Lectures on the Worldline Formalism

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School on *Spinning Particles in Quantum Field Theory: Worldline Formalism, Higher Spins, and Conformal Geometry*

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Preface

These are notes of lectures on the worldline formalism given at the School on *Spinning Particles in Quantum Field Theory: Worldline Formalism, Higher Spins, and Conformal Geometry*, held at Morelia, Mexico, from 19. - 23. November, 2012. Those lectures were addressed to graduate level students with a background in relativistic quantum mechanics and at least a rudimentary knowledge of field theory. They were given in coordination with Olindo Corradini's lectures on path integrals at the same School, where many of the concepts which are used in the relativistic worldline formalism are introduced in the somewhat simpler context of quantum mechanics. Thus my lecture notes are meant to be used together with his [6].

Chapter 1

Scalar QED

1.1 History of the worldline formalism

In 1948, Feynman developed the path integral approach to nonrelativistic quantum mechanics (based on earlier work by Wentzel and Dirac). Two years later, he started his famous series of papers that laid the foundations of relativistic quantum field theory (essentially quantum electrodynamics at the time) and introduced Feynman diagrams. However, at the same time he also developed a representation of the QED S-matrix in terms of relativistic particle path integrals. It appears that he considered this approach less promising, since he relegated the information on it to appendices of [1] and [2]. And indeed, no essential use was made of those path integral representations for many years after; and even today path integrals are used in field theory mainly as integrals over fields, not over particles. Excepting an early brilliant application by Affleck et al. [3] in 1984, the potential of this particle path integral or "worldline" formalism to improve on standard field theory methods, at least for certain types of computations, was recognized only in the early nineties through the work of Bern and Kosower [4] and Strassler [5]. In these lectures, I will concentrate on the case of the one-loop effective actions in QED and QCD, and the associated photon/gluon amplitudes. since here the formalism has been shown to be particularly efficient, and to offer some distinct advantages over the usual Feynman diagram approach. At the end I will shortly treat also the gravitational case, which so far has been much less explored. This is due to a number of mathematical difficulties which arise in the construction of path integrals in curved space, and which have been resolved satisfactorily only recently.

1.2 The free propagator

We start with the free scalar propagator, that is, the Green's function for the Klein-Gordon operator equation:

$$D_0^{xx'} \equiv \langle 0|\mathrm{T}\phi(x)\phi(x')|0\rangle = \langle x|\frac{1}{-\Box + m^2}|x'\rangle\,. \tag{1.1}$$

We work with euclidean conventions, defined by starting in Minkowski space with the metric $(-+++)$ and performing a Wick rotation (analytic continuation)

$$\begin{aligned} E = k^0 = -k_0 &\to ik_4\,, \\ t = x^0 = -x_0 &\to ix_4\,. \end{aligned} \tag{1.2}$$

Thus the wave operator $\Box$ turns into the four-dimensional Laplacian,

$$\Box = \sum_{i=1}^{4} \frac{\partial^2}{\partial {x_i}^2}\,. \tag{1.3}$$

We will also set $\hbar = c = 1$.

We exponentiate the denominator using a Schwinger proper-time parameter T. This gives

$$\begin{aligned} D_0^{xx'} &= \langle x|\int_0^\infty dT\,\exp\Bigl[-T(-\Box + m^2)\Bigr]|x'\rangle \\ &= \int_0^\infty dT\;\mathrm{e}^{-m^2T}\langle x|\exp\Bigl[-T(-\Box)\Bigr]|x'\rangle\,. \end{aligned} \tag{1.4}$$

Rather than working from scratch to transform the transition amplitude appearing under the integral into a path integral, let us compare with the formula derived by Olindo for the transition amplitude of the free particle in quantum mechanics:

$$\langle \vec{x}, t | \vec{x}', 0 \rangle \equiv \langle \vec{x} | \, \mathrm{e}^{-itH} | \vec{x}' \rangle = \int_{x(0)=\vec{x}'}^{x(t)=\vec{x}} \mathcal{D}x(\tau) \; \mathrm{e}^{i \int_0^t d\tau \frac{m}{2} \dot{x}^2} \tag{1.5}$$

where $H = -\frac{1}{2m}\nabla^2$. Using (1.5) with the formal replacements

$$\begin{aligned} \nabla^2 &\rightarrow \Box \,, \\ m &\rightarrow \frac{1}{2} \,, \\ \tau &\rightarrow -i\tau \,, \\ t &\rightarrow -iT \,. \end{aligned} \tag{1.6}$$

we get

$$D_0^{xx'} = \int_0^\infty dT \; \mathrm{e}^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) \, e^{-\int_0^T d\tau \frac{1}{4} \dot{x}^2} \,. \tag{1.7}$$

This is the *worldline path integral representation* of the relativistic propagator of a scalar particle in euclidean spacetime from x' to x. Note that now $\dot{x}^2 = \sum_{i=1}^4 \dot{x}_i^2$. The parameter T for us will just be an integration variable, but as discussed by Olindo also has a deeper mathematical meaning related to one-dimensional diffeomorphism invariance.

Having found this path integral, let us calculate it, as a consistency check and also to start developing our technical tools. First, let us perform a change of variables from $x(\tau)$ to $q(\tau)$, defined by

$$x^\mu(\tau) = x^\mu_{cl}(\tau) + q^\mu(\tau) = \Big[x^\mu + \frac{\tau}{T}(x^\mu - x'^\mu) \Big] + q^\mu(\tau) \,. \tag{1.8}$$

That is, $x_{cl}(\tau)$ is the free (straight-line) classical trajectory fulfilling the path integral boundary conditions, $x_{cl}(0) = x'$, $x_{cl}(T) = x$, and $q(\tau)$ is the fluctuating quantum variable around it, fulfilling the Dirichlet boundary conditions ("DBC" in the following)

$$q(0) = q(T) = 0\,. \tag{1.9}$$

This is still very familiar from quantum mechanics. From (1.8) we have

$$\dot{x}^\mu(\tau) = \dot{q}^\mu(\tau) + \frac{1}{T}(x^\mu - x'^\mu)\,. \tag{1.10}$$

Plugging this last equation into the kinetic term of the path integral, and using that $\int_0^T d\tau \dot{q}(\tau) = 0$ on account of the boundary conditions (1.9), we find

$$\int_0^T d\tau \frac{1}{4}\dot{x}^2 = \int_0^T d\tau \frac{1}{4}\dot{q}^2 + \frac{(x-x')^2}{4T}\,. \tag{1.11}$$

Implementing this change of variables also in the path integral (this is just a linear shift and thus does not induce a Jacobi determinant factor) we get

$$D_0^{xx'} = \int_0^\infty dT\; \mathrm{e}^{-m^2T}\,\mathrm{e}^{-\frac{(x-x')^2}{4T}} \int_{DBC} \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau \frac{1}{4}\dot{q}^2}\,. \tag{1.12}$$

The new path integral over the fluctuation variable $q(\tau)$ depends only on T, not on x, x'. And again we need not calculate it ourselves; using eq. (2.26) of [6] and our formal substitutions (1.6), and taking into account that our path integral has D components, we get

$$\int_{DBC} \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau \frac{1}{4}\dot{q}^2} = (4\pi T)^{-\frac{D}{2}}\,. \tag{1.13}$$

Thus we have

$$D_0^{xx'} = \int_0^\infty dT\,(4\pi T)^{-\frac{D}{2}}\,\mathrm{e}^{-m^2T}\,\mathrm{e}^{-\frac{(x-x')^2}{4T}}\,. \tag{1.14}$$

This is indeed a representation of the free propagator in x-space, but let us now Fourier transform it (still in D dimensions) to get the more familiar momentum space representation:

$$
\begin{aligned}
D_0^{pp'} &\equiv \int\int dx dx'\, e^{ip\cdot x} e^{ip'\cdot x'} D_0^{xx'} \\
&= \int_0^\infty dT\,(4\pi T)^{-\frac{D}{2}}\, \mathrm{e}^{-m^2 T} \int\int dx dx'\, e^{ip\cdot x} e^{ip'\cdot x'}\, \mathrm{e}^{-\frac{(x-x')^2}{4T}} \\
&\overset{x'\to x'+x}{=} \int_0^\infty dT\,(4\pi T)^{-\frac{D}{2}}\, \mathrm{e}^{-m^2 T} \int dx\; \mathrm{e}^{i(p'+p)\cdot x} \int dx'\, e^{ip'\cdot x'}\, \mathrm{e}^{-\frac{x'^2}{4T}} \\
&= (2\pi)^D \delta^D(p+p') \int_0^\infty dT\, e^{-T(p'^2+m^2)} \\
&= (2\pi)^D \delta^D(p+p')\, \frac{1}{p^2+m^2}\ . \qquad (1.15)
\end{aligned}
$$

1.3 Coupling to the electromagnetic field

We are now ready to couple the scalar particle to an electromagnetic field $A_\mu(x)$. The worldline action then becomes [6]

$$
S[x(\tau)] = \int_0^T d\tau \Big(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\Big)\,. \qquad (1.16)
$$

This is also what one would expect from classical Maxwell theory (the factor i comes from the analytic continuation). We then get the "full" or "complete" propagator $D^{xx'}[A]$ for a scalar particle, that interacts with the background field A continuously while propagating from x' to x:

$$
D^{xx'}[A] \;=\; \int_0^\infty dT\; \mathrm{e}^{-m^2 T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\right)}\ . \qquad (1.17)
$$

Similarly, Olindo introduced already the *effective action* for the particle in the background field:

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int_{x(T)=x(0)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\right)} . \tag{1.18}$$

Note that we now have a dT/T, and that the path integration is over closed loops; those trajectories can therefore belong only to virtual particles, not to real ones. The effective action contains the quantum effects caused by the presence of such particles in the vacuum for the background field. In particular, it causes electrodynamics to become a nonlinear theory at the one-loop level, where photons can interact with each other in an indirect fashion.

