22-nd Canadian Mathematical Olympiad 1990

April 4, 1990

1. A competition involving $n \geq 2$ players was held over k days. On each day, the players received scores of $1, 2, \ldots, n$ points with no two players receiving the same score. At the end of the k days, it was found that each player had exactly 26 points in total. Find all pairs (n, k) for which this is possible.

2. A set of $\frac{n(n+1)}{2}$ distinct numbers is arranged at random in a triangular array:

$$\begin{array}{c} * \\ * \ * \\ * \ * \ * \\ \cdots\cdots\cdots \\ * \ * \ \cdots \ * \ * \end{array}$$

   If M_k denotes the largest number in the k-th row from the top, find the probability that $M_1 < M_2 < \cdots < M_n$.

3. The diagonals AC and BD of a cyclic quadrilateral $ABCD$ meet at X. Let A', B', C', D' be the feet of the perpendiculars from X to AB, BC, CD, DA, respectively. Prove that
$$A'B' + C'D' = A'D' + B'C'.$$

4. A particle can travel at speeds up to $2m/s$ along the x-axis, and up to $1m/s$ elsewhere in the plane. Provide a labelled sketch of the region which can be reached within one second by the particle starting at the origin.

5. A function f is defined on $\mathbb{N}$ as follows:
$$f(1) = 1, \qquad f(2) = 2;$$
$$f(n+2) = f(n+2-f(n+1)) + f(n+1-f(n)) \quad \text{for each } n \geq 1.$$

   (a) Prove that $0 \leq f(n+1) - f(n) \leq 1$, and that $f(n+1) = f(n) + 1$ if $f(n)$ is odd.

   (b) Find, with proof, all values of n for which $f(n) = 2^{10} + 1$.