The Sub-exponential Upper Bound for On-line Chain Partitioning

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***Abstract*—The main question in the on-line chain partitioning problem is to determine whether there exists an algorithm that partitions on-line posets of width at most w into polynomial number of chains – see Trotter's chapter *Partially ordered sets* in the *Handbook of Combinatorics*. So far the best known on-line algorithm of Kierstead used at most $(5^w - 1)/4$ chains; on the other hand Szemerédi proved that any on-line algorithm requires at least $\binom{w+1}{2}$ chains. These results were obtained in the early eighties and since then no progress in the general case has been done.**

We provide an on-line algorithm that partitions orders of width w into at most $w^{16 \log w}$ chains. This yields the first sub-exponential upper bound for on-line chain partitioning problem.

I. Introduction

We define the on-line chain partitioning problem in terms of the game between two players: Spoiler and Algorithm. The game is played in rounds. In each single round of the game Spoiler introduces a new point x to a poset and establishes comparabilities of x with points presented so far; the relation between x and older points, now established, can not be changed in the future. The Algorithm's task is to maintain a chain partition of the growing order: when Spoiler presents the new point x Algorithm responds either by incorporating x to an existing chain or by creating a new chain for x itself. The objectives of Algorithm and Spoiler are opposed: Spoiler builds a poset so as to force Algorithm to use as many chains as possible; Algorithm tries to stay with the fewest number of chains.

The value $\text{val}(w)$ of the game on orders of width w is the least integer n such that Algorithm has a strategy to partition any growing order of width w into at most n chains. It may be verified $\text{val}(w)$ is also the largest n for which Spoiler has a strategy to force any algorithm to use n chains on an order of width w. Note that Dilworth's Theorem asserts each order of width w may be partitioned, in the off-line way, into w chains. Therefore the value of the parameter $\text{val}(w)$ estimates how good on-line solutions may be obtained by on-line algorithms in comparison to the best off-line solution. The game presented on Fig. 1 shows that the luck of knowledge on future points results on-line solutions are not optimal.

The first question we should ask is whether the value $\text{val}(w)$ is bounded? Kierstead [Kie81] was the first one who answered this question in the affirmative way. He showed a strategy for Algorithm that uses exponentially many chains.

Theorem 1.1: $\text{val}(w) \leq (5^w - 1)/4$.

A good outline of the proof of the theorem one may find in Trotter's chapter [Tro95] in the *Handbook of Combinatorics*.

There are also known some lower bounds on $\text{val}(w)$. Kierstead [Kie81] proved that $\text{val}(w) \geq 4w - 3$, Szemerédi (published in [Kie86]) showed $\text{val}(w) \geq \binom{w+1}{2}$. Very recently, the quadratic lower bound was improved by a constant by Bosek at al. [BKK+10]. The current record is

Theorem 1.2: $\text{val}(w) \geq (2 - o(1)) \binom{w+1}{2}$.

The main question in the on-line chain partitioning problem is whether $\text{val}(w)$ is bounded by a polynomial of w? In this paper we present a strategy that uses $w^{16 \log w}$ many chains. This yields the first sub-exponential upper bound on $\text{val}(w)$.

Theorem 1.3: $\text{val}(w) \leq w^{16 \log w}$.

The exact value of $\text{val}(w)$ is settled only for $w \leq 2$. Obviously $\text{val}(1) = 1$. Felsner [Fel97] proved that any order of width 2 may be partitioned on-line into 5 chains. This, together with Kierstead's lower bound $4w - 3$ yields $\text{val}(2) = 5$. The value of $\text{val}(3)$ is not known. Again, the best known lower bound $\text{val}(3) \geq 9$ follows by Kiesrtead's bound. Recently, Bosek [Bos08] proved that $\text{val}(3) \leq 16$.

As the main problem has been proved to be resistant for improvements, many of its restricted versions were considered. In particular, there are many results concerning on-line chain partition games in which Spoiler is limited to introduce a poset from a given class $\mathcal{P}$. The following classes of posets were investigated: interval orders and their generalizations $(t{+}t)$-free orders, semi-orders, d-dimensional orders. In these cases the corresponding versions in which Spoiler introduces a poset by revealing its representation (intervals, unit intervals, linear extensions of d-dimensional posets) were also considered. For more details we refer the reader to the recent survey [BKK+10].

Nevertheless, here we bring up two results we shall essentially use in the proof of the main theorem. In [Fel97] Felsner introduced a variant of the chain partition game in which the Spoiler's power is limited by the condition the new point introduced is maximal in the order consisting of all points presented so far. A poset built this way is called *up-growing*. In [Fel97] Felsner settled the precise value of this game.

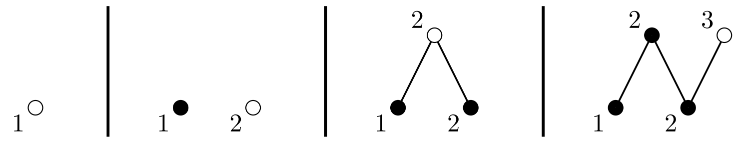

Fig. 1. Spoiler forces 3 chains on an order of width 2. The white element is the new element of a round. 1, 2, 3 are chains. In the third round Algorithm has three choices for x: The case $x = 3$ is an immediate win for Spoiler, the other two cases are symmetric and lead to Spoiler's win in round 4.

Theorem 1.4: The value of the chain partition game for up-growing orders of width w is $\binom{w+1}{2}$.

