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No occurrence obstructions in geometric complexity theory

Peter Bürgisser
Technische Universität Berlin
Email: pbuerg@math.tu-berlin.de

Christian Ikenmeyer
Texas A&M University
Email: ciken@math.tamu.edu

Greta Panova
University of Pennsylvania
panova@math.upenn.edu

***Abstract*—The permanent versus determinant conjecture is a major problem in complexity theory that is equivalent to the separation of the complexity classes $\mathrm{VP}_{\mathrm{ws}}$ and VNP. Mulmuley and Sohoni [32] suggested to study a strengthened version of this conjecture over the complex numbers that amounts to separating the orbit closures of the determinant and padded permanent polynomials. In that paper it was also proposed to separate these orbit closures by exhibiting *occurrence obstructions*, which are irreducible representations of $\mathrm{GL}_{n^2}(\mathbb{C})$, which occur in one coordinate ring of the orbit closure, but not in the other. We prove that this approach is impossible. However, we do not rule out the approach to the permanent versus determinant problem via *multiplicity obstructions* as proposed in [32].**

I. Introduction

A central problem in algebraic complexity theory is to prove that there is no efficient algorithm to evaluate the *permanent* $\mathrm{per}_n := \sum_{\pi\in S_n} x_{1\pi(1)}\cdots x_{n\pi(n)}$. The natural model of computation to study this question is the one of straight-line programs (or arithmetic circuits), which perform arithmetic operations $+,-,*$ in the polynomial ring, starting with the variables X_{ij} and complex constants. Efficient means that the number of arithmetic operations is bounded by a polynomial in n. The permanent arises in combinatorics and physics as a generating function. Its relevance for complexity theory derives from Valiant's discovery [40], [39] that the evaluation of the permanent is a complete problem in the complexity class VNP (and also in the class #P in the model of Turing machines); see [3], [28] for more information.

The *determinant* $\det_n := \sum_{\pi\in S_n} \mathrm{sgn}(\pi)\, x_{1\pi(1)}\cdots x_{n\pi(n)}$ is known to have an efficient algorithm. Its evaluation is complete for the complexity class $\mathrm{VP}_{\mathrm{ws}}$; cf. [39], [38]. From the definition it is clear that $\mathrm{VP}_{\mathrm{ws}} \subseteq \mathrm{VNP}$ and proving the separation $\mathrm{VP}_{\mathrm{ws}} \neq \mathrm{VNP}$ is the flagship problem in algebraic complexity theory. It can be seen as an "easier" version of the famous $\mathsf{P} \neq \mathsf{NP}$ problem; cf. [4].

The conjecture $\mathrm{VP}_{\mathrm{ws}} \neq \mathrm{VNP}$ can be restated without any reference to complexity classes by directly comparing permanents and determinants. The *determinantal complexity* $\mathrm{dc}(f)$ of a polynomial $f \in \mathbb{C}[X_1,\dots,X_N]$ is defined as the smallest integer $n \in \mathbb{N}$ such that f can be written as a determinant of an n by n matrix whose entries are affine linear forms in the variables X_i. It is known [39] that $\mathrm{dc}(f) \le s+1$ if f if has a formula of size s. Valiant [39], [41] and Toda [38] proved that $\mathrm{VP}_{\mathrm{ws}} \neq \mathrm{VNP}$ is equivalent to the following conjecture.

I.1 Conjecture (Valiant 1979). *The determinantal complexity* $\mathrm{dc}(\mathrm{per}_n)$ *grows superpolynomially in* n.

It is known [17] that $\mathrm{dc}(\mathrm{per}_n) \le 2^n - 1$. Finding lower bounds on $\mathrm{dc}(\mathrm{per}_n)$ is an active area of research [30], [13], [25], [20], [2], [43], but the best known lower bounds are only $\Omega(n^2)$.

A. An attempt via algebraic geometry and representation theory

Towards answering Conj. I.1, Mulmuley and Sohoni [32], [33], [31] proposed an approach based on algebraic geometry and representation theory, for which they coined the name geometric complexity theory.

We consider $\mathsf{Sym}^n\mathbb{C}^{n^2}$ as the space of homogeneous polynomials of degree n in n^2 variables. Clearly, $\det_n \in \mathsf{Sym}^n\mathbb{C}^{n^2}$. The group GL_{n^2} acts on $\mathsf{Sym}^n\mathbb{C}^{n^2}$ by linear substitution. The *orbit* $\mathrm{GL}_{n^2}\cdot\det_n$ is obtained by applying all possible invertible linear transformations to $\det_n$. Consider the closure

$$\Omega_n := \overline{\mathrm{GL}_{n^2}\cdot\det_n} \subseteq \mathsf{Sym}^n\mathbb{C}^{n^2} \tag{I.2}$$

of this orbit with respect to the Euclidean topology. By a general principle, this is the same as the closure with respect to the Zariski topology; see [34, §2.C]. It is easy to see that $\Omega_2 = \mathsf{Sym}^2\mathbb{C}^4$. For $n=3$, the boundary of Ω_n has been determined recently [18], but for $n=4$ it is already unknown.

For $n \ge m$ we consider the *padded permanent* defined as $X_{11}^{n-m}\mathrm{per}_m \in \mathsf{Sym}^n\mathbb{C}^{m^2}$. (Sometimes the padding is achieved by using a variable not appearing in per_m, but this is irrelevant, cf. [22, Appendix].)

I.3 Conjecture (Mulmuley and Sohoni 2001). *For all* $c \in \mathbb{N}_{\ge 1}$ *we have* $X_{11}^{m^c-m}\mathrm{per}_m \notin \Omega_{m^c}$ *for infinitely many* m.

This conjecture was stated in [32]. We refer to [12, Prop. 9.3.2] for an equivalent formulation in terms of complexity classes that goes back to [5]. (In particular see [5, Problem 4.3].)

Conj. I.3 implies Conj. I.1. Indeed, using that GL_{n^2} is dense in $\mathbb{C}^{n^2\times n^2}$, one shows (e.g., see [6]) that $\mathrm{dc}(\mathrm{per}_m) \le n$ implies $X_{11}^{n-m}\mathrm{per}_m \in \Omega_n$. (The latter must be a point in the boundary of Ω_n if $n > m$.)

The following strategy towards Conj. I.3 was proposed in [32]. The action of the group $G := \mathrm{GL}_{n^2}$ on Ω_n induces a corresponding action on the coordinate ring $\mathbb{C}[\Omega_n]$. It is well

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known [16] that the irreducible polynomial representations of G can be labeled by partitions λ into at most n^2 parts. The coordinate ring $\mathbb{C}[\Omega_n]$ is a direct sum of its irreducible submodules since G is reductive. We say that λ occurs in $\mathbb{C}[\Omega_n]$ if it contains (the dual of) an irreducible G-module of type λ. (We will identify spaces with their duals to avoid negative weights.)

Let $Z_{n,m}$ denote the orbit closure of the padded permanent $X_{11}^{n-m}\mathrm{per}_m \in \mathsf{Sym}^n\mathbb{C}^{n^2}$. If the latter is contained in Ω_n, then $Z_{n,m} \subseteq \Omega_n$, and the restriction defines a surjective G-equivariant homomorphism $\mathbb{C}[\Omega_n] \to \mathbb{C}[Z_{n,m}]$ of the coordinate rings. Schur's lemma implies that if λ occurs in $\mathbb{C}[Z_{n,m}]$, then it must also occur in $\mathbb{C}[\Omega_n]$. A partition λ violating this condition is called an *occurrence obstruction*. Its existence thus proves that $Z_{n,m} \not\subseteq \Omega_n$. It is known that occurence obstructions λ must satisfy $|\lambda| = nd$ and $\ell(\lambda) \leq m^2$, cf. [32], [33], [12]. Here $|\lambda| := \sum_i \lambda_i$ denotes the *size* of λ and $\ell(\lambda)$ denotes the *length* of λ, which is defined as the number of nonzero parts of λ. We write $\lambda \vdash |\lambda|$, so in our case $\lambda \vdash nd$.

In [32], [33] it was suggested to prove Conj. I.3 by exhibiting occurrence obstructions. More specifically, the following conjecture was made.

I.4 Conjecture (Mulmuley and Sohoni 2001)**.** *For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m, there exists a partition λ occurring in $\mathbb{C}[Z_{m^c,m}]$ but not in $\mathbb{C}[\Omega_{m^c}]$.*

This conjecture implies Conj. I.3 by the above reasoning. Conj. I.4 on the existence of occurrence obstructions has stimulated a lot of research and has been the main focus of researchers in geometric complexity theory in the past years, see Section I (c). Unfortunately, this conjecture is false! This is the main result of this work. More specifically, we show the following.

I.5 Theorem. *Let n, d, m be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[Z_{n,m}]$, then λ also occurs in $\mathbb{C}[\Omega_n]$. In particular, Conj. I.4 is false.*

One can likely improve the bound on n by a more careful analysis.