1.4 Gaussian integrals

As was already mentioned, the path integral formulas (1.17) and (1.18) were found by Feynman already in 1950 [1]. Techniques for their efficient calculation were, however, developed only much later. Presently there are three different methods available, namely

1. The analytic or "string-inspired" approach, based on the use of worldline Green's functions.
2. The semi-classical approximation, based on a stationary trajectory ("worldline instanton").
3. A direct numerical calculation of the path integral ("worldline Monte Carlo").

In these lectures we will treat mainly the "string-inspired" approach; only in chapter three will we shortly discuss also the "semi-classical" approach.

In the "string-inspired" approach all path integrals are brought into gaussian form; usually this requires some expansion and truncation. They are then calculated by a formal extension of the n-dimensional gaussian integration formulas to infinite dimensions. This is possible because gaussian integration involves only very crude information on operators, namely their determinants and inverses (Green's functions). We recall that, in n dimensions,

$$\int d^n x \ \mathrm{e}^{-\frac{1}{4}x\cdot M\cdot x} = \frac{(4\pi)^{n/2}}{(\det M)^{1/2}}\,, \tag{1.19}$$

$$\frac{\int d^n x \ \mathrm{e}^{-\frac{1}{4}x\cdot M\cdot x + x\cdot j}}{\int d^n x \ \mathrm{e}^{-\frac{1}{4}x\cdot M\cdot x}} = \mathrm{e}^{j\cdot M^{-1}\cdot j}\,, \tag{1.20}$$

where the $n \times n$ matrix M is assumed to be symmetric and positive definite. Also, by multiple differentiation of the second formula with respect to the components of the vector j one gets

$$\begin{aligned}
\frac{\int d^n x \, x_i x_j \ \mathrm{e}^{-\frac{1}{4}x\cdot M\cdot x}}{\int d^n x \ \mathrm{e}^{-\frac{1}{4}x\cdot M\cdot x}} &= 2M^{-1}_{ij}\,, \\
\frac{\int d^n x \, x_i x_j x_k x_l \ \mathrm{e}^{-\frac{1}{4}x\cdot M\cdot x}}{\int d^n x \ \mathrm{e}^{-\frac{1}{4}x\cdot M\cdot x}} &= 4\Big(M^{-1}_{ij}M^{-1}_{kl} + M^{-1}_{ik}M^{-1}_{jl} + M^{-1}_{il}M^{-1}_{jk}\Big)\,, \\
\vdots \qquad & \qquad \vdots
\end{aligned} \tag{1.21}$$

(an odd number of x_i's gives zero by antisymmetry of the integral). Note that on the right hand sides we always have one term for each way of grouping all the $x_i's$ into pairs; each such grouping is called a "Wick contraction". In the canonical formalism the same combinatorics arises from the canonical commutator relations.

As will be seen, in flat space calculations these formulas can be generalized to the worldline path integral case in a quite naive way, while in curved space there arise considerable subtleties. Those subtleties are, to some extent, present already in nonrelativistic quantum mechanics.

1.5 The N-photon amplitude

We will focus on the closed-loop case in the following, since it turns out to be simpler than the propagator one. Nevertheless, it should be emphasized that everything that we will do in the following for the effective action can also be done for the propagator.

We could use (1.18) for a direct calculation of the effective action in a derivative expansion (see [8]), but let us instead apply it to the calculation of the one-loop N-photon amplitudes in scalar QED. This means that we will now consider the special case where the scalar particle, while moving along the closed trajectory in spacetime, absorbs or emits a fixed but arbitrary number N of quanta of the background field, that is, photons of fixed momentum k and polarization ε. In field theory, to implement this we first specialize the background $A(x)$, which so far was an arbitrary Maxwell field, to a sum of N plane waves,

$$A^\mu(x) = \sum_{i=1}^{N} \varepsilon_i^\mu \ \mathrm{e}^{ik_i\cdot x} \,. \tag{1.22}$$

We expand the part of the exponential involving the interaction with A as a power series, and take the term of order A^N. This term looks like

$$\frac{(-ie)^N}{N!}\Bigl(\int_0^T d\tau \sum_{i=1}^{N} \varepsilon_i \cdot \dot{x}(\tau)\ \mathrm{e}^{ik_i\cdot x(\tau)}\Bigr)^N \,. \tag{1.23}$$

Of these N^N terms we are to take only the $N!$ “totally mixed” ones that involve all N different polarizations and momenta. Those are all equal by symmetry, so that the $1/N!$ just gets cancelled, and we remain with

$$(-ie)^N \ \int_0^T d\tau_1 \varepsilon_1 \cdot \dot{x}(\tau_1)\ \mathrm{e}^{ik_1\cdot x(\tau_1)} \cdots \int_0^T d\tau_N \varepsilon_N \cdot \dot{x}(\tau_N)\ \mathrm{e}^{ik_N\cdot x(\tau_N)} \,. \tag{1.24}$$

This can also be written as

$$(-ie)^N V^\gamma_{\rm scal}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm scal}[k_N,\varepsilon_N] \tag{1.25}$$

where we have introduced the *photon vertex operator*

$$V^\gamma_{\rm scal}[k,\varepsilon] \ \equiv \ \int_0^T d\tau\, \varepsilon\cdot \dot{x}(\tau)\ \mathrm{e}^{ik\cdot x(\tau)} \,. \tag{1.26}$$

This is the same vertex operator which is used in (open) string theory to describe the emission or absorption of a photon by a string. We can thus write the N-photon amplitude as

$$\begin{aligned}\Gamma_{\rm scal}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) &= (-ie)^N\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int_{x(0)=x(T)}\mathcal{D}x(\tau)\,e^{-\int_0^T d\tau\frac{1}{4}\dot{x}^2}\\ &\quad\times V^\gamma_{\rm scal}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm scal}[k_N,\varepsilon_N]\end{aligned} \tag{1.27}$$

(where we now abbreviate $x(\tau_i) =: x_i$). Note that each vertex operator represents the emission or absorption of a single photon, however the moment when this happens is arbitrary and must therefore be integrated over.

To perform the path integral, note that it is already of Gaussian form. Doing it for arbitrary N the way it stands would still be difficult, though, due to the factors of $\dot{x}_i$. Therefore we first use a little formal exponentiation trick, writing

$$\varepsilon_i\cdot\dot{x}_i = \mathrm{e}^{\varepsilon_i\cdot\dot{x}_i}\Big|_{\mathrm{lin}(\varepsilon_i)}\,. \tag{1.28}$$

Now the path integral is of the standard Gaussian form (1.20). Since

$$\int_0^T d\tau\dot{x}^2 = \int_0^T d\tau x\big(-\frac{d^2}{d\tau^2}\big)x \tag{1.29}$$

(the boundary terms vanish because of the periodic boundary condition) we have the correspondence $M\leftrightarrow -\frac{d^2}{d\tau^2}$ Thus we now need the determinant and the inverse of this operator. However, we first have to solve a little technical problem: positive definiteness does not hold for the full path integral $\mathcal{D}x(\tau)$, since there is a "zero mode"; the path integral over closed trajectories includes the constant loops, $x(\tau) = const.$, on which the kinetic term vanishes, corresponding to a zero eigenvalue of the matrix M [1].
To solve it, we define the loop center-of-mass (or average position) by

$$x_0 := \frac{1}{T}\int_0^T d\tau\, x^\mu(\tau)\,. \tag{1.30}$$

[1]No such problem arose for the propagator, since $q(0) = q(T) = 0$ and $q \equiv const.$ implies that $q\equiv 0$.

We then separate off the integration over x_0, thus reducing the path integral to an integral over the relative coordinate q:

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau), \qquad \int \mathcal{D}x(\tau) = \int d^D x_0 \int \mathcal{D}q(\tau)\,. \tag{1.31}$$

It follows from (1.30),(1.31) that the variable $q(\tau)$ obeys, in addition to periodicity, the constraint equation

$$\int_0^T d\tau\, q^\mu(\tau) \;=\; 0\,. \tag{1.32}$$

The zero mode integral can be done immediately, since it factors out as (since $\dot{x} = \dot{q}$)

$$\int d^D x_0 \;\mathrm{e}^{i\sum_{i=1}^N k_i\cdot x_0} = (2\pi)^D \delta\Bigl(\sum_{i=1}^N k_i\Bigr)\,. \tag{1.33}$$

This is just the expected global delta function for energy-momentum conservation.

In the reduced space of the $q(\tau)$'s the operator $M = -\frac{d^2}{d\tau^2}$ has only positive eigenvalues. In the exercises you will show that, in this space,

$$\mathrm{det}M = (4T)^D \tag{1.34}$$

and that the Green's function of M corresponding to the above treatment of the zero mode is

$$G_B^{\mathrm{c}}(\tau,\tau') \equiv 2\langle\tau|\Bigl(\frac{d^2}{d\tau^2}\Bigr)^{-2}|\tau'\rangle_{SI} = |\tau-\tau'| - \frac{(\tau-\tau')^2}{T} - \frac{T}{6}\,. \tag{1.35}$$

Note that this Green's functions is the inverse of $\frac{1}{2}\frac{d^2}{d\tau^2}$ rather than $-\frac{d^2}{d\tau^2}$, a convention introduced in [5]. Note also that it is a function of the difference $\tau - \tau'$ only; the reason is that the boundary condition (1.32) does not break the translation invariance in τ. The subscript "B" stands for "bosonic" (later on we will also introduce a "fermonic" Green's function) and the subscript

"SI" stands for "string-inspired", since in string theory the zero mode of the worldsheet path integral is usually fixed analogously. The superscript "c" refers to the inclusion of the constant (= coincidence limit) $-T/6$. It turns out that, in flat space calculations, this constant is irrelevant and can be omitted. Thus in flat space we will usually use instead

$$G_B(\tau,\tau') \equiv |\tau-\tau'| - \frac{(\tau-\tau')^2}{T}\,. \tag{1.36}$$

This replacement does not work in curved space calculations, though. Below we will also need the first and second derivatives of this Green's function, which are

$$\dot G_B(\tau,\tau') = \mathrm{sign}(\tau-\tau') - 2\frac{\tau-\tau'}{T}\,, \tag{1.37}$$

$$\ddot G_B(\tau,\tau') = 2\delta(\tau-\tau') - \frac{2}{T}\,. \tag{1.38}$$

Here and in the following a "dot" always means a derivative with respect to the first variable; since $G_B(\tau,\tau')$ is a function of $\tau-\tau'$, we can always rewrite $\frac{\partial}{\partial\tau'} = -\frac{\partial}{\partial\tau}$.

