that $\mathscr{C}$ is an idempotent complete additive category, but $X \ncong \mathsf{Im}\, p \oplus \mathsf{Im}(\mathsf{id}_X - p)$. In fact, the cokernel of p, and therefore $\mathsf{Im}\, p$, does not exist.

What options are there for $\mathsf{Coker}\, p$? The pair (K, r) is not an option since this implies $rp = 0$ and consequently $p = 0$. For any object Y, the pair $(Y, X \xrightarrow{0} Y)$ is not an option as c and c' are non-zero. The pair (C, c) is not an option as there is no map $f : C \to C'$ such that $c' = fc$. Similarly, $(C, -c), (C', c'), (C', -c')$ and $(K, -r)$ are not options. This exhausts all the possibilities.

Corollary 14. *If $p : X \to X$ is an idempotent in a pseudo-abelian category then $X \cong \mathsf{Ker}\, p \oplus \mathsf{Im}\, p$.*

Proof. The kernel of an idempotent p is the image of $\mathsf{id}_X - p$ (and visa versa). Why? The cokernel of p is $(X, \mathsf{id}_X - p)$, so the cokernel of $\mathsf{id}_X - p$ is (X, p). Thus, $\mathsf{Im}(\mathsf{id}_X - p) = \mathsf{Ker}\,\mathsf{Coker}(\mathsf{id}_X - p) = \mathsf{Ker}\, p$. □

Definition 15. A non-zero object X in an additive category $\mathscr{C}$ is *indecomposable* if it does not admit a non-trivial decomposition into a direct sum of subobjects.

It is obvious that every object is the direct sum of indecomposable objects. (So every additive category is 'semi-indecomposable'.)

Corollary 16. *In a pseudo-abelian category, an object X is indecomposable if and only if the only idempotents $p : X \to X$ are the zero map and the identity.*

Proof. If X is decomposable as a non-trivial sum of objects, then the identity map on the first summand is a non-trivial idempotent of X.

If there is a non-trivial idempotent $p : X \to X$ then $X \cong \mathsf{Ker}\, p \oplus \mathsf{Im}\, p$. We claim that $\mathsf{Ker}\, p$ and $\mathsf{Im}\, p$ are both non-zero.

If $\mathsf{Ker}\, p$ is zero then p is monic. Since $p(\mathsf{id}_X - p) = 0$, this implies $p = \mathsf{id}_X$. The image $\mathsf{Im}\, p$ is non-zero since otherwise p would factor through 0, implying $p = 0$. □

5 Definitions

Before proceeding to prove the various equivalences, we provide the different formal definitions of a semisimple category. These are provided here for reference.

Recall that, in an abelian category, a subobject of X is an object Y together with a monic $i : Y \to X$.

Definition 17. A non-zero object X in an abelian category $\mathscr{C}$ is *abelian simple* if 0 and X are its only subobjects (up to isomorphism). An object X in $\mathscr{C}$ is *semisimple* if it is a direct sum of simple objects. The category $\mathscr{C}$ is *abelian semisimple* if all of its objects are *semisimple*.

In an abelian category, a simple object is indecomposable but a indecomposable object is not necessarily simple. Moreover, while every object is the sum of indecomposables, it is not necessarily the sum of simple objects.

Definition 18. In a k-linear category $\mathscr{C}$, an object X is *linear simple* if $\mathscr{C}(X \to X) \cong k$. Two objects X and Y are *disjoint* if $\mathscr{C}(X \to Y) \cong \mathscr{C}(Y \to X) \cong 0$.

Definition 19. A k-linear category $\mathscr{C}$ is *Müger semisimple* if

1. it has direct sums,
2. it is idempotent complete
3. there is a collection of mutually disjoint linear simple objects $\{X_i\}_{i \in I}$,

such that the composition map

$$\bigoplus_{i \in I} \mathscr{C}(X_i \to Z) \otimes_k \mathscr{C}(Y \to X_i) \longrightarrow \mathscr{C}(Y \to Z)$$

is an isomorphism for all objects Y and Z.

Definition 20. A k-linear category is *object semisimple* if

1. it has direct sums,
2. there is a collection of mutually disjoint linear simple objects $\{X_i\}_{i \in I}$,

such that every object is isomorphic to a direct sum of objects in $\{X_i\}_{i \in I}$.

Definition 21. A category $\mathscr{C}$ is *endomorphism semisimple* if $\mathsf{End}(X)$ is a (finite dimensional) semisimple ring for all objects $X \in \mathscr{C}$.