B. Proof outline

The rough outline of the proof of Thm. I.5 closely follows the structure of the proof of the main theorem in the recent paper [22], which is concerned with the positivity of rectangular Kronecker coefficients. Recall that the *Kronecker coefficient* $k(\lambda, \mu, \nu)$ of three partitions λ, μ, ν of the same size d can be defined as the dimension of the space of S_d-invariants of $[\lambda] \otimes [\mu] \otimes [\nu]$, where $[\lambda]$ denotes the irreducible S_d-module of type λ; see [27, I§7, internal product]. We write $n \times d$ for the rectangular partition $(d, \dots, d)$ of size nd and call $k_n(\lambda) := k(\lambda, n \times d, n \times d)$ the *rectangular Kronecker coefficient* of $\lambda \vdash nd$. In [32] it was realized that if $k_n(\lambda) = 0$, then λ does not occur in $\mathbb{C}[\Omega_n]$. This was proposed as a potential method of proving Conj. I.4, which has stimulated research in algebraic combinatorics on these quantities. The main result in [22] says that proving Conj. I.4 in this way is not possible:

I.6 Theorem ([22])**.** *Let $n, d, m \in \mathbb{N}$ with $n > 3m^4$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[Z_{n,m}]$, then $k_n(\lambda) > 0$.*

The *body* $\bar{\lambda}$ of a partition λ is obtained from λ by removing its first row. The proofs of Thm. I.5 and Thm. I.6 crucially use the following observation by Kadish and Landsberg [23] (see Prop. IV.1). If $\lambda \vdash nd$ appears in $\mathbb{C}[Z_{n,m}]$, then $|\bar{\lambda}| \leq md$. This is equivalent to $\lambda_1 \geq (n-m)d$, which implies that λ must have a very long first row if n is substantially larger than m. This fact can be used to prove a lower bound on all possible degrees d, although in the two proofs this lower bound is obtained using very different techniques. Then λ gets decomposed by cutting it vertically into several long rectangles of short height, treating the first row separately. For these rectangles it is shown independently that they appear in $\mathbb{C}[\Omega_n]$ (for the proof of Thm. I.5) or that their rectangular Kronecker coefficient is positive (for the proof of Thm. I.6). Besides taking care of several technicalities, the main result then follows from glueing λ back together via the so-called *semigroup property*. Although this rough outline is shared among both proofs, the details and techniques differ vastly.

A crucial technique in the present paper is the encoding of a generating system of highest weight vectors v_T in plethysms $\mathsf{Sym}^d\mathsf{Sym}^nV$ by (classes of) tableaux with contents $d \times n$, as well as the analysis of their evaluation at tensors of rank one in a combinatorial way. This is similar to [7], [19]. A further technique is the "lifting" of highest weight vectors of $\mathsf{Sym}^d\mathsf{Sym}^nV$, when increasing the inner degree n, as introduced by Kadish and Landsberg [23]. This is closely related to a stability property of the plethysm coefficients [42], [14], [29]. A concrete understanding of the stability property in terms of lifting highest weight vectors is important for the proof of the main result. Remarkably, for the proof of Thm. I.5, the only information we need about the orbit closures Ω_n is that they contain certain padded power sums (cf. Thm. II.8), see also [11]. Several proofs had to be omitted in this extended abstract due to lack of space. We refer to the full version on the arXiv for missing details: arXiv:1604.0643v1.

C. Related work

Let $\tilde{k}_n(\lambda)$ denote the multiplicity by which the irreducible GL_{n^2}-module of type $\lambda \vdash nd$ occurs in $\mathbb{C}[\Omega_n]_d$. We call the numbers $\tilde{k}_n(\lambda)$ the *GCT-coefficients*. Note that an occurrence obstruction for $Z_{n,m} \not\subseteq \Omega_n$ is a partition λ for which $\tilde{k}_n(\lambda) = 0$ and such that λ occurs in $\mathbb{C}[Z_{n,m}]$.

In [32] it was realized that GCT-coefficients can be upper bounded by rectangular Kronecker coefficients: we have $\tilde{k}_n(\lambda) \leq k_n(\lambda)$ for $\lambda \vdash nd$. In fact, the multiplicity of λ in the coordinate ring of the orbit $\mathrm{GL}_{n^2} \cdot \det_n$ equals the so-called symmetric rectangular Kronecker coefficient [12], which is upper bounded by $k_n(\lambda)$.

Since Kronecker coefficients are fundamental quantities that have been the object of study in algebraic combinatorics for a long time [35], much of the the research in geometric complexity has focused on these quantities, with the emphasis on understanding when they vanish for rectangular formats. A difficulty in the study of Kronecker coefficients is that there is no known counting interpretation of them [37].

A first attempt towards finding occurence obstructions was by asymptotic considerations (moment polytopes): this was ruled out in [8], where it was proven that for all $\lambda \vdash nd$ there exists a stretching factor $\ell \geq 1$ such that $k_n(\ell\lambda) > 0$. In [21] it was shown that deciding positivity of Kronecker coefficients in general is NP-hard, but this proof fails for rectangular formats.

Clearly, $\tilde{k}_n(\lambda)$ is bounded from above by the multiplicity $a_\lambda(d[n])$ of the partition λ in the plethysm $\mathsf{Sym}^d\mathsf{Sym}^n\mathbb{C}^{n^2}$. Kumar [24] ruled out asymptotic considerations for the GCT-coefficients: assuming the Alon-Tarsi Conj. [1], Kumar derived that $\tilde{k}_n(n\lambda) > 0$ for all $\lambda \vdash nd$ and even n. A similar conclusion, unconditional, although with less information on the stretching factor, was obtained in [10].

D. Future directions

While our main result, Thm. I.5, rules out the possibility of proving the Conj. I.3 via occurrence obstructions, there still remains the possibility that one may succeed so by comparing multiplicities. If the orbit closure $Z_{n,m}$ of the padded permanent $X_{11}^{n-m}\mathrm{per}_m$ is contained in Ω_n, then the restriction defines a surjective G-equivariant homomorphism $\mathbb{C}[\Omega_n] \to \mathbb{C}[Z_{n,m}]$ of the coordinate rings, and hence the multiplicity of the type λ in $\mathbb{C}[Z_{n,m}]$ is bounded from above by the GCT-coefficient $\tilde{k}_n(\lambda)$. Thus, proving that $\tilde{k}_n(\lambda)$ is strictly smaller than the latter multiplicity implies that $Z_{n,m} \not\subseteq \Omega_n$. We note that the paper [15] rules out one natural asymptotic method for achieving this. Mulmuley pointed out to us a paper by Larsen and Pink [26] that is of potential interest in this connection.

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II. PRELIMINARIES

A. Preliminaries on highest weight vectors

Let $U_N \subseteq \mathrm{GL}_N(\mathbb{C})$ be the subgroup of upper triangular matrices with 1s on the main diagonal. For $\alpha_1, \ldots, \alpha_N \in \mathbb{C}^\times$ let $\mathrm{diag}(\alpha_1, \ldots, \alpha_N)$ denote the diagonal matrix with α_i on the diagonal. In a GL_N-representation $\mathscr{V}$, a *highest weight vector* (or HWV) of weight $\lambda \in \mathbb{Z}^N$ is a vector $f \in \mathscr{V}$ such that $u \cdot f = f$ for all $u \in U_N$ and $\mathrm{diag}(\alpha_1, \ldots, \alpha_N) \cdot f = \alpha_1^{\lambda_1} \cdots \alpha_N^{\lambda_N} f$ for all $\alpha_i \in \mathbb{C}^\times$. Every irreducible $\mathscr{V}$ of type λ contains, up to scaling, a unique HWV of weight λ, and no other HWVs, cf. [16]. In this section we will describe a hands-on construction for HWVs.

Assume $V := \mathbb{C}^N$ and let $\mathfrak{S}_d$ denote the symmetric group on d symbols. The famous Schur-Weyl duality (cf. [16]) states that the following $\mathrm{GL}(V) \times \mathfrak{S}_d$-representations are isomorphic:

$$\bigotimes^d V \simeq \bigoplus_{\lambda \vdash d} \{\lambda\} \otimes [\lambda].$$

Here $\{\lambda\}$ denotes an irreducible $\mathrm{GL}(V)$-representation of type λ (Schur-Weyl module) and $[\lambda]$ denotes an irreducible $\mathfrak{S}_d$-representation of type λ (Specht module). Since every irreducible $\mathrm{GL}(V)$-representation $\{\lambda\}$ contains a unique line of HWVs of weight λ and no HWV of any other weight, it follows that the vector space $\mathsf{HWV}_\lambda(\bigotimes^d V)$ of HWVs of weight λ in $\bigotimes^d V$ is isomorphic to $[\lambda]$ as an $\mathfrak{S}_d$-representation. Thus the $\mathfrak{S}_d$-orbit of a nonzero element $v_\lambda \in \mathsf{HWV}_\lambda(\bigotimes^d V)$ generates $\mathsf{HWV}_\lambda(\bigotimes^d V)$ as a vector space.

