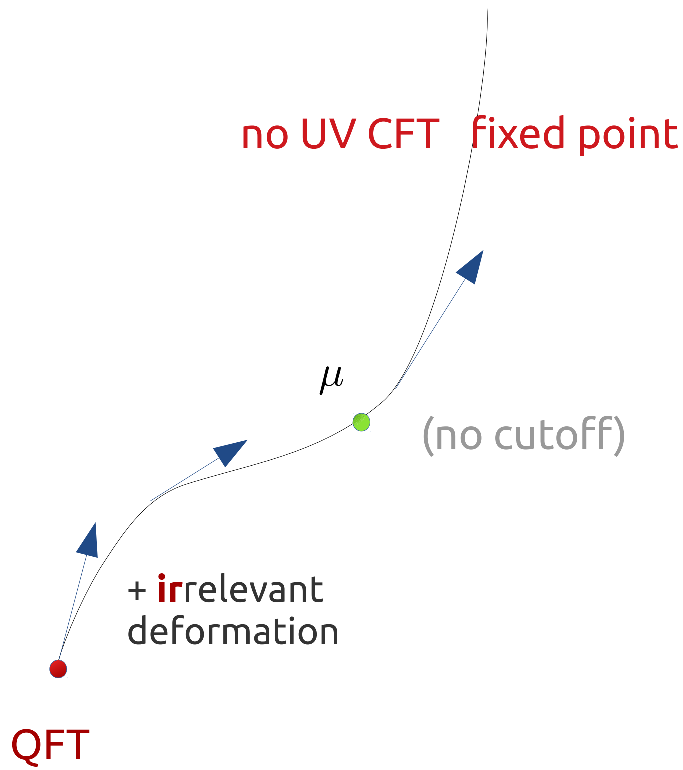
$T\bar{T}$ deformations and holography: review and open questions

Monica Guica

IphT, CEA Saclay

What is the $T\bar{T}$ deformation?

- irrelevant deformation of 2d QFTs → UV complete QFTs that are non-local



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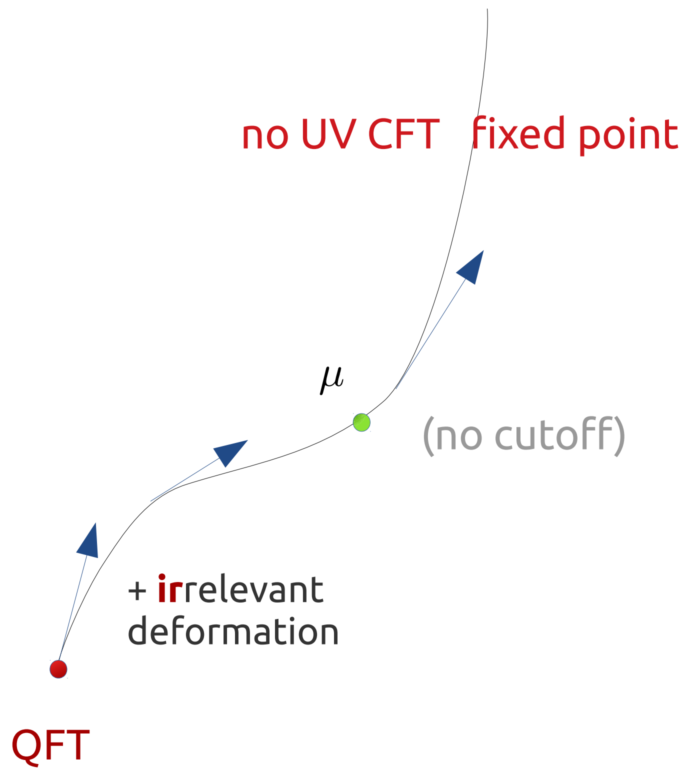
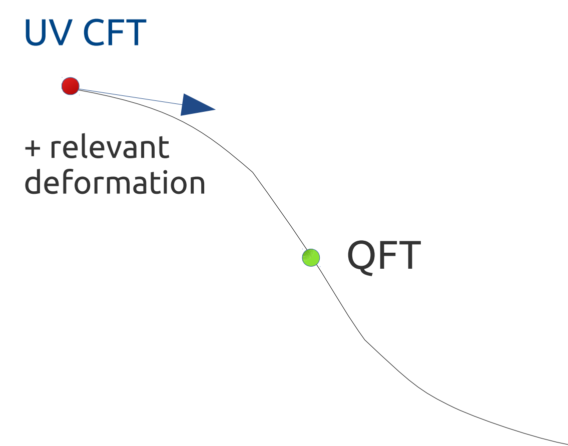


- finely tuned irrelevant flow

  integrability preserved

- well-defined S-matrix → UV completeness

- not an RG flow



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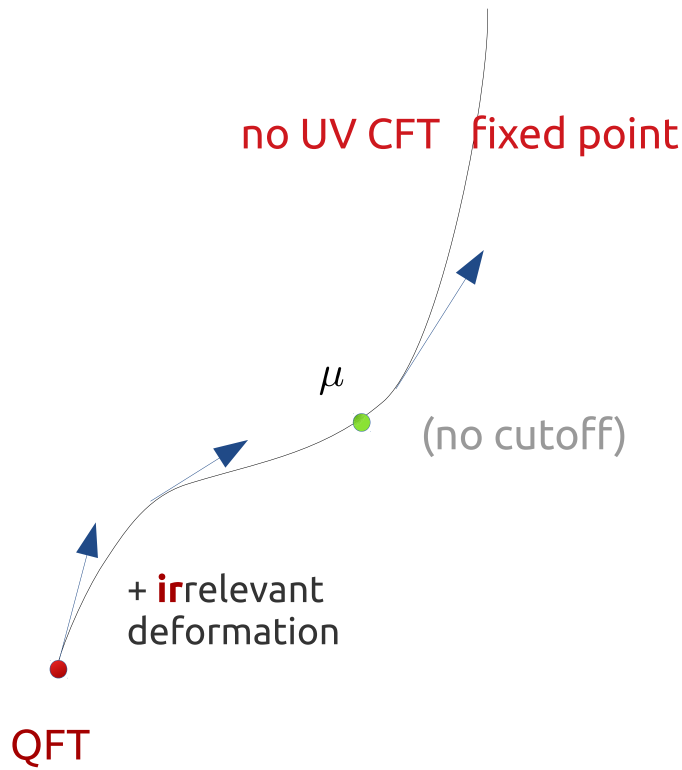

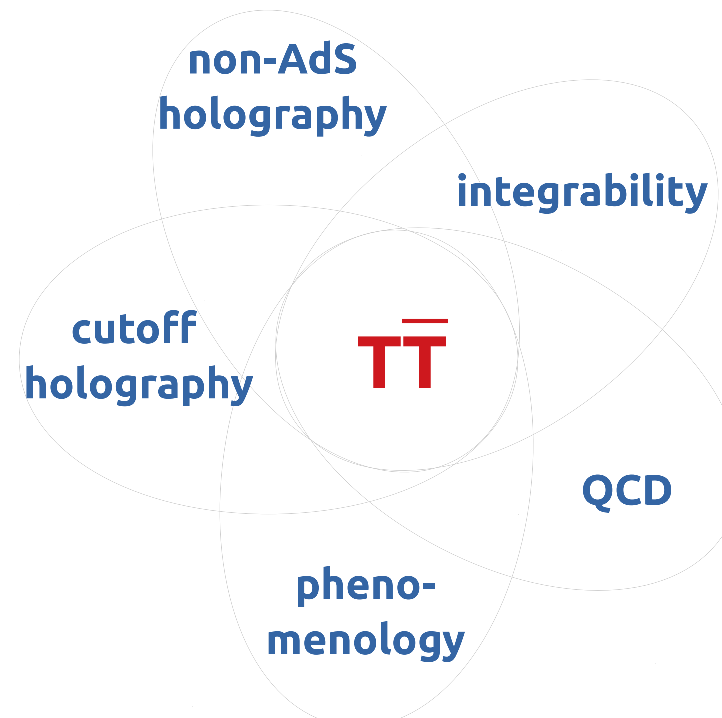

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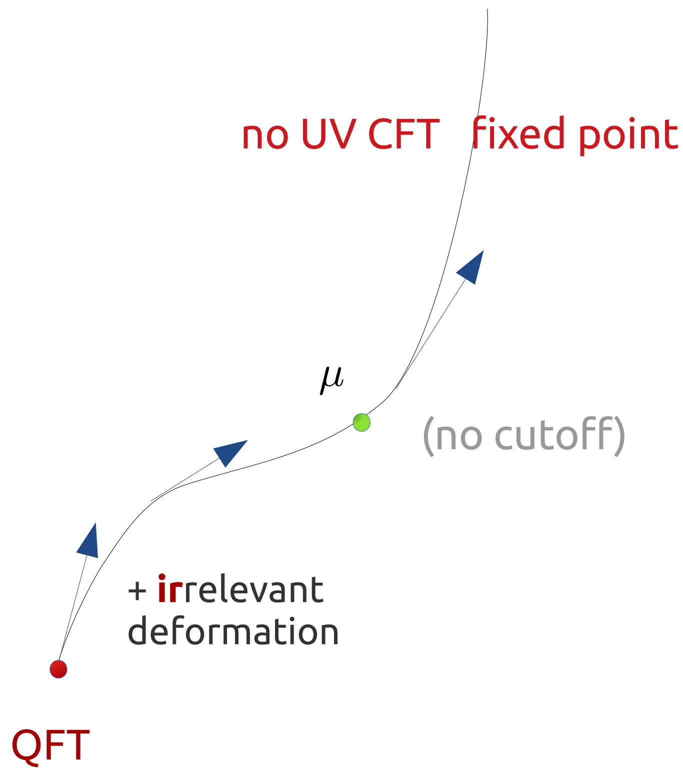