To use the gaussian integral formula (1.20), we define

$$j(\tau) \equiv \sum_{i=1}^N \Bigl(i\delta(\tau-\tau_i)k_i - \delta'(\tau-\tau_i)\varepsilon_i\Bigr)\,. \tag{1.39}$$

This enables us to rewrite (recall that $\int_0^T \delta'(\tau-\tau_i)q(\tau) = -q(\tau_i)$)

$$\mathrm{e}^{\sum_{i=1}^N (ik_i\cdot q_i + \varepsilon_i\cdot\dot q_i)} = \mathrm{e}^{\int_0^T d\tau j(\tau)\cdot q(\tau)}\,. \tag{1.40}$$

Now the formal application of (1.20) yields

$$\frac{\int \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau \frac14 \dot q^2}\ \mathrm{e}^{\sum_{i=1}^N\left(ik_i\cdot q_i+\varepsilon_i\cdot\dot q_i\right)}}{\int \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau\frac14\dot q^2}} = \exp\Bigl[-\frac12\int_0^T d\tau\int_0^T d\tau' G_B(\tau,\tau')j(\tau)\cdot j(\tau')\Bigr]$$
$$= \exp\left\{\sum_{i,j=1}^N\left[\frac12 G_{B_Bij}k_i\cdot k_j - i\dot G_{Bij}\varepsilon_i\cdot k_j + \frac12\ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\right]\right\}. \tag{1.41}$$

Here we have abbreviated $G_{Bij} \equiv G_B(\tau_i, \tau_j)$, and in the second step we have used the antisymmetry of $\dot{G}_{Bij}$. Note that a constant added to G_B would have no effect, since it would modify only the first term in the exponent, and by a term that vanishes by momentum conservation. This justifies our replacement of G_B^c by G_B.

Finally, we need also the absolute normalization of the free path integral, which turns out to be the same as in the DBC case:

$$\int \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau \frac{1}{4}\dot{q}^2} = (4\pi T)^{\frac{D}{2}}\,. \tag{1.42}$$

Putting things together, we get the famous "Bern-Kosower master formula":

$$\begin{aligned}\Gamma_{\rm scal}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) \;&=\; (-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i\\ &\times\exp\biggl\{\sum_{i,j=1}^N\Bigl[\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\Bigr]\biggr\}\,\Big|_{\rm lin(\varepsilon_1,\ldots,\varepsilon_N)}\;.\end{aligned} \tag{1.43}$$

This formula (or rather its analogue for the QCD case, see below) was first derived by Bern and Kosower from string theory [4], and rederived in the present approach by Strassler [5]. As it stands, it represents the one-loop N- photon amplitude in scalar QED, but Bern and Kosower also derived a set of rules which allows one to construct, starting from this master formula and by purely algebraic means, parameter integral representations for the N-photon amplitudes with a fermion loop, as well as for the N-gluon amplitudes involving a scalar, spinor or gluon loop [4, 7, 8]. However, there is a part of those rules that is valid only after imposing on-shell conditions, while the master formula itself is still valid completely off-shell.

1.6 The vacuum polarization

Let us look at the simplest case of the two-point function (the photon propagator or vacuum polarization). For $N = 2$ we get from the master formula (1.43), after expanding out the exponential,

$$\Gamma_{\rm scal}(k_1,\varepsilon_1;k_2,\varepsilon_2) = (-ie)^2(2\pi)^D\delta(p_1+p_2)\int_0^\infty \frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T} \times \int_0^T d\tau_1 \int_0^T d\tau_2\,(-i)^2 P_2\, e^{G_{B12}k_1\cdot k_2} \tag{1.44}$$

where

$$P_2 = \dot G_{B12}\varepsilon_1\cdot k_2 \dot G_{B21}\varepsilon_2\cdot k_1 - \ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\,. \tag{1.45}$$

We could now perform the parameter integrals straight away. However, it is advantageous to first remove the term involving $\ddot G_{B12}$ by an IBP. This transforms P_2 into Q_2,

$$Q_2 = \dot G_{B12}\dot G_{B21}\bigl(\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1 - \varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2\bigr)\,. \tag{1.46}$$

We use momentum conservation to set $k_1 = -k_2 =: k$, and define

$$\Gamma_{\rm scal}(k_1,\varepsilon_1;k_2,\varepsilon_2) \equiv (2\pi)^D\delta(p_1+p_2)\varepsilon_1\cdot\Pi_{\rm scal}\cdot\varepsilon_2\,. \tag{1.47}$$

Then we have

$$\Pi^{\mu\nu}_{\rm scal}(k) = e^2(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty \frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T} \times \int_0^T d\tau_1\int_0^T d\tau_2 \dot G_{B12}\dot G_{B21}\, e^{G_{B12}k_1\cdot k_2} \,. \tag{1.48}$$

Note that the IBP has had the effect to factor out the usual transversal projector $\delta^{\mu\nu}k^2 - k^\mu k^\nu$ already at the integrand level.

We rescale to the unit circle, $\tau_i = Tu_i, i = 1,2$, and use the translation invariance in τ to fix the zero to be at the location of the second vertex operator, $u_2 = 0, u_1 = u$. We have then

$$
\begin{aligned}
G_B(\tau_1,\tau_2) &= Tu(1-u)\,.\\
\dot{G}_B(\tau_1,\tau_2) &= 1-2u\,.
\end{aligned}
\tag{1.49}
$$

$$
\begin{aligned}
\Pi^{\mu\nu}_{\rm scal}(k) &= -\frac{e^2}{(4\pi)^{\frac{D}{2}}}(\delta^{\mu\nu}k^2-k^\mu k^\nu)\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}T^{2-\frac{D}{2}}\\
&\quad\times\int_0^1 du(1-2u)^2\,\mathrm{e}^{-Tu(1-u)k^2}\,.
\end{aligned}
\tag{1.50}
$$

Using the elementary integral

$$
\int_0^\infty \frac{dx}{x}\,x^\lambda\,\mathrm{e}^{-ax} = \Gamma(\lambda)a^{-\lambda}
\tag{1.51}
$$

(for $a>0$) we get finally

$$
\begin{aligned}
\Pi^{\mu\nu}_{\rm scal}(k) &= -\frac{e^2}{(4\pi)^{\frac{D}{2}}}(\delta^{\mu\nu}k^2-k^\mu k^\nu)\Gamma\left(2-\frac{D}{2}\right)\\
&\quad\times\int_0^1 du(1-2u)^2\Big[m^2+u(1-u)k^2\Big]^{\frac{D}{2}-2}\,.
\end{aligned}
\tag{1.52}
$$

This should now be renormalized, but we need not pursue this here.

Our result agrees, of course, with a computation of the two corresponding Feynman diagrams, fig. 1.1.

In this simple case our integrand before the IBP, (1.44), would have still allowed a direct comparison with the Feynman diagram calculation; namely, the diagram involving the quartic vertex matches with the contribution of the $\delta(\tau_1-\tau_2)$ part of the $\ddot{G}_{B12}$ contained in P_2.

Finally, it should be mentioned that, although any gaussian integral can be brought to the standard form of (1.20) by formal exponentiations such as (1.28), this is not always the most efficient way to proceed. Alternatively, one can use the following set of rules for Wick contractions involving elementary fields as well as exponentials of fields:

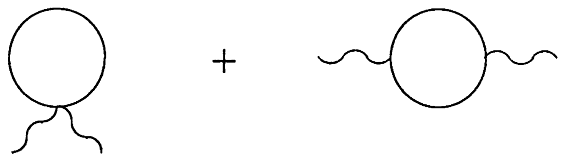

Figure 1.1: Vacuum polarization diagrams in Scalar QED

1. The basic Wick contraction of two fields is

$$\langle q^\mu(\tau_1) q^\nu(\tau_2)\rangle = -G_B(\tau_1,\tau_2)\delta^{\mu\nu}\,. \tag{1.53}$$

2. Wick contract fields among themselves according to (1.21), e.g.,

$$\Big\langle q^\kappa(\tau_1) q^\lambda(\tau_2) q^\mu(\tau_3) q^\nu(\tau_4)\Big\rangle = G_{B12}G_{B34}\delta^{\kappa\lambda}\delta^{\mu\nu} + G_{B13}G_{B24}\delta^{\kappa\mu}\delta^{\lambda\nu} + G_{B14}G_{B23}\delta^{\kappa\nu}\delta^{\lambda\mu}\,. \tag{1.54}$$

3. Contract fields with exponentials according to

$$\Big\langle q^\mu(\tau_1)\ \mathrm{e}^{ik\cdot q(\tau_2)}\Big\rangle = i\langle q^\mu(\tau_1) q^\nu(\tau_2)\rangle k_\nu\,\mathrm{e}^{ik\cdot q(\tau_2)} \tag{1.55}$$

(the field disappears, the exponential remains).