In the chain partition game, among all strategies for Algorithm, probably First-Fit is one of the simplest. First-Fit identifies chains with natural numbers and assigns the incoming point the lowest possible number. Kierstead [Kie86] showed that Spoiler may build an up-growing order of width 2 on which First-Fit uses arbitrarily many chains. Nevertheless, Bosek, Krawczyk and Szczypka [BKS10] showed that First-Fit works surprisingly well in partitioning posets into chains provided they exclude two incomparable chains of height t.

Theorem 1.5: First-Fit partitions $(t+t)$-free orders of width w into at most $3tw^2$ chains.

II. The general idea for the main theorem

The proof of Theorem 1.3 is split into two parts. First we reduce the general chain partition problem to a family of instances of a more structured problem that we call *a regular game*. The second part contains a description and an analysis of the algorithm for the regular game.

A regular game of width w is played between two players: Spoiler and Algorithm. The description of the game is based on the notion of *a regular board*. After each round of the game a regular board is determined by a pair $(\mathcal{A}, \leq)$ which satisfies:

(B_1) $(\bigcup \mathcal{A}, \leq)$ is a poset of width w unless $\mathcal{A} = \emptyset$.

(B_2) Each $A \in \mathcal{A}$ is an antichain of size w.

(B_3) Elements in $\mathcal{A}$ are pairwise disjoint.

(B_4) The set $\mathcal{A}$ is linearly ordered with respect to $\sqsubseteq$-relation, where $X \sqsubseteq Y$ if for any $x \in X$ there is $y \in Y$ with $x \leq y$.

(B_5) For every two consecutive antichains L, H in $(\mathcal{A}, \sqsubseteq)$ the comparability graph $(L \cup H, <|_{L \cup H})$ (shortly written $(L, H, <)$) is *regular*, i.e., each edge $(x < y)$ in $(L, H, <)$ is extendable to a matching of size w in $(L, H, <)$.

Each antichain $A \in \mathcal{A}$ is introduced by Spoiler during a single round of the game as a one atomic move. For convenience we assume that during the first two rounds of the game Spoiler introduces two antichains $\perp$ and $\top$ such that $x < y$ for all $x \in \perp$, $y \in \top$. The antichains $\perp$ and $\top$ are fixed to be the borders of the board, i.e., further antichains are placed in between $\perp$ and $\top$ with respect to $\sqsubseteq$ relation; see Fig. 2.

In a single round of the regular game its board $(\mathcal{A}, \leq)$ from the previous round is transformed into $(\mathcal{A}^+, \leq^+)$ according to the following rules. First, Spoiler chooses two consecutive antichains L, H in $(\mathcal{A}, \sqsubseteq)$. Then, he introduces a set M of size w, i.e. it sets $\mathcal{A}^+ = \mathcal{A} \cup \{M\}$, and extends the relation $\leq$ to $\leq^+$ so that:

- $\leq^+|_{\bigcup \mathcal{A}}$ equals $\leq$,
- for any $x \in M$ all immediate predecessors (successors) of x are contained in L (H).

Obviously, $(\mathcal{A}^+, \leq^+)$ must satisfy (B_1)-(B_5). It means, in particular, that $(L, M, <^+)$ and $(M, H, <^+)$ are regular. Note also that the condition $\leq^+|_{\bigcup \mathcal{A}} = \leq$ means in consecutive rounds of the game the order $(\bigcup \mathcal{A}, \leq)$ is extended in the on-line way; see Fig. 2.

The task of Algorithm is to maintain a coloring of the elements of the poset $(\bigcup \mathcal{A} \setminus (\perp \cup \top), \leq)$ such that the points with the same color form a chain.

The reduction from the general chain partition problem to the regular game is done in two steps. First the on-line order $(P, \leq)$ is split into a sequence of w suborders $P_1, \ldots, P_w$ such that the width of $P_1 \cup \ldots \cup P_i$ is at most i. This is done on-line by assigning the new point x to the first P_i where it does not violate the above width constraints. Each P_i is then used to construct a regular game of width i. The coloring produced in this regular game yields a chain partition of P_i. The essence of the reduction is captured in the following proposition:

Proposition 2.1: If Algorithm has a strategy which uses at most $\mathrm{reg}(v)$ colors on a regular game of width v for $v = 1, \ldots, w$, then there is an on-line algorithm that partitions posets of width w into at most $\mathrm{reg}(1) + \ldots + \mathrm{reg}(w)$ chains. In the next section we will prove the following.

Proposition 2.2: Suppose that there is a strategy that partitions on-line orders of width $v < w$ into at most $\mathrm{alg}(v)$ colors. Then there exists a strategy for Algorithm in the regular game of width w that uses at most

$$\left(2\binom{w^3+1}{2}+2\right) \cdot 3\,(2w-1)\left(w^3\right)^2 \cdot w^2 \cdot \mathrm{alg}(w/2)$$

colors.

The combination of the previous two propositions yields the main result of the paper:

$$\begin{aligned}&\mathrm{alg}(w) \leq \\ &\leq w \cdot \left(2\binom{w^3+1}{2}+2\right) \cdot 3\,(2w-1)\left(w^3\right)^2 \cdot w^2 \cdot \mathrm{alg}(w/2) \\ &\leq \mathit{poly}(w) \cdot w^{16 \log w}\end{aligned}$$

that may be easily strengthen to $\mathrm{alg}(w) \leq w^{16 \log w}$.