1) theoretical interest → classify integrable 2d QFTs

Smirnov & Zamolodchikov '16

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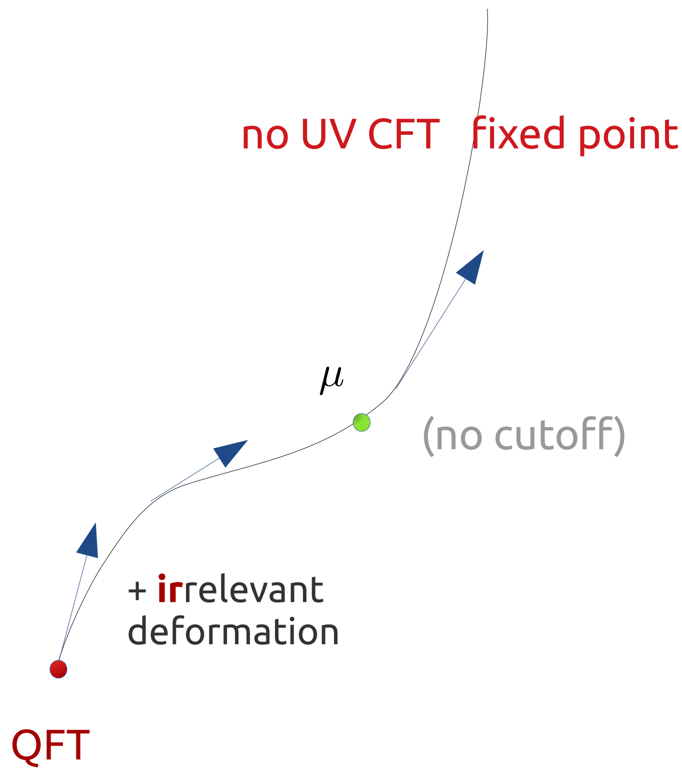

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- 2→2 S-matrix → unitary, analytic & crossing-symm.

$$S_{\{\alpha\}}(\theta) = S_{\{0\}}(\theta)\, e^{-i\sum_s \alpha_s \sinh(s\theta)}\,, \qquad s \in 2\mathbb{Z}+1$$

rel. rapidity

CDD ambiguity

- $T\overline{T}$ ↔ $s=1$

→ effect of changing S-matrix asymptotics on QFT?

- study effect on gnd. state energy $E_0(R)$ via TBA

→ square root singularity @ finite R → Hagedorn

→ this behaviour may be generic

Camilo, Fleury, Lencses, Negro & Zamolodchikov '21

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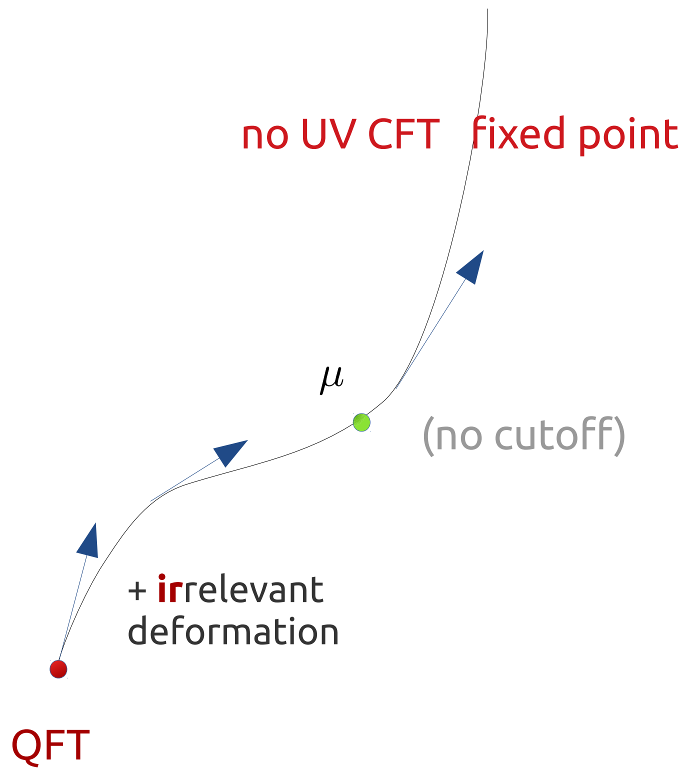

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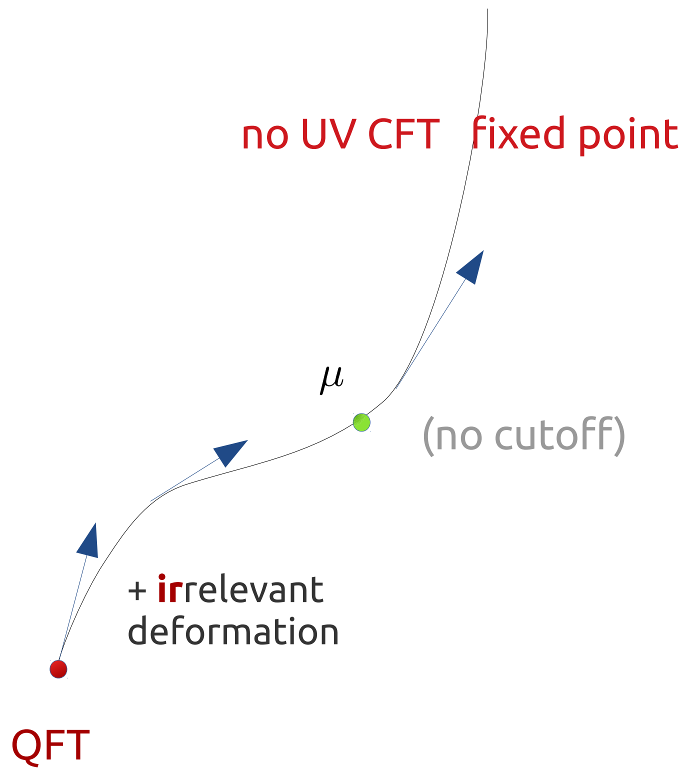

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2) QCD string — Dubovsky, Flauger, Gorbenko '12-'14; Dubovsky et al. '12-

   - $T\bar{T}$ – deformed free bosons → Nambu-Goto

     = lowest term in effective string action

     Caselle, Fioravanti, Gliozzi, Tateo '13

   - integrability broken at higher order

   - good agreement with lattice simulations

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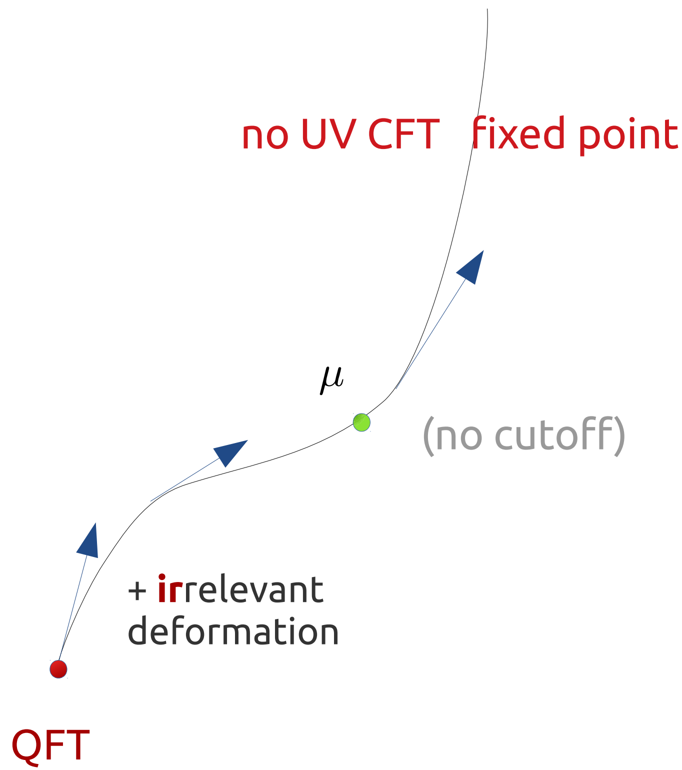