4. Once all elementary fields have been eliminated, the contraction of the remaining exponentials yields a universal factor

$$\begin{aligned}\Big\langle \mathrm{e}^{ik_1\cdot q_1}\cdots \mathrm{e}^{ik_N\cdot q_N}\Big\rangle &= \exp\Big[-\frac{1}{2}\sum_{i,j=1}^{N} k_{i\mu}\langle q^\mu(\tau_i) q^\nu(\tau_j)\rangle k_{j\nu}\Big]\\ &= \exp\Big[\frac{1}{2}\sum_{i,j=1}^{N} G_{Bij}k_i\cdot k_j\Big]\,.\end{aligned} \tag{1.56}$$

It is assumed that Wick-contractions commute with derivatives (since no matter how the path integral is constructed, it must obey linearity).

Chapter 2

Spinor QED

2.1 Feynman's vs Grassmann representation

In [2], Feynman presented the following generalization of the formula (1.18) for the effective action to the Spinor QED case:

$$\Gamma_{\rm spin}[A] \;\; = \;\; -\frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int_P \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\right)} {\rm Spin}[x(\tau), A]\,. \tag{2.1}$$

Here ${\rm Spin}[x(\tau), A]$ is the "spin factor"

$$ {\rm Spin}[x(\tau), A] \;\; = \;\; {\rm tr}_\gamma \mathcal{P} \exp\left[i\frac{e}{4}\,[\gamma^\mu, \gamma^\nu] \int_0^T d\tau\, F_{\mu\nu}(x(\tau))\right] \tag{2.2}$$

where ${\rm tr}_\gamma$ denotes the Dirac trace, and $\mathcal{P}$ is the path ordering operator. The comparison of (1.18) with (2.1) makes it clear that the x-path integral, which is the same as we had for the scalar case, represents the contribution to the effective action due to the orbital degree of freedom of the spin $\frac{1}{2}$ particle, and that all the spin effects are indeed due to the spin factor. The minus sign in front of the path integral implements the Fermi statistics.

A more modern way of writing the same spin factor is in terms of an additional Grassmann path integral [10, 11, 12, 13]. As shown already in Olindo's lectures, one has

$$\mathrm{Spin}[x(\tau), A] \;=\; \int_A \mathcal{D}\psi(\tau)\,\exp\left[-\int_0^T d\tau\left(\frac{1}{2}\psi\cdot\dot{\psi} - ie\psi^\mu F_{\mu\nu}(x(\tau))\psi^\nu\right)\right]. \tag{2.3}$$

Here the path integration is over the space of anticommuting functions antiperiodic in proper-time, $\psi^\mu(\tau_1)\psi^\nu(\tau_2) = -\psi^\nu(\tau_2)\psi^\mu(\tau_1)$, $\psi^\mu(T) = -\psi^\mu(0)$ (which is indicated by the subscript 'A' on the path integral). The exponential in (2.3) is now an ordinary one, not a path-ordered one. The ψ^μ's effectively replace the Dirac matrices γ^μ, but are functions of the proper-time, and thus will appear in all possible orderings after the expansion of the exponential. This fact is crucial for extending to the Spinor QED case the above-mentioned ability of the formalism to combine the contributions of Feynman diagrams with different orderings of the photon legs around the loop. Another advantage of introducing this second path integral is that, as it turns out, there is a "worldline" supersymmetry between the coordinate function $x(\tau)$ and the spin function $\psi(\tau)$ [12]. Namely, the total worldline Lagrangian

$$L = \frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A + \frac{1}{2}\psi\cdot\dot{\psi} - ie\psi^\mu F_{\mu\nu}\psi^\nu \tag{2.4}$$

is invariant under

$$\begin{aligned}\delta x^\mu &= -2\eta\psi^\mu\,,\\ \delta\psi^\mu &= \eta\dot{x}^\mu\,,\end{aligned} \tag{2.5}$$

with a constant Grassmann parameter η. Although this "worldline supersymmetry" is broken by the different periodicity conditions for x and ψ, it still has a number of useful computational consequences.

2.2 Grassmann gauss integrals

We need to generalize the formulas for the usual gaussian integrals to the case of Grassmann (anticommuting) numbers. For the treatment of QED we can restrict ourselves to real Grassmann variables.

First, for a single Grassmann variable ψ we define the Grassmann integratiion by setting

$$\int d\psi \psi = 1 \tag{2.6}$$

and imposing linearity. Since $\psi^2 = 0$ the most general function of ψ is of the form $f(\psi) = a + b\psi$, and

$$\int d\psi f(\psi) = b\,. \tag{2.7}$$

For two Grassmann variables ψ_1, ψ_2 we have the most general function $f(\psi_1, \psi_2) = a + b\psi_1 + c\psi_2 + d\psi_1\psi_2$, and

$$\int d\psi_1 \int d\psi_2 f(\psi_1\psi_2) = -d\,. \tag{2.8}$$

We can then also form a gaussian integral: let $\psi = (\psi_1, \psi_2)$ and M a real antisymmetric matrix. Then

$$\mathrm{e}^{-\frac{1}{2}\psi^T\cdot M\cdot\psi} = \mathrm{e}^{-M_{12}\psi_1\psi_2} = 1 - M_{12}\psi_1\psi_2$$

(note that a symmetric part added to M would cancel out, so that we can restrict ourselves to the antisymmetric case from the beginning) and

$$\int d\psi_1 \int d\psi_2 \; \mathrm{e}^{-\frac{1}{2}\psi^T\cdot M\cdot\psi} = M_{12}\,. \tag{2.9}$$

On the other hand,

$$\det M = -M_{12}M_{21} = M_{12}^2\,. \tag{2.10}$$

Thus we have

$$\int d\psi_2 \int d\psi_1 \; \mathrm{e}^{-\frac{1}{2}\psi^T\cdot M\cdot\psi} = \pm(\det M)^{\frac{1}{2}}\,. \tag{2.11}$$

It is easy to show show that this generalizes to any even dimension: let $\psi_1, \ldots, \psi_{2n}$ be Grassmann variables and M an antisymmetric $2n \times 2n$ matrix. Then

$$\int d\psi_1 \cdots \int d\psi_{2n} \; \mathrm{e}^{-\frac{1}{2}\psi^T \cdot M \cdot \psi} = \pm(\mathrm{det} M)^{\frac{1}{2}} \,. \tag{2.12}$$

Thus we have a determinant factor in the numerator, instead of the denominator as we had for the ordinary gauss integral (1.19). The formula (1.20) also generalizes to the Grassmann case, namely one finds

$$\frac{\int d\psi_1 \cdots \int d\psi_{2n} \; \mathrm{e}^{-\frac{1}{2}\psi^T \cdot M \cdot \psi + \psi \cdot j}}{\int d\psi_1 \cdots \int d\psi_{2n} \; \mathrm{e}^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}} \quad = \quad \mathrm{e}^{\frac{1}{2} j \cdot M^{-1} \cdot j} \,. \tag{2.13}$$

By differentiation with respect to the components of j one finds the Grassmann analogue of the Wick contraction rules (1.21):

$$\begin{aligned}
\frac{\int d\psi_1 \cdots \int d\psi_{2n} \psi_i \psi_j \; \mathrm{e}^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}}{\int d\psi_1 \cdots \int d\psi_{2n} \; \mathrm{e}^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}} \quad &= \quad M^{-1}_{ij} \,, \\
\frac{\int d\psi_1 \cdots \int d\psi_{2n} \psi_i \psi_j \psi_k \psi_l \; \mathrm{e}^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}}{\int d\psi_1 \cdots \int d\psi_{2n} \; \mathrm{e}^{-\frac{1}{2}\psi^T \cdot M \cdot \psi}} \quad &= \quad M^{-1}_{ij} M^{-1}_{kl} - M^{-1}_{ik} M^{-1}_{jl} + M^{-1}_{il} M^{-1}_{jk} \,, \\
\vdots \quad & \quad \vdots
\end{aligned} \tag{2.14}$$

Note that we have now alternating signs in the permutations in the second formula in (2.14); this is because now also j must be Grassmann-valued.

Returning to our Grassmann path integral (2.3), we see that the matrix M now corresponds to the first derivative operator $\frac{d}{d\tau}$, acting in the space of antiperiodic functions. Thus we need its inverse, which is very simple: one finds

$$G_F(\tau, \tau') \equiv 2\langle \tau | \Bigl(\frac{d^2}{d\tau^2}\Bigr)^{-1} | \tau' \rangle = \mathrm{sign}(\tau - \tau') \,. \tag{2.15}$$

Note that in the antiperiodic case there is no zero mode problem. Thus we find the Wick contraction rules

$$
\begin{aligned}
\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= \frac{1}{2} G_F(\tau_1,\tau_2)\delta^{\mu\nu}\,,\\
\Bigl\langle \psi^\kappa(\tau_1)\psi^\lambda(\tau_2)\psi^\mu(\tau_3)\psi^\nu(\tau_4)\Bigr\rangle &= \frac{1}{4}\Bigl[G_{F12}G_{F34}\delta^{\kappa\lambda}\delta^{\mu\nu} - G_{F13}G_{F24}\delta^{\kappa\mu}\delta^{\lambda\nu} + G_{F14}G_{F23}\delta^{\kappa\nu}\delta^{\lambda\mu}\Bigr]\,,\\
\vdots \quad &\quad \vdots
\end{aligned}
\tag{2.16}
$$

The only other information which we will need in a perturbation evaluation of (2.3) is the free path integral normalization. But this was (with somewhat different conventions) already calculated by Olindo, eq. (3.21). It is

$$
\int_A \mathcal{D}\psi(\tau)\ \mathrm{e}^{-\int_0^T d\tau \frac{1}{2}\psi\cdot\dot\psi} = 2^{\frac{D}{2}}\,.
\tag{2.17}
$$

Here D is any *even* spacetime dimension, and we recognize the factor $2^{\frac{D}{2}}$ as the number of real degrees of freedom of a Dirac spinor in such dimension.