We construct such a v_λ as follows. Let μ denote the transpose of λ, so μ_i denotes the number of boxes in the i-th column of λ. For $j \in \mathbb{N}$ we define the antisymmetric product

$$v_{j\times 1} := X_1 \wedge X_2 \wedge \cdots \wedge X_j \in \mathsf{HWV}_{j\times 1}(\textstyle\bigotimes^j V),$$

where $X_1, \ldots, X_N$ are the standard basis vectors of $V = \mathbb{C}^N$. Then we define

$$v_\lambda := \bigotimes_{i=1}^{\lambda_1} v_{\mu_i \times 1} \in \mathsf{HWV}_\lambda(\textstyle\bigotimes^d V). \tag{II.1}$$

So $\{\pi v_\lambda \mid \pi \in \mathfrak{S}_d\}$ generates the vector space $\mathsf{HWV}_\lambda(\bigotimes^d V)$.

B. Plethysms

Again assume $V = \mathbb{C}^N$. We study here the plethysm $\mathsf{Sym}^d\mathsf{Sym}^n V$ and give an explicit construction of a set of generators for its spaces of highest weight vectors (Prop. II.7). For $\lambda \vdash dn$ we define the *plethysm coefficient*

$$a_\lambda(d[n]) := \dim \mathsf{HWV}_\lambda(\mathsf{Sym}^d\mathsf{Sym}^n V), \tag{II.2}$$

which is the multiplicity of $\{\lambda\}$ in $\mathsf{Sym}^d\mathsf{Sym}^n V$. (If $\lambda \vdash dn$ violated, then the multiplicity is zero.)

The position set $[dn] := \{1, \ldots, dn\}$ is partitioned into the blocks $B_1, \ldots, B_d$, where $B_u := \{(u-1)n+v \mid 1 \leq v \leq n\}$. The *wreath product* $\mathfrak{S}_d \wr \mathfrak{S}_n$ is the subgroup of $\mathfrak{S}_{dn}$ generated by the permutations leaving the blocks invariant, and the permutations of the form $(u-1)n+v \mapsto (\pi(u)-1)n+v$ with $\pi \in \mathfrak{S}_d$, which simultaneously permute the blocks. Structurally, the wreath product is a semidirect product $\mathfrak{S}_d \wr \mathfrak{S}_n \simeq (\mathfrak{S}_n)^d \rtimes \mathfrak{S}_d$. Symmetrizing over $\mathfrak{S}_d \wr \mathfrak{S}_n$ yields the projection

$$\Sigma_{d,n} \colon \bigotimes^{dn} V \twoheadrightarrow \mathsf{Sym}^d\mathsf{Sym}^n V, \quad w \mapsto \frac{1}{(n!)^d d!} \sum_{\sigma \in \mathfrak{S}_n \wr \mathfrak{S}_d} \sigma w \tag{II.3}$$

which maps $\mathsf{HWV}_\lambda(\bigotimes^{dn} V)$ to $\mathsf{HWV}_\lambda(\mathsf{Sym}^d\mathsf{Sym}^n V)$, which is thus generated by $\{\Sigma_{d,n}(\pi v_\lambda) \mid \pi \in \mathfrak{S}_{dn}\}$.

The *column-standard Young tableau* T_λ^{std} *of shape* λ is the Young tableau that contains numbers $1, \ldots, |\lambda|$ ordered columnwise, from top to bottom and left to right. For example,

$$T_{(4,2)}^{\mathrm{std}} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

is column-standard. The symmetric group $\mathfrak{S}_{dn}$ acts on the set of numberings of shape λ by replacing each entry i with $\pi(i)$. For example, for $\pi = (2453)$, we obtain

$$\pi T_{(4,2)}^{\mathrm{std}} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}. \tag{II.4}$$

By the definition of v_λ in (II.1), if i and j are in the same column in $\pi T_\lambda^{\mathrm{std}}$, then $\tau\pi v_\lambda = -\pi v_\lambda$ for the transposition $\tau = (i\ j)$. Moreover, if τ is the permutation that switches two columns of the same length in $\pi T_\lambda^{\mathrm{std}}$, then $\tau\pi v_\lambda = \pi v_\lambda$.

In the same way in which the tableau $\pi T_\lambda^{\mathrm{std}}$ provides a short notation for πv_λ, we use a short notation for the projection (II.3) as follows: in $\pi T_\lambda^{\mathrm{std}}$ we replace the numbers in the block $B_1 = \{1, 2, \ldots, n\}$ with a letter, the numbers in the block $B_2 = \{n+1, \ldots, 2n\}$ with a different letter, and so on, until we have replaced every entry with one of d distinct letters. We think of the set of letters as *not* being ordered and hence identify tableaux that arise by permuting the letters. For example, for the tableau in (II.4) we get for $n = 3, d = 2$,

$$\begin{array}{|c|c|c|c|}\hline a & a & a & b \\ \hline b & b \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline b & b & b & a \\ \hline a & a \\ \cline{1-2}\end{array}.$$

We call theses objects *tableaux T of shape λ with rectangular content $d \times n$*. Formally, this is a tableau T together with a partition of the set of its boxes, $\mathrm{boxes}(T) = C_1 \cup \ldots \cup C_d$, such that $|C_u| = n$ for all u (C_u stands for the set of boxes labelled with the letter u). The numbering of the classes C_u is irrelevant.

For a tableau T with rectangular content $d \times n$, we choose $\pi \in \mathfrak{S}_{dn}$ such that T is obtained from $\pi T_\lambda^{\mathrm{std}}$ by the above procedure, and define

$$v_T := \Sigma_{d,n}(\pi v_\lambda) \quad \text{where } T \text{ results from } \pi T_\lambda^{\mathrm{std}}. \qquad \text{(II.5)}$$

This is well defined since $\pi T_\lambda^{\mathrm{std}}$ and $\pi' T_\lambda^{\mathrm{std}}$ define the same T iff $(\mathfrak{S}_n \wr \mathfrak{S}_d) \cdot \pi = (\mathfrak{S}_n \wr \mathfrak{S}_d) \cdot \pi'$. Moreover, $\Sigma_{d,n}(\pi v_\lambda) = \Sigma_{d,n}(\pi' v_\lambda)$ if π and π' are in the same coset.

II.6 Lemma.
1) *Let T be a tableau of shape λ with rectangular content $d \times n$. If the same letter appears in a column of T more than once, then $v_T = 0$.*
2) *Let T and T' be two tableaux of shape λ with rectangular content $d \times n$ that can be obtained from each other by switching two columns that have the same length. Then $v_T = v_{T'}$.*

We will need part two of Lemma II.6 only for columns of length one, which we will call *singleton columns*.

We will focus on tableaux of shape λ with rectangular content $d \times n$, where no letter appears more than once in a column. We call two such tableaux *equivalent* if they differ only by a reordering of their singleton columns and shall call an equivalence class of such tableaux an *(d, n, λ)-tableau class*. For example

$$\begin{array}{|c|c|c|c|}\hline a & a & a & b \\ \hline b & b \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline b & b & b & a \\ \hline a & a \\ \cline{1-2}\end{array} \simeq \begin{array}{|c|c|c|c|}\hline b & b & a & b \\ \hline a & a \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline a & a & b & a \\ \hline b & b \\ \cline{1-2}\end{array}.$$

By Lemma II.6(2), v_T is well-defined for a (d, n, λ)-tableau class T. We obtain with Lemma II.6(1):

II.7 Proposition. *A generating set of $\mathsf{HWV}_\lambda(\mathsf{Sym}^d\mathsf{Sym}^n V)$ is given by the v_T, where T ranges over all (d, n, λ)-tableaux classes.*

In Section III we shall explain how to combinatorially evaluate v_T at tensors of rank one.

C. Padded power sum embedding

Recall from (I.2) that Ω_n denotes the orbit closure of $\det_n$. The following result goes back to Valiant [39].

II.8 Theorem. *For positive integers n, s, d such that $n \geq sd$ we have $X_1^{n-s}(\ell_1^s + \cdots + \ell_d^s) \in \Omega_n$ for arbitrary linear forms $\ell_1, \ldots, \ell_d$ in $X_1, \ldots, X_N$.*

Proof. Writing the power sum $X_1^s + \cdots + X_d^s$ as a formula requires at most $(s-1)d + d - 1 = sd - 1$ many additions and multiplications. Valiant's construction [39] implies that $X_1^s + \cdots + X_d^s$ has the determinantal complexity at most $sd \leq n$, i.e., it can be written as the determinant of an $n \times n$-matrix with affine linear entries in $X_1, \ldots, X_N$. The determinantal complexity is invariant under invertible linear transformations. Hence the determinantal complexity of $\ell_1^s + \cdots + \ell_d^s$ is at most n, for any linearly independent system $\ell_1, \ldots, \ell_d$ of linear forms in $X_1, \ldots, X_N$. By homogenizing with respect to a new variable Y and then substituting Y by X_1, we see that $X_1^{n-s}(\ell_1^s + \cdots + \ell_d^s) \in \Omega_n$. Since Ω_n is closed, we can go over to the limit and drop the assumption that the ℓ_i are linearly independent. $\square$

D. Evaluation at low rank points

In our work it is essential to interpret homogeneous polynomials as symmetric tensors. Let W denote a finite dimensional $\mathbb{C}$-vector space and $d \geq 1$. The symbol $f \in \mathsf{Sym}^d W^*$ denotes a symmetric tensor $f \in \otimes^d W^* \simeq (\otimes^d W)^*$. It defines the polynomial function $f\colon W \to \mathbb{C}$, which is homogeneous of degree d (and we denote it by the same symbol). By the polarization formula, any homogeneous polynomial function $f\colon W \to \mathbb{C}$ of degree d can be obtained from a symmetric tensor.