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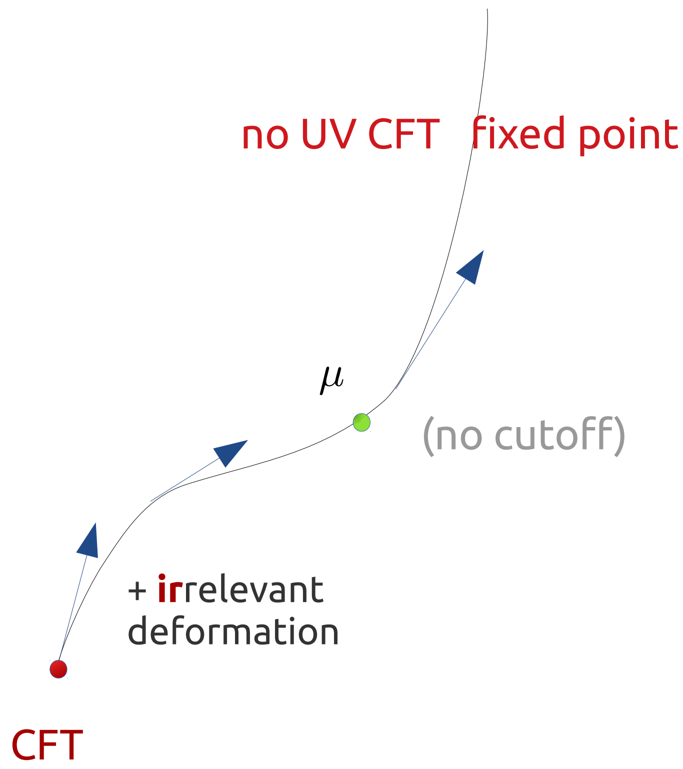

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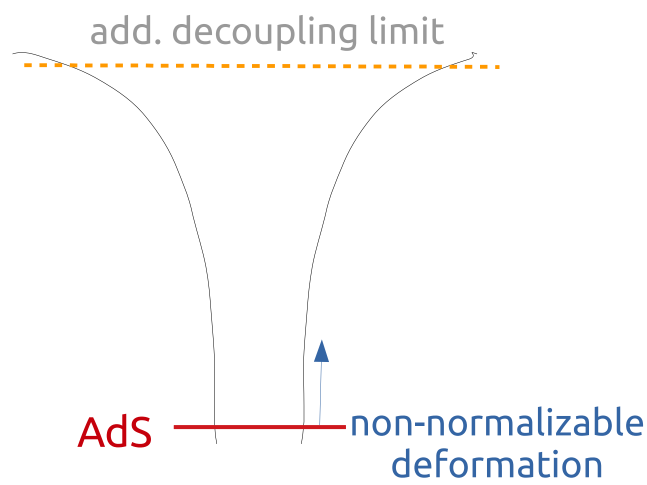

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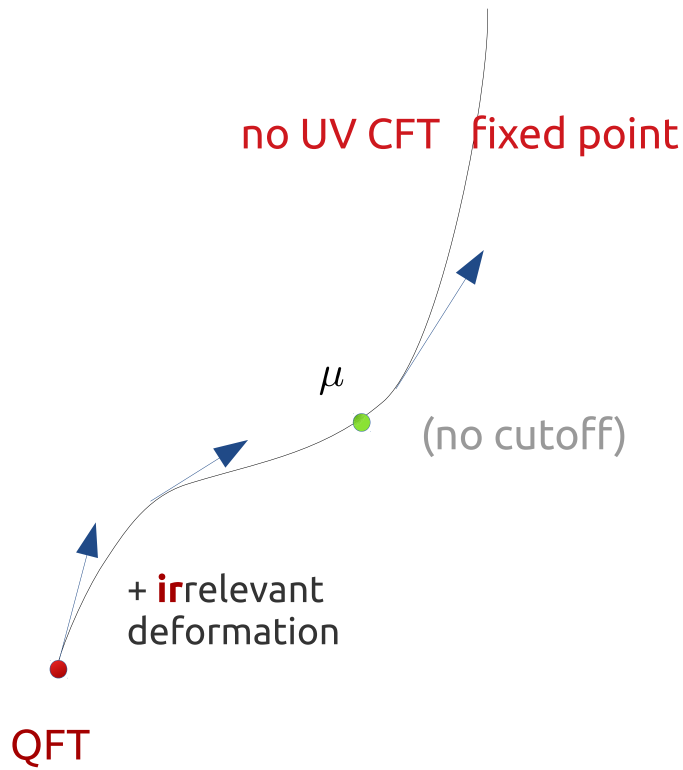

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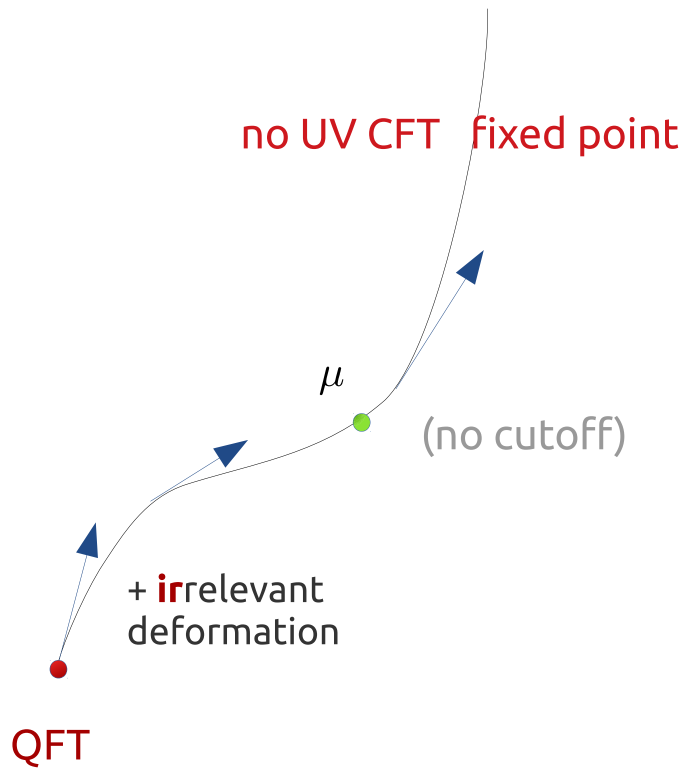

0) highly tractable : exact f.s. spectrum, S-matrix, etc.

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Plan

- review of the basic field-theoretical propeties of $T\bar{T}$

- holographic dictionary & its extensions

- the "single-trace" $T\bar{T}$ deformation & non-AdS holography

- will concentrate on the $T\bar{T}$ deformation (in particular of 2d CFTs).

  Other deformations will be mentioned only if something qualitatively new can be learned from them.

Smirnov-Zamolodchikov deformations

- irrelevant deformations of 2d QFTs → bilinears of two (higher spin) conserved currents $J^A,\ J^B$

- define $\mathcal{O}_{J^A J^B}$:

$$\lim_{y\to x} \epsilon^{\alpha\beta} J^A_\alpha(x) J^B_\beta(y) = \mathcal{O}_{J^A J^B}(x) + \text{derivative terms}$$

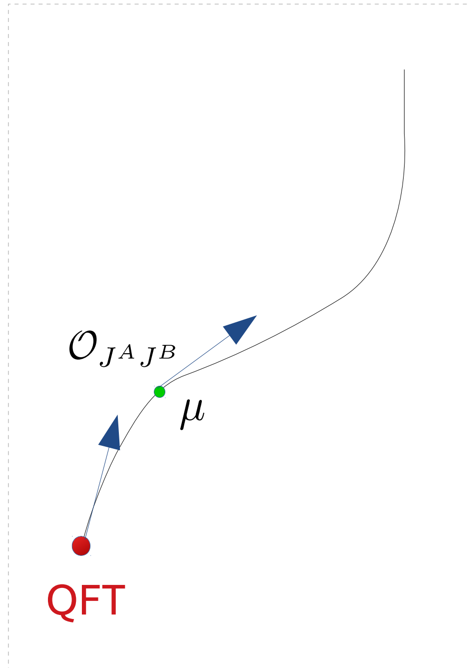
Zamolodchikov '04

SZ '16

nice factorization properties

- deformation :

$$\frac{\partial S(\mu)}{\partial \mu} = \int d^2x\, \mathcal{O}_{J^A J^B}(\mu)$$



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- deformation :

$$\frac{\partial S(\mu)}{\partial \mu} = \int d^2x\, \mathcal{O}_{J^A J^B}(\mu) = \int d^2z \, (T_{zz} T_{\bar{z}\bar{z}} - T^2_{z\bar{z}})$$

"$T\bar{T}$" "$\det T$"

$[\mu] = (length)^2$

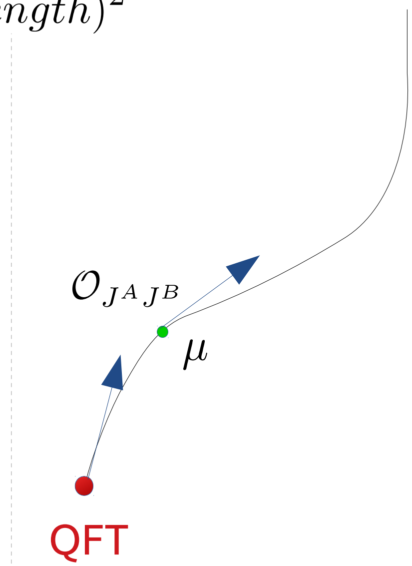
- examples:

$T\bar{T}$: $J^A_\alpha = T_\alpha{}^A \,, \quad J^B_\beta = T_\beta{}^B \;\; (\times \epsilon_{AB})$ (2,2)