2.3 The N-photon amplitudes

The procedure for obtaining the N - photon amplitude in spinor QED from the effective action (2.1) with (2.3) is completely analogous to the Scalar QED case treated in section 1.5, and we proceed straight away to the analogue of (1.27):

$$
\begin{aligned}
\Gamma_{\rm spin}(k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N) &= -\frac{1}{2}(-ie)^N\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int_{x(0)=x(T)}\mathcal{D}x(\tau)\,e^{-\int_0^T d\tau\frac{1}{4}\dot x^2}\\
&\quad\times\int_A \mathcal{D}\psi(\tau)\ \mathrm{e}^{-\int_0^T d\tau\frac{1}{2}\psi\cdot\dot\psi} V^\gamma_{\rm spin}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm spin}[k_N,\varepsilon_N]\,.
\end{aligned}
\tag{2.18}
$$

Now $V^\gamma_{\rm spin}$ is the vertex operator for the emission or absorption of a photon by a spinor,

$$
V^\gamma_{\rm spin}[k,\varepsilon] \equiv \int_0^T d\tau\,\Bigl[\varepsilon\cdot\dot x(\tau) + 2i\varepsilon\cdot\psi(\tau)k\cdot\psi(\tau)\Bigr]\,\mathrm{e}^{ik\cdot x(\tau)}\,,
\tag{2.19}
$$

the second term in brackets being the momentum-space version of the $\psi \cdot F \cdot \psi$ - term in (2.3).

Now one would like to obtain a closed formula for general N, that is a generalization of the Bern-Kosower master formula. This is possible, but requires a more elaborate use of the worldline supersymmetry; see section 4.2 of [8]. But as we will see, there is a more efficient way to proceed. We will first revisit the case of the photon propagator.

2.4 The vacuum polarization

For $N = 2$, (2.18) becomes

$$\Gamma_{\rm spin}(k_1,\varepsilon_1;k_2,\varepsilon_2) = -\frac{1}{2}(-ie)^2\int_0^\infty \frac{dT}{T}\,{\rm e}^{-m^2T}\int \mathcal{D}x\int\mathcal{D}\psi\int_0^T d\tau_1\int_0^T d\tau_2 \times\varepsilon_{1\mu}\Bigl(\dot{x}_1^\mu + 2i\psi_1^\mu\psi_1\cdot k_1\Bigr)\,{\rm e}^{ik_1\cdot x_1}\varepsilon_{2\nu}\Bigl(\dot{x}_2^\nu + 2i\psi_2^\nu\psi_2\cdot k_2\Bigr)\,{\rm e}^{ik_2\cdot x_2}{\rm e}^{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+\frac{1}{2}\psi\dot{\psi}\right)}\,. \tag{2.20}$$

Since the Wick contractions do not mix the x and ψ fields, the calculation of $\mathcal{D}x$ is identical with the scalar QED calculation. Only the calculation of $\mathcal{D}\psi$ is new, and amounts to a use of the four-point Wick contraction (2.16),

$$(2i)^2\Bigl\langle\psi_1^\mu\psi_1\cdot k_1\psi_2^\nu\psi_2\cdot k_2\Bigr\rangle = G_{F12}^2\Bigl[\delta^{\mu\nu}k_1\cdot k_2 - k_2^\mu k_1^\nu\Bigr]\,. \tag{2.21}$$

Adding this term to the integrand for the scalar QED case, (1.48) one finds

$$\Pi^{\mu\nu}_{\rm spin}(k) = -2e^2(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty\frac{dT}{T}(4\pi T)^{-D/2}e^{-m^2T} \times\int_0^T d\tau_1\int_0^T d\tau_2(\dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21})\,e^{G_{B12}k_1\cdot k_2}\,. \tag{2.22}$$

Thus we see that, up to the normalization, the parameter integral for the spinor loop is obtained from the one for the scalar loop simply by replacing, in eq.(1.50),

$$\dot{G}_{B12}\dot{G}_{B21} \rightarrow \dot{G}_{B12}\dot{G}_{B21} - G_{F12}G_{F21}\,. \tag{2.23}$$

Proceeding as in the scalar case, one obtains the final parameter integral representation

$$\Pi^{\mu\nu}_{\rm spin}(k) = -8\frac{e^2}{(4\pi)^{\frac{D}{2}}}\Big[g^{\mu\nu}k^2 - k^\mu k^\nu\Big]\Gamma\Big(2-\frac{D}{2}\Big)\int_0^1 du\, u(1-u)\Big[m^2 + u(1-u)k^2\Big]^{\frac{D}{2}-2} \tag{2.24}$$

and at this level one can easily verify the equivalence with the standard textbook calculation of the vacuum polarization diagram in spinor QED.

2.5 Integration-by-parts and the replacement rule

We have made a big point out of the substitution (2.23) because it is actually only the simplest instance of a general "replacement rule" due to Bern and Kosower [4]. Namely, performing the expansion of the exponential factor in (1.43) will yield an integrand $\sim P_N\, e^{(\cdot)}$, where we abbreviated

$$e^{(\cdot)} := \exp\biggl\{\frac{1}{2}\sum_{i,j=1}^{N} G_{Bij}p_i\cdot p_j\biggr\}\,, \tag{2.25}$$

and P_N is a polynomial in $\dot{G}_{Bij}, \ddot{G}_{Bij}$ and the kinematic invariants. It is possible to remove all second derivatives $\ddot{G}_{Bij}$ appearing in P_N by suitable integrations-by-parts, leading to a new integrand $\sim Q_N\, e^{(\cdot)}$ depending only on the $\dot{G}_{Bij}$s. Look in Q_N for "τ-cycles", that is, products of $\dot{G}_{Bij}$'s whose indices form a closed chain. A τ-cycle can thus be written as $\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1}$ (to put it into this form may require the use of the antisymmetry of $\dot{G}_B$, e.g. $\dot{G}_{B12}\dot{G}_{B12} = -\dot{G}_{B12}\dot{G}_{B21}$). Then the integrand for the spinor loop case can be obtained from the one for the scalar loop simply by simultaneously replacing every τ-cycle appearing in Q_N by

$$\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} \rightarrow \dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1}\,,$$

and supplying the global factor of -2 which we have already seen above.

This "replacement rule" is very convenient, since it means that we do not have to really compute the Grassmann Wick contractions. However, the objective of removing the $\ddot{G}_{Bij}$s does not fix the IBP procedure, nor the final integrand Q_N, and it is not at all obvious how to proceed in a systematic way for arbitrary N. Moreover, our two-point computations above suggest that the IBP procedure may also be useful for achieving transversality at the integrand level. Thus ideally one might want to have an algorithm for passing from P_N to (some) Q_N that, besided removing all $\ddot{G}_{Bij}$s, has also the following properties:

1. It should maintain the permutation symmetry between the photons.
2. It should lead to a Q_N where each polarization vector ε_i is absorbed into the corresponding field strength tensor f_i, defined by

$$f_i^{\mu\nu} \equiv k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu \,. \tag{2.26}$$

   This assures manifest transversality at the integrand level.
3. It should be systematic enough to be computerizable.

Only very recently an algorithm has been developed that has all these properties [9].

Chapter 3

Spinor QED in a constant field

In QED, an important role is played by processes in an external electromagnetic field that can be approximated as constant in space and time. This is not only due to the phenomenological importance of such fields, but also to the mathematical fact that such a field is one of a very few configurations for which the Dirac equation can be solved in closed form. In the worldline formalism, the corresponding mathematical fact is that, under the addition of the constant external field, the worldline Green's functions (1.35), (2.15) generalize to simple trigonometric expressions in the field [8]. The resulting formalism is extremely powerful and has already found a number of interesting applications [15, 14, 16, 17]. Here we must be satisfied with considering just the simplest possible constant field problem, namely the effective action in such a field itself.

3.1 The Euler-Heisenberg Lagrangian

Thus we now assume that the background field $A(x)$ has a constant field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It will be convenient to use Fock-Schwinger gauge. This gauge is defined by fixing a "center-point" x_c, and the gauge condition

$$(x - x_c)^\mu A_\mu(x) \equiv 0\,. \tag{3.1}$$

The great advantage of this gauge choice is that it allows one to express derivatives of the gauge potential A_μ at the point x_c in terms of derivatives of the field strength tensor at this point. Namely, it follows from (3.1) that the Taylor expansion of $A(x)$ at x_c becomes

$$A_\mu(x_c+q) = \frac{1}{2}q^\nu F_{\nu\mu} + \frac{1}{3}q^\lambda q^\nu \partial_\lambda F_{\nu\mu} + \ldots \tag{3.2}$$

(in particular, $A(x_c) = 0$). For a constant field we have only the first term on the rhs of (3.2). With the obvious choice of $x_c = x_0$ we have along the loop (remember the definition of x_0 (1.31))

$$A_\mu(x(\tau)) = \frac{1}{2}q^\nu(\tau)F_{\nu\mu}\,. \tag{3.3}$$

Thus from (2.1), (2.3) we have, for this constant field case,

$$\begin{aligned}\Gamma_{\rm spin}(F) \;\; = \;\; & -\frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\int d^4x_0 \int_P \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{q}^2+\frac{1}{2}ie\,q\cdot F\cdot\dot{q}\right)} \\ & \times \int_A \mathcal{D}\psi(\tau)\; \mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{2}\psi\cdot\dot{\psi}-ie\psi^\mu F_{\mu\nu}\psi^\nu\right)}\,.\end{aligned} \tag{3.4}$$