III. The strategy for Algorithm in the regular game

In this section we present techniques and main ideas that allow us to provide a strategy for Algorithm in the regular

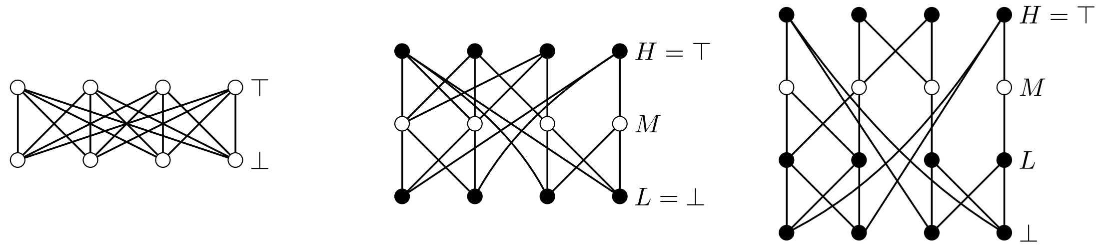

Fig. 2. Two moves of Spoiler in a regular game of width 4.

game claimed in Proposition 2.2. Throughout this section we assume that $(\mathcal{A}, \leq)$ is a board of a regular game of width w.

We begin with a simple but crucial observation that links the notions of optimal (of size w) chain partitions of $(\bigcup\mathcal{A}, \leq)$ with matchings between consecutive antichains in $(\mathcal{A}, \sqsubseteq)$. So suppose that $\mathcal{C} = \{C_1, \ldots, C_w\}$ is an optimal chain partition of $(\bigcup\mathcal{A}, \leq)$. Let L, H be two consecutive antichains in $(\mathcal{A}, \sqsubseteq)$. Then $\mathcal{C}$ yields a unique matching $\mathcal{M}$ in the regular graph $(L, H, <)$ defined $(x < y) \in \mathcal{M}$ if $x, y \in C_i$ for some i. On the other hand, if for any two consecutive antichains L, H in $(\mathcal{A}, \sqsubseteq)$ a matching $\mathcal{M}$ in $(L, H, <)$ is fixed, then the transitive closure of the union of all those matchings yields an optimal chain partition of $(\bigcup\mathcal{A}, \leq)$. Therefore we have also the following observation.

Observation 3.1: Suppose that $A \sqsubseteq B \sqsubseteq C$ are three antichains from $\mathcal{A}$, let $X \subseteq B$. Then

$$|A \cap X{\Downarrow}| \geq |X| \text{ and } |C \cap X{\Uparrow}| \geq |X|\,.$$

A. Nodes and their characteristics.

In this subsection we will introduce a concept of a *node*; this is somehow the basic notion that will help us to describe changes done on the board $(\mathcal{A}, \leq)$ of the regular game.

Definition 3.1: $(X, Y, <)$ is a node in $(L, H, <)$ if after some round of the regular game L, H are consecutive in $(\mathcal{A}, \sqsubseteq)$ and $X \cup Y$ is a maximal connected component of $(L, H, <)$.

The next observations collects some simple but important facts concerning nodes in $(L, H, <)$ (see Fig. 3).

Observation 3.2: Let $(X, Y, <)$ be a node in $(L, H, <)$. Then:

1) $(X, Y, <)$ is regular, i.e., $|X| = |Y|$ and each edge $(x < y)$ in $(X, Y, <)$ is extendable to a matching from X to Y,
2) for all $\emptyset \neq S \subseteq X$ the inequality $|S{\Uparrow} \cap Y| \geq \min\{|S| + 1, |Y|\}$ is satisfied.

Proof: The statements follow by the facts $(L, H, <)$ is regular and $X \cup Y$ is a minimal connected component in $(L, H, <)$. ∎

The *characteristics* of a node $(X, Y, <)$ in $(L, H, <)$ consists of its width $w(X, Y, <) = |X| = |Y|$ and its surplus (in the Hall's condition) $s(X, Y, <)$ which is the largest k such that for all non-empty $S \subseteq X$ we have

$$|S{\Uparrow} \cap Y| \geq \min\{|S| + k, |Y|\}\,.$$

For $(X, Y, <)$ being a complete bipartite graph the above condition is true for every k and we put $s(X, Y, <) = \infty$. For a node $(X, Y, <)$ its characteristics is denoted as a pair $(w(X, Y, <), s(X, Y, <))$ (see Fig. 3). Note that by Observation 3.2.(2) we have $s(X, Y, <) \geq 1$ for any node $(X, Y, <)$. So far we defined two parameters of a given node $(X, Y, <)$: the width and the surplus. Here we define the third one: *vitality*. A *cross* of a node $(X, Y, <)$ is a set $\{x_1, x_2, y_1, y_2\}$, where $x_1, x_2 \in X$, $y_1, y_2 \in Y$, with the four edges $(x_i < y_j)$ for $i, j = 1, 2$ such that both edges $(x_1 < y_1)$, $(x_2 < y_2)$ are extendable to a matching $\mathcal{M}$ in $(X, Y, <)$. A node is *vital* if it contains a cross; otherwise a node is *non-vital*. We assume that for each vital node $(X, Y, <)$ a representative cross $R(X, Y, <)$ is fixed (see Fig. 3).