SZ '16

Cavaglia, Negro, Szecsenyi, Tateo '16

universal



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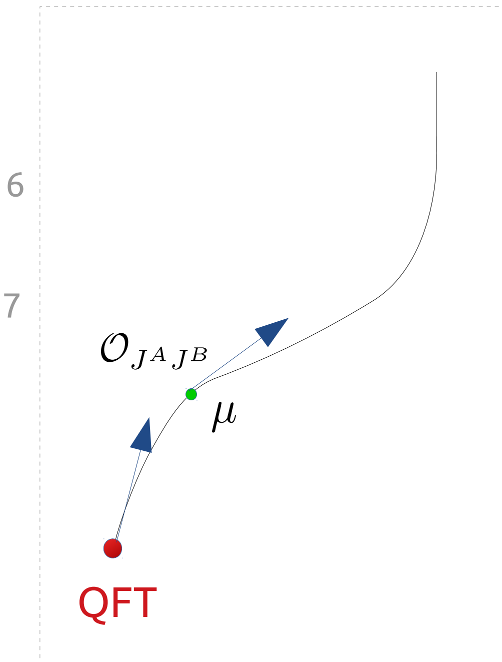
$J\bar{T}:\quad J^A_\alpha = J_\alpha\,,\quad J^B_\beta = T_{\beta\bar{z}}$ ~~Lorentz~~ (1,2) MG '17

$$\underbrace{SL(2,\mathbb{R})_L}_{\text{local \& conformal}} \times \underbrace{U(1)_R}_{\text{non-local!}}$$



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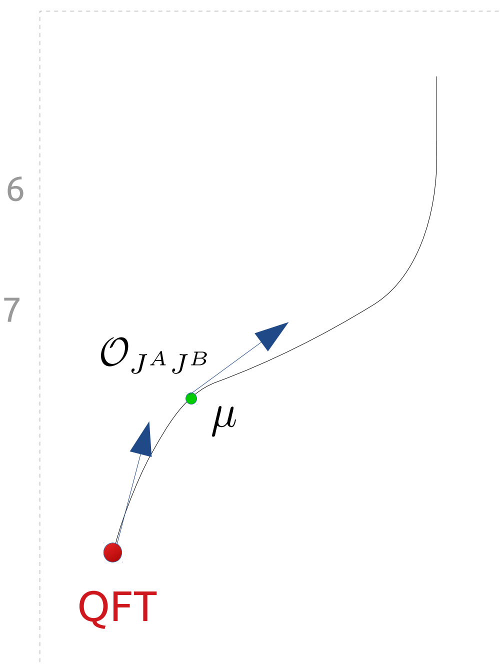
$SL(2,\mathbb{R})_L \times U(1)_R$

simpler than TT̄ ! ← local & conformal; non-local!



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- examples:

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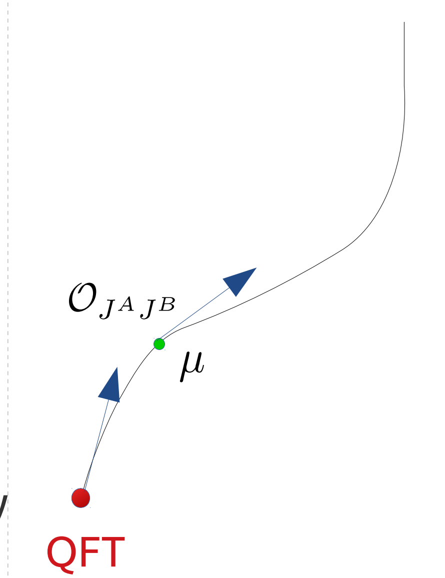
  - $T\bar{T}:\quad J^A_\alpha = T_\alpha{}^A\ ,\quad J^B_\beta = T_\beta{}^B\ \ (\times\epsilon_{AB})$ — (2,2) — SZ '16
  - $J\bar{T}:\quad J^A_\alpha = J_\alpha\ ,\quad J^B_\beta = T_{\beta\bar{z}}$ — ~~Lorentz~~ — (1,2) — MG '17
  - arbitrary combin. of $T\bar{T},\ JT_a$, etc. — Lefloch & Mezei; Frolov '19

  integrable
  - $T\bar{T}^{(s)}$ (generalized $T\bar{T}$) : higher spin currents ← integrability — SZ '16

- deformed theory non-local (scale $\mu^{\#}$) but argued UV complete



Basic observables

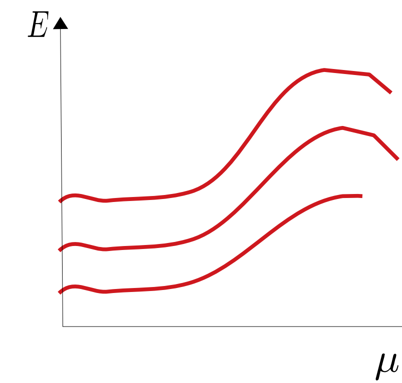
- place $T\bar{T}$ - deformed theory on a cylinder (R) → Hilbert space unchanged, only $H(\mu)$ and its eigenvalues

$$\partial_\mu E_n^{(\mu)} = \langle n_\mu | \partial_\mu H | n_\mu \rangle \qquad \partial_\mu |n_\mu\rangle = \sum_{m\neq n} \frac{\langle m_\mu | \partial_\mu H | n_\mu\rangle}{E_n^{(\mu)} - E_m^{(\mu)}} |m_\mu\rangle \equiv \mathcal{X}_{T\bar{T}} |n_\mu\rangle$$

factorization

$$\partial_\mu H = \frac{1}{R} \epsilon^{\alpha\beta} \epsilon_{AB} \int d\sigma\, T_\alpha^A \int d\sigma\, T_\beta^B + [H, i \int d\sigma d\tilde{\sigma} G(\sigma - \tilde{\sigma}) \epsilon_{AB} T_t^A(\sigma) T_t^B(\tilde{\sigma})]$$

$cls.$



Kruthoff, Parrikar '20

- Burger's eqn → universal deformed finite-size energies $E_n^{(\mu)}(R)$ determined only by the initial ones

$(P = 0)$

$$E_n^{(\mu)}(R) = E_n^{(0)}(R + \mu E_n^{(\mu)})$$

Thermodynamic Bethe Ansatz

(integrability)

$$S_{2\to 2}^{(\mu)}(\theta) = e^{i\mu m^2 \sinh\theta} S_{2\to 2}^{(0)}(\theta)$$

$e^{\frac{i\mu s}{2}}$ massless

- similar exact results for $J\bar{T}$ spectrum and of arbitrary combinations of $T\bar{T}$ and JT_a M.G. '17

Chakraborty, Giveon, Kutasov '18; LeFloch, Mezei '19; Frolov '19

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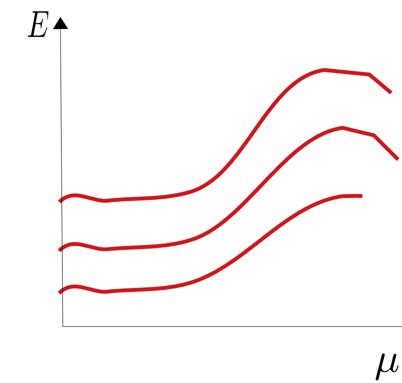

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at the core of many exact results

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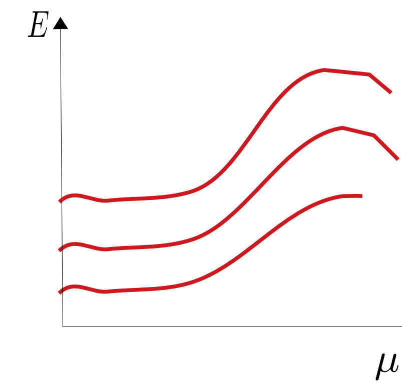

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$(P = 0)$

at the core of many exact results

(beyond integrability)

Dubovsky, Gorbenko, Mirbabayi '17

$$E_n^{(\mu)}(R) = E_n^{(0)}(R + \mu E_n^{(\mu)})$$

$$\mathcal{S}_\mu(p_i^\alpha) = e^{i\mu \sum_{i,j} \epsilon_{\alpha\beta} p_i^\alpha p_j^\beta} \mathcal{S}_0(p_i^\alpha)$$

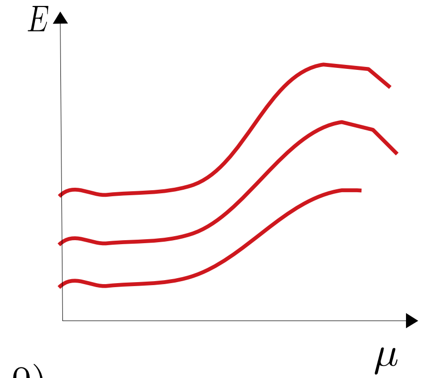
Thermodynamic Bethe Ansatz

(integrability)

- classically, closed-form expression for the deformed Hamiltonian density $\mathcal{H} = \frac{\sqrt{1 + 4\mu \mathcal{H}_{CFT}^{(0)} + 4\mu^2 \mathcal{P}^2} - 1}{2\mu}$

Jorjadze, Theisen '20

$T\bar{T}$ – deformed CFT spectrum & thermodynamics



- if the seed theory is a CFT $E_n^{(0)} = \dfrac{\Delta}{R}$, then $E_n^{(\mu)}(R) = E_n^{(0)}(R + \mu E_n^{(\mu)})$ yields

$$E_n^{(\mu)}(R) = \frac{R}{2\mu}\left(\sqrt{1 + \frac{4\mu E_n^{(0)}}{R} + \frac{4\mu^2 P^2}{R^2}} - 1\right)$$