We note that the zero mode integral is empty - nothing in the integrand depends on it. For a constant field this is expected, since by translation invariance the effective action must contain an infinite volume factor. Factoring out this volume factor we obtain the effective Lagrangian $\mathcal{L}$, which, as we will see, is well-defined. And this time we will not need any expansions to get the worldline path integrals into gaussian form - they are already gaussian! Using our formulas (1.42) and (2.17) for the free path integrals (we can now set $D = 4$), and the formulas (1.19), (2.12) for the ordinary and Grassmann gaussian integrals, we can right away write the integrand in terms of determinant factors:

$$
\begin{aligned}
\mathcal{L}(F) &= -\frac{1}{2}\cdot 4\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-2}\frac{\int_P \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{q}^2+\frac{1}{2}ie\,q\cdot F\cdot\dot{q}\right)}}{\int_P \mathcal{D}q(\tau)\, e^{-\int_0^T d\tau\frac{1}{4}\dot{q}^2}} \\
&\quad\times\frac{\int_A \mathcal{D}\psi(\tau)\ \mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{2}\psi\cdot\dot{\psi}-ie\psi^\mu F_{\mu\nu}\psi^\nu\right)}}{\int_A \mathcal{D}\psi(\tau)\ \mathrm{e}^{-\int_0^T d\tau\frac{1}{2}\psi\cdot\dot{\psi}}} \\
&= -2\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-2}\frac{\mathrm{Det}'^{-\frac{1}{2}}_P\left(-\frac{1}{4}\frac{d^2}{d\tau^2}+\frac{1}{2}ieF\frac{d}{d\tau}\right)}{\mathrm{Det}'^{-\frac{1}{2}}_P\left(-\frac{1}{4}\frac{d^2}{d\tau^2}\right)}\frac{\mathrm{Det}^{+\frac{1}{2}}_A\left(\frac{d}{d\tau}-2ieF\frac{d}{d\tau}\right)}{\mathrm{Det}^{+\frac{1}{2}}_A\left(\frac{d}{d\tau}\right)} \\
&= -2\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-2}\mathrm{Det}'^{-\frac{1}{2}}_P\left(\mathbb{1}-2ieF\Big(\frac{d}{d\tau}\Big)^{-1}\right)\mathrm{Det}^{+\frac{1}{2}}_A\left(\mathbb{1}-2ieF\Big(\frac{d}{d\tau}\Big)^{-1}\right).
\end{aligned}
\tag{3.5}
$$

Thus we now have to calculate the determinant of the same operator,

$$
\mathcal{O}(F) \equiv \mathbb{1}-2ieF\Big(\frac{d}{d\tau}\Big)^{-1} \tag{3.6}
$$

acting once in the space of periodic functions (but with the zero mode eliminated, which we have indicated by a 'prime' on the 'Det' as is customary) and once in the space of antiperiodic functions [1]. As we will now see, both determinants can be calculated in the most direct manner, by an explicit computation of the eigenvalues of the operator and their infinite product, and moreover these infinite products are convergent. This is a rather rare case for a quantum field theory computation!

This could be done using the operator $\mathcal{O}(F)$ as it stands, but we can use a little trick to replace it by an operator that is diagonal in the Lorentz indices: since the determinant of $\mathcal{O}(F)$ must be a Lorentz scalar, and it is not possible to form such a scalar with an odd number of field strength tensors F, it is clear that the determinant can depend also on e only through e^2. Thus we can write (abbreviating now $\mathrm{Det}\mathcal{O} \equiv |\mathcal{O}|$)

[1] Note that, if there was no difference in the boundary conditions, the two determinants would cancel each other; this cancellation is related to the worldline supersymmetry (2.5).

$$
\begin{aligned}
|\mathcal{O}(F)|^2 &= \left|\mathbb{1} - 2ieF\Big(\frac{d}{d\tau}\Big)^{-1}\right|\left|\mathbb{1} + 2ieF\Big(\frac{d}{d\tau}\Big)^{-1}\right| \\
&= \left|\mathbb{1} + 4e^2F^2\Big(\frac{d}{d\tau}\Big)^{-2}\right| .
\end{aligned}
\tag{3.7}
$$

Next, we can use the fact from classical electrodynamics that, for a generic constant electromagnetic field, there is a Lorentz frame such that both the electric and the magnetic field point along the z-axis. The euclidean field strength tensor then takes the form

$$
\mathbf{F} = \begin{pmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & iE \\ 0 & 0 & -iE & 0 \end{pmatrix} \tag{3.8}
$$

so that

$$
\mathbf{F}^2 = \begin{pmatrix} -B^2 & 0 & 0 & 0 \\ 0 & -B^2 & 0 & 0 \\ 0 & 0 & E^2 & 0 \\ 0 & 0 & 0 & E^2 \end{pmatrix} \tag{3.9}
$$

(in Minkowski space the same relative sign between B^2 and E^2 would arise through the raising of one index necessary in the multiplication of $F_{\mu\nu}$ with itself). Using (3.9) in (3.7), and taking the square root again, we get

$$
|\mathcal{O}(F)| = \left|\mathbb{1} + 4e^2E^2\Big(\frac{d}{d\tau}\Big)^{-2}\right|\left|\mathbb{1} - 4e^2B^2\Big(\frac{d}{d\tau}\Big)^{-2}\right| . \tag{3.10}
$$

Thus we have managed to reduce the original matrix operator to usual (one-component) operators.

Next we determine the spectrum of the operator $-\frac{d^2}{d\tau^2}$ for the two boundary conditions. Thus we need to solve the eigenvalue equation

$$
-\frac{d^2}{d\tau^2} f(\tau) = \lambda f(\tau) . \tag{3.11}
$$

In the periodic case, a basis of eigenfunctions is given by

$$
\begin{aligned}
e_n(\tau) &= \cos(2\pi n\tau/T), \quad n = 1, 2, \ldots \\
\tilde{e}_n(\tau) &= \sin(2\pi n\tau/T), \quad n = 1, 2, \ldots
\end{aligned}
\tag{3.12}
$$

with

$$
\lambda_n = \frac{(2\pi n)^2}{T^2} \tag{3.13}
$$

and in the antiperiodic case by

$$
\begin{aligned}
p_n(\tau) &= \cos\bigl(2\pi(n+1/2)\tau/T\bigr), \quad n = 0, 1, 2, \ldots \\
\tilde{p}_n(\tau) &= \sin\bigl(2\pi(n+1/2)\tau/T\bigr), \quad n = 0, 1, 2, \ldots
\end{aligned}
\tag{3.14}
$$

with

$$
\lambda_n = \frac{\bigl(2\pi(n+1)\bigr)^2}{T^2}\,. \tag{3.15}
$$

Using these eigenvalues in (3.10), we see that the infinite products which we encounter are just Euler's famous infinite product representations of the elementary trigonometric functions,

$$
\begin{aligned}
\frac{\sin x}{x} &= \prod_{n=1}^{\infty}\Bigl(1 - \frac{x^2}{(n\pi)^2}\Bigr)\,, \\
\frac{\sinh x}{x} &= \prod_{n=1}^{\infty}\Bigl(1 + \frac{x^2}{(n\pi)^2}\Bigr)\,, \\
\cos x &= \prod_{n=0}^{\infty}\Bigl(1 - \frac{x^2}{\bigl((n+1/2)\pi\bigr)^2}\Bigr)\,, \\
\cosh x &= \prod_{n=0}^{\infty}\Bigl(1 + \frac{x^2}{\bigl((n+1/2)\pi\bigr)^2}\Bigr)\,.
\end{aligned}
\tag{3.16}
$$

Putting things together, we get

$$\begin{aligned}\mathcal{L}(F) &= -2\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-2}\frac{eET}{\sin(eET)}\frac{eBT}{\sinh(eBT)}\cos(eET)\cosh(eBT) \\ &= -2\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-2}\frac{eET}{\tan(eET)}\frac{eBT}{\tanh(\mathrm{eBT})}\,. \qquad (3.17)\end{aligned}$$

This is the famous Lagrangian found by Euler and Heisenberg in 1936 as one of the first nontrivial results in quantum electrodynamics. Its existence tells us that, although in electrodynamics there is no interaction between photons at the classical level, such interactions do arise after quantization indirectly through the interaction of photons with the virtual electrons and positrons in the vacuum. Moreover, it encodes this information in a form which is convenient for the actual calculation of such processes; see, e.g., [18].

3.2 Schwinger pair production and worldline instantons

The effective Lagrangian (3.17) contains a singularity in the T-integral at $T = 0$. This is an ultraviolet-divergence, and needs to be removed by renormalization (see, e.g., [19]). Moreover, except for the purely magnetic field case this integral shows further poles at

$$T_n = \frac{n\pi}{eE}, \quad n = 1, 2, \ldots \qquad (3.18)$$

where the $\sin(eET)$ contained in the denominator vanishes. These poles generate an imaginary part for the effective Lagrangian. Restricting ourselves now to the purely electric field case, $B = 0$, it is a simple application of complex analysis to show that

$$\mathrm{Im}\mathcal{L}_{\mathrm{spin}}(E) = \frac{(eE)^2}{8\pi^3}\sum_{n=1}^\infty \exp\left[-n\pi\frac{m^2}{eE}\right]. \qquad (3.19)$$

Here the nth term in the sum is generated by the nth pole in (3.17).

The physical interpretation of this imaginary part was already anticipated by F. Sauter in 1932: in the presence of an electric field the vacuum becomes instable, since a virtual electron-positron pair can, by a statistical fluctuation, gain enough energy from the field to turn real. However, the probability for this to happen becomes significant only at about $E \approx 10^{16}\mathrm{V/cm}$, which is presently still out of the reach of experiment.