B. A single round of a regular game

Assume that $(\mathcal{A}, \leq)$ determines the board of a regular game after t rounds, $t \geq 3$. Suppose that in the $t-$th round Spoiler has put an antichain M between L and H, i.e. $L \sqsubseteq M \sqsubseteq H$ are consecutive in $(\mathcal{A}, \sqsubseteq)$.

We begin with the observation that describes the mutual location of nodes in $(L, H, <)$ and of new nodes that have appeared either in $(L, M, <)$ or in $(M, H, <)$ (see Fig. 4). For this purpose, for a node $(X, Y, <)$ we set

$$\overline{int}(X, Y, <) = \\ = \left\{z \in \bigcup\mathcal{A} : x \leq z \leq y \text{ for some } x \in X, y \in Y\right\}.$$

Observation 3.3: The following statements hold:

1) for each node $(X, Y, <)$ in $(L, H, <)$ we have $X{\Uparrow} \cap M = Y{\Downarrow} \cap M$, and this set is of size $|X| = |Y|$,
2) for each node $(Z, T, <)$ in $(L, M, <)$ $[(M, H, <)]$ there exists a unique node $(X, Y, <)$ in $(L, H, <)$ such that $\overline{int}(Z, T, <) \subseteq \overline{int}(X, Y, <)$.

Proof: By Observation 3.1 and by the fact $(X{\Uparrow} \cap M){\Uparrow} \cap H \subseteq Y$ we get

$$|X| \leq |X{\Uparrow} \cap M| \leq |(X{\Uparrow} \cap M){\Uparrow} \cap H| \leq |Y|$$

and, since $|X| = |Y|$, we imply all the above sets are of the same size $|X|$. Similarly, we have

$$|Y| = |Y{\Downarrow} \cap M| = |(Y{\Downarrow} \cap M){\Downarrow} \cap L| = |X|\,.$$

It follows $|X| = |X{\Uparrow} \cap M| = |Y{\Downarrow} \cap M| = |Y|$. But we have also $X{\Uparrow} \cap M = Y{\Downarrow} \cap M$ as otherwise $X{\Uparrow} \cap H \not\subseteq Y$ or $Y{\Downarrow} \cap L \not\subseteq X$, which would contradict $(X \cup Y)$ is a

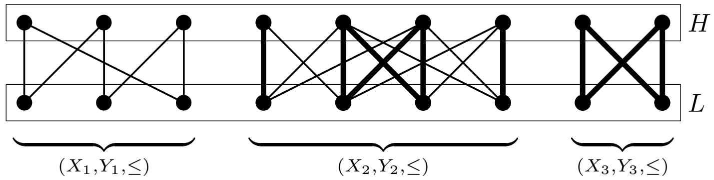

Fig. 3. A regular graph $(L, H, <)$ and its three nodes $(X_1, Y_1, <)$, $(X_2, Y_2, <)$ and $(X_3, Y_3, <)$. Their characteristics are equal $(3, 1)$, $(4, 1)$ and $(2, \infty)$, respectively. The node $(X_1, Y_1, <)$ is non-vital. The nodes $(X_2, Y_2, <)$ and $(X_3, Y_3, <)$ are vital; their crosses and matchings extending the crosses are drawn with bolded lines.

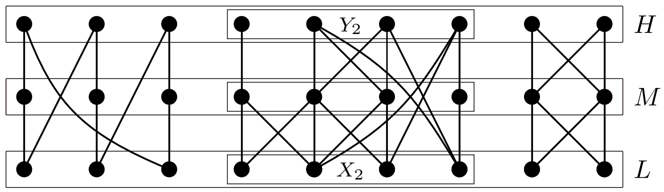

Fig. 4. Spoiler introduces M between L and H. The node $(X_2, Y_2, <)$ has five subnodes.

maximal connected component in $(L, H, <)$. It proves (1). The statement (2) is an immediate consequence of (1). ■

C. A regular game tree

Observation 3.3.(2) asserts all nodes appearing during the regular game can be organized in a tree, called *a regular game tree*.

Definition 3.2: A regular game tree is a pair $(TG, \vdash)$, where TG is a set of all nodes appearing during a regular game and $\vdash$ is a binary relation on TG defined $(Z, T, <) \vdash (X, Y, <)$ if for $(Z, T, <)$ $(X, Y, <)$ is a unique node that satisfies the statement (2) of Observation 3.3.
Obviously, the root of $(TG, \vdash)$ is the node $(\bot, \top, <)$ and the leafs of $(TG, \vdash)$ are nodes that are settled between some two consecutive antichains in $(\mathcal{A}, \sqsubseteq)$.

The next observation says that the characteristics (width, surplus) of nodes are weakly decreasing with respect to the lex-order along paths in the regular game tree.

Observation 3.4: If $(X, Y, <)$ is an ancestor of $(Z, T, <)$ in $(TG, \vdash)$ then:

1) $w(Z, T, <) \leq w(X, Y, <)$,
2) if $w(Z, T, <) = w(X, Y, <)$ then $s(Z, T, <) \leq s(X, Y, <)$,
3) if $(Z, T, <)$ is vital then $(X, Y, <)$ is vital as well.

