A separator theorem in minor-closed classes

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***Abstract*—It is shown that for each t, there is a separator of size $O(t\sqrt{n})$ in any n-vertex graph G with no K_t-minor.**

This settles a conjecture of Alon, Seymour and Thomas (J. Amer. Math. Soc., 1990 and STOC'90), and generalizes a result of Djidjev (1981), and Gilbert, Hutchinson and Tarjan (J. Algorithm, 1984), independently, who proved that every graph with n vertices and genus g has a separator of order $O(\sqrt{gn})$, because K_t has genus $\Omega(t^2)$.

The bound $O(t\sqrt{n})$ is best possible because every 3-regular expander graph with n vertices is a graph with no K_t-minor for $t = cn^{1/2}$, and with no separator of size dn for appropriately chosen positive constants c, d.

In addition, we give an $O(n^2)$ time algorithm to obtain such a separator, and then give a sketch how to obtain such a separator in $O(n^{1+\varepsilon})$ time for any $\varepsilon > 0$. Finally, we discuss several algorithm aspects of our separator theorem, including a possibility to obtain a separator of order $g(t)\sqrt{n}$, for some function g of t, in an n-vertex graph G with no K_t-minor in $O(n)$ time.

***Keywords*-separator, excluded minor, and divide and conquer.**

I. Introduction

A. Graph Separators

"Divide and conquer" is one of the oldest and most widely used techniques for designing efficient algorithms. Divide-and-conquer algorithms partition their inputs into two or more independent subproblems, solve those subproblems recursively, and then combine the solutions to those subproblems to obtain their final output. This strategy can be successfully applied to several graph problems, provided we can quickly separate the graph into roughly equal subgraphs. An s-separator of an n-vertex graph $G = (V, E)$ is a subset $S \subset V$ such that each connected component of $G - S$ has at most sn vertices. Our goal is to find s-separators, for some constant $1/2 \leq s < 1$, that have few vertices. For example, any path has a $1/2$-separator consisting of a single vertex; any binary tree has a $2/3$-separator consisting of a single vertex; and any outerplanar graph has a $2/3$-separator consisting of at most two vertices.

In the late 1970s, Lipton and Tarjan [15] proved the following seminal result.

The Planar Separator Theorem. Any n-vertex planar graph has a $2/3$-separator of order $O(\sqrt{n})$.

This theorem has attracted a lot of attention by many researchers, because there are many algorithmic applications, see [16]. In fact, Lipton and Tarjan proved that there is a linear time algorithm to find such a separator, see [16].

The original proof of Lipton and Tarjan [15] is simplified significantly. Alon, Seymour and Thomas [2] give a much shorter proof, based on graph theoretical tools, while Spielman and Teng [27] do the same, based on a well-known geometric characterization of Koebe [13].

Djidjev [4], and Gilbert, Hutchinson and Tarjan [8], independently, generalized the planar separator theorem to bounded genus graphs. Namely, they proved that every graph with n vertices and genus g has a $2/3$-separator of order $O(\sqrt{gn})$. The bound is best possible, up to constant factor. It is natural to ask whether or not the bounded genus result can be extended to minor-closed classes. In fact, Linial conjectured that there is a separator of order $O(\sqrt{n})$ in minor-closed classes. This conjecture was answered in the positive by Alon, Seymour and Thomas [1]. Namely, they proved the following.

Theorem 1.1: Let G be a graph with n vertices and with no K_t-minor for some integer t. Then G has a $2/3$-separator of order $O(t^{3/2}\sqrt{n})$. Moreover, such a separator can be found in $O(n^{3/2})$ time.

On the other hand, Alon, Seymour and Thomas suspected that the expression $O(t^{3/2}\sqrt{n})$ in Theorem 1.1 is not best possible. They conjectured that $O(t\sqrt{n})$ would be the correct answer. If true, this would generalize the above mentioned result of Djidjev, and Gilbert, Hutchinson and Tarjan, independently, because K_t has genus $\Omega(t^2)$. The bound $O(t\sqrt{n})$ would be best possible because every 3-regular expander graph with n vertices is a graph with no K_t-minor for $t = cn^{1/2}$, and with no $2/3$-separator of size dn for appropriately chosen positive constants c, d.

B. Main Result

In this paper, we settle a 20 years old conjecture of Alon, Seymour and Thomas. Namely, we prove the following.

Theorem 1.2: Let G be a graph with n vertices and with no K_t-minor for some integer t. Then G has a $2/3$-separator of order $O(t\sqrt{n})$.

As mentioned above, the expression $O(t\sqrt{n})$ is best possible. We have a couple of algorithmic remarks which will be given in Section X. We can find such a $2/3$-separator in $O(n^2)$ time for fixed t. In fact, we can make it in $O(n^{1+\varepsilon})$ time, if we use our recent result with Z. Li (which is not fully written yet). Remarks about our algorithm will be given in Section X. We point out that all the algorithmic applications mentioned in [1] (in STOC'90 version) can be also carried over using our algorithm too.

The proof of Theorem 1.1 given in [1] is actually simple and elegant. In fact, Alon, Seymour and Thomas did not use any structural result from Graph Minor Theory [23]. On the other hand, Kotlov [14] and G. Tardos, independently, proved that one of the lemmas for the proof of Theorem 1.1 (Lemma (2.1) in [1]) is best possible up to a constant factor. This means that Alon, Seymour and Thomas' proof would not generalize to prove Theorem 1.2. Specifically, we would need some topological structure for graphs with no K_t-minor. Therefore, we shall use the main structure theorem given in [23].

The structure theorem in [23], roughly, says that every graph with no K_t-minor has a tree-decomposition so that each piece can be embedded into a surface in which K_t cannot not be embedded, with small number of non-planar areas, which are called *vortices*. But the structure theorem itself is only an approximate version. This is not enough to prove Theorem 1.2, as far as we can see. Specifically we need to give better bounds on genus and the number of vortices. In addition, the tree structure is somehow troublesome. We need to focus on one piece of the decomposition. This needs a little more work.

After taking care of these issues, at a high level, we would like to use the $2/3$-separator theorem by Djidjev [4], and Gilbert, Hutchinson and Tarjan [8], independently, for bounded genus graphs to get a $2/3$-separator of order $O(t\sqrt{n})$ in the graph minor structure. But vortices are still troublesome. So a large part of this paper is devoted to handle vortices. We shall give overview of our proof in the next subsection.