But let us now return to the worldline path integral for the constant field case, eq. (3.4). Clearly in the presence of an electric field component something special must happen to the path integral when T takes one of the values (3.18). What is it? Looking at eqs. (3.10) - (3.13) we can see that the reason for the pole in the effective Lagrangian at T_n is a zero of the operator $\mathcal{O}(F)$ with periodic boundary conditions (the one coming from the x - path integral) due to the eigenfunctions $e_n(\tau)$ and $\tilde{e}_n(\tau)$. Now let us return to Feynman's original worldline path integral for Spinor QED (2.1) and consider the classical equations of motion following from the coordinate worldline action, eq. (1.16). It is

$$\ddot{x}^\mu = 2ieF^{\mu\nu}\dot{x}_\nu \,. \tag{3.20}$$

This is, of course, simply the Lorentz force equation, only written in unfamiliar conventions and using unphysical (from a classical point of view) boundary conditions. Now in our case of a purely electric constant field, after putting $T = T_n$ we have $e_n(\tau) = \cos(2eE\tau), \tilde{e}_n(\tau) = \sin(2eE\tau)$, and we can use these functions to construct the following circular solution of (3.20):

$$x(\tau) = \big(x_1, x_2, \mathcal{N}e_n(\tau), \mathcal{N}\tilde{e}_n(\tau)\big)\,. \tag{3.21}$$

Here x_1, x_2 and $\mathcal{N} > 0$ are constants. Such a worldline trajectory that obeys both the classical field equations and the periodic boundary conditions is called a *worldline instanton* [3]. On this trajectory the worldline action (1.16) gets minimized, namely it vanishes, leaving only the exponential factor e^{-m^2T} with $T = T_n$ which reproduces the nth Schwinger exponential in (3.19). This suggests that Schwinger's formula (3.19), including the prefactors, can be obtained by a semiclassical (stationary path) approximation of the worldline path integral, and indeed this can be made precise [3, 20, 21]. In the constant field case this approximation actually turns out to be even exact; for more general fields the method still works as a large-mass approximation.

Note that the electron spin does not play a role in the determination of the Schwinger exponent. In the constant field case, the spin factor decouples from the x path integral and yields, evaluated at T_n, a global factor of

$$4\cos(eET_n) = 4\cos(n\pi) = 4(-1)^n\,. \tag{3.22}$$

In an early but highly nontrivial application of the worldline formalism [3], this semiclassical approximation was, for the Scalar QED case and in the weak-field approximation, even extended to higher loop orders, yielding the following formula for the total (summed over all loop orders) Schwinger pair creation exponential:

$$\mathrm{Im}\mathcal{L}_{\rm scal}^{(\rm all-loop)}(E) \overset{\frac{eE}{m^2}\to 0}{\approx} \frac{(eE)^2}{16\pi^3}\exp\left[-\pi\frac{m^2}{eE} + \alpha\pi\right] \tag{3.23}$$

(only the first Schwinger exponential is relevant in the weak-field limit). This formula is presently still somewhat conjectural, though, since some of the arguments of [3] are not completely rigorous.

Chapter 4

Other gauge theories

After having so far focused totally on the case of QED, in this last lecture let us very shortly discuss the most important things to know about the generalization to the nonabelian case, as well as to gravitation.

4.1 The QCD N-gluon amplitudes

It is easy to guess what has to be done to generalize Feynman's Scalar QED formula (1.18) to the nonabelian case, i.e. to the case where the gauge field $A_\mu(x)$ is a Yang-Mills field with an arbitrary gauge group G, and the loop scalar transforms in some representation of that group. We can then write

$$A_\mu(x) = A_\mu^a(x) T^a \tag{4.1}$$

where the $T^a, a = 1, \ldots, \dim(G)$ are a basis for the Lie algebra of G in the representation of the scalar (e.g. they could be the eight Gell-Mann matrices for the case of $SU(3)$ and a scalar in the fundamental representation).

Thus the $A_\mu(x)$ are now matrices, and the $A_\mu(x(\tau))$ along the loop for different τ will in general not commute with each other. As is well known from quantum mechanics, in such a case the ordinary exponential function has to be replaced by a "path-ordered exponential" (the coupling constant g in our conventions corresponds to $-e$):

$$\begin{aligned}\mathcal{P}\,\mathrm{e}^{ig\int_0^T d\tau \dot{x}\cdot A(x(\tau))} &\equiv 1 + ig\int_0^T d\tau\, \dot{x}\cdot A(x(\tau)) \\ &\quad +(ig)^2\int_0^T d\tau_1\, \dot{x}_1\cdot A(x(\tau_1))\int_0^{\tau_1} d\tau_2\, \dot{x}_2\cdot A(x(\tau_2)) + \dots\end{aligned} \tag{4.2}$$

And since the effective action must be a scalar, we will clearly also need a global color trace tr_c. Thus our nonabelian generalization of (1.18) becomes [5]

$$\Gamma_{\rm scal}[A] = \mathrm{tr}_c\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\int_P \mathcal{D}x\,\mathcal{P}\exp\Big[-\int_0^T d\tau\Big(\frac{1}{4}\dot{x}^2 - ig\,\dot{x}\cdot A\Big)\Big]\,. \tag{4.3}$$

Moving on to the N - gluon amplitude, it is clear that the vertex operator for a gluon should carry, in addition to a definite momentum and a definite polarization, also a definite color assignment; thus we supplement the one for the photon (1.26) with one of the basis matrices T^a. Thus our *gluon vertex operator* will be

$$V^g_{\rm scal}[k,\varepsilon,a] \;\equiv\; T^a\int_0^T d\tau\,\varepsilon\cdot\dot{x}(\tau)\;\mathrm{e}^{ik\cdot x(\tau)}\,. \tag{4.4}$$

It is also easy to figure out what will happen to the Bern-Kosower master formula (1.43): due to the path ordering, the vertex operators will have to appear in the path integral in a fixed ordering. Thus their color trace will also factor out of the τ - integrals, leading to the nonabelian master formula

$$\begin{aligned}\Gamma_{\rm scal}(k_1,\varepsilon_1,a_1;\dots;k_N,\varepsilon_N,a_N) &= (ig)^N\mathrm{tr}_c(T^{a_1}T^{a_2}\cdots T^{a_N})\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T} \\ &\quad\times\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\cdots\int_0^{\tau_{N-2}}d\tau_{N-1} \\ &\quad\times\exp\Bigg\{\sum_{i,j=1}^N\Big[\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\Big]\Bigg\}\,\Big|_{\mathrm{lin}(\varepsilon_1,\dots,\varepsilon_N)}\end{aligned} \tag{4.5}$$

(we have eliminated one integration by choosing the zero on the loop to be at the position of the Nth vertex operator).

Contrary to the abelian formula (1.43), this multi-integral does not represent the complete amplitude yet, rather one has to still sum over all $(N-1)!$ non-cyclic permutations. A further difference to the QED case is that the one-loop N-gluon amplitudes have, in contrast to the photon ones, also reducible amplitudes, which are not yet included in the master formula. However, it is one of the remarkable features of the Bern-Kosower rules that it allows one to construct also the reducible contributions knowing only the integrand of the master formula.

As in the photonic case, one can use the IBP procedure and the replacement rule (2.26) to get the integrand for the spin half loop directly from the one for the scalar loop. Moreover, for the gluon amplitudes there is also diagram with a gluon loop, and its integrand, too, can be obtained from the scalar loop one by a rule similar to (2.26); see [8].

However, in the nonabelian case there will, in general, be boundary terms, since the τ - integrals do not run over the full loop any more. Had we restricted - unnecessarily - our integrals to run over ordered sectors in the abelian case, we would have found that the total derivative terms added in the IBP produced boundary terms, but those would have cancelled out between adjacent sectors (where "adjacent sectors" means differing only by an interchange of two neighbouring vertex operators). Here instead of cancelling those boundary terms will combine into color commutators. For example, assume that we are calculating the four-gluon amplitude, and we are doing an IBP in the variable τ_3. For the sector with the "standard" ordering $\tau_1 > \tau_2 > \tau_3 > \tau_4$ this may produce a boundary term at the upper limit $\tau_3 = \tau_2$. The same total derivative term used in the adjacent sector with ordering $\tau_1 > \tau_3 > \tau_2 > \tau_4$ will yield the same boundary term as a lower limit. The color factors of both terms will thus combine as $\mathrm{tr}_c(T^{a_1}[T^{a_2}, T^{a_3}]T^{a_4})$.

The role of those terms is easier to analyze for the effective action than for the gluon amplitudes. The nonabelian effective action can, in principle, be written as a series involving only Lorentz and gauge-invariant expressions such as $\mathrm{tr}_c(D_\mu F_{\alpha\beta} D^\mu F^{\alpha\beta})$, where F is the full nonabelian field strength tensor

$$F_{\mu\nu} \equiv F^a_{\mu\nu} T^a = F^0_{\mu\nu} - ig[A^b_\mu T^b, A^c_\nu T^c] \tag{4.6}$$

where by $F^0_{\mu\nu}$ we denote its "abelian part",

$$F^0_{\mu\nu} \equiv (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)T^a \,. \tag{4.7}$$

Each such invariant has a "core term" that would exist already in the abelian case; in the example, this is $\partial_\mu F^0_{\alpha\beta}\partial^\mu F^{0\alpha\beta}$. For the core term, the IBP goes from bulk-to-bulk, and has a covariantizing effect in making it possible to combine vector potentials A into field strength tensors F^0. As one can easily check for examples (see section 4.10 of [8]) the addition of the boundary terms will complete this "covariantization" process by providing the commutator terms that are necessary to complete all F^0s to full Fs, and all partial derivatives to covariant ones. While this covariantization will ultimately happen after the computation of all integrals also using any other valid computation method, the interesting thing about the string-inspired formalism is that one can achieve manifest gauge invariance at the integrand level, before doing any integrals.

This property of manifest gauge invariance is somewhat less transparent in momentum space, but still very useful for analyzing the tensor structure of the off-shell N-gluon amplitudes; see [22] for the three-gluon case.