$(P \neq 0)$

- thermodynamics: smoothly deformed levels → unchanged density of states

$(P = 0)$

$$S(E) = S_{Cardy}(E^{(0)}(E)) = \sqrt{\frac{2\pi c}{3}(ER + \mu E^2)}$$

$\mu > 0$: ground state energy $E_0 = -\dfrac{c}{12R}$ becomes complex for $R < R_{min} = \#\sqrt{\mu c}$

- Hagedorn behaviour $S \propto E$ at high energy $T_H = R_{min}^{-1}$

$\mu < 0$: all states with $E_0 > \dfrac{R}{4|\mu|}$ acquire imaginary energies → finite # of real-energy states

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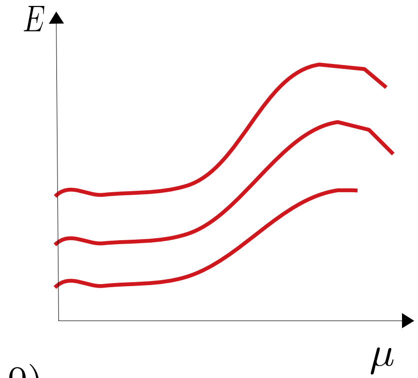

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Open Q:

- interpretation:
  - instability ?
  - truncate away imaginary energies?
    McGough, Mezei, Verlinde '16
  - superluminal propagation → CTCs in compact space
    Cooper, Dubovsky, Mohsen '13
    - $\mu > 0$ no sense for $R < R_{min}$ $(\sim T > T_H)$
    - $\mu < 0$ no sense for $\forall\, R$ finite

$T\bar{T}$ & string worldsheet

- D-2 $T\bar{T}$ - deformed free bosons = Nambu-Goto action for string in D – dim target space in static gauge

  → true classically for any D , and QM for D=3,26 $\quad X^0 = t \quad X^1 = \sigma$

- $T\bar{T}$ deformation = change of gauge in the NG action (conformal → static)

  Dubovsky, Flauger, Gorbenko '12

- deformed (U,V) and undeformed (u,v) theories related by a field-dependent coord. transformation

  onshell

$$U = u - 2\mu \int^v dv' T_{vv}(v') \,, \qquad V = v - 2\mu \int^u du' T_{uu}(u')$$

"dynamical coord."

- general definition of $T\bar{T}$ -deformed QFTs by coupling to topological 2d gravity

$$Z_{T\bar{T}} = \int \mathcal{D}e^a_\alpha \mathcal{D}X^a \mathcal{D}\varphi \, e^{-\frac{1}{2\mu}\int d^2x\, \epsilon^{\alpha\beta}\epsilon_{ab}(\partial_\alpha X^a - e^a_\alpha)(\partial_\beta X^b - e^b_\beta) + S_{QFT}(\varphi, e^a_\alpha)}$$

Dubovsky, Gorbenko, Hernandez-Chifflet '18

- can be derived by applying Hubbard-Stratonovich trick to $T\bar{T}$ Cardy '18

- alternate def'n: JT gravity + cosmo. constant $\Lambda \propto 1/\mu$ / interpretation: 2d ghost-free massive gravity

Dubovsky, Gorbenko, Mirbabayi '17 Tolley '19

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  fully non-perturbative
  reproduces flow energy
  reproduces S-matrix

  Dubovsky, Gorbenko, Hernandez-Chifflet '18

- can be derived by applying Hubbard-Stratonovich trick to $T\bar{T}$ Cardy '18

- alternate def'n: JT gravity + cosmo. constant $\Lambda \propto 1/\mu$ / interpretation: 2d ghost-free massive gravity

  Dubovsky, Gorbenko, Mirbabayi '17 Tolley '19

$T\bar{T}$: non-local 2d QFT or Quantum Gravity?

Open Q:

- does the worldsheet/coupling to topological gravity description imply $T\bar{T}$ = 2d quantum gravity?

- absence of propagating graviton →  Dubovsky, Flauger, Gorbenko'12

- S-matrix: $\mathcal{S}_\mu(p_i^\alpha) = e^{i\mu \sum_{i,j} \epsilon_{\alpha\beta} p_i^\alpha p_j^\beta} \mathcal{S}_0(p_i^\alpha)$ ~ gravitational
  - time delay $\propto$ energy ✓
  - minimum length $\Delta x_L \Delta x_R \geq \mu$ ✓

- off-shell observables:
  - → correlation functions of (quasi-local) operators
  - → deformation ~ attaches gravitational Wilson line Cardy '19

- $T\bar{T}$ – deformed CFTs : Virasoro x Virasoro symmetry (bulk) $T\bar{T}$ $\approx$ non-local CFT
  - → can this symmetry fix the correlation functions of special operators (primary analogues?)

$T\bar{T}$ - deformed CFTs as non-local CFTs

- symmetries: flow of energy eigenstates on the cylinder $\partial_\mu |n_\mu\rangle = \mathcal{X}_{T\bar{T}} |n_\mu\rangle$

  → define $\tilde{L}_m^\mu$ etc. via $\partial_\mu \tilde{L}_m^\mu = [\mathcal{X}_{T\bar{T}}, \tilde{L}_m^\mu]$ with $\tilde{L}_m^\mu(\mu=0) = L_m^{CFT}$

  → well-defined quantum-mechanically, satisfy Virasoro algebra (c undef) by construction

  → conserved (using $H_{T\bar{T}} = f(\tilde{L}_0^\mu, \tilde{\bar{L}}_0^\mu)$) $\Rightarrow$ $\tilde{L}_m^\mu$ are symmetries MG'21

  LeFloch, Mezei '19

- classical limit : $\tilde{L}_n^{\mu,cls} = R_u L_n^{cls} = (R + 2\mu H_R) \int d\sigma e^{inu} \mathcal{H}_L$ & similarly for the right-movers (!!! on u)

  w.i.p with R. Monten, I. Tsiares

- in $J\bar{T}$ – deformed CFTs, the analogous relation $\tilde{L}_n^\mu = R\, L_n - \lambda H_R J_n + \frac{\lambda^2 H_R^2}{4} \delta_{n,0}$ + RM

  assumed at full quantum level

  → definition of analogues of primary operators (Ward identities w.r.t. L_n)

  → their (momentum-space) correlation func. are entirely fixed by those of the undeformed CFT MG'21

  e.g. 2 & 3 – point functions = CFT momentum-space correlators, but with $\tilde{h} \to h(\bar{p})$, $\tilde{\bar{h}} \to \bar{h}(\bar{p})$

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  → can this symmetry fix the correlation functions of special operators (primary analogues?)

  → if yes, then $T\bar{T}$ = original CFT seen though the "prism of the dynamical coord."

  1 ↔ 1 correspondence all observables

Holography

Holographic dictionary for $T\bar{T}$ - deformed CFTs

- seed CFT : large c, large gap, dual to Einstein gravity + low-lying matter fields

- $T\bar{T}$: double trace → mixed boundary conditions for dual bulk field (non-dynamical graviton)

  1. use Hubbard-Stratonovich trick/variational principle to relate $Z_\mu[J] = Z_{\mu=0}[J + vevs]$

  only uses large N field theory

  2. interpret result in terms of bulk field data (using undeformed dictionary)

- asymptotic AdS$_3$ metric

universal non-univ

$$ds^2 = \frac{\ell^2 d\rho^2}{4\rho^2} + \left(\frac{g^{(0)}_{\alpha\beta}}{\rho} + g^{(2)}_{\alpha\beta} + \ldots \right) dx^\alpha dx^\beta$$

$T_{\alpha\beta} - \gamma_{\alpha\beta} T$

$$g^{(0)}_{\alpha\beta} \leftrightarrow \gamma_{\alpha\beta}(0)\,, \qquad g^{(2)}_{\alpha\beta} \leftrightarrow 8\pi G \ell\, \hat{T}_{\alpha\beta}(0)$$

- result:

$$\gamma_{\alpha\beta}(\mu) \overset{1.}{=} \gamma_{\alpha\beta}(0) - \mu \hat{T}_{\alpha\beta}(0) + \mu^2 \hat{T}_\alpha{}^\gamma \hat{T}_{\gamma\beta}(0) \qquad \langle T_{\alpha\beta}(\mu) \rangle = \ldots$$

$$\overset{2.}{=} g^{(0)}_{\alpha\beta} - \frac{\mu}{4\pi G\ell} g^{(2)}_{\alpha\beta} + \frac{\mu^2}{(8\pi G\ell)^2} g^{(2)}_{\alpha\gamma} g^{(0)\gamma\delta} g^{(2)}_{\delta\beta} \quad = \text{fixed}$$

MG, Monten '19

(Chern-Simons): Llabres '19

- for $\gamma_{\alpha\beta}(\mu) = \eta_{\alpha\beta}$ build phase space & compute the deformed energy spectrum

  → perfect match to field-theory formula (both signs of μ, matter field vevs on → universal!)

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  - both signs of μ
  - other (matter) vevs can be on
  - only depend on asymptotics

- for $\gamma_{\alpha\beta}(\mu) = \eta_{\alpha\beta}$ build phase space & compute the deformed energy spectrum — MG, Monten '19

  → perfect match to field-theory formula (both signs of μ, matter field vevs on → universal!)