The restriction to finite dimensional algebras is artificial, but makes things a lot simpler, because finite dimensional semisimple algebras are significantly more well-behaved than their infinite dimensional analogues.

Definition 22. A category is *morphism semisimple* if it has direct sums and $\bigoplus_{X,Y \in \mathscr{C}} \mathscr{C}(X \to Y)$ is a semisimple ring.

Definition 23. A k-linear category is *representation semisimple* if every functor $F : \mathscr{C} \to \mathbf{fdVec}$ is naturally isomorphic to $\mathscr{C}(X \to -)$ for some $X \in \mathscr{C}$.

A functor $F : \mathscr{C} \to \mathbf{fdVec}$ is a representation of $\mathscr{C}$ [nR]. If $F \cong \mathscr{C}(X \to -)$ for some $X \in \mathscr{C}$ then F is said to be representable [Le, chapter 4]. Therefore, a more concise definition of representation semisimple is that every representation must be representable. Since functors $C^{\mathrm{op}} \to \mathbf{fdVec}$ are functors $C \to \mathbf{fdVec}^{\mathrm{op}}$ and $\mathbf{fdVec} \cong \mathbf{fdVec}^{\mathrm{op}}$, we can equivalently define a representation as a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{fdVec}$, which allows for a more easy application of Yoneda's lemma.

We will largely ignore these last two definition in this report.

6 Abelian Semisimple Categories

In this section, we will show how abelian semisimplicity implies many of the other notions of semisimplicity. (Note that in abelian categories, we can recover a number of key theorems of representation theory, including Schur's lemma, and the Jordan Holder and the Krull-Schmidt theorems.)

Proposition 24 (Schur's lemma)**.** *Suppose $\mathscr{C}$ is abelian semisimple and k-linear, with k algebraically closed. If all the hom-spaces are finite dimensional, then $\mathscr{C}$ is object semisimple.*

Proof. We need to show that the (non-isomorphic) abelian simple objects are mutually disjoint and object semisimple. First, we will show that any morphism between simple objects X, Y is an isomorphism or zero. If this was not the case, then there would be a morphism with a non-trivial kernel or cokernel. This would mean X or Y has a non-trivial subobject.

Thus, $\mathscr{C}(X \to Y) = 0$ when X and Y are not isomorphic. Moreover, all non-zero morphisms in $\mathsf{End}(X)$ are isomorphisms. So $\mathsf{End}(X)$ is a division algebra. It is also finite-dimensional over k by assumption. But every finite dimensional division algebra over an algebraically closed field k is actually k: if $\alpha \notin k$ was an element of such a division algebra, then $k(\alpha)$ would be a finite (hence algebraic) field extension of k and this cannot happen since k is algebraically closed. ☐

Proposition 25 [Mat]**.** *If $\mathscr{C}$ is object semisimple then $\mathscr{C}$ is endomorphism semisimple.*

Proof. Decompose X as a sum of simple objects $\bigoplus_i n_i X_i$. The hom-space of $n_i X_i$ is ring-isomorphic to a matrix algebra $M_{n_i}(k)$. Moreover, $\mathscr{C}(n_i X_i \to n_j X_j) \cong 0$ for $i \neq j$. Thus, $\mathsf{End}(X)$ is isomorphic to a product of matrix algebras and consequently $\mathsf{End}(X)$ is semisimple by Corollary 8. $\square$

Proposition 26 (Lemma 2 of [Ja]). *If $\mathscr{C}$ is k-linear, pseudo-abelian and endomorphism semisimple then $\mathscr{C}$ is abelian semisimple.*

Proof. By Wedderburn's theorem, $\mathsf{End}(X)$ is a finite product of matrix algebras over division rings. That is, $\mathsf{End}(X) \cong \bigoplus_i M_{n_i}(D_i)$ for positive n_i and division rings D_i. This implies X is indecomposable if and only if $\mathsf{End}(X)$ is a division ring. Why? First, we will show that if $\mathsf{End}(X)$ is not a division ring, then X is decomposable. There are two cases here:

1. $\mathsf{End}(X)$ is the sum of two or more matrix algebra,
2. $\mathsf{End}(X) \cong M_n(D)$ for some $n \geq 2$ and division ring D.

In the first case, take the identity matrix ι in some summand. Since ι is a non-trivial idempotent, Corollary 16 implies X is decomposable.

In the second case, let $\iota = e_{1,1}$ be the matrix with a one in the top left cell and apply the same argument as in case 1.

The converse is easy: if $X \cong \oplus_i X_i$ is decomposable then there exist non-invertible endomorphisms. For example $e_{1,1}$ is not invertible. Thus, $\mathsf{End}(X)$ is not a division ring.