Proof: To complete the proof it suffices to show the statements (1), (2), (3) are satisfied for $(Z, T, <)$ being a subnode of $(X, Y, <)$. Without loss of generality, assume that $Z \subseteq X \subseteq L$, $T \subseteq M$ and $Y \subseteq H$ for some three consecutive antichains $L \sqsubseteq M \sqsubseteq H$ in $(\mathcal{A}, \sqsubseteq)$. By Observation 3.3 there exists a matching from $X\Uparrow \cap M$ to Y. Then, by transitivity of $<$ and by the inclusion $T \subseteq (X\Uparrow \cap M)$ we deduce (1), (2), (3) hold. ■

D. An edge coloring function.

For a node $(X, Y, <) \in TG$ by $\mathcal{E}(X, Y, <)$ we denote all edges $(x < y)$ in the node $(X, Y, <)$.

Now, we make a first remark how Algorithm in the regular game works. During round t, just before coloring the points of an incoming antichain M, Algorithm assigns a non-empty set of colors to each comparability edge of the incoming regular orders $(L, M, <)$ and $(M, H, <)$ such that:

$**$ for each color γ the set of points incident to an edge colored with γ forms a chain in $\leq$.

Thus Algorithm maintains a coloring $c : \mathcal{E}(TG) \longrightarrow \mathcal{P}^+(\Gamma)$. The next step is easy. To every $x \in M$ Algorithm assigns a color of any edge incidental to x. Condition $**$ guarantees that all points with the same color lie in one chain.

E. An edge order.

In the next two subsections we will prove the lemmas that bases the strategy of coloring the edges from $\mathcal{E}(TG)$. In particular, we will be working with subsets of TG that satisfy so-called *ancestor-free* property. We say $\mathcal{N} \subseteq TG$ is *ancestor-free* if for all two nodes $N_1, N_2 \in \mathcal{N}$ the node N_1 is not an ancestor of N_2.

We begin with a short technical lemma; but earlier for $(X, Y, <) \in TG$ we set

$$int(X, Y, <) = \\ = \left\{ z \in \bigcup \mathcal{A} : x < z < y \text{ for some } x \in X, y \in Y \right\}.$$

Lemma 3.1: Let $\mathcal{N} \subseteq TG$ be an ancestor-free set. Assume that for all N in $\mathcal{N}$ a matching $\mathcal{M}(N)$ in N is fixed. Let

$$P^- = \bigcup \mathcal{A} \setminus \bigcup \{int(N) : N \in \mathcal{N}\}.$$

Then there exists a chain partition $\{C_1, \ldots, C_w\}$ of $(P^-, \leq)$ such that for all $(a < b)$ in $\mathcal{M}(N)$, $N \in \mathcal{N}$, the points a and b are in a same chain C_i. In addition, b immediately succeeds a in C_i (see Fig. 5).

Proof: Let

$$TG^- = TG \setminus \{N' : N' \text{ is a descendant of a node from } \mathcal{N}\}.$$

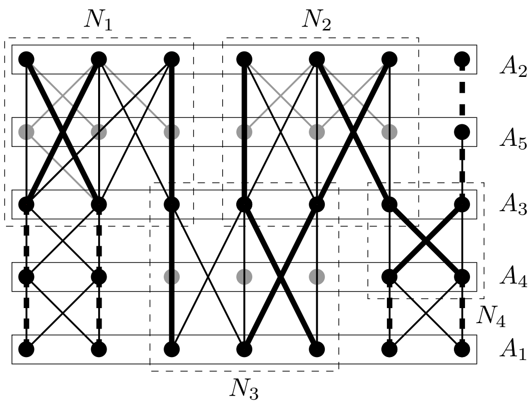

Fig. 5. $\{N_1, N_2, N_3, N_4\}$ is ancestor-free. P^- contains all black points. For any leaf N in $\left(TG^-, \vdash|_{TG^-}\right)$ the edges of $\mathcal{M}(N)$ are: bolded if $N \in \mathcal{N}$ or dotted if $N \notin \mathcal{N}$.

Clearly, $(TG^-, \vdash|_{TG^-})$ is a subtree of $(TG, \vdash)$; the nodes from $\mathcal{N}$ are leafs in $(TG^-, \vdash|_{TG^-})$. For any leaf node N in $(TG^-, \vdash|_{TG^-})$, $N \notin \mathcal{N}$, we fix a matching $\mathcal{M}(N)$ in N. Then we note that the transitive closure of

$$\bigcup \left\{\mathcal{M}(N) : N \text{ is a leaf in } \left(TG^-, \vdash|_{TG^-}\right)\right\}$$

yields a desired chain partition of $(P^-, \leq)$. ■

Let $\mathcal{N} \subseteq TG$ be an ancestor-free set. On the set $\mathcal{E}(\mathcal{N})$ we define the binary relation $<_e$:

$$(a < b) <_e (c < d) \quad \text{iff} \quad b \leq c.$$

Lemma 3.2: Assume $\mathcal{N} \subseteq TG$ is ancestor-free. Then the pair $(\mathcal{E}(\mathcal{N}), \leq_e)$ is a partial order of width at most w^3.

Proof: It is straightforward to check $(\mathcal{E}(\mathcal{N}), <_e)$ is irreflexive and transitive. Thus $(\mathcal{E}(\mathcal{N}), \leq_e)$ is a partial order.

Let U be an antichain in $(\mathcal{E}(\mathcal{N}), \leq_e)$. We will show $|U| \leq w^3$. It yields $(\mathcal{E}(\mathcal{N}), \leq_e)$ is of width at most w^3.