II.9 Proposition. *Let V be a finite dimensional $\mathbb{C}$-vector space and $d, n \geq 1$. If $f \in \mathsf{Sym}^d\mathsf{Sym}^n V \setminus \{0\}$ then f does not vanish on $\ell_1^n + \cdots + \ell_d^n$, for Zariski almost all $(\ell_1, \ldots, \ell_d) \in V^d$.*

III. Evaluation of highest weight vectors in plethysms

A. Tensor contraction

Recall the definition (II.1) of the highest weight vector $v_\lambda \in \otimes^{dn} V$ for $\lambda \vdash dn$ and $V = \mathbb{C}^N$. Let μ denote the the transpose of λ. Contracting v_λ with a rank one tensor $t = t_1 \otimes \cdots \otimes t_{dn} \in \bigotimes^{dn} V$ yields the following product of determinants:

$$\langle v_\lambda, t\rangle = \prod_{i=1}^{\ell_1} \widetilde{\det}(t_{\mu_1+\ldots+\mu_{i-1}+1}, \ldots, t_{\mu_1+\ldots+\mu_i}),$$

where $\widetilde{\det}(t_{i+1}, \ldots, t_{i+j})$ is the determinant of top $j \times j$ minor of the $N \times j$ matrix whose columns are $t_{i+1}, \ldots, t_{i+j}$. The group $\mathfrak{S}_{dn}$ acts on $\bigotimes^{dn} V$ by permuting the positions, $\pi^{-1}t = t_{\pi(1)} \otimes \cdots \otimes t_{\pi(dn)}$, for $\pi \in \mathfrak{S}_{dn}$. Noting that $\pi(1), \pi(2), \ldots, \pi(\mu_1)$ are the numbers in the first column of $\pi T_\lambda^{\mathrm{std}}$ etc. this implies

$$\begin{aligned} \langle \pi v_\lambda, t\rangle &= \langle v_\lambda, \pi^{-1}t\rangle & \text{(III.1)} \\ &= \prod_{\text{column } c \text{ of } \pi T_\lambda^{\mathrm{std}}} \widetilde{\det}(t_{\text{1st entry in } c}, \ldots, t_{\text{last entry in } c}). \end{aligned}$$

Pictorially, for each k we place the vector t_k into the box of $\pi T_\lambda^{\text{std}}$ with entry k and then take the product over the determinants given by the columns. We can use this pictorial interpretation to understand $\langle v_T, t\rangle$ for a (d,n,λ)-tableau class T as follows.

Recall from Section II (b) the decomposition $[dn] = B_1 \cup \ldots \cup B_d$ into blocks of size n. We say that an assignment $\vartheta\colon \text{boxes}(T) \to [dn]$ *respects the* (d,n,λ)*-tableau class* T if it is bijective and the boxes in T with the same letter get assigned numbers that belong to the same block B_u, i.e., we require $\lceil \vartheta(i,i')/n \rceil = \lceil \vartheta(j,j')/n \rceil$ whenever T has the same letter at the positions (i,i') and (j,j').

Suppose the tableau T with rectangular content $d \times n$ results from $\pi T_\lambda^{\text{std}}$ by substituting the entries in the block B_u by the letter u. Then we get from the definition of v_T in (II.5) and (II.3) that

$$\langle v_T, t\rangle = \langle \Sigma_{d,n}(\pi v_\lambda), t\rangle = \frac{1}{n!^d d!} \sum_{\sigma \in \mathfrak{S}_n \wr \mathfrak{S}_d} \langle \pi v_\lambda, \sigma^{-1} t\rangle.$$

Using (III.1), this can be expressed in the following way:

$$\langle v_T, t\rangle = \frac{1}{n!^d d!} \cdot \sum_{\substack{\vartheta:\text{boxes}(T)\to[dn] \\ \text{respecting } T}} \prod_{\text{column } c \text{ of } T} \widetilde{\det}(t_{\text{1st entry in } c}, \ldots, t_{\text{last entry in } c}). \tag{III.2}$$

To illustrate this, note that the map ϑ assigns to each box in T (the index i of) a vector t_i such that $\{t_1, t_2, \ldots, t_n\}$ are assigned to a letter ℓ_1, $\{t_{n+1}, t_{n+2}, \ldots, t_{2n}\}$ are assigned to a different letter ℓ_2, etc.

B. Combinatorial Contraction

Let $X_1, \ldots, X_N$ denote the standard basis of $V = \mathbb{C}^N$ and $s\colon [dn] \to [N]$ be a map. If $t = X_{s(1)} \otimes \cdots \otimes X_{s(dn)}$, then the contraction (III.2) can be expressed in simpler terms.

We define the *sign* of a list of integers $j = (j_1, \ldots, j_k)$ to be zero if $\{j_1, \ldots, j_k\} \neq \{1, 2, \ldots, k\}$ and otherwise let $\text{sgn}(j_1, \ldots, j_k)$ denote the sign of the permutation $(j_1, \ldots, j_k)$.

Again let T be a tableau of shape λ and let μ denote the partition transposed to λ. An assignment $\vartheta\colon \text{boxes}(T) \to [dn]$ of the boxes of T with integers defines for each column c of T the list $(s(\vartheta(1,c)), \ldots, s(\vartheta(\mu_c, c))$ of integers, whose sign we briefly denote by $\text{sgn}(s \circ \vartheta)_{|c}$. We define the *value* $\text{val}_\vartheta(s)$ of the assignment ϑ at the list s by

$$\text{val}_\vartheta(s) := \prod_{\text{column } c \text{ of } T} \text{sgn}(s \circ \vartheta)_{|c}.$$

In particular, if $\text{val}_\vartheta(s) \neq 0$, then ϑ must assign the value 1 to all boxes in singleton columns of T.

Noting that in (III.2) we take determinants of permutation matrices, we arrive at the following important rule. We shall use it in Section IV to understand the Kadish-Landsberg lifting, and in the proof of Thm. V.4 to show that certain v_T are nonzero.

III.3 Proposition. *Let T be a (d,n,λ)-tableau and $s\colon [dn] \to [N]$ be an assignment. Then*

$$\langle v_T, X_{s(1)} \otimes \ldots \otimes X_{s(dn)}\rangle = \frac{1}{n!^d d!} \sum_{\substack{\vartheta:\text{boxes}(T)\to[dn] \\ \text{respecting } T}} \text{val}_\vartheta(s).$$

III.4 Corollary. *Let n be even and consider the unique $(d,n,d \times n)$-tableau class T in which in each row the letters are the same, but different rows have different letters. The evaluation of $v_T \in \mathsf{Sym}^d\mathsf{Sym}^n V$ at the power sum $w = X_1^n + \ldots + X_d^n$ yields $v_T(w) = d!$.*

Proof. According to Section II (d), the evaluation is done via polarization: $v_T(w) = \langle v_T, w^{\otimes d}\rangle$. If we interpret $X_i^n = X_i^{\otimes n}$, we have

$$w^{\otimes d} = \sum_{i_1,\ldots,i_d} X_{i_1}^{\otimes n} \otimes \ldots \otimes X_{i_d}^{\otimes n},$$

where the sum is over all $1 \le i_1, \ldots, i_d \le d$.

Assume first $(i_1, \ldots, i_d) = (1, \ldots, d)$. We apply Prop. III.3 to compute $\langle v_T, X_1^{\otimes n} \otimes \ldots \otimes X_d^{\otimes n}\rangle$. Using a rowwise enumeration of the boxes of T, the bijections $\vartheta : \text{boxes}(T) \to [dn]$ respecting T are in one to one correspondence with the elements of the wreath product $\mathfrak{S}_d \wr \mathfrak{S}_n$. They are given by permutations $\sigma \in \mathfrak{S}_d$ of the rows and permutations of the numbers within the rows. Such a bijection ϑ contributes $\text{val}_\vartheta(s) = \text{sgn}(\sigma)^n$, where $s = (1^n, \ldots, d^n)$. Hence we obtain

$$\langle v_T, X_1^{\otimes n} \otimes \ldots \otimes X_d^{\otimes n}\rangle = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma)^n.$$

This equals 1 if n is even, and 0 if n is odd and $d > 1$.

We turn now to the contributions of an arbitrary sequence $(i_1, \ldots, i_d)$. If it is a permutation of $1, \ldots, d$, then, by the same argument as before, we see that we get the same contribution as for $(1, \ldots, d)$. On the other hand, if the sequence is not a permutation of $1, \ldots, d$, we get zero. Altogether, we obtain $v_T(X_1^n + \ldots + X_d^n) = d!$ if n is even. $\square$

IV. Lifting highest weight vectors in plethysms

Recall that $Z_{n,m}$ denotes the orbit closure of the padded permanent $X_{11}^{n-m}\text{per}_m \in \mathsf{Sym}^n\mathbb{C}^{n^2}$ for $n > m$. Moreover, $\bar\lambda$ denotes the partition λ with its first row removed. The following insight is due to Kadish and Landsberg [23].