Since every object is a finite sum of indecomposables, we can chose a collection $\{X_i\}$ of non-isomorphic indecomposable objects such that every object $X \cong \oplus_i n_i X_i$. Write D_i for the division ring $\mathsf{End}(X_i)$. Any morphism $f : X \to Y$ can then be thought of as a map $\oplus_i n_i X_i \to \oplus_i m_i X_i$, where $X \cong \oplus_i n_i X_i$ and $Y \cong \oplus_i m_i X_i$. Write $f_{i,j}$ for the component of f mapping $n_j X_j$ onto $m_i X_i$. Note that $f_{i,i} : n_i X_i \to m_i X_i$ is an $m_i \times n_i$ matrix with elements in $\mathsf{End}(X_i) \cong D_i$. Thus, $f_{i,i}$ can be thought of as a map $D^{n_i} \to D^{m_i}$.

We will prove that if X and Y are indecomposable and $\mathscr{C}(X \to Y) \neq \{0\}$, then $X \cong Y$. Thus, for any morphism $f : \oplus_i n_i X_i \to \oplus_i m_i X_i$, the component $f_{i,j}$ is the zero map for all $i \neq j$. It follows that the functor $F : \mathscr{C} \to \oplus_i \mathbf{fdVec}_{D_i}$ defined by

$$\begin{aligned} \textstyle\bigoplus_i n_i X_i &\longmapsto \textstyle\bigoplus_i D_i^{n_i}, \\ (f : \textstyle\bigoplus_i n_i X_i \to \bigoplus_i m_i X_i) &\longmapsto \textstyle\bigoplus_i \left(f_{i,i} : D_i^{n_i} \to D_i^{m_i}\right), \end{aligned}$$

is fully faithful and essentially surjective. This implies $\mathscr{C}$ is semisimple abelian.

It remains to show that $X \cong Y$ for X, Y indecomposable with $\mathscr{C}(X \to Y) \neq \{0\}$. If this homset is non-zero then one of the composition maps

$$\begin{aligned} \mathscr{C}(Y \to X) \otimes \mathscr{C}(X \to Y) &\longrightarrow \mathsf{End}(X), \\ (f, g) &\longmapsto f \circ g, \end{aligned}$$

and

$$\begin{aligned} \mathscr{C}(X \to Y) \otimes \mathscr{C}(Y \to X) &\longrightarrow \mathsf{End}(Y), \\ (g, f) &\longmapsto g \circ f, \end{aligned}$$

is non-zero otherwise

$$\begin{pmatrix} 0 & 0 \\ \mathscr{C}(X \to Y) & 0 \end{pmatrix} \subset \begin{pmatrix} \mathsf{End}(X) & \mathscr{C}(Y \to X) \\ \mathscr{C}(X \to Y) & \mathsf{End}(Y) \end{pmatrix} = \mathsf{End}(X \oplus Y)$$

is a non-trivial nilpotent two-sided ideal, contradicting Corollary 8. Suppose the first composition is non-zero and choose $f : Y \to X$ and $g : X \to Y$ with $fg \neq 0$. (The case when the second composition is non-zero is analogous.) As $\mathsf{End}(X)$ is a division ring, fg is invertible and $(fg)^{-1} \circ f$ is a left inverse of g.

Finally, $g(fg)^{-1}f$ is idempotent. Since Y is indecomposable, $g(fg)^{-1}f$ is either the zero map or the identity by proposition 13. But $g(fg)^{-1}f \neq 0$ otherwise $g = g(fg)^{-1}fg = 0g = 0$ and so $fg = f0 = 0$. Therefore, $(fg)^{-1} \circ f$ is a right inverse of g. $\square$

7 Müger Semisimple Categories

In 2008, Müger developed a new definition of semisimple categories [Mü]. He wanted a general definition of semisimple categories, which avoided the unneccesary requirement that the category must be abelian.

Example 27. Every fusion category is Müger semisimple [nS].

Proposition 28 (section 3 of [nS])**.** *If a category $\mathscr{C}$ is Müger semisimple and homspaces are finite dimensional, then it is object semisimple. That is, every object V is a direct sum of simple objects in $\{X_i\}_{i\in I}$:*

$$V = \bigoplus_{i\in I} n_i X_i$$

where $n_i = \dim \mathscr{C}(X_i \to V) = \dim \mathscr{C}(V \to X_i)$.