To prove $|U| \leq w^3$, assume that $\mathcal{U}$ is the set of all nodes from $\mathcal{N}$ that contain at least one edge from U. We will show $|\mathcal{U}| \leq w$ and then, since each node contains at most w^2 edges, we will have $|U| \leq w^3$. Assume contrary that $|\mathcal{U}| > w$. For each node $N \in \mathcal{U}$ let $\mathcal{M}(N)$ be a matching in N extending at least one edge from U – such a matching exists as N is regular. By Lemma 3.1 there is a chain partition $\{C_1, \ldots, C_w\}$ of $(\bigcup \mathcal{A} \setminus \bigcup \{int(N) : N \in \mathcal{U}\}, \leq)$ such that for each $(x < y) \in \mathcal{M}(N)$, $N \in \mathcal{U}$, the points x and y are in a same chain C_i and y immediately succeeds x in C_i. By the pigeonhole principle there exists a chain C_j such that two edges from U share a common chain C_j. Of course, these two edges are comparable with respect to $\leq_e$ relation. It contradicts U is an antichain in $(\mathcal{E}(\mathcal{N}), \leq_e)$. ■

F. Recursive partial orders.

A node N with characteristics (u, s) is called *active* if it is vital (thus a representative cross $R(N)$ is fixed) and it has no ancestor in $(TG, \vdash)$ with the same characteristics. On the set $P_{(u,s)}$ of all active nodes with characteristics (u, s) we define a binary relation $<_{(u,s)}$ as follows (see Fig. 6):

$$N <_{(u,s)} N' \text{ iff}$$
$$\exists x \in \max(R(N)) \;\; \exists y \in \min(R(N')) \;\; \text{with } x \leq y.$$

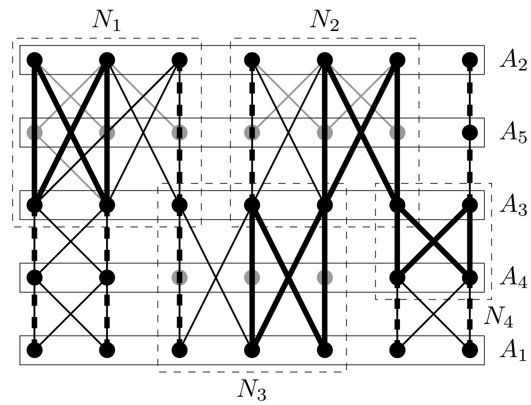

Fig. 6. $\{N_1, N_2, N_3, N_4\} \subseteq P_{(3,1)}$. The set $\{N_1, N_3, N_4\}$ forms a maximum antichain in $(P_{(3,1)}, \leq_{(3,1)})$. We have $N_3 \leq_{(3,1)} N_2$.

Lemma 3.3: The pair $\left(P_{(u,s)}, \leq_{(u,s)}\right)$ is a partial order of width at most $\lfloor w/2 \rfloor$.

Proof: One may easily verify that $<_{(u,s)}$ is transitive and irreflexive. Thus $\left(P_{(u,s)}, \leq_{(u,s)}\right)$ is a partial order.

For $N \in P_{(u,s)}$ let $\mathcal{M}(N)$ be a matching that extends two edges from $R(N)$; such a matching exists by definition of $R(N)$. As $P_{(u,s)}$ is ancestor-free, by Lemma 3.1 there exists a chain partition of $(\bigcup \mathcal{A} \setminus \bigcup \{\overline{int}(N) : N \in P_{(u,s)}\}, \leq)$ such that for each edge $(a < b) \in \mathcal{M}(N)$, $N \in P_{(u,s)}$, a and b are in a same chain and in this chain b immediately succeeds a. As two edges of any representant $R(N)$ of N, $N \in P_{(u,s)}$, are settled on two different chains, we easily deduce that a maximum antichain in $(P_{(u,s)}, \leq_{(u,s)})$ is of size at most $\lfloor w/2 \rfloor$. Thus the width of $(P_{(u,s)}, \leq_{(u,s)})$ is at most $\lfloor w/2 \rfloor$. ■

Lemma 3.4: Let $\mathcal{C} \subseteq P_{(u,s)}$ be a chain in $\left(P_{(u,s)}, \leq_{(u,s)}\right)$. Then the partial order $(\mathcal{E}(\mathcal{C}), \leq_e)$ is $((2w-1)+(2w-1))$-free and the width of $(\mathcal{E}(\mathcal{C}), \leq_e)$ does not exceed w^3.

Proof: The set $\mathcal{C}$, as a subset of $P_{(u,s)}$, is ancestor-free. Thus, by Lemma 3.2 we have $(\mathcal{E}(\mathcal{C}), \leq_e)$ is a partial order of width at most w^3.

To prove $(\mathcal{E}(\mathcal{C}), \leq_e)$ is $((2w-1)+(2w-1))$-free we assume contrary that $(\mathcal{E}(\mathcal{C}), \leq_e)$ contains two incomparable chains

$$(a_1 < b_1) <_e \ldots <_e (a_{2w-1} < b_{2w-1})$$
$$\text{and}$$
$$(c_1 < d_1) <_e \ldots <_e (c_{2w-1} < d_{2w-1}),$$

where $(a_i < b_i) \in N_i$, $(c_i < d_i) \in M_i$ for some $N_i, M_i \in \mathcal{C}$, $i = 1, \ldots, 2w-1$. Assume that $a_i \in A_i$, $b_i \in B_i$, $c_i \in C_i$, $d_i \in D_i$ for some $A_i, B_i, C_i, D_i \in \mathcal{A}$. First, we will prove the following claim.