IV.1 Proposition ([23])**.** *If $\lambda \vdash nd$ occurs in $\mathbb{C}[Z_{n,m}]$, then $\ell(\lambda) \le m^2$ and $|\bar\lambda| \le md$.*

In the following we analyze two ways of "lifting" highest weight vectors in $\mathsf{Sym}^d\mathsf{Sym}^m V$ by raising either the *inner degree* m or the *outer degree* d. If d and m are sufficiently large in comparison with μ, then these liftings provide isomorphisms of the spaces of highest weight vectors. In particular, the multiplicity $a_\mu(d[m])$ does not increase, which is known as the stability property of the plethysm coefficients [42], [14], [29]. A detailed understanding of the lifting is crucial for the proof of the main result.

A. Lifting the inner degree

Let $X_1,\dots,X_N$ denote the standard basis of $V=\mathbb{C}^N$ and K be the span of $X_2,\dots,X_N$. Then $\mathbb{C}^N=\mathbb{C}X_1\oplus K$. For $n\ge m$ we have the direct decomposition

$$\mathsf{Sym}^n V = X_1^{n-m}\mathsf{Sym}^m V \oplus \bigoplus_{i=0}^{n-m-1} X_1^i \mathsf{Sym}^{n-i} K \qquad \text{(IV.2)}$$

corresponding to $q=X_1^{n-m}p+r$ with $\deg_{X_1} r < n-m$, and the corresponding projection $\pi\colon \mathsf{Sym}^n V\to \mathsf{Sym}^m V$, $q\mapsto p$.

The space of degree d homogeneous polynomials $(\mathsf{Sym}^m V)^*\to\mathbb{C}$ can be canonically identified with the space $\mathsf{Sym}^d(\mathsf{Sym}^m V)^*$. Thus, composing with the projection $\pi\colon \mathsf{Sym}^n V\to\mathsf{Sym}^m V$ from above gives the following injective linear map

$$\mathsf{Sym}^d(\mathsf{Sym}^m V)^*\to\mathsf{Sym}^d(\mathsf{Sym}^n V)^*,\ f\mapsto f^{\sharp n}:=f\circ\pi, \qquad \text{(IV.3)}$$

which we call the *Kadish-Landsberg lifting*. Using the above notation, we have $f^{\sharp n}(q)=f(p)$. We note that the lifting is canonical in the sense that $(f^{\sharp n})^{\sharp n'}=f^{\sharp n'}$ for $m\le n\le n'$.

To avoid negative weights and to simplify notation, we are going to identify $(\mathsf{Sym}^m V)^*$ with $\mathsf{Sym}^m V$ in a natural way as follows.

The standard hermitian inner product $\langle v,w\rangle:=\sum_j z_j\bar w_j$ on $V:=\mathbb{C}^N$ is invariant under unitary transformations. Note that the standard basis $X_1,\dots,X_N$ of V is orthonormal. The inner product naturally extends to a hermitian inner product on $\otimes^m V$, which is characterized by

$$\langle v_1\otimes\dots v_m, w_1\otimes\dots w_m\rangle=\langle v_1,w_1\rangle\cdots\langle v_m,w_m\rangle.$$

This inner product is also invariant under the action of the unitary group of V. For a list $I=(i_1,\dots,i_m)\in[N]^m$ we define

$$X_I:=X_{I_1}\otimes X_{I_2}\otimes\cdots\otimes X_{I_m}\in\bigotimes{}^m V.$$

These X_I form an orthonormal basis of $\bigotimes^m V$. For $i\in[N]$ let $\zeta(I)_i$ denote the number of appearances of i in I. We call $\zeta(I)\in\mathbb{N}^N$ the *type* of a I. The monomial corresponding to the coefficient vector $\alpha\in\mathbb{N}^N$ with $|\alpha|=m$ is given by

$$X^\alpha=\frac{1}{\binom{m}{\alpha}}\sum_{I:\zeta(I)=\alpha}X_I. \qquad \text{(IV.4)}$$

Since the X_I are pairwise orthonormal, we obtain

$$\|X^\alpha\|^2=\frac{1}{\binom{m}{\alpha}^2}\sum_{I:\zeta(I=\alpha}\|X_I\|^2=\frac{1}{\binom{m}{\alpha}}.$$

Moreover, it is easy to check that different monomials are orthogonal. The restriction of the inner product $\langle\ ,\ \rangle$ to $\mathsf{Sym}^m V$ is sometimes called Weyl's inner product, cf. [9, §16.1].

We are going to identify $\mathsf{Sym}^m V$ with its dual space with respect to Weyl's inner product. More specifically, the identification isomorphism is given by

$$\mathsf{Sym}^m V\to(\mathsf{Sym}^m V)^*,\ X^\alpha\mapsto\binom{m}{\alpha}(X^*)^\alpha, \qquad \text{(IV.5)}$$

where $X_1^*,\dots,X_N^*$ denotes the dual basis of $X_1,\dots,X_N$ and $(X^*)^\alpha:=(X_1^*)^{\alpha_1}\cdots(X_N^*)^{\alpha_N}$. The justification for this is provided by the fact that $\left\langle\binom{m}{\alpha}(X^*)^\alpha,X^\beta\right\rangle=\delta_{\alpha,\beta}$.

After composing the lifting (IV.3) with the identification maps (IV.5) for $\mathsf{Sym}^m V$ and $\mathsf{Sym}^n V$, we obtain another lifting map

$$\mathsf{Sym}^d\mathsf{Sym}^m V\to\mathsf{Sym}^d\mathsf{Sym}^n V, \qquad \text{(IV.6)}$$

which we also denote by $f\mapsto f^{\sharp n}$.

We interpret now the effect of this map in terms of tableaux. Recall the definition of the highest weight vectors v_T from (II.5). We denote $(k):=1\times k$.

IV.7 Lemma. 1) *For $\mu\vdash md$, $n\ge m$, $\lambda:=\mu+(d(n-m))$, the lifting* (IV.6) *induces injective linear maps*

$$\mathsf{HWV}_\mu(\mathsf{Sym}^d\mathsf{Sym}^m V)\to\mathsf{HWV}_\lambda(\mathsf{Sym}^d\mathsf{Sym}^n V),$$

and we have $f^{\sharp n}(X_1^{n-m}\cdot p)=f(p)$ for $p\in\mathsf{Sym}^m V$.

2) *Let T be a (d,m,μ)-tableau class and the tableau class T' be obtained from T by adding $n-m$ copies of each of the d letters in the first row (in any order). Then $v_{T'}(X_1^{n-m}p)=v_T(p)$ for all $p\in\mathsf{Sym}^m V$.*

Proof. (1) Note that the identification isomorphism (IV.5) maps highest weight vectors of weight μ to highest weight vectors of weight $-\mu$. Therefore, it suffices to prove that if $f\in\mathsf{Sym}^d(\mathsf{Sym}^m V)^*$ is a highest weight vector of weight $-\mu$, then its image $f^{\sharp n}\in\mathsf{Sym}^d(\mathsf{Sym}^n V)^*$ under (IV.6) is a highest weight vector of weight $-\mu-(d(n-m))$. This is a straightforward verification that we leave to the reader.

(2) For the second assertion, we note that the lifting from inner degree m to n can be obtained as a composition of liftings that increase the inner degree by one only. Hence we may assume without loss of generality that $n=m+1$.

We need to prove that for all $p\in\mathsf{Sym}^m V$,

$$\langle v_{T'},(X_1p)^{\otimes d}\rangle=\langle v_T,p^{\otimes d}\rangle. \qquad \text{(IV.8)}$$

Let $C_N(m)$ denote the set of all $\beta\in\mathbb{N}^N$ with $|\beta|=m$. A symmetric tensor $p\in\mathsf{Sym}^m V$ can naturally be expanded in the monomial basis: $p=\sum_{\beta\in C_N(m)}c_\beta X^\beta$, cf. (IV.4). The image of p under the isomorphism (IV.5) is given by

$$p^*=\sum_{\beta\in C_N(m)}c_\beta\binom{m}{\beta}(X^*)^\beta=\sum_{\beta\in C_N(m)}c_\beta\sum_{I:\zeta(I)=\beta}X_I^*. \qquad \text{(IV.9)}$$

We take the dth tensor power and expand:

$$(p^*)^{\otimes d}=\sum_{\alpha:[d]\to C_N(m)}C_\alpha\sum_{I^1,\dots I^d:\zeta(I^k)=\alpha(k)}X_{I^1}^*\otimes\cdots\otimes X_{I^d}^*,$$

where $C_\alpha:=\prod_{k=1}^d c_{\alpha(k)}$. Therefore, using Prop. III.3, we see that $\langle v_T,(p^*)^{\otimes d}\rangle$ equals

$$\sum_{\alpha:[d]\to C_N(m)}\sum_{\zeta(I^k)=\alpha(k)}C_\alpha\langle v_T,X_{I^1}\otimes\cdots\otimes X_{I^d}\rangle$$

$$=\sum_{\alpha:[d]\to C_N(m)}C_\alpha\frac{1}{d!(m!)^d}\sum_{\zeta(I^k)=\alpha(k)}\sum_\vartheta \mathrm{val}_\vartheta(I^1,\dots,I^d), \qquad \text{(IV.10)}$$

where the last sum is over all assignments ϑ : boxes$(T) \to [dm]$ respecting T.