4.2 Graviton amplitudes

Finally, let us discuss the inclusion of gravity at least for the simplest possible case, which is a scalar particle coupled to background gravity. Naively, it seems clear what to do: in the presence of a background metric field $g_{\mu\nu}(x)$ we should replace the free kinetic part of the worldline Lagrangian by the geodesic one:

$$L_{\rm free} = \frac{\dot{x}^2}{4} \longrightarrow L_{\rm geo} \equiv \frac{1}{4}\,\dot{x}^\mu g_{\mu\nu}(x(\tau))\dot{x}^\nu \,. \tag{4.8}$$

As you probably know from General Relativity (or remember from Andrew's lectures at this School) this action yields the classical equations of motion for a spinless particle in a background gravitational field. So we would be tempted to write down the following formula for the one-loop effective action due to a scalar particle in quantum gravity:

$$\Gamma_{\rm scal}[g] \stackrel{?}{=} \int_0^\infty \frac{dT}{T}\, {\rm e}^{-m^2T} \int_P \mathcal{D}x \; e^{-\int_0^T d\tau L_{\rm geo}} \,. \tag{4.9}$$

As is usual in Quantum Gravity, we could then introduce gravitons as small plane wave perturbations of the metric around flat space,

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \kappa h_{\mu\nu}(x) \tag{4.10}$$

where κ is the gravitational coupling constant and

$$h_{\mu\nu}(x) = \varepsilon_{\mu\nu}\ \mathrm{e}^{ik\cdot x} \tag{4.11}$$

with a symmetric polarization tensor $\varepsilon_{\mu\nu}$. The usual perturbative evaluation of the path integral will then yield graviton amplitudes in terms of Wick contractions with the usual bosonic Green's function G_B, and each graviton presented by a vertex operator

$$V^h_{\mathrm{scal}}(k,\varepsilon) \stackrel{?}{=} \varepsilon_{\mu\nu}\int_0^T d\tau \dot{x}^\mu \dot{x}^\nu\ \mathrm{e}^{ik\cdot x(\tau)}\,. \tag{4.12}$$

However, one can immediately see that, contrary to the gauge theory case, the Wick contractions will now lead to mathematically ill-defined expressions, due to the $\delta(\tau-\tau')$ contained in $\ddot{G}_B(\tau,\tau')$. For example, a Wick contraction of two vertex operator will produce a term with a $\delta^2(\tau_1-\tau_2)$, and even the one of just a single vertex operator will already contain an ill-defined $\delta(0)$. These ill-defined terms signal UV divergences in our one-dimensional worldline field theory. However, they are of a spurious nature. To get rid of them we have to take the nontrivial background metric into account not only in the Lagrangian, but also in the path integral measure. In general relativity the general covariance requires that each spacetime integral should contain a factor of $\sqrt{\det g}$. Therefore the measure which should be used in (4.9) is of the form [23]

$$\mathcal{D}x = Dx \prod_{0\le\tau<T}\sqrt{\det g_{\mu\nu}(x(\tau))} \tag{4.13}$$

where $Dx = \prod_\tau d^D x(\tau)$ is the standard translationally invariant measure. However, in our string-inspired approach we clearly cannot use these metric factors the way they stand; they ought to be somehow exponentiated. A convenient way of doing this [24, 25] is by introducing commuting a^μ and anticommuting b^μ, c^μ worldline ghost fields with periodic boundary conditions

$$\mathcal{D}x = Dx \prod_{0\le\tau<1} \sqrt{\det g_{\mu\nu}(x(\tau))} = Dx \int_{PBC} DaDbDc\, \mathrm{e}^{-S_{gh}[x,a,b,c]} \tag{4.14}$$

where the ghost action is given by

$$S_{gh}[x,a,b,c] = \int_0^T d\tau\, \frac{1}{4} g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu)\,. \tag{4.15}$$

Thus the final version of the path integral representation (4.9) is

$$\Gamma_{\text{scal}}[g] = \int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2T} \int_P DxDaDbDc\, \mathrm{e}^{-\frac{1}{4}\int_0^T d\tau\, g_{\mu\nu}(x(\tau))\left(\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)+a^\mu(\tau)a^\nu(\tau)+b^\mu(\tau)c^\nu(\tau)\right)}\,. \tag{4.16}$$

After the split (4.10) we will now get the graviton vertex operator in its final form,

$$V^h_{\text{scal}}[k,\varepsilon] \;=\; \varepsilon_{\mu\nu}\int_0^T d\tau \Big[\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau)\Big]\, \mathrm{e}^{ik\cdot x(\tau)}\,. \tag{4.17}$$

The ghost fields have trivial kinetic terms, so that their Wick contractions involve only δ - functions:

$$\begin{aligned} \langle a^\mu(\tau_1)a^\nu(\tau_2)\rangle &= 2\delta(\tau_1-\tau_2)\delta^{\mu\nu}\,, \\ \langle b^\mu(\tau_1)c^\nu(\tau_2)\rangle &= -4\delta(\tau_1-\tau_2)\delta^{\mu\nu}\,. \end{aligned} \tag{4.18}$$

The extra terms arising from the ghost action will remove all the UV divergences. Let us quickly check this for the self-contraction of the graviton vertex operator (4.17):

$$\begin{aligned}\Big\langle \dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau)\Big\rangle &= \Big(\ddot{G}_B(\tau,\tau) + 2\delta(0) - 4\delta(0)\Big)\delta^{\mu\nu} \\ &= -\frac{2}{T}\delta^{\mu\nu}\,. \end{aligned} \tag{4.19}$$

However, not all troubles are over yet. As it often happens in field theory, the cancellation of UV divergences leaves behind a finite ambiguity: some of the remaining integrals require a regularization to be assigned a definite value. This regularization dependence must then be removed by the addition of appropriate finite counterterms to the worldline Lagrangian. If our one-dimensional worldline theory was an autonomous one, this would introduce some new free parameters into the game. However our goal here is just to reproduce the same results as would otherwise be obtained in the standard Feynman diagram approach to quantum gravity. By comparing with some known quantities, such as conformal anomalies, one can establish [26] that consistency with the standard formalism can be achieved by adding, for any given regularization scheme, just three such counterterms,

$$L_{\text{counter}} = c_1 R + c_2 g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} + c_3 g^{\mu\nu}g^{\alpha\beta}g_{\lambda\rho}\Gamma^\lambda_{\mu\alpha}\Gamma^\rho_{\nu\beta} \tag{4.20}$$

with the appropriate (regularization dependent) coefficients c_i.

Once these coefficients have been computed for a given regularization (which for the most important regularization choices has been done) there are no more singularities or ambiguities, and one can can now go ahead and compute the effective action, or the corresponding N-graviton amplitudes, as in the gauge theory case [1]. See [26, 28, 29] for the generalization to the spinor loop case and for some applications. Unfortunately, it is presently not known how to treat a graviton in the loop in the worldline formalism.

[1] To be precise, in curved space there are a few more issues related to operator ordering and zero mode fixing, but those have been completely resolved, too [26, 27].

Chapter 5

Conclusions

As I said at the beginning, in this short introduction to the worldline formalism I have concentrated on the case of gauge boson amplitudes, or the corresponding effective actions, since it is for those calculations that the formalism has been developed to the point where it offers distinct advantages over the standard Feynman diagram approach. This does not mean that the formalism cannot be applied to, e.g., amplitudes involving external fermions, or other couplings such as Yukawa or axial interactions; however there is still a dearth of convincing state-of-the-art applications. Moreover, we have restricted ourselves to the one-loop case, despite of the fact that at least for QED Feynman already had given a path integral construction of the full S-matrix to all loop orders. Here the reason was lack of time rather than lack of nice examples; the worldline formalism has been applied to a recalculation of the two-loop QED β function as well as of the two-loop correction to the Euler-Heisenberg Lagrangian (3.17) (these calculations are summarized in [8]). This is unfortunate, since the property of the formalism to enable one to combine various different Feynman diagrams into a single integral representation, which I mentioned in connection with the N-photon amplitudes, becomes much more interesting at higher loops. See [30] and [8] for a detailed discussion of these multiloop QED issues and calculations.

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Chapter 6

Exercises

1. One can often compute, or define, the determinant of an infinite-dimensional operator $\mathcal{O}$ using *ζ-function regularization*, defined as follows: assume that $\mathcal{P}$ has a discrete spectrum of real eigenvalues $\lambda_1 < \lambda_2 < \dots$. Define the ζ-function $\zeta_{\mathcal{O}}$ of $\mathcal{O}$ by

$$\zeta_{\mathcal{O}}(z) \equiv \sum_{n=1}^{\infty} \lambda_n^{-z}$$

   where $z \in \mathbf{C}$. Show that *formally*

$$\mathrm{Det}(\mathcal{O}) \equiv \prod_{n=1}^{\infty} \lambda_n = \exp\Bigl(-\frac{d}{dz}\zeta_{\mathcal{O}}(0)\Bigr).$$

   Use this method to verify (1.34) (you will need some properties of the Riemann ζ-function).

2. Derive the Green's function (1.35), using a direct eigenfunction expansion.

3. Verify (1.41) in detail.

4. Compute P_3 and Q_3.

5. Verify that the first of the Scalar QED diagrams of fig. 1.1 matches with the contribution of the $\delta(\tau_1 - \tau_2)$ part of the $\ddot{G}_{B12}$ contained in P_2.

6. Verify that (2.4) is invariant under (2.5) up to a total derivative term (you will need the Bianchi identity for the field strength tensor).
7. Show (2.12).
8. Verify (2.13).
9. Show (2.15).
10. Show (2.17) using ζ-function regularization (you will need some properties of the Hurwitz ζ-function).
11. Verify (2.22).
12. Using (3.1), obtain the first two terms on the rhs of (3.2).
13. Derive (3.19) from (3.17) (circumvent the poles displacing the integration contour above).
14. Verify (3.20).
15. Verify that (3.21) fulfills (3.20).
16. Verify (3.22).
17. Show (4.14), (4.15) using our formal rules for bosonic and fermionic (grassmann) gaussian integrations.
18. Verify, that in the Wick contraction of two graviton vertex operators all terms involving $\delta(0)$ or $\delta^2(\tau_1 - \tau_2)$ cancel out after the inclusion of the terms from the ghosts.

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