Claim 3.1: The relation $b_1 < x$ holds for some $x \in \min(R(N_w))$ (see Fig. 8).

Proof: The sentence of the claim is equivalent to $(b_1\Uparrow \cap A_w) \cap \min(R(N_w)) \neq \emptyset$. First, we note that if for some $i \in \{2, \dots, w-1\}$ we had $(b_1\Uparrow \cap A_i) \cap R(N_i) \neq \emptyset$ then by $N_i \leq_{(u,s)} N_w$ we would get $(b_1\Uparrow \cap A_w) \cap \min(R(N_w)) \neq \emptyset$ and our claim would be proved. So we may assume that for $i = 2, \dots, w-1$ we have

$$(b_1\Uparrow \cap A_i) \cap \min(R(N_i)) = \emptyset. \tag{1}$$

With this assumption we claim that for $i = 1, \dots, w-1$

$$|b_1\Uparrow \cap B_i| \geq i. \tag{2}$$

We prove (2) by induction on i. For $i = 1$ the inequality (2) is obvious. Assume $|b_1\Uparrow \cap B_{i-1}| \geq i-1$ (see Fig. 7). Let $N_i = (X_i, Y_i, <)$. In the set A_i we distinguish two sets:

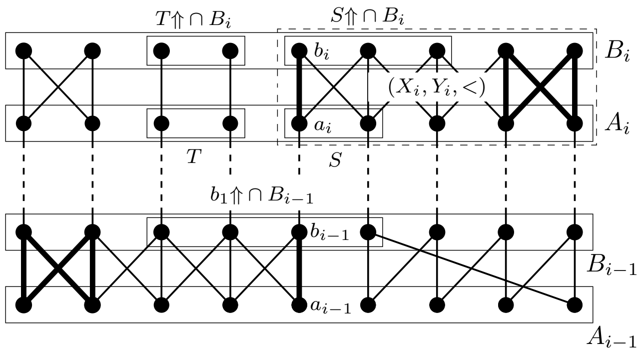

Fig. 7.

$S = b_1\Uparrow \cap X_i$ and $T = b_1\Uparrow \cap (A_i \setminus X_i)$. The sets S and T are disjoint. By $|b_1\Uparrow \cap B_{i-1}| \geq i-1$ and by Observation 3.1 we imply $S \cup T$ contains at least $i-1$ elements. Now we consider the sets $S\Uparrow \cap B_i$ and $T\Uparrow \cap B_i$. The sets $S\Uparrow \cap B_i$ and $T\Uparrow \cap B_i$ are disjoint as N_i is a node in $(A_i, B_i, <)$. Again, by Observation 3.1 the set $T\Uparrow \cap B_i$ contains at least $|T|$ elements. Note that $S \cap \min(R(N_i)) = \emptyset$ by (1). Hence $S \subsetneq X_1$. But, $a_i \in S$ as $(a_1 < b_1) <_e (a_i < b_i)$. Then, by Observation 3.2.(2) we have $S\Uparrow \cap B_i$ contains at least $|S| + 1$ elements. Thus $(S\Uparrow \cap B_i) \cup (T\Uparrow \cap B_i) \subseteq (b_1\Uparrow \cap B_i)$ has at least i elements. It proves (2).

By (2) we conclude the set $(b_1\Uparrow \cap B_{w-1})$ contains at least $w-1$ elements. Again, by Observation 3.1 we imply $(b_1\Uparrow \cap A_w) \cap \min(R(N_w)) \neq \emptyset$. It proves $b_1 < x$ for some $x \in \min(R(N_w))$. ■

Quite similar we may prove the following claims (see Fig. 8):

Claim 3.2: The relation $a_{2w-1} > x$ holds for some $x \in \max(R(N_w))$.

Claim 3.3: The relation $d_1 < x$ holds for some $x \in \min(R(M_w))$.

Claim 3.4: The relation $c_{2w-1} > x$ holds for some $x \in \max(R(M_w))$.

Thus, as $\mathcal{C}$ is a chain in $\left(P_{(u,s)}, \leq_{(u,s)}\right)$ then either $N_w \leq_{(u,s)} M_w$ and then $(a_1 < b_1) \leq_e (c_{2w-1} < d_{2w-1})$ or $M_w \leq_{(u,s)} N_w$ and then $(c_1 < d_1) \leq_e (a_{2w-1} < b_{2w-1})$; see Fig. 8. Hence the chains

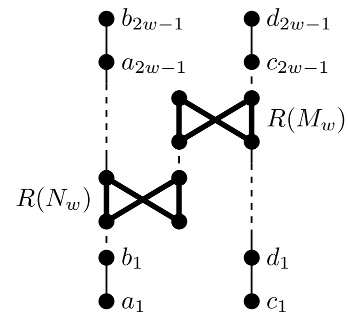

Fig. 8.

$$(a_1 < b_1) <_e \dots <_e (a_{2w-1} < b_{2w-1})$$
and
$$(c_1 < d_1) <_e \dots <_e (c_{2w-1} < d_{2w-1})$$

are not incomparable – a contradiction. ■

G. Algorithm in the regular game.

We begin with a description of how Algorithm colors the edges of all active nodes appearing in $(TG, \vdash)$:

- Algorithm generates on-line a partition of $\left(P_{(u,s)}, \leq_{(u,s)}\right)$ into at most $\mathrm{alg}(w/2)$ chains (Lemma 3.3);
- For each chain $\mathcal{C} \in \left(P_{(u,s)}, \leq_{(u,s)}\right)$ First-Fitpartitions $(\mathcal{E}(\mathcal{C}), \leq_e)$ into at most $3(2w-1)\left(w^3\right)^2$ (Lemmas 3.2 and 3.4, Theorem 1.5).