In the following we write $\beta + e_1 := \beta + (1, 0, \dots, 0)$. Multiplying p with X_1 yields $X_1 p = \sum_{\beta \in C_N(m)} c_\beta X^{\beta+e_1}$, hence, after applying (IV.5),

$$(X_1 p)^* = \sum_{\beta \in C_N(m)} c_\beta \binom{m}{\beta + e_1} (X^*)^{\beta+e_1} = \sum_{\beta \in C_N(m)} c_\beta \sum_{J:\zeta(J)=\beta+e_1} X_J^*.$$

Taking the dth tensor power and expanding yields

$$\big((X_1p)^*\big)^{\otimes d} = \sum_{\alpha:[d]\to C_N(m)} \sum_{\substack{J^1,\dots,J^d \\ \zeta(J^k)=\alpha(k)+e_1}} C_\alpha X_{J^1}^* \otimes \cdots \otimes X_{J^d}^*.$$

Using Prop. III.3, we see that $v_{T'}, (X_1 p)^{\otimes d}\rangle$ equals

$$\sum_{\alpha:[d]\to C_N(m)} C_\alpha \sum_{\zeta(J^k)=\alpha(k)+e_1} \langle v_{T'}, X_{J^1} \otimes \cdots \otimes X_{J^d}\rangle = \sum_{\alpha:[d]\to C_N(m)} \frac{C_\alpha}{d!((m+1)!)^d} \sum_{\zeta(J^k)=\alpha(k)+e_1} \sum_{\vartheta'} \mathrm{val}_{\vartheta'}(\underline{J}),$$

where we abbreviated $\underline{J} := (J^1, \dots, J^d)$ and the last sum is over all assignments ϑ': boxes$(T') \to [d(m+1)]$ respecting T'.

The following claim, together with (IV.10) and the above, implies that $\langle v_{T'}, (X_1 p)^{\otimes d}\rangle = \langle v_T, p^{\otimes d}\rangle$, which is the asserted equality.

Claim. We have for all α: $[d] \to C_N(m)$,

$$\sum_{\zeta(J^k)=\alpha(k)+e_1} \sum_{\vartheta'} \mathrm{val}_{\vartheta'}(J^1, \dots, J^d) = (m+1)^d \sum_{\zeta(I^k)=\alpha(k)} \sum_{\vartheta} \mathrm{val}_{\vartheta}(I^1, \dots, I^d).$$

It remains to show the claim. Suppose we have an asssignment ϑ': boxes$(T') \to [d(m+1)]$ respecting T', and a tuple $(J^1, \dots, J^d)$ of lists in $[N]^m$ such that $\mathrm{val}_{\vartheta'}(J^1, \dots, J^d) \neq 0$. The d singleton boxes of T' that have been added to T are mapped to different blocks by ϑ'. Moreover, those boxes are mapped to positions that are assigned the value 1 by $(J^1, \dots, J^d)$. Now we remove these positions to obtain a new tuple $(I^1, \dots, I^d)$ of lists. This process is best explained by an example. For instance, suppose that $d = 2, m = 3$, and $J^1 = (2, 1, 1, 2)$, $J^2 = (1, 1, 2, 2)$. Let $\square_1$ and $\square_2$ denote the singleton boxes added to T. Assume that $\vartheta'(\square_1) = 2$ and $\vartheta'(\square_2) = 5$. This means that $\square_1$ is mapped to the second position of the block $B_1 = \{1, 2, 3, 4\}$ and $\square_2$ is mapped to the first position of the block $B_2 = \{5, 6, 7, 8\}$. In both positions, J^1 and J^2 provide the value 1. After removing the two positions we get the lists $I^1 = (2, 1, 2)$, $J^2 = (1, 2, 2)$.

By adjusting for the positions shifts, we obtain from ϑ' an assignment ϑ: boxes$(T') \to [dm]$ that clearly respects T. A moment's thought shows that the values are preserved: $\mathrm{val}_{\vartheta'}(J^1, \dots, J^d) = \mathrm{val}_{\vartheta}(I^1, \dots, I^d)$. In addition, if $\zeta(J^k) = \alpha(k) + e_1$, then $\zeta(I^k) = \alpha(k)$.

Summarizing, we have set up a map from the set of pairs $(\vartheta', (J^1, \dots, J^d))$, such that ϑ' respects T', $\mathrm{val}_{\vartheta'}(J^1, \dots, J^d) \neq 0$, and $\zeta(J^k) = \alpha(k) + e_1$ for all k, to the set of pairs $(\vartheta, (I^1, \dots, I^d))$, such that ϑ respects T, $\mathrm{val}_\vartheta(I^1, \dots, I^d) \neq 0$, and $\zeta(I^k) = \alpha(k)$ for all k.

It is easily checked that that all the fibers of this map have the cardinality $(m + 1)^d$. The reason is that in each of the d blocks, we have $m + 1$ possibilities to insert a new position, which is then assigned the value 1 under the J^k to be defined.

This completes the proof of the claim. □

IV.11 Proposition. *Suppose $\mu \vdash md$ such that $\mu_2 \le m$ and let $n \ge m$.*

1) *We have $a_\mu(d[m]) = a_{\mu+(d(n-m))}(d[n])$ and thus the lifting in Lemma IV.7(1) is an isomorphism.*
2) *Suppose that $\lambda \vdash nd$ satisfies $\lambda_2 \le m$ and $\lambda_2 + |\bar\lambda| \le md$. Then every highest weight vector F of weight λ in $\mathsf{Sym}^d\mathsf{Sym}^n V$ is obtained by lifting a highest weight vector f in $\mathsf{Sym}^d\mathsf{Sym}^m V$ of weight $\mu \vdash md$ such that $\bar\mu = \bar\lambda$.*

Proof. (1) This is shown in [42, Cor. 1.8] or [14].

(2) Note that $\lambda_2 + |\bar\lambda| \le md$ is the number of boxes of λ that appear in columns of that are not singleton columns. We can therefore shorten the given λ to a partition $\mu \vdash md$ by removing singleton columns. Then $\bar\mu = \bar\lambda$ and $\lambda = \mu + (d(n - m))$ and we conclude with part one. □

B. Lifting the outer degree

We keep the notation $V = \mathbb{C}^N$ with the standard basis $X_1, \dots, X_N$ from the previous section. If we view an element $p = cX_1^m + \cdots$ in $\mathsf{Sym}^m V$ as a polynomial in $X_1, \dots, X_N$, then $\langle X_1^{\otimes m}, p\rangle = c$ extracts the *leading coefficient* of p. For this reason, we shall write $\mathrm{lc} := X_1^{\otimes m} \in \mathsf{Sym}^m V$. We note that lc is a highest weight vector with the weight (m). We claim that for $v_1, \dots, v_m \in V$

$$\langle \mathrm{lc}, v_1 \otimes \ldots \otimes v_m\rangle = \frac{1}{m!} \sum_{\pi \in \mathfrak{S}_m} \langle X_1, v_{\pi(1)}\rangle \cdots \langle X_1, v_{\pi(m)}\rangle. \tag{IV.12}$$

Indeed, with the symmetrizer $\Pi_m := \frac{1}{m!} \sum_{\pi\in\mathfrak{S}_m} \pi$ defining the projection Π_m: $\otimes^m V \to \mathsf{Sym}^m V$, we have

$$\langle \mathrm{lc}, v_1 \otimes \ldots \otimes v_m\rangle = \langle \mathrm{lc}, \Pi_m(v_1 \otimes \ldots \otimes v_m)\rangle = \frac{1}{m!} \sum_{\pi\in\mathfrak{S}_m} \langle X_1^{\otimes m}, v_{\pi(1)} \otimes \ldots \otimes v_{\pi(m)}\rangle,$$

since the kernel of Π_m is orthogonal to $\mathsf{Sym}^m V$.

For a vector space W, the *symmetric product* (cf. [36])

$$\mathsf{Sym}^a W \times \mathsf{Sym}^b W \to \mathsf{Sym}^{a+b} W,\ (p, q) \mapsto p{\cdot}q := \Pi_{a+b}(p \otimes q)$$

is defined by symmetrizing $p \otimes q$. (This is just the usual multiplication in the polynomial ring.)

For $k \le d$ we consider the injective linear map

$$\mathsf{Sym}^k\mathsf{Sym}^m V \to \mathsf{Sym}^d\mathsf{Sym}^m V,\ g \mapsto \mathrm{lc}^{d-k} \cdot g, \tag{IV.13}$$

where the multiplication refers to the symmetric product (with

$W = \mathsf{Sym}^m V$).