Remarks

- 1st instance of mixed bnd. conditions on AdS_3 metric → bulk & boundary have independent definitions

  → standard situation: given bulk + consistency → infer properties of boundary theory

- change bnd. conditions on AdS_3 → radical modification of the boundary theory: local → non-local

- possibility of precision holography, despite irrelevant deformation

  → perfect match of bulk/boundary spectrum

  → symmetries : ASG $\approx$ Virasoro x Virasoro (subtleties in matching)

  Open Q:

  → match correlation functions ?

  → entanglement entropy?

- field theory feedback → test & sharpen rules for ASG & holographic dictionary

- $J\bar{T}$ → mixed bnd. conditions b/w metric & CS gauge field ~ Compere-Song-Strominger in metric sector

Bzowski, MG '18

Pure gravity

- the FG expansion terminates $ds^2 = \frac{\ell^2 d\rho^2}{4\rho^2} + \frac{g^{(0)}_{\alpha\beta} + \rho\, g^{(2)}_{\alpha\beta} + \rho^2\, g^{(2)\gamma}_{\alpha}\, g^{(2)}_{\gamma\beta}}{\rho}\, dx^\alpha dx^\beta$

- $\gamma_{\alpha\beta}(\mu)$ coincides with the induced metric at $\rho_c = -\frac{\mu}{4\pi G\ell}$ $\mu < 0$

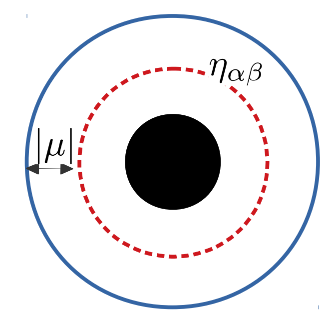
- $\langle T_{\alpha\beta}(\mu)\rangle$ coincides with the Brown-York stress tensor at ρ_c

- $\rho = \rho_c$ in mixed phase space can be mapped to $r_c^2 = \rho_c^{-1}$ in standard BTZ (independently of the mass)

- in agreement with observation that $T\bar{T}$ – deformed energies

  = energy of ``black hole in a box''

$$E(\mu) = \frac{R}{2|\mu|}\left(1 - \sqrt{1 - \frac{4|\mu|M}{R} + \frac{4\mu^2 J^2}{R^2}}\right)$$



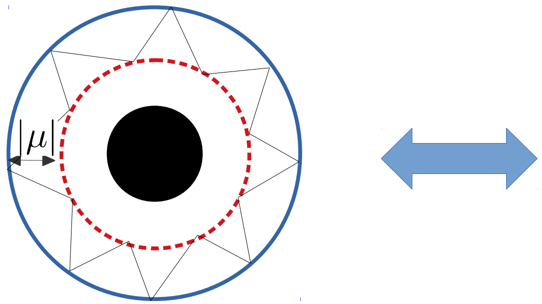
McGough, Mezei, Verlinde '16

- observation: onset of imaginary energies coincides with $r_{Schw} = r_c$ ($@M_{max} = \frac{R}{4|\mu|}$ mixed UV-IR)

| $T\bar{T}$ – deformed CFTs (dual to pure gravity) w/ $\mu < 0$ & truncated to real energy states | ⟺ | Quantum (pure) gravity in AdS_3 with a sharp radial cutoff |

Applications of the finite cutoff idea

- MMV proposal → stunningly simple holographic realization of QG with a finite cutoff (vs. Wilsonian RG)



$\mu < 0 \quad T\bar{T}$

with ImE truncated

- $T\bar{T}$ trace relation $Tr(T) = -\mu \det T$ maps to radial constraint Einstein equation (3d pure gravity)

- generalizations: higher-dim'l $T\bar{T}$ → engineered to reproduce higher D pure gravity with Dirichlet at r_c

  factorization ← large N, has a cutoff — M. Taylor '18; Hartman, Kruthoff, Shaghoulian, Tajdini '18

  - de Sitter cosmology: $T\bar{T} + \Lambda_2$ → engineered to reproduce flow from AdS_3 to dS_3

    dS microstates — Gorbenko, Silverstein, Torroba '18, Silverstein et al. '21

  - "bulk reconstruction" (QI) — Caputa, Kruthoff, Parrikar '20; Chandra, de Boer, Flory, Heller, Hortner

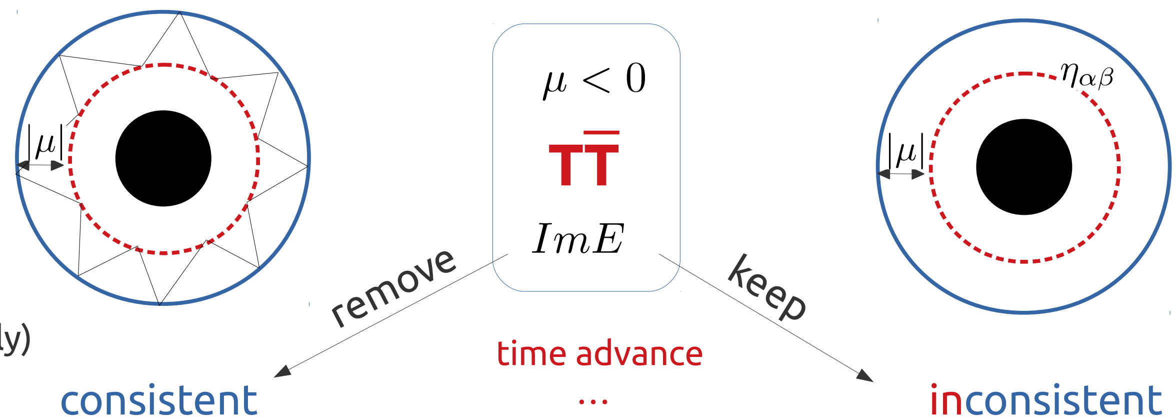
- ~~precise specif.~~, ~~UV complenetness~~ ~ definite/tractable QM system dual to gravity in a finite volume

Open questions

- what is the precise relation between

mixed boundary conditions

general (matter)



(non-perturbatively)

- in-depth study of pure 3d gravity at finite cutoff & comparison to $\mu < 0$ $T\bar{T}$ Kraus, Monten, Myers '21

Ebert, Hijano, Kraus, Monten, Myers '22 ; Kraus, Monten, Roumpedakis '22

- observables that distinguish between the two possibilities?

→ energy, stress tensor correlation functions match, but do not distinguish

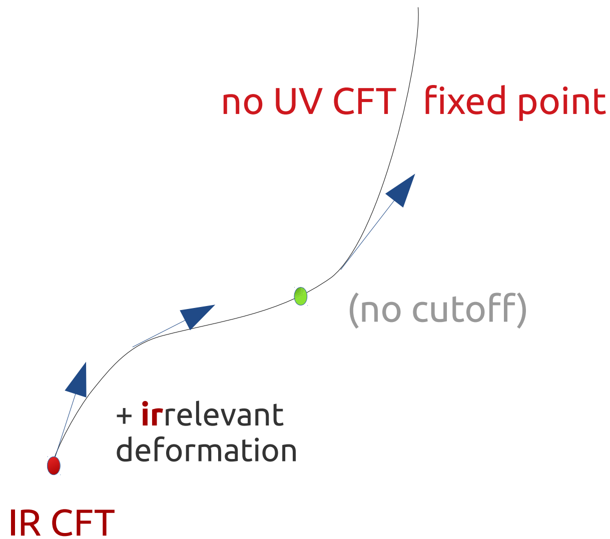
→ entanglement entropy? first-principles derivation & comparison to the bulk

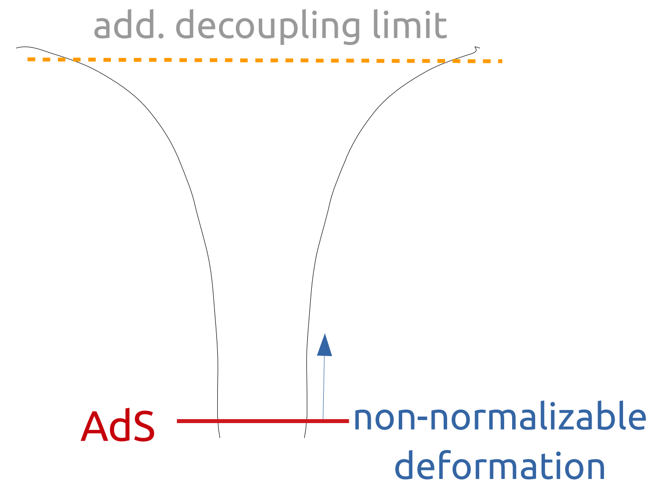
(in single-trace $T\bar{T}$, the EE is divergent)

Holography II

The “single -trace” $T\bar{T}$ deformation

Irrelevant flows and non-AdS holography





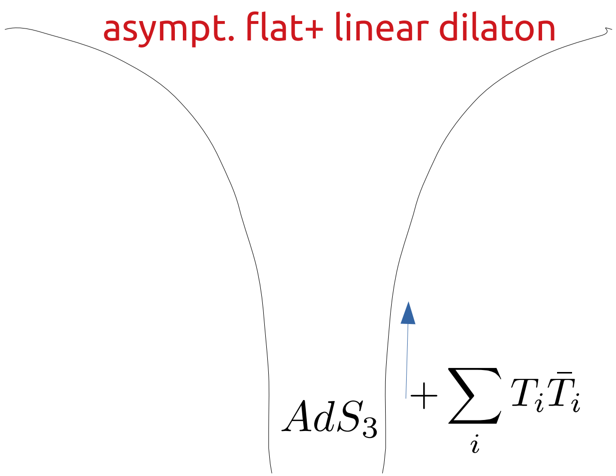
- $T\bar{T}$ ↛ non-AdS geometry because it is double-trace → need single-trace irrelevant deformation
- AdS_3/CFT_2 gauge group: S_p (permutations)