As there are at most w^2 feasible characteristics (u, s), to complete this task Algorithm needs $\lambda(w) = 3(2w-1)\left(w^3\right)^2 \cdot w^2 \cdot \mathrm{alg}(w/2)$ colors.

Here we sketch the general idea how the coloring of edges from active nodes is extended on the whole set $\mathcal{E}(TG)$ so that $**$ is preserved.

With an active node N we associate a set $D(N)$ of *dependent nodes*. It is the set of nodes N' such that N is the first active node on the path from N' to the root $(\bot, \top, <)$ of $(TG, \vdash)$. Since $(\bot, \top, <)$ is active, the set $\{D(N) : N \text{ is active}\}$ forms a partition of all nodes in TG.

The basic idea is to replace each of the $\lambda(w)$ colors used for the edges of active node by a bundle of μ colors. Then the colors in the bundles associated with the edges of an active node N are used to color all the edges from $\mathcal{E}(D(N))$.

Observation 3.4.(3) yields the following property of non-vital nodes:

- All descendants of a non-vital node are also non-vital and therefore if N' is a non-vital node in $D(N)$ then all descendants of N' are also in $D(N)$.

Although a non-vital $N' \in D(N)$ may have a lot of descendants the fact it does not have a cross results in:

- There is a greedy strategy that extends an edge coloring of a non-vital node N' to an edge coloring with property $**$ of all edges of descendants of N'. This extension does not require additional colors.

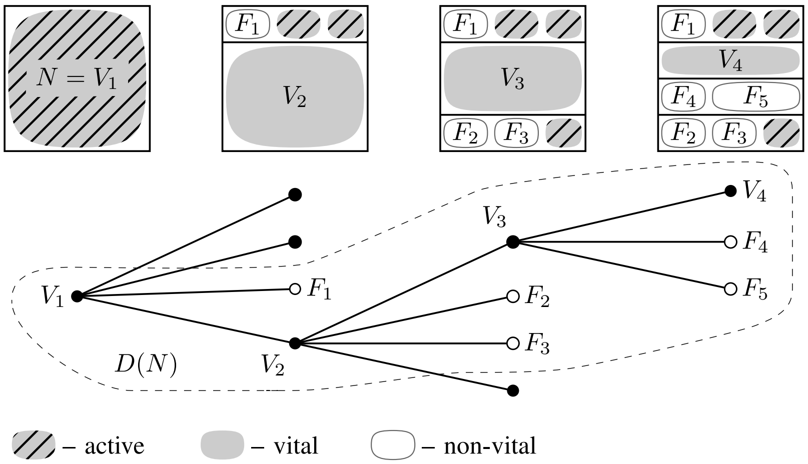

Fig. 9. The tree-structure and the path of vital nodes in $D(N)$.

Now, in order to color all the edges in $D(N)$ it remains to deal with the edges of vital nodes in $D(N)$ and with their non-vital sons (we briefly call them *first-non-vital nodes*). Unless N represents a complete bipartite graph we have:

- All vital nodes in $D(N)$ have the same characteristics as N and they form a path in $(TG, \vdash)$ (see Fig. 9).

Note that consecutive vital nodes on a path in $D(N)$ split first-non-vital nodes in $D(N)$ into two subsets. Let V be the last vital node on this path and let $A(V)$ [respectively $B(V)$] be the set of first-non-vital nodes F such that

$$\overline{int}(F) \subseteq \max(V)\Uparrow \quad [\overline{int}(F) \subseteq \min(V)\Downarrow].$$

Obviously, $A(V)$ and $B(V)$ are ancestor-free. Now, consider $\left(\mathcal{E}(A(V)), \leq_e\right)$ $[\left(\mathcal{E}(B(V)), \leq_e\right)]$ as on-line orders:

- $\left(\mathcal{E}(A(V)), \leq_e\right)$ $[\left(\mathcal{E}(B(V)), \leq_e\right)]$ is down-growing [up-growing] order of width at most w^3, see Lemma 3.2. Hence, it can be partitioned on-line into $\binom{w^3+1}{2}$ chains, see Theorem 1.4.

To an edge $(z < t) \in \mathcal{E}(A(V)) \cup \mathcal{E}(B(V))$ we want to assign a color that is used on some edge $(x < y)$ in N such that property $**$ is preserved. That is we need $x \leq z < t \leq y$. Such a color assignment is certainly possible if only every edge $(x < y) \in N$ has $2\binom{w^3+1}{2}$ colors in its bundle.

It remains to color edges of vital nodes and possibly first-non-vital nodes that appear as sons of the last node on a path of vital nodes in $D(N)$. To take care of all those edges it is sufficient to have two additional colors in the bundle of every edge $(x < y) \in N$.

The case where N represents a complete bipartite graph can be handled with similar ideas.

We deduce that the strategy presented above requires $\left(2\binom{w^3+1}{2} + 2\right) \cdot 3\,(2w-1)\left(w^3\right)^2 \cdot w^2 \cdot \mathrm{alg}(w/2)$ colors as it was claimed in Proposition 2.2.

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