IV.14 Lemma. 1) *For $\nu \vdash mk$, $d \geq k$, $\mu := \nu + ((d-k)m)$, the map* (IV.13) *induces an injective linear map*

$$\mathsf{HWV}_\nu(\mathsf{Sym}^k\mathsf{Sym}^m V) \to \mathsf{HWV}_\mu(\mathsf{Sym}^d\mathsf{Sym}^m V).$$

2) *Let T be a (k,m,ν)-tableau class and the tableau class T'' be obtained from T by adding m copies of $d-k$ new letters to the first row (in any order). Then $v_{T''}$ is obtained as the image of v_T under above map.*

IV.15 Proposition. *Suppose $\nu \vdash mk$ such that $\nu_2 \leq k$ and let $d \geq k$.*

1) *Put $\mu := \nu + ((d-k)m)$. Then any (d,m,μ)-tableau class T'' is obtained from a (k,m,ν)-tableau class T as in Lemma IV.14(2).*
2) *The lifting in Lemma IV.14(1) is an isomorphism and thus $a_\nu(k[m]) = a_{\nu+((d-k)m)}(d[m])$.*
3) *Suppose that $\mu \vdash md$ satisfies $\mu_2 + |\bar{\mu}| \leq k$. Then every highest weight vector f in $\mathsf{HWV}_\mu(\mathsf{Sym}^d\mathsf{Sym}^m V)$ is of the form $f = \mathrm{lc}^{d-k} \cdot g$ for some $g \in \mathsf{HWV}_\nu(\mathsf{Sym}^k\mathsf{Sym}^m V)$ with $\nu \vdash mk$ and $\bar{\nu} = \bar{\mu}$.*

Proof. (1) Let T'' be a (d,m,μ) tableau class. Since a letter can occur at most once in a column of T'', there are at least $d-\mu_2 = d-\nu_2 \geq d-k$ many letters in T'' that appear in singleton columns only. Let T denote the tableau class obtained by removing all m occurrences of of these $d-k$ letters from the singleton columns. Then T has the shape ν and T'' is obtained from T as in Lemma IV.14(2).

(2) This follows by combining part one with Lemma IV.14(2) and Prop. II.7.

(3) Since $\mu_2 + |\bar{\mu}| \leq k$ is the number of boxes of μ that are not singleton columns, we can shorten μ to a partition $\nu \vdash km$ by removing singleton columns. Then $\bar{\nu} = \bar{\lambda}$ and $\mu = \nu + ((d-k)m)$. We conclude with part one. ☐

We remark that the stability of plethysm in Prop. IV.15(1) was first shown in [29] with a geometric method.

V. Proof of Thm. I.5

A. Small degrees or extremely long first rows

To warm up, we first show that (n) occurs in $\mathbb{C}[\Omega_n]_1$. Indeed, the highest weight vector $\mathrm{lc} \in \mathsf{Sym}^1\mathsf{Sym}\mathbb{C}^{n^2}$ of weight (n) from Section IV (b) satisfies $\mathrm{lc}(X_{11}^n) = 1$ (where X_{11} is the first variable), and clearly $X_{11}^n \in \Omega_n$.

The following result deals with the case of of partitions $\lambda \vdash nd$ where d is small.

V.1 Proposition. *Let $\lambda \vdash nd$ such that there exists a positive integer m satisfying $|\bar{\lambda}| \leq md$ and $md^2 \leq n$. Then every $F \in \mathsf{HWV}_\lambda(\mathsf{Sym}^d\mathsf{Sym}^n V)$ does not vanish on Ω_n. In particular, if λ occurs in $\mathsf{Sym}^d\mathsf{Sym}^n V$, then λ occurs in $\mathbb{C}[\Omega_n]_d$.*

Proof. The case $d=1$ is trivial as (n) occurs in $\mathbb{C}[\Omega_n]_1$, as noted before. Suppose $d \geq 2$. We consider the lifting map $\mathsf{Sym}^d\mathsf{Sym}^{md}V \to \mathsf{Sym}^d\mathsf{Sym}^n V$, cf. (IV.6). Since $\lambda_2 \leq |\bar{\lambda}| \leq md$ and $\lambda_2 + |\bar{\lambda}| \leq 2|\bar{\lambda}| \leq 2md \leq md \cdot d$, the assumptions of Prop. IV.11(2) are satisfied and we conclude that $F = f^{\sharp n}$ for some $f \in \mathsf{HWV}_\mu(\mathsf{Sym}^d\mathsf{Sym}^{md}V)$. Moreover, $f(p) = F(X_1^{n-md}p)$ for all $p \in \mathsf{Sym}^{md}V$. By Proposition II.9, f does not vanish on $p := \ell_1^{md} + \cdots + \ell_d^{md}$ for some linear forms $\ell_1, \ldots, \ell_d$. Hence F does not vanish on $X_1^{n-md}p$. By Thm. II.8 we have $X_1^{n-md}p \in \Omega_n$, since $n \geq md \cdot d$. Therefore, F does not vanish on Ω_n. ☐

The next result deals with the extreme situation, where the body $\bar{\lambda}$ is very small compared with the size of λ. (In this situation, the splitting strategy in the proof of Prop. V.5 below would fail.)

V.2 Proposition. *Let $\lambda \vdash nd$ and assume there exist positive integers s, m such that $\ell(\lambda) \leq m^2$, $\lambda_2 \leq s$, $m^2s^2 \leq n$, and $m^2 s \leq d$. Then every $F \in \mathsf{HWV}_\lambda(\mathsf{Sym}^d\mathsf{Sym}^n V)$ of weight λ does not vanish on Ω_n.*

Proof. We first consider the inner degree lifting $\mathsf{Sym}^d\mathsf{Sym}^s V \to \mathsf{Sym}^d\mathsf{Sym}^n V$. Since $\lambda_2 \leq s$ and

$$\lambda_2 + |\bar{\lambda}| \leq \lambda_2 + (\ell(\lambda)-1)\lambda_2 = \ell(\lambda)\lambda_2 \leq m^2 s \leq d \leq ds,$$

the assumptions of Prop. IV.11(2) are satisfied and we conclude that $F = f^{\sharp n}$ for some $f \in \mathsf{HWV}_\mu(\mathsf{Sym}^d\mathsf{Sym}^s V)$ with $\mu \vdash ds$ and $\bar{\mu} = \bar{\lambda}$.

By assumption, $k := m^2 s \leq d$. We continue with the outer degree lifting map $\mathsf{Sym}^k\mathsf{Sym}^s V \to \mathsf{Sym}^d\mathsf{Sym}^s V$, see (IV.13). We have, using the above,

$$\mu_2 + |\bar{\mu}| = \lambda_2 + |\bar{\lambda}| \leq m^2 s = k,$$

hence the assumptions of Prop. IV.15(2) are satisfied and we have $f = \mathrm{lc}^{d-k} \cdot g$ for some highest weight vector $g \in \mathsf{Sym}^k\mathsf{Sym}^s V$ of weight $\nu \vdash ks$ such that $\bar{\nu} = \bar{\mu}$.

By Proposition II.9 we have $g(\ell_1^s + \cdots + \ell_k^s) \neq 0$ for generic $\ell_1, \ldots, \ell_k \in V$. Moreover, $\mathrm{lc}(\ell_1^s + \cdots + \ell_k^s) \neq 0$ for generic ℓ_i. Using $f = \mathrm{lc}^{d-k} \cdot g$, we see that $f(\ell_1^s + \cdots + \ell_k^s) \neq 0$ for generic ℓ_i. Since $F = f^{\sharp n}$, we have

$$F(X_1^{n-s}(\ell_1^s + \ell_2^s + \cdots + \ell_k^s)) = f(\ell_1^s + \ell_2^s + \cdots + \ell_k^s) \neq 0.$$

On the other hand, by Thm. II.8, the padded polynomial $X_1^{n-s}(\ell_1^s + \ell_2^s + \cdots + \ell_k^s)$ is contained in Ω_n, as $n \geq sk$ by assumption. ☐

B. Building blocks and splitting technique

We construct as "building blocks" certain partitions that occur in $\mathbb{C}[\Omega_n]$. Due to Thm. II.8 and Lemma **??** this boils down to prove that certain plethyms coefficients are nonzero. We achieve this by providing explicit tableaux constructions and showing that the corresponding highest weight vectors do not vanish on a certain tensor.

In a first step we consider the case of even row length. Let $\lambda^{\sharp M}$ denote the partition $\lambda + (M - |\lambda|)$, which is λ with a prolonged first row so that $\lambda^{\sharp M}$ has M boxes.

V.3 Proposition. *Let N be even and $n \geq Nd$. Then $(d \times N)^{\sharp nd}$ occurs in $\mathbb{C}[\Omega_n]_d$.*

Proof. We use the construction from [7]. Consider the unique

$(d, N, d \times N)$-tableau class T in which each letter appears in only a single row and each row contains N coinciding letters. Corollary (III.4) states that $v_T(X_1^N + \cdots + X_d^N)$ is nonzero since N is even. By lifting $f := v_T \in \mathsf{Sym}^d\mathsf{Sym}^N V$ (see Lemma IV.7), we get the HWV $f^{\sharp n} \in \mathsf{Sym}^d\mathsf{Sym}^n V$ of weight $(d \times N)^{\sharp nd}$, which does not vanish on the padded power sum $X_1^{n-N}(X_1^N + \cdots + X_d^N)$. The latter is contained in Ω_n by Thm. II.8, since $n \geq dN$. Therefore, $f^{\sharp n}$ does not vanish on Ω_n. □

In order to handle partitions with odd parts, we use as further building blocks partitions obtained from rectangles by adding a single row and a single column.