Giveon, Itzhaki, Kutasov '17

seed symmetric product orbifold CFT ⟺ "single-trace $T\bar{T}$" deformation (finite μ)

$$\mathcal{M}^p/S_p \qquad \sum_{i=1}^{p} T_i\bar{T}_i \quad \Rightarrow \quad (T\bar{T}_{def.}\,\mathcal{M})^p/S_p$$

The GIK proposal



N_5 NS5 and N_1 F1 strings in the NS5 decoupling limit

$g_s \to 0\,,\quad \alpha'$ fixed

N_1 large

UV: Little String Theory

non-gravitational, non-local theory with Hagedorn growth

IR: AdS_3 ~ descr. by $(\mathcal{M}_{6N_5})^{N_1}/S_{N_1}$ symmetric orbifold CFT

- worldsheet σ - model : exactly marginal deformation of the WZW model describing AdS_3 by $J^-\bar{J}^-$

  → dual to CFT source for a $(2,2)$ single-trace operator $\sum_{i=1}^{N_1} T_i\bar{T}_i$

Giveon, Itzhaki, Kutasov '17

$$Z_{string}[\text{NS5- F1}] = Z\left[(T\bar{T} - \text{def. CFT}_{6N_5})^{N_1}/S_{N_1}\right]$$

✓ finite deformation $\mu = \pi\sqrt{\alpha'}$

- similar construction for single-trace $J\bar{T}$ → NS-NS warped AdS_3 (concrete micro. realization of Kerr/CFT)

Apolo, Song; Chakraborty, Giveon, Kutasov '18

Checks and predictions

$$Z_{string}[\text{NS5- F1}] = Z\left[(T\bar{T} - \text{def. CFT}_{6N_5})^{N_1}/S_{N_1}\right]$$

- spectrum of long string excitations exactly matches single-trace $T\bar{T}$ spectrum GIK '17
- black hole entropy $S(E)$ agrees with $T\bar{T}$ entropy (Cardy → Hagedorn)

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≠ !

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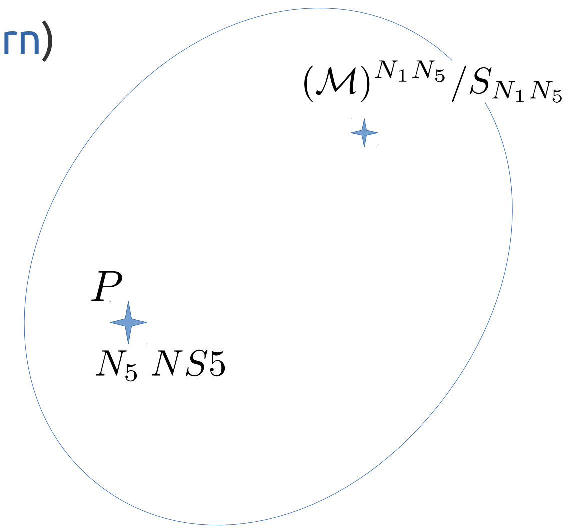
  long strings

  Eberhardt '21



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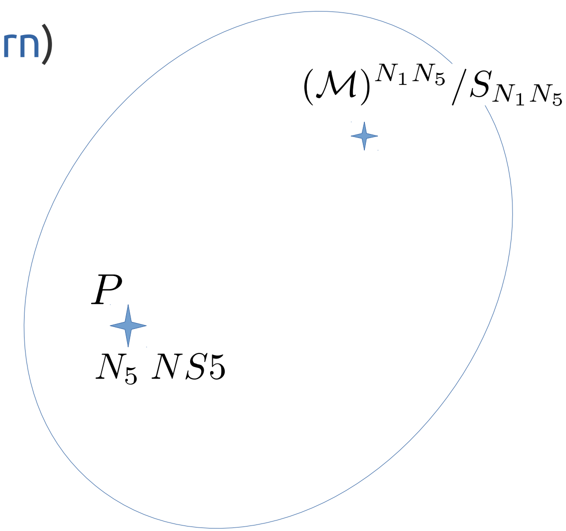
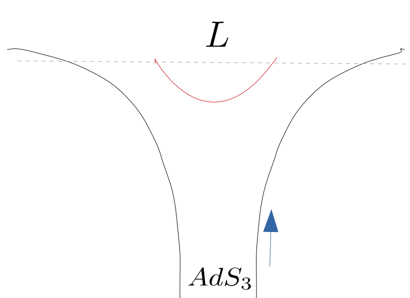
  long strings
  Eberhardt '21
- correlation functions $\langle\mathcal{O}(p)\mathcal{O}(-p)\rangle$ → compute using worldsheet
  Asrat, Giveon, Itzhaki, Kutasov '17; Giribet '17
  ~ do not match prediction from symm. prod. orb. (J$\bar{\text{T}}$)
  w.i.p. w/ S. Chakraborty, S. Georgescu
- holographic entanglement entropy





- logarithmically divergent with L Chakraborty, Giveon, Itzhaki, Kutasov '17
- is not defined for $L < L_{min} = \frac{\pi}{2}\sqrt{N_5\alpha'}$
- 2 intervals: mutual information diverges when distance = L_{min}

Asrat, Kudler-Flam '20

Infinite symmetries

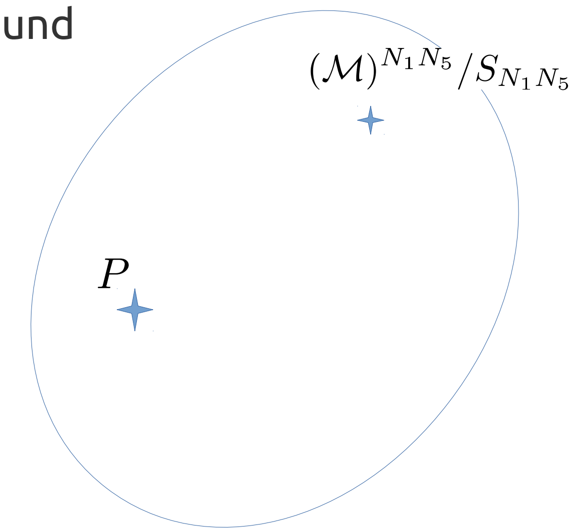
- Virasoro symmetries of $T\bar{T}$ -deformed CFTs survive the symmetric product orbifold

  w.i.p. w/ S. Chakraborty, S. Georgescu

- asymptotic symmetry analysis of the asympt. linear dilaton background

  ~ Virasoro x Virasoro w.i.p. w/ S. Georgescu

  → does not follow from the above result on the symmetries of symmetric product orbifolds of $T\bar{T}$



- possible explanation:

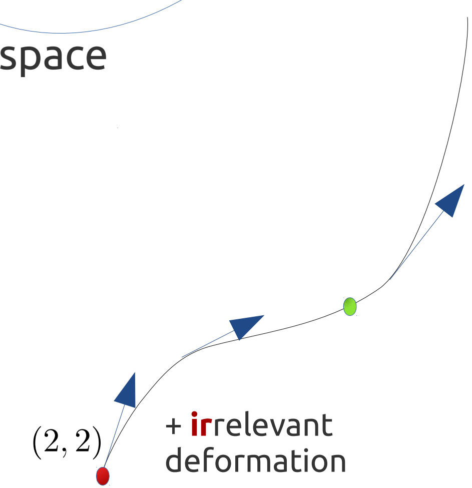
  → $\exists$ analogues of the single-trace $T\bar{T}$ deformation over the entire moduli space