Proof. We prove that $\mathscr{C}(X_i \to V)$ and $\mathscr{C}(V \to X_i)$ are dual vector spaces. Then choosing a basis

$$\{f_{i,p} : X_i \to V\}_{p=1}^{n_i}$$

for $\mathscr{C}(X_i \to V)$, we have a corresponding dual basis[†]

$$\{f_i^p : V \to X_i\}_{p=1}^{n_i}$$

for $\mathscr{C}(V \to X_i)$. By construction, we have $f_i^p f_{j,q} = \delta_{i,j}\delta_{p,q}$ and $\sum_{i,p} f_{i,p} f_i^p = \mathsf{id}_V$. This gives an isomorphism between V and $\oplus_{i\in I} n_i X_i$: The map $V \to \oplus_{i\in I} n_i X_i$ is the row vector consisting of the f_i^p's and the map $\oplus_{i\in I} n_i X_i \to V$ is the column vector consisting of the $f_{i,p}$'s.

Now we just need to prove that $\mathscr{C}(X_i \to V)$ and $\mathscr{C}(V \to X_i)$ are dual. Recall (c.f. [nD]) that a dual of an object X in a strict monoidal category is an object X^* such that there are morphisms $\mathrm{ev} : X^* \otimes X \to 1$ and $\mathrm{coev} : 1 \to X \otimes X^*$ satisfying

$$(\mathrm{ev} \otimes \mathsf{id}_X) \circ (id_X \otimes \mathrm{coev}) = \mathsf{id}_X \tag{1}$$

and

$$(\mathsf{id}_{X^*} \otimes \mathrm{coev}) \circ (\mathrm{ev} \otimes \mathsf{id}_{X^*}) = \mathsf{id}_{X^*}. \tag{2}$$

Define $\mathrm{ev} : \mathscr{C}(V \to X_i) \otimes \mathscr{C}(X_i \to V) \longrightarrow k$ by sending $f \otimes g$ to the scalar associated with $f \circ g \in \mathsf{End}(X_i) \cong k$. Next consider the image of id_V under the isomorphism

$$\begin{aligned} \mathscr{C}(V \to V) &\longrightarrow \bigoplus_{i\in I} \mathscr{C}(X_i \to V) \otimes \mathscr{C}(V \to X_i), \\ \mathsf{id}_V &\longmapsto \bigoplus_{i\in I} f_i \otimes g_i. \end{aligned}$$

Define $\mathrm{coev} : k \longrightarrow \mathscr{C}(V \to X_i) \otimes \mathscr{C}(X_i \to V)$ by sending the scalar associated with $\mathsf{id}_{X_i} \in \mathsf{End}(X_i) \cong k$ to the product $f_i \otimes g_i$ from the image of id_V. It is straightforward to check that ev and coev satisfy equations 1 and 2 and therefore $\mathscr{C}(X_i \to V)$ and $\mathscr{C}(V \to X_i)$ are dual. □

An object semisimple category $\mathscr{C}$ is not necessarily Müger semisimple, even if it is idempotent complete. For example, let X be a linear simple object and consider the homset $\mathscr{C}(X \oplus X \to X \oplus X \oplus X)$. It is isomorphic to the vector space of 3×2 matrices over k. However, $\mathscr{C}(X \oplus X \to X)$ is isomorphic to the space of 1×2 matrices and $\mathscr{C}(X \to X \oplus X \oplus X)$ is isomorphic to the space of 3×1 matrices. So we do not necessarily have the isomorphism in condition 3 of the definition of Müger semisimple categories. We run into similar problems when attempting to prove that abelian semisimple implies Müger semisimple.

Bruce Bartlett in [Bar] gives a nice explanation of why condition 3 of Müger semisimplicity is better than requiring every object to be a direct sum of simple objects:

[†]Here we need that $\mathscr{C}(X_i \to V)$ is finite dimensional.

We ask ourselves: given a linear category with direct sums and subobjects, and a chosen maximal collection $\{X_i\}_{i \in I}$ of non-isomorphic simple objects, how can we check if its semisimple? In the one way, we have to check whether a certain canonically defined map is an isomorphism. In the other way, we have to check if each object V can be expressed as a direct sum of the X_i's. Actually finding such a decomposition would be a non-canonical operation.[Bar]

8 Discussion

Scott Morrison has shown that representation semisimple implies object semisimple, given some finiteness assumptions (unpublished).

There are additional characterisations of semisimple categories not discussed in this report, which are equivalent to some of our characterisations, under certain conditions. See section 1 of [Ha], section 8.3 and 13.1 of [Ka] and page 6 of [Mü].

This report has only provided partial results. Further work is required to properly understand the relationships between all of these different notions of semisimplicity.

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