The proof of the following technical result is based on explicit constructions of highest weight vectors and omitted for lack of space.

V.4 Theorem. *Let $2 \leq b, c \leq m^2$ and let $n \geq 24m^6$. Then there exists an even $i \leq 2m^4$, such that*

$$\lambda = b \times 1 + c \times i + 1 \times j$$

occurs in $\mathbb{C}[\Omega_n]_{3m^4}$ for $j = 3m^4 n - b - ic$.

The splitting strategy in the following proof is a refinement of the one in [22].

V.5 Proposition. *Given a partition λ with $|\lambda| = nd$ such that there exists $m \geq 2$ with $\ell(\lambda) \leq m^2$, $m^{10} \leq |\bar{\lambda}| \leq md$, $n \geq 24m^6$, and $d > 4m^6$. Then λ occurs in $\mathbb{C}[\Omega_n]_d$.*

Proof. Let $L := \ell(\lambda)$ and c_k denote the number of columns of length k in λ for $1 \leq k \leq L$. Let K be the index $k \geq 2$, for which c_k is maximal, i.e. $c_K = \max(c_k; k = 2, \ldots, L)$. By assumption, we have $2 \leq K \leq m^2$ and

$$m^{10} \leq |\bar{\lambda}| = \sum_{k=2}^{L}(k-1)c_k \leq c_K \sum_{k=2}^{L}(k-1) \leq c_K \frac{L^2}{2} \leq c_K \frac{m^4}{2},$$

hence $c_K \geq 2m^6$.

The columns of odd length of λ need a special treatment: let S denote the set of integers $k \in \{2, \ldots, L\}$ for which c_k is odd. For $k \in S$ we define the partition $\omega_k := k \times 1 + K \times i_k$, where the even integer $i_k \leq 2m^4$ is taken from Thm. V.4, so that $\omega_k^{\sharp 3nm^4}$ occurs in $\mathbb{C}[\Omega_n]_{3m^4}$. (Here we have used the assumption $n \geq 24m^6$.)

Assume first that $K \notin S$, that is, c_K is even. Then we can split λ vertically in rectangles as follows:

$$\lambda = 1 \times c_1 + \sum_{\substack{k=2 \\ k \notin S \cup \{K\}}}^{L} k \times c_k + \sum_{\substack{k=2 \\ k \in S}}^{L} k \times c_k + K \times c_K$$

$$= 1 \times c_1 + \sum_{\substack{k=2 \\ k \notin S \cup \{K\}}}^{L} k \times c_k + \sum_{\substack{k=2 \\ k \in S}}^{L} k \times (c_k - 1) + \sum_{k \in S} \omega_k + K \times \left(c_K - \sum_{k \in S} i_k\right)$$

If, for $k \leq L$, we set $d_k := c_k$ if $k \notin S \cup \{K\}$ and $d_k := c_k - 1$ if $k \in S$, and define $d_K := c_K - \sum_{k \in S} i_k$, then the above can be briefly written as

$$\lambda = 1 \times c_1 + \sum_{k=2}^{L} k \times d_k + \sum_{k \in S} \omega_k. \tag{V.6}$$

By construction, all d_k are even. It is crucial to note that

$$d_K = c_K - \sum_{k \in S} i_k \geq c_K - (m^2 - 1) \cdot 2m^4 \geq c_K - 2m^6 \geq 0,$$

where we have used $i_k \leq 2m^4$ for the first inequality.

In the case where $K \in S$, we achieve the same decomposition as in (V.6) with the modified definition $d_K := c_K - 1 - \sum_{k \in S} i_k$. Here, as well $d_K \geq 0$ and all d_k are even.

We need to round down rational numbers to the next even number, so for $a \in \mathbb{Q}$ we define $\llbracket a \rrbracket := 2\lfloor a/2 \rfloor$. Note that $\llbracket a \rrbracket \geq a - 2$ for all $a \in \mathbb{Q}$. Hence $\llbracket n/k \rrbracket \geq n/k - 2 \geq 2$ for all $2 \leq k \leq m^2$, since $n \geq 4m^2$.

Using division with remainder, let us write $d_k = q_k \llbracket \frac{n}{k} \rrbracket + r_k$ with $0 \leq r_k < \llbracket \frac{n}{k} \rrbracket$. Then we split $k \times d_k = q_k(k \times \llbracket \frac{n}{k} \rrbracket) + k \times r_k$. Since d_k is even and $\llbracket n/k \rrbracket$ is even, r_k is even as well. From (V.6) we obtain that the partition

$$\mu := \sum_{k=2}^{L} q_k(k \times \llbracket n/k \rrbracket)^{\sharp nk} + \sum_{k=2}^{L}(k \times r_k)^{\sharp nk} + \sum_{k \in S} \omega_k^{\sharp 3nm^4} \tag{V.7}$$

coincides with λ in all but possibly the first row.

Since $\llbracket n/k \rrbracket \leq n/k$, $r_k \leq n/k$, and both $\llbracket n/k \rrbracket$ and r_k are even, Prop. V.3 implies that $(k \times \llbracket n/k \rrbracket)^{\sharp nk}$ and $(k \times r_k)^{\sharp nk}$ occur as highest weights in $\mathbb{C}[\Omega_n]_k$. Moreover, Thm. V.4 tells us that $\omega_k^{\sharp 3nm^4}$ occurs as a highest weight in $\mathbb{C}[\Omega_n]_{3m^4}$. The semigroup property implies that μ occurs in $\mathbb{C}[\Omega_n]$.

Claim. $|\mu| \leq dn$.

Let us finish the proof assuming the claim. If $|\mu| \leq dn$, we can obtain λ from μ by adding boxes to the first row of μ. Note that $|\lambda| - |\mu|$ is a multiple of n. Since $(n) \in \mathbb{C}[\Omega_n]$, the semigroup property implies that λ occurs in $\mathbb{C}[\Omega_n]_d$.

It remains to verify the claim. From (V.7) we get

$$|\mu| \;\leq\; \sum_{k=2}^{L}(q_k nk + nk + 3nm^4).$$

Noting $\llbracket a \rrbracket \geq a - 2$ we get $q_k \leq \frac{d_k}{\llbracket n/k \rrbracket} \leq \frac{kd_k}{n-2k}$. This implies

$$|\mu| \;\leq\; n \sum_{k=2}^{L} \Big(\frac{k^2 d_k}{n - 2k} + k + 3m^4\Big).$$

Using $d_k \leq c_k$ and $L \leq m^2$, we get

$$|\mu| \;\leq\; n \sum_{k=2}^{L} \frac{m^2}{n - 2m^2} k c_k + n \sum_{k=2}^{m^2} k + 3nm^4(m^2 - 1).$$

Noting that $\sum_{k=2}^{L} k c_k = |\bar{\lambda}| + \lambda_2 \leq 2|\bar{\lambda}|$, we continue with

$$\begin{aligned} |\mu| \;&\leq\; \frac{nm^2}{n - 2m^2} \cdot 2|\bar{\lambda}| + n\Big(\frac{m^2(m^2+1)}{2} + 3m^4(m^2 - 1)\Big) \\ &\leq\; \frac{nm^2}{12m^6 - m^2} \cdot |\bar{\lambda}| + n\Big(3m^6 - \frac{5}{2}m^4 + \frac{1}{2}m^2\Big), \end{aligned}$$

where we have used $n > 24m^6$ for the second inequality.

Plugging in the assumptions $|\bar{\lambda}| \leq dm$ and $d > 4m^6$, we obtain

$$|\mu| \ \leq \ \frac{dnm^3}{11m^6} + 3nm^6 \leq \frac{dn}{11} + 3nm^6 \leq \frac{dn}{11} + \frac{3dn}{4} < dn,$$

which shows the claim and completes the proof. □

We can now complete the proof of our main result.

Proof of Thm. I.5. We may assume that $m \geq 2$, as the case $m = 1$ is trivial. Suppose that $\lambda \vdash nd$ occurs in $\mathbb{C}[Z_{n,m}]$ and $n \geq m^{25}$. Prop. IV.1 implies that $|\bar{\lambda}| \leq md$ and $\ell(\lambda) \leq m^2$.

In the case of "small degree", where $n \geq md^2$, Prop. V.1 implies that λ occurs in $\mathbb{C}[\Omega_n]$.

So we may assume that $d > \sqrt{n/m}$. In this case we have $d \geq \sqrt{m^{25}/m} = m^{12}$. We conclude by two further case distinctions.

If $|\bar{\lambda}| < m^{10}$, we can apply Prop. V.2 with $s := m^{10}$ since $\lambda_2 \leq |\bar{\lambda}| \leq s$, $m^2 s^2 = m^{22} \leq n$, and $m^2 s = m^{12} \leq d$. Thus λ occurs in $\mathbb{C}[\Omega_n]_d$.

Finally, if $|\bar{\lambda}| \geq m^{10}$, then the above Prop. V.5 tells us that λ occurs in $\mathbb{C}[\Omega_n]_d$. □

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