C. Overview

We now give overview of our proof of Theorem 1.2. For our convenience, we shall give a proof by step by step. This would give a polynomial time algorithm to construct a $2/3$-separator in Theorem 1.2.

Step 1. A complete graph K_t may have crossing number $\Omega(t^4)$. This seems a big problem for us, because we would like to bound non-planar crosses by at most t^2. Thus we shall define the "cross-grid" graph CT of height t. Roughly, the graph CT of height t can be obtained from a grid of height $t+2$ by giving a non-planar single-cross at each vertex of degree 4 in the grid. In other words, each vertex of degree 4 in the grid is replaced by a non-planar single cross that consists of two disjoint paths P_1 and P_2 of length 2. We also add an edge between P_1 and P_2. For more details, we refer the reader to Section IV. The point here is that the cross-grid CT of height t has K_t as a minor. Thus graphs with no K_t as a minor do not contain CT of height t as a minor.

The cross-grid graph CT of height t has several advantages. Besides containing K_t as a minor, it has crossing number exactly t^2, as easily seen. Since K_t may have crossing number $\Omega(t^4)$, this bound is much better. This is one of our key ideas. Step 1 will be discussed in Section IV.

Step 2. We shall apply the main graph minor structure theorem of Robertson and Seymour to a given graph G to exclude the cross-grid graph CT of height t as a minor (For more details, we refer the reader to Section III). This structure theorem no longer involves the tree structure. However, in order to apply this structure theorem to G, we need to confirm that tree-width of G is large. We rather work on the similar concept, called "bramble". Note that if a bramble in a given graph G is of large order then tree-width of G is large (see Section II for more details).

We are interested in the following bramble; the bramble β_G that consists of trees of G containing more than half the vertices of G. Note that any two such trees intersect, so this set β_G satisfies the definition of a bramble (see Section II). We first confirm that such a bramble β_G has large order so that we can apply the main structure theorem of Robertson and Seymour (see Theorem 3.1 and Section III). Otherwise, we can easily find a $2/3$-separator of order $O(t\sqrt{n})$ (for more details, we refer the reader to Section II). The advantage of this bramble β_G is the following.

> Each piece of the decomposition of each vortex in G contains at most $n/3$ vertices.

We will use this fact many times.

We actually apply a more precise structure theorem with respect to this bramble β_G of large order, which is given in [6], to exclude the cross-grid CT of height t as a minor. It follows from this structure theorem, together with a result in [22], [24], that genus g of the surface part of G plus the number of vortices in G is at most t^2. Again, the key for this fact is that the cross-grid CT of height t has crossing number t^2, and hence genus of CT is t^2. Let g be genus of the surface part of G. So $g \leq t^2$.

Step 3. By using a result for bounded genus graphs by Djidjev [4], and Gilbert, Hutchinson and Tarjan [8], independently, we can find a subgraph W' of order at most $2\sqrt{gn} \leq 2t\sqrt{n}$ (since genus g is at most t^2), such that $G - W'$ consists of one planar graph, up to 3-separations, with still a bounded number of vortices. Moreover, by our choice of the bramble β_G, each piece of the decomposition of each vortex in G contains at most $n/3$ vertices.

Step 4. As in Section VII, we can prove that there is a vertex set T of order at most $2t\sqrt{n}$ such that $G - W' - T$ consists of components $Q_1, \dots, Q_x$, where $x < f'(t)$ for some function f' of t, such that each Q_i either

1) has at most $n/3$ vertices, or
2) is a planar graph, up to 3-separations, with at most one vortex. Moreover, by our choice of the bramble β_G, each piece of the decomposition of the vortex contains at most $n/3$ vertices.

In addition, if there are two Q_i that have at least $n/3$ vertices or if there is no such Q_i, we can find a desired $2/3$-separator $W' \cup T$ of order $O(t\sqrt{n})$ as in Theorem 1.2. Thus we can assume that Q_1 has at least $n/3$ vertices, and satisfies the second structure.

Step 5. We use a generalization of the planar separator theorem given by Alon, Seymour and Thomas [2] to confirm that Q_1 has a $2/3$-separator S of order at most $O(\sqrt{|Q_1|})$. This implies that G has a $2/3$-separator $W' \cup T \cup S$ of order $O(t\sqrt{n})$.

In Section X, we give our algorithm to find a $2/3$-separator of order $O(t\sqrt{n})$ for fixed t. We shall also consider the case when we find a $2/3$-separator of order $g(t)\sqrt{n}$ for some function g of t.

The next two sections concern notations for our proof of Theorem 1.2. Section II gives basic terminology in this paper. Then in Section III, we give a complete description of the seminal structure theorem in graph minor theory [23]. These parts may be tedious for the reader who is familiar with graph minor notations. In this case, it would be better to skip the next two sections, and go to Section IV directly.

II. Preliminaries for Theorem 1.2

A separation (A, B) is that $G = A \cup B$, and there are no edges between $A - B$ and $B - A$. The order of the separation (A, B) is $|A \cap B|$.

We omit the definitions of tree-width, bramble, and wall, as they are given in [10]. Recall that Reed [18] gives an $O(n \log n)$ algorithm to construct a tree decomposition of width at most $4k$ for a graph of tree-width k for any fixed integer k. In his proof, the following theorem, which we find very important for our purpose, is proved.

Theorem 2.1 (Reed, [18]): For any fixed integer k and for some vertex set X (X could be $V(G)$), there is a linear time algorithm which, given a graph G, either:

1) finds a cutset Y of vertices of G with $|Y| \leq k$ such that no component of $G - Y$ contains more than $\frac{2}{3}|X - Y|$ vertices, or
2) determines that for any set Y of vertices of G with $|Y| \leq k$, there is a component of $G - Y$ which contains more than $\frac{1}{2}|X - Y|$ vertices.

It is easy to see that if the outcome is 2, then we have determined that the bramble β_X, consisting of all the connected subgraphs containing more than half the vertices of X, has order at least $k/2$. This means the following.

Corollary 2.2: For any fixed integer k, either $\beta_G \geq k/2$ or there is a cutset Y of vertices of G with $|Y| \leq k$ such that no component of $G - Y$ contains more than $\frac{2}{3}|G|$ vertices.

Moreover, if $\beta_G \geq k/2$, then for any set Y of vertices of G with $|Y| \leq k/2$, there is a unique component of $G - Y$ which contains an element of the bramble β_G. Therefore this unique component contains more than $\frac{1}{2}|G - Y|$ vertices. In addition, any component of $G - Y$ that does not contain an element of the bramble β_G contains at most $\frac{1}{3}|G|$ vertices.

Note that the last conclusion holds because if there is a component C in $G - Y$ that does not contain an element of the bramble β_G, but contains at least $\frac{1}{3}|G|$ vertices, then there is no component in $G - Y$ that contains more than $\frac{2}{3}|G - Y|$ vertices. Note that C contains at most $\frac{1}{2}|G|$ vertices.

A related result of the grid theorem is proved by Robertson, Seymour and Thomas [25], and Reed [19], independently. The second part of the following result, namely the algorithmic result, is obtained by Reed, see in [10], [12]. Namely:

Theorem 2.3: For any fixed integer k, there is a function $f(k)$ such that, for any $D \geq f(k)$, given any bramble β_X of order $f(k)$ for some vertex set X which either has size at most D or is $V(G)$, there is a wall W of height k, which is controlled by β_X, i.e, for every set Y of vertices of G of small order (say order less than k), the unique component of $G - Y$ containing a path from the top row of W to the bottom row of W contains more than half the vertices of X. It follows that this unique component contains an element of the bramble β_X.

Moreover, given the bramble β_X of order $f(k)$, there is a linear time algorithm to find the above wall W, which is controlled by β_X.

When we apply the algorithmic part of Theorem 2.3, we need to make clear how the bramble in Theorem 2.3 will be given. If we perform Theorem 2.1 and the conclusion is 2, then this implies that the bramble β_X for some small set X or $X = V(G)$ has order at least $k/2$. This bramble is one of keys for our algorithm, and will be used in Theorem 2.3.

III. Graphs without H-minors: Use of Bramble

A *vortex* is a pair $V = (G, \Omega)$, where G is a graph and $\Omega =: \Omega(V)$ is a linearly ordered set $(w_1, \dots, w_n)$ of vertices in G. These vertices are the *society vertices* of the vortex; their number n is its *length*. We do not always distinguish notationally between a vortex and its underlying graph or vertex set; for example, a *subgraph of V* is just a subgraph of G, a subset of V is a subset of $V(G)$, and so on. Also, we will often use Ω to refer both to the linear order of the vertices $w_1, \dots, w_n$ as well as the set of vertices $\{w_1, \dots, w_n\}$.

A path-decomposition $\mathcal{D} = (X_1, \ldots, X_m)$ of G is a *decomposition of* V if $m = n$ and $w_i \in X_i$ for all i. The *depth* of the vortex V is the minimum width of a path-decomposition of G that is a decomposition of V.

The *adhesion* of our decomposition $\mathcal{D}$ of V is the maximum value of $|X_{i-1} \cap X_i|$, taken over all $1 < i \leq n$. We define the *adhesion* of a vortex V as the minimum adhesion of a decomposition of that vortex. When $\mathcal{D}$ is a decomposition of a vortex V as above, we write $Z_i := (X_i \cap X_{i+1}) - \Omega$, for all $1 \leq i < n$. These Z_i are the *adhesion sets* of $\mathcal{D}$.

Let β be a bramble of order θ. For each separation (A, B) of order at most $\theta - 1$, we can distinguish one of A, B, say B, which we call a "big" side, i.e, by the definition of β, we may assume that B always contains one element of β. So when we talk about a separation (A, B) of order at most $\theta - 1$, we always assume that B is a big side. We sometimes call that A is a "small side".

For a bramble β of order θ in a graph G, let $Z \subseteq V(G)$ be a vertex set with $|Z| < \theta$. For the set of all separations (A', B') of $G - Z$ of order less than $\theta - |Z|$, there exists a separation (A, B) with $Z \subseteq A \cap B$, $A - Z = A'$ and $B - Z = B'$ such that B is a big side. It is shown in [19] that $G - Z$ has a bramble β' of order $\theta - |Z|$ in $G - Z$ which is obtained from each element of β by deleting Z. Thus we can also say that B' is a "big side".

Robertson and Seymour's main theorem is concerning the structure capturing a big side with respect to a bramble. Let us now state the result.

We remark that when we speak about *genus* of a surface Σ, we always mean its euler genus.

For a positive integer α, a graph G is *α-nearly embeddable* in a surface Σ if there is a subset $A \subseteq V(G)$ with $|A| \leq \alpha$ such that there are integers $\alpha' \leq \alpha$ and $n \geq \alpha'$ for which $G - A$ can be written as the union of $n+1$ edge-disjoint graphs $G_0, \ldots, G_n$ with the following properties: (i)

1) For all $1 \leq i \leq j \leq n$ and $\Omega_i := V(G_i \cap G_0)$, the pairs $(G_i, \Omega_i) =: V_i$ are vortices and $G_i \cap G_j \subseteq G_0$ when $i \neq j$.
2) The vortices $V_1, \ldots, V_{\alpha'}$ are disjoint and have adhesion at most α; we denote the set of these vortices by $\mathcal{V}$. We will sometimes refer to these vortices as *large* vortices.
3) The vortices $V_{\alpha'+1}, \ldots, V_n$ have length at most 3; we denote the set of these vortices by $\mathcal{W}$. These are the *small* vortices of the near-embedding. We sometimes say $G_0 \cup \mathcal{W}$ has a planar embedding, up to 3-separations.
4) There are closed discs in Σ with disjoint interiors $D_1, \ldots, D_n$ and an embedding $\sigma : G_0 \hookrightarrow \Sigma - \bigcup_{i=1}^{n} D_i$ such that $\sigma(G_0) \cap \partial D_i = \sigma(\Omega_i)$ for all i and the generic linear ordering of Ω_i is compatible with the natural cyclic ordering of its image (i.e., coincides with the linear ordering of $\sigma(\Omega_i)$ induced by $[0, 1)$ when ∂D_i is viewed as a suitable homeomorphic copy of $[0, 1]/\{0, 1\}$). For $i = 1, \ldots, n$ we think of the disc D_i as *accommodating* the (nonembedded) vortex V_i, and denote D_i as $D(V_i)$. In addition, we assume that V_i cannot be embedded into the disk D_i, up to 3-separations, with Ω_i in the natural cyclic order.

We call $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ an *α-near embedding* of G in Σ or just *near-embedding* if the bound is clear from the context. It *captures* a bramble β if the "big side" B of a separation $(A, B) \in G - Z$ is never contained in a vortex.

Let G'_0 be the graph resulting from G_0 by joining any two nonadjacent vertices $u, v \in G_0$ that lie in a common vortex $V \in \mathcal{W}$; the new edge uv of G'_0 will be called a *virtual edge*. By embedding these virtual edges disjointly in the discs $D(V)$ accommodating their vortex V, we extend our embedding $\sigma\colon G_0 \hookrightarrow \Sigma$ to an embedding $\sigma'\colon G'_0 \hookrightarrow \Sigma$. We shall not normally distinguish G'_0 from its image in Σ under σ'.

A vortex (G_i, Ω_i) is *properly attached* to G_0 if it satisfies the following two requirements. First, for every pair of distinct vertices $u, v \in \Omega_i$ the graph G_i must contain an Ω_i-path (one with no inner vertices in Ω_i) from u to v. Second, whenever $u, v, w \in \Omega_i$ are distinct vertices (not necessarily in this order), there are two internally disjoint Ω_i-paths in G_i linking u to v and v to w, respectively.

The *distance in* Σ of two vertices $x, y \in \Sigma$ is the minimal value of $|G'_0 \cap C|$ taken over all curves C in the surface that link x to y and meet the graph only in vertices. The *distance in* Σ of two vortices V and W is the minimum distance in Σ of a vertex in $\Omega(V)$ from a vertex in $\Omega(W)$. Similarly, the *distance in* Σ of two subgraphs H and H' of G'_0 is the minimum distance in Σ of a vertex in H from a vertex in H'.

A cycle C in Σ is *flat* if C bounds an open disc $D(C)$ in Σ. A flat triangle is a *boundary triangle* if it bounds a disc that is a face of G'_0 in Σ. Disjoint cycles $C_1, \ldots, C_n$ in Σ are *concentric* if they bound discs $D_1 \supseteq \ldots \supseteq D_n$ in Σ. A path system $\mathcal{P}$ (i.e, a set of disjoint paths) intersects $C_1, \ldots, C_n$ *orthogonally* if every path P in $\mathcal{P}$ intersects each of the cycles in a (possibly trivial) subpath of P.

For a near-embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of some graph G in a surface Σ and a vortex $V \in \mathcal{V}$, let $C_1, \ldots, C_n$ be cycles in G'_0 that are concentric in Σ. The cycles $C_1, \ldots, C_n$ *enclose* V if $D(C_n) \setminus \partial D(C_n)$ contains $\Omega(V)$. They *tightly enclose* V if the following holds:

[c] For every vertex $v \in V(C_k)$ and for all $1 \leq k \leq n$, there is a vertex $w \in \Omega(V)$ such that the distance of v and w in Σ is at most $n - k + 2$.

A closed curve C in Σ is *genus-reducing* if the (one or two) surfaces obtained by capping the holes of the components of $\Sigma \setminus C$ have smaller genus than Σ. Note that if C separates Σ and one of the two resulting surfaces is homeomorphic to S^2, the other is homeomorphic to Σ.

Hence in this case C is not genus-reducing.

The *representativity* of an embedding $G \hookrightarrow \Sigma \not\approx S^2$ is the smallest integer k such that every genus-reducing curve C in Σ that meets G only in vertices meets it in at least k vertices.

An α-near embedding $(\sigma, G_0, A, \mathcal{V}, \mathcal{W})$ of a graph G in some surface Σ is *δ-rich* for some integer δ if the following statements hold: (i)

1) G_0' contains a flat r-wall H for an integer $r \geq \delta$.
2) The representativity of G_0' in Σ is at least 3δ.
3) For every vortex $V \in \mathcal{V}$ there are δ concentric cycles $C_1(V), \ldots, C_\delta(V)$ in G_0' tightly enclosing V and bounding open discs $D_1(V) \supseteq \ldots \supseteq D_\delta(V)$, such that $D_\delta(V)$ contains $\Omega(V)$ and $D(H)$ does not intersect $D_1(V) \cup C_1(V)$. For distinct vortices $V, W \in \mathcal{V}$, the discs $\overline{D_1(V)}$ and $\overline{D_1(W)}$ are disjoint. (Sometime, the cycle $C_1(V)$ is called the outermost cycle, and the cycle $C_\delta(V)$ is called the innermost cycle.)
4) Every two vortices in $\mathcal{V}$ have distance at least 3δ in Σ.
5) For every vortex $V \in \mathcal{V}$, its set of society vertices $\Omega(V)$ is linked in G_0' to nails of H by a path system $\mathcal{P}(V)$ of δ disjoint paths having no inner vertices in H.
6) For every vortex $V \in \mathcal{V}$, the path system $\mathcal{P}(V)$ intersects the cycles $C_1(V), \ldots, C_\delta(V)$ orthogonally.
7) All vortices in $\mathcal{W}$ are properly attached to G_0.

We are now ready to state the Robertson and Seymour's main theorem, Theorem (3.1), in [23]. This theorem is concerning the structure relative to big sides of separations of small order, with respect to a given bramble β of large order. Actually, we use the following more subtle version of Theorem (3.1) in [23], which is shown in [6]. This version is already used in [3].

Theorem 3.1: For every graph R there is an integer α such that for every integer δ there is an integer $w = w(R, \delta)$ such that the following holds. Every graph G with a bramble β of order at least w that does not contain R as a minor has an α-near, δ-rich embedding in some surface Σ in which R cannot be embedded. Moreover, this α-near, δ-rich embedding captures the bramble β.

IV. Start of Proof

We now start proving our main theorem, Theorem 1.2. Suppose G is a graph with no K_t as a minor. Hereafter, n always means the number of vertices of G.

In this section, we shall give some useful concepts that are needed in our proof of Theorem 1.2.

A. Defining a cross-grid minor

We have to exclude a K_t-minor, but for our convenience, we will exclude a graph which contains K_t as a minor. Specifically, We construct this graph.

We start with a grid T of height $t+2$ (thus T has $(t+2)^2$ vertices and $2(t+1)(t+2)$ edges).

For each vertex $v_{i,j}$ (with $2 \leq i, j \leq t+1$) in T, we split it into four vertices $v_{i,j,1}, v_{i,j,2}, v_{i,j,3}, v_{i,j,4}$, such that

1) there is an edge between $v_{i,j,1}$ and $v_{i,j-1,3}$,
2) there is an edge between $v_{i,j,2}$ and $v_{i+1,j,4}$,
3) there is an edge between $v_{i,j,3}$ and $v_{i,j+1,1}$, and
4) there is an edge between $v_{i,j,4}$ and $v_{i-1,j,2}$.

Intuitively, i corresponds to the "row" of the grid T, while j corresponds to the "column" of the grid T. We then add the edges $v_{i,j,1}v_{i,j,3}, v_{i,j,2}v_{i,j,4}, v_{i,j,1}v_{i,j,2}$. We call the resulting grid T a *cross-grid CT of height t*.

Let $P_1, \ldots, P_{t+2}$ be the "column" of the grid T, and $P_1', \ldots, P_{t+2}'$ be the "low" of the grid T. Even after "splitting each vertex of $v_{i,j}$" as above, it is clear that such columns and lows can be defined. Thus we can define the the "column" P_i of the cross-grid CT and the "low" P_i' of the cross-grid CT in a natural way.

Let $T_i = P_i \cup P_i'$ for $i = 2, \ldots, t+1$. Then clearly $T_2, \ldots, T_{t+1}$ give rise to a K_t-minor. Thus any graph with no K_t-minor has no cross-grid CT of height t as a minor.

There is a big advantage to consider the graph CT of height t rather than K_t. Crossing number of the graph CT of height t is t^2 as easily seen. On the other hand, crossing number of K_t could be $\Omega(t^4)$. This difference makes our proof much easier. We shall see this advantage in the next section, Section V.

B. Defining a bramble and applying Theorem 3.1

We shall define the bramble β_G, which also plays a key role in our proof. By Corollary 2.2 with $k \geq 2w$, where w comes from Theorem 3.1 with $R = CT$ of height t, either there is a $2/3$-separator of order $2w$ (in which case, we are done, as this separator is as desired in Theorem 1.2), or the bramble β_G has order at least w.

Thus we can apply Theorem 3.1 with $R = CT$ of height t with respect to this bramble β_G. Since the α-near, δ-rich embedding in Theorem 3.1 captures the bramble β_G, thus by Corollary 2.2, we may assume that each small vortex in G has at most $n/3$ vertices and each piece of the decomposition of each large vortex in G has at most $n/3$ vertices. We shall use this fact many times in this paper.

For the rest of the paper, we assume the notations in Theorem 3.1 and the notations in Section III.

V. Bounding genus plus vortices

In this section we shall take advantage of the cross-grid CT of height t. Recall that we are given the structure as in Theorem 3.1 with $R = CT$ of height t (with respect to the bramble β_G). We first observe that the cross-grid CT of height t has crossing number t^2, and hence genus of CT is t^2. Let g be genus of G_0'.

Each large vortex V_i in $\mathcal{V}$ cannot be embedded into the disk D_i, up to 3-separations, with Ω_i in the boundary. Thus by the result in [26] or [28], for each large vortex $V \in \mathcal{V}$ and its tightly enclosing outermost cycle $C_1(V)$ in G'_0, we may assume that there are four vertices a, b, c, d that appear in $C_1(V)$ in this order, in such a way that there are two disjoint paths P_1, P_2 in $D_1(V) \cup C_1(V)$, such that P_1 joins a and c, and P_2 joins b and d. Note that the distance between $C_1(V')$ and $C_1(V'')$ in G'_0 is at least δ for any two vortices $V', V'' \in \mathcal{V}$ by the property (iv) of the δ-rich. Since the paths P_1 and P_2 have to pass through all the cycles $C_\delta(V), \ldots, C_1(V)$ in G'_0 that tightly enclose a large vortex $V \in \mathcal{V}$ in this order (because the graph bounded by two cycles $C_1(V)$ and $C_\delta(V)$ in G'_0 is planar with the outer cycle $C_1(V)$ and the inner cycle $C_\delta(V)$), thus we can assume that a, b, c, d are contained in the paths system $\mathcal{P}(V)$ in the property (vi) of the δ-rich. Hence by the properties (v) and (vi) of the δ-rich, there are four disjoint paths from a, b, c, d to some nails of H in $G'_0 - D_1(V)$. Therefore, we may assume that a, b, c, d satisfy the notion "free" in the language of (4.5) in [22] (or in the language of (9.4) in [24]). Let us call this configuration *eye* (the reader may figure out the reason why we call it *eye*, see [23]). We can now use a result in [22].

Theorem 5.1: If $\alpha' + g > t^2$, then G has a K_t-minor.

Proof. It is easy to see that the cross-grid CT of height t can be embedded in a surface S of genus g with q crossings, such that each of q crossings is single-crossing and pairwise disjoint, where $g+q = t^2$. Suppose G has the structure as in Theorem 3.1. By the above remarks, there are α' eyes that are pairwise distance at least δ. It follows from (4.5) in [22] (or (9.4) in [24]) that if $g + \alpha' \geq t^2$ (with $\delta \geq \theta$, where θ is given in (4.5) in [22]. We assume $\delta \geq \theta$ hereafter), then G has the cross-grid CT of height t as a minor. Since the cross-grid CT of height t has K_t as a minor, thus G has K_t-minor as a minor as well.■

VI. Planarizing subgraphs

In this section, we shall consider a graph H on a fixed surface with genus g, and find a subgraph W such that $H - W$ is planar. Such a graph W is called a *planarizing* graph.

Let H be a graph embedded into a surface S of genus g. Recall that a *noncontractible* curve C in H is a curve C hitting only the vertices of H such that if we delete all the vertices that hit C (we shall refer to this vertex set as $V(C)$) from H, then genus of the resulting graph of H is less than g. Such a noncontractible curve is called *surface separating* if it divides the surface S into two regions, none of which is sphere. Otherwise, we call it *surface nonseparating*. It is well-known that there are $2g - 2$ different types of surface nonseparating noncontractible curves of S, and there are $g - 1$ different types of surface separating noncontractible curves of S (see [17]).

Let W be a planarizing subgraph of H that is embedded into a surface S of genus g. Then by taking W minimal, we may assume that W consists of at most $2g-2$ different types of minimal surface nonseparating noncontractible curves of S, and at most $g - 1$ different types of minimal surface separating noncontractible curves in S (thus there are at most $3g - 3$ noncontractible curves in W. Note that these curves are not necessarily disjoint). Moreover, we may assume that each of these at most $3g-3$ curves passes through each face of H at most once (otherwise, we can "shorten" the curve, see Lemma 10 in [6] for more details).

Having given the above remarks, we can state the main result in Djidjev [4], and Gilbert, Hutchinson and Tarjan [8], independently. The best known constant is given by Eppstein [7].

Theorem 6.1: Any n-vertex graph H embedded into a surface S of genus g has a planarizing subgraph that consists of at most $2g-2$ different types of minimal surface nonseparating noncontractible curves of S, and at most $g-1$ different types of minimal surface separating noncontractible curves of S (thus there are at most $3g - 3$ noncontractible curves in W. Note that these curves are not necessarily disjoint), such that $|W| \leq 2\sqrt{g|H|}$. Moreover, given an embedded graph H in S of genus g, there is an $O(n)$ time algorithm to find such a planarizing subgraph W.

Let us recall that by Theorem 5.1, we may assume that $g < t^2$, where g is genus of G_0.

We now apply Theorem 6.1 to G'_0 that is embedded into a surface S of genus g. Recall the definitions of G_0 and G'_0 from Section III. Let W be a planarizing subgraph of G'_0 that is obtained from Theorem 6.1. Let us remind that G_0 has an embedded graph, up to 3-separations. Since each small vortex is attached to G_0 at most three vertices in a face of G'_0 and since G'_0 can be embedded into S, so it follows that $G_0 - W$ is a planar graph, up to 3-separations. We point out the reader that the fact that $G_0 - W$ is planar, up to 3-separations, if W is a planarizing subgraph of G'_0, will be used in this paper many places without specifically mentioning.

We have to deal with the α' disks $D_1, \ldots, D_{\alpha'}$ in G that large vortices are attached to. By the above remark,

1) Each of minimal surface nonseparating noncontractible curves of S in W passes through each disk D_i at most once for $i = 1, \ldots, \alpha'$. There are at most $2g - 2$ different types of such curves.
2) Each of minimal surface separating noncontractible curves of S in W passes through each disk D_i at most once for $i = 1, \ldots, \alpha'$. There are at most $g - 1$ different types of such curves.

Let us observe that for each above minimal noncontractible curve C that hits a disk D_i (and hence it also hits a large vortex V_i), if we delete at most two adhesion sets of the decomposition of V_i, then the curve C can separate the vortex V_i in $G_0 \cup G_1 \cup \ldots, \cup G_{\alpha'}$. Therefore, after deleting at most $3g \times \alpha' \times 2\alpha$ vertices of adhesion sets of large vortices, and after deleting the apex set A that has at most α vertices, we get a subgraph W' that is obtained from W by adding these vertices, such that $G - W'$ consists of one planar graph, up to 3-separations, with at most $2^{3g} \times \alpha'$ large vortices. Note that each above minimal noncontractible curve could divide each large vortex into two parts. Thus we have at most $2^{3g} \times \alpha'$ large vortices in total in $G - W'$. In summary;

Theorem 6.2: G has a subgraph W' of order at most $2\sqrt{gn} + 3g \times \alpha' \times 2\alpha + \alpha$ such that $G - W'$ consists of one planar graph G_0, up to 3-separations, with at most $2^{3g} \times \alpha'$ large vortices. Moreover, by our choice of the bramble β_G, both each small vortex in $G - W'$ and each piece of the decomposition of each large vortex in $G - W'$ have at most $n/3$ vertices by Corollary 2.2.

By Theorem 5.1, $g \leq t^2$. Thus $|W'| \leq 2t\sqrt{n} + (3t^2 \times 2\alpha' + 1)\alpha$.

VII. Merging and cutting off vortices

In this section, we shall start with Theorem 6.2. Then we would obtain the δ-rich structure in $G - W'$. We shall show here how to obtain the δ-rich structure in the sphere in $G - W'$. Let us point out that all the arguments here are given in [6] (c.f, Lemma 12). So we just give a sketch.

We proceed as follows: First, recall the definition G_0' which is a planar graph (Note that $G - W'$ is a planar graph, up to 3-separations, with at most $2^{3g} \times \alpha'$ large vortices).

We then apply Lemma 12 in [6] to a planar graph $G - W'$, up to 3-separations, with at most $2^{3g} \times \alpha'$ large vortices. More precisely, if two vortices in $G - W'$ have distance at most 3δ in G_0', then we cut through the shortest curve between these large vortices, and merge them into one. Also, if there is a contractible curve C of order at most 3δ in G_0' such that both sides divided by this curve C have at least one vortex, then we cut through this curve, and reduce the number of vortices. Since all the details are exactly the same as those given in the proof of Lemma 12 in [6], we omit the details of the proof. In summary, since there are at most $2^{3g} \times \alpha'$ large vortices, we can confirm the following:

Lemma 7.1: After deleting at most $3\delta \times 2^{3g} \times \alpha'$ vertices and adding them to W', $G - W'$ consists of components $Q_1, \dots, Q_l$, where $l \leq f(t)$ for some function f of t, such that each satisfies an α-near, δ-rich embedding in the sphere.

Moreover by our choice of the bramble β_G, both each small vortex in Q_i and each piece of the decomposition of each large vortex in Q_i have at most $n/3$ vertices for $i = 1, \dots, l$ by Corollary 2.2.

In addition, the order of W' is at most $2\sqrt{gn} + 3g \times \alpha' \times 2\alpha + \alpha + 3\delta \times 2^{3g} \times \alpha' \leq 2t\sqrt{n} + (6t^2\alpha + 1 + 3\delta \times 2^{3t^2})\alpha$ (note that $\alpha' \leq \alpha$, and $g < t^2$ by Theorem 5.1).

Let us observe that if there are two Q_i that have at least $n/3$ vertices or if there is no such Q_i, we can find a desired $2/3$-separator W' of order $O(t\sqrt{n})$ as in Theorem 1.2. Thus we can assume that Q_1 only has at least $n/3$ vertices.

So Q_1 consists of a planar graph, up to 3-separations, together with large vortices $V_1, \dots, V_q$. Moreover by our choice of the bramble β_G, both each small vortex in Q_1 and each piece of the decomposition of each large vortex in Q_1 have at most $n/3$ vertices by Corollary 2.2. By Theorem 5.1, $q < t^2$.

We now prove the following claim.

Claim 7.2: If $q \geq 2$, then one of the following holds;

1) $Q_{1,0}'$ has a contractible curve C of order at most $\sqrt{n/q}$ such that both disks bounded by C contain at least one large vortex in Q_1' (Recall the definition of the graph $Q_{1,0}'$, which corresponds to G_0' for G in Section III).
2) There are two large vortices that have distance at most $2\sqrt{n/q}$ in $Q_{1,0}'$, i.e, there is a curve of order at most $2\sqrt{n/q}$ joining these two vortices in $Q_{1,0}'$.

Proof. Suppose none of them holds. For each large vortex $V \in \mathcal{V}$, we pick up $\sqrt{n/q}$ concentric cycles tightly enclosing V in $Q_{1,0}'$. By the definition of *tightly enclosing* V, we may assume that the outermost cycle of tightly enclosing V does not hit the outermost cycle of tightly enclosing any other large vortex in $Q_{1,0}'$, for otherwise the second conclusion holds. Let $C_1(V), C_{\sqrt{n/q}}(V)$ be the outermost cycle and the innermost cycle of tightly enclosing V in $Q_{1,0}'$, respectively.

We claim that there are at least $\sqrt{n/q} + 1$ vertex-disjoint paths between $C_1(V)$ and $C_{\sqrt{n/q}}(V)$ in the cylinder in $Q_{1,0}'$ bounded by $C_1(V)$ and $C_{\sqrt{n/q}}(V)$. Otherwise there is a cutset of order at most $\sqrt{n/q}$ in $Q_{1,0}'$ that separates $C_1(V)$ and $C_{\sqrt{n/q}}(V)$, which implies that there is a contractible curve C of order at most $\sqrt{n/q}$ in $Q_{1,0}'$ such that both disks divided by C contain at least one large vortex in Q_1. Then the first conclusion holds.

Thus there are at least $\sqrt{n/q} + 1$ vertex-disjoint paths between C_1 and $C_{\sqrt{n/q}}$ in the cylinder in $Q_{1,0}'$ bounded by $C_1(V)$ and $C_{\sqrt{n/q}}(V)$. Since each path must intersect all the cycles $C_{\sqrt{n/q}}(V), \dots, C_1(V)$, this means that $D_1(V) \cup C_1(V)$ contains at least $n/q + 1$ vertices for each large vortex $V \in \mathcal{V}$, which implies that $Q_{1,0}'$ contains more than n vertices, a contradiction. This completes the proof.■

Let us observe that, as remarked in Section VI, all the curves in $Q_{1,0}'$ mentioned in this proof can be extended to $Q_{1,0}$ and hence to Q_1. Thus hereafter, we will talk about such curves in Q_1.

Suppose Q_1 contains at least two large vortices. Let C' be the curve obtained in one of the conclusions in Claim 7.2. So C' has order at most $2\sqrt{n/q}$. We then delete C' from Q_1.

We then keep applying Claim 7.2 to component(s) of $Q_1 - C'$ that contain(s) at least $n/3$ vertices, and then deleting the curves obtained from Claim 7.2 from this component(s), until each component contains either at most $n/3$ vertices or at most one large vortex. We claim that if there are two components that have at least $n/3$ vertices in some iteration of this process, or if there is no such a component in some iteration of this process, we can find a desired 2/3-separator of order $O(t\sqrt{n})$ as in Theorem 1.2. Indeed, since there are $q \leq t^2$ large vortices in Q_1 and since each time we perform Claim 7.2, the number of large vortices goes down for each component (and hence there are at most $q-1$ iterations, and we delete at most $\sum_{j=1}^{q} 2\sqrt{n/j} \leq 2\sqrt{qn} \leq 2t\sqrt{n}$ vertices in total (by applying Claim 7.2) in this process), thus it follows that we have exactly one component that has at least $n/3$ vertices in each iteration of this process (otherwise, as remarked just after Lemma 7.1, we can find a desired 2/3-separator of order $O(t\sqrt{n})$ as in Theorem 1.2, which is obtained from W' and at most $q-1$ curves obtained from Claim 7.3). Therefore, in the end, we obtain the following;

Lemma 7.3: After applying Claim 7.2 at most $q-1 \leq t^2$ times, and deleting at most $\sum_{j=1}^{q} 2\sqrt{n/j} \leq 2\sqrt{qn} \leq 2t\sqrt{n}$ vertices (letting them be T), $G - W' - T$ consists of components $Q'_1, \dots, Q'_x$, where $x < f'(t)$ for some function f' of t, such that all Q'_i, except for Q'_1, contain at most $n/3$ vertices (hence only Q'_1 contains at least $n/3$ vertices), and Q'_1 is a planar graph, up to 3-separations, with at most one large vortex. Moreover by our choice of the bramble β_G, both each small vortex in Q'_1 and each piece of the decomposition of the large vortex in Q'_1 have at most $n/3$ vertices by Corollary 2.2.

VIII. Planar graph with one vortex

In this section, we shall find a 2/3-separator in a planar graph, up to 3-separations, with exactly one large vortex. In order to do so, we shall use the following result by Alon, Seymour and Thomas [2].

Theorem 8.1: Let G be a plane graph with n vertices and for each vertex $v \in V(G)$, let $w(v) > 0$ be a real number. There is a contractible curve C hitting only vertices of G with $|C| \leq 3/2\sqrt{n}$ such that, letting two disks divided by C be D_1, D_2, $\sum_{v \in D_i} w(v) + 1/2 \sum_{v \in C} w(v) \leq 2/3 \sum_{v \in V(G)} (v)$ for $i = 1, 2$.

We are now ready to prove the following main result in this section.

Theorem 8.2: Let H be a planar n-vertex graph, up to 3-separations, with at most one large vortex H_1 of adhesion α'. Moreover both each small vortex in H and each piece of the decomposition of the large vortex H_1 have at most $n/3$ vertices. Then H has a 2/3-separator of order $7/2\sqrt{n} + 6\alpha$ vertices.

Proof of Theorem 8.2 will be given in the full paper.

IX. Putting together

We now give a full proof of Theorem 1.2. For our convenience, we shall give a proof by step by step. This would give a polynomial time algorithm to construct a 2/3-separator in Theorem 1.2. Suppose a graph G with no K_t as a minor is given.

Step 1. We first confirm that the bramble β_G has large order so that we can apply the main structure theorem (Theorem 3.1) of Robertson and Seymour. Otherwise, we can easily find a 2/3-separator of order $O(t\sqrt{n})$ (for more details, see Corollary 2.2).

We then apply Theorem 3.1 with the bramble β_G and $R = CT$ of height t to G. As discussed in Section IV, the cross grid CT of height t has K_t as a minor. Thus graphs with no K_t as a minor do not contain CT of height t as a minor. Let us observe that CT of height t has crossing number t^2, and hence genus of CT of height t is t^2. Let g be genus of G_0. By Theorem 5.1, $g + \alpha' \leq t^2$.

Step 2. By Theorem 6.2, G has a subgraph W' of order at most $2\sqrt{gn} + 3g \times \alpha' \times 2\alpha + \alpha$ such that $G - W'$ consists of one planar graph, up to 3-separations, with at most $2^{3g} \times \alpha'$ large vortices. Moreover, by our choice of the bramble β_G, both each small vortex in $G - W'$ and each piece of the decomposition of each large vortex in $G - W'$ have at most $n/3$ vertices by Corollary 2.2.

Step 3. As in Lemma 7.1, after deleting at most $3\delta \times 2^{3g} \times \alpha'$ vertices, $G - W'$ consists of several components $Q_1, \dots, Q_l$, where $l \leq f(t)$ for some function f of t, such that each satisfies an α-near, δ-rich embedding in the sphere. Moreover, by our choice of the bramble β_G, both each small vortex in each Q_i and each piece of the decomposition of each large vortex in each Q_i have at most $n/3$ vertices by Corollary 2.2.

By Lemma 7.3, there is a vertex set T of order at most $2t\sqrt{n}$ such that $G - W' - T$ consists of components $Q'_1, \dots, Q'_x$, where $x \leq f'(t)$ for some function f' of t, in such a way that all Q'_i, except for Q'_1, contain at most $n/3$ vertices (and hence only Q'_1 contains at least $n/3$ vertices), and Q'_1 is a planar graph, up to 3-separations, with at most one large vortex. Moreover, by our choice of the bramble β_G, both each small vortex in Q'_1 and each piece of the decomposition of the large vortex in Q'_1 have at most $n/3$ vertices by Corollary 2.2.

Step 4. By Theorem 8.2, Q'_1 has a 2/3-separator S of order at most $7/2\sqrt{|Q'_1|} + 6\alpha' \leq 7/2\sqrt{n} + 6\alpha'$. This implies that G has a 2/3-separator that consists of $W' \cup T \cup S$, with order at most

$$\begin{aligned}
|W'| + |T| + 7/2\sqrt{n} + 6\alpha' + \delta \times 2^{3g} \times \alpha' \\
\leq (2\sqrt{gn} + 3g \times \alpha' \times 2\alpha + \alpha) \\
+2t\sqrt{n} + (7/2\sqrt{n} + 6\alpha') + 3\delta \times 2^{3g} \times \alpha' \\
\leq (2t + 2\sqrt{g} + 7/2)\sqrt{n} + (6g\alpha + 6 + 1 + 3\delta \times 2^{3g})\alpha \\
\leq (4t + 7/2)\sqrt{n} + (2^{3t^2} \times 3\delta + 6t^2\alpha + 7)\alpha.
\end{aligned}$$

Note that $\alpha' \leq \alpha$ and by Theorem 5.1, $g \leq t^2$. This completes the proof of Theorem 1.2.■

X. Concluding remarks

We now make the proof in Section IX algorithmic. Suppose a graph G with no K_t as a minor is given. Let us first mention that for our algorithm, we assume that t is fixed.

First, the first half of Step 1 can be done by Theorem 2.1. We can also find a desired subgraph W' in Theorem 6.2 in $O(n)$ time by Theorem 6.1. Thus Step 2 can be done in $O(n)$ time.

Both the proof of Lemma 7.1 and the proof of Claim 7.2 can be implemented in $O(n)$ time because $q \leq t^2$, and

1) we can find a shortest curve between any two large vortices in the proof of Lemma 7.1 and in the proof of Claim 7.2, in $O(n)$ time, and
2) each contractible curve in the proof of Lemma 7.1 and in the proof of Claim 7.2, can be found in $O(n)$ time by finding a minimum cut. More precisely, for each face C that a large vortex is attached to, we add a vertex v to C, and add edges between v and all the society vertices in C. Then we try to find a smallest separation that cuts off v from the rest of the graph. This can be clearly computed in $O(n)$ time by using the standard max-flow method.

Since we apply Lemma 7.1 once, and Claim 7.2 at most t^2 times, to obtain a desired vertex set T in Lemma 7.3, thus Step 3 can be done in $O(n)$.

The proof of Lemma 8.1 in [2] can be implemented in $O(n^2)$ time. Thus a desired 2/3-separator in Theorem 8.2 can be found in $O(n^2)$ time, and hence Step 4 can be done in $O(n^2)$ time because we only need algorithms for Lemma 8.1 and Theorem 6.1 (which can be done in $O(n)$ time), respectively.

All other steps actually depend on Theorems 3.1.

In [5], a polynomial time algorithm to construct the structure as in Theorem 3.1 is given for fixed t. This result is further extended in [11]. Namely, the structure as in Theorem 3.1 can be found in $O(n^2)$ time for fixed t. Thus we can find a 2/3-separator in Theorem 1.2 in $O(n^2)$ time, if we fix t.

We can actually do it faster. First, in [9], Gu and Tamaki gave an $O(n^{1+\varepsilon})$ time algorithm to give a constant-factor approximation of the tallest height of a grid minor for planar graphs for any $\varepsilon > 0$. Note that, as pointed by Alon, Seymour and Thomas [1], tree-width is essentially the same as the order of a 2/3-separator for planar graphs, and hence the tallest height of a grid minor is essentially the same as the order of a 2/3-separator for planar graphs by the result in [25] (which says that every planar graph with tree-width $6r$ contains a grid minor of height r) . In addition, if we get a constant-factor approximation of the tallest height of a grid minor for planar graphs, this can be used to find a contractible curve C as in Lemma 8.1 by taking some cycle of the grid minor. Thus using the algorithm in [9], we can get a 2/3-separator of order $O(\sqrt{n})$ as in Theorem 8.2 in $O(n^{1+\varepsilon})$ time. This implies that Step 4 can be done in $O(n^{1+\varepsilon})$ time for any $\varepsilon > 0$.

Recently, together with Z. Li, we have made all the graph minor proofs algorithmic. Moreover, we develop some further arguments beyond graph minor theory in [12] in order to give a faster algorithm for the minor testing problem. At the moment, we believe that we have an $O(npoly(\log n))$ algorithm to construct the structure as in Theorems 3.1