  → irrelevant deformations by $(2,2)$ operators → UV-complete theories

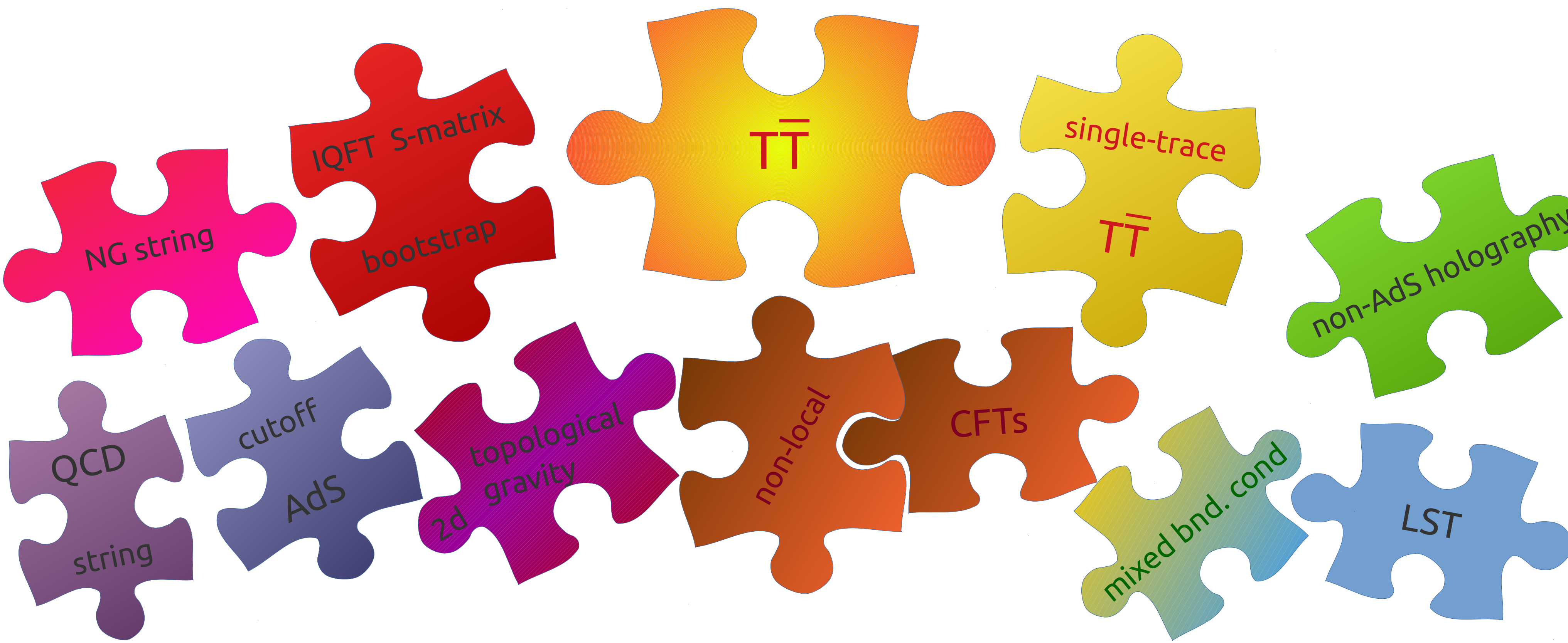
  (how to define them appropriately?)

  → all theories so defined would possess Virasoro x Virasoro symmetry

  $\approx$ non-local 2d CFTs



Thank you !



IQFT S-matrix
bootstrap
$T\bar{T}$
single-trace
$T\bar{T}$
NG string
non-AdS holography
cutoff
AdS
topological
2d gravity
non-local
CFTs
QCD
string
mixed bnd. cond
LST

Conclusions

- $T\bar{T}$, $J\bar{T}$ are a set of well-defined and highly tractable irrelevant deformations of 2d QFTs

  → UV complete non-local QFTs

  → deformed spectrum, S-matrix, thermodynamics extensively studied

  → relevant for QCD string, non-AdS holography in their single-trace version

- $T\bar{T}$, $J\bar{T}$ - deformed CFTs correspond to non-local CFTs ← Virasoro symmetries (f-dep. coord. transf.)

  → it seems possible to define an analogue of primary operators

  → correlation functions fixed in terms of those of the undeformed CFT

- what are the most general non-local CFTs (at large N)?

  → axiomatic definition? Bootstrap?

  → physical applications?

The primary condition

- main idea: use interplay of the two sets of symmetry generators

$$\begin{cases} \tilde{L}^\mu_n = R\, L_n - \lambda H_R J_n + \dfrac{\lambda^2 H_R^2}{4}\delta_{n,0}\,, \qquad \tilde{J}^\mu_n = J_n - \dfrac{\lambda H_R}{2}\delta_{n,0} \\ \tilde{\bar{L}}^\mu_n = R_v \bar{L}_n - \lambda : H_R \bar{J}_n : + \dfrac{\lambda^2 H_R^2}{4}\delta_{n,0}\,, \qquad \tilde{\bar{J}}^\mu_n = \bar{J}_n - \dfrac{\lambda H_R}{2}\delta_{n,0} \end{cases}$$

assumed

full quantum

- algebra LM (L_n, J_n) : Virasoro-Kac-Moody ; algebra RM $(\bar{L}_n, \bar{J}_n)$: non-linear modification of Vir.-KM
- LM: operators should be primary w.r.t. L_n, J_n → primary Ward identities w/ $h = \tilde{h} + \lambda \bar{p}\tilde{q} + \dfrac{\lambda^2 \bar{p}^2}{4}$
- introduce auxiliary ops. $\tilde{\mathcal{O}}(w,\bar{w})$ defined via $\partial_\lambda \tilde{\mathcal{O}}(w,\bar{w}) = [\mathcal{X}_{J\bar{T}}, \tilde{\mathcal{O}}(w,\bar{w})]$ ← identical correlation functions and Ward identities w.r.t. $\tilde{L}_n$ etc., as the operators in the undeformed CFT
- RM: momentum space $\bar{p}$, primary condition w.r.t. $\bar{L}_n$??? → guess !

$$\mathcal{O}(p,\bar{p}) = \int dw d\bar{w}\, e^{-pw-\bar{p}\bar{w}} e^{Aw+B\bar{w}} e^{\lambda\bar{p}\sum_{n=1}^\infty (e^{nw}\tilde{J}_{-n} + e^{n\bar{w}}\tilde{\bar{J}}_{-n})}\tilde{\mathcal{O}}(w,\bar{w}) e^{-\lambda\bar{p}\sum_{n=1}^\infty (e^{-nw}\tilde{J}_n + e^{-n\bar{w}}\tilde{\bar{J}}_n)}$$

Correlation functions

$$\mathcal{O}(p,\bar p) = \int dw d\bar w\, e^{-pw-\bar p\bar w} e^{Aw+B\bar w} e^{\lambda\bar p\sum_{n=1}^\infty (e^{nw}\tilde J_{-n} + e^{n\bar w}\tilde{\bar J}_{-n})}\tilde{\mathcal{O}}(w,\bar w) e^{-\lambda\bar p\sum_{n=1}^\infty(e^{-nw}\tilde J_n + e^{-n\bar w}\tilde{\bar J}_n)}$$

- Ward identities w.r.t $\bar L_n, \bar J_n$ → CFT Ward identities in the decompactification limit $R \to \infty$

$$h = \tilde h + \lambda\bar p\tilde q + \frac{\lambda^2\bar p^2}{4} \qquad \bar h = \tilde{\bar h} + \lambda\bar p\tilde{\bar q} + \frac{\lambda^2\bar p^2}{4}$$

- arbitrary correlation functions → $\tilde{\mathcal{O}}$ correlators = undeformed CFT correlators in flowed vacuum

➡ all correlation functions of $\mathcal{O}(p,\bar p)$ are entirely determined by original CFT correlators

- e.g., 2 & 3 – point functions = CFT momentum-space correlators, but with $\tilde h \to h(\bar p)$, $\tilde{\bar h} \to \bar h(\bar p)$

same behaviour as seen in black holes !!!

Comments

- precision holography, despite the deformation being irrelevant

- mixed metric boundary conditions keep full dynamics of matter fields → unchanged b.c.

- change bnd. conditons on AdS3 metric → radical modification of the dual theory: local → non-local

- asymptotic symmetries

    → expect : $T\bar{T}$ deformation breaks CFT conformal symmetries to $U(1)_L \times U(1)_R$

    → find: $\boxed{Virasoro(u) \times Virasoro(v)}$ with same **c** as in the undeformed CFT

    u, v → field-dependent coordinates $U = u - \mu \int T_{vv} dv$

    → suggest $T\bar{T}$ – deformed CFTs possess Virasoro symmetry, despite being non-local **NEW!**

The $J\bar{T}$ holographic dictionary

MG, Bzowski '18

- introduce sources: $J^\alpha \leftrightarrow a_\alpha \qquad T^a{}_\alpha \leftrightarrow e^a{}_\alpha$

- variational principle:

$$\delta S_\mu = \delta S_{CFT} - \delta S_{J\bar{T}} = \int d^2x\, [\underbrace{eT^a{}_\alpha \delta e^\alpha_a + eJ^\alpha \delta a_\alpha}_{\text{CFT}} - \delta(\underbrace{\mu_a T^a{}_\alpha J^\alpha e}_{\text{deformation}})] = \int d^2x\, \tilde{e}(\underbrace{\tilde{T}^a{}_\alpha \delta \tilde{e}^\alpha_a + \tilde{J}^\alpha \delta a_\alpha}_{\text{new sources \& vevs}})$$

- new sources $\qquad \tilde{e}^\alpha_a = e^\alpha_a - \mu_a \langle J^\alpha \rangle \,, \qquad \tilde{a}_\alpha = a_\alpha - \mu_a \langle T^a_\alpha \rangle$
- new vevs $\qquad \tilde{T}^a{}_\alpha = T^a{}_\alpha + (e^a_\alpha + \mu_\alpha J^a)\, \mu_b T^b{}_\beta J^\beta \,, \qquad \tilde{J}^\alpha = J^\alpha$

} **large N field theory**

Holography:

$(T^a{}_\alpha, e^a{}_\alpha)$ modelled by 3d Einstein gravity

(J^α, a_α) $U(1)$ Chern-Simons gauge field

} non-dynamical

- AdS_3 gravity with mixed boundary conditions (CSS-like, but allowing full dynamics)

- perfect match between energies of black holes and the defomed CFT spectrum ✓

- asymptotic symmetry group: $\underbrace{SL(2,\mathbb{R})_L \times U(1)_L}_{\text{Virasoro - Kac-Moody}} \times U(1)_R$ (J; non-local) → Virasoro - Kac-Moody $\times$ Virasoro$_R$ ← $f(x^- - \lambda \int J)$

non-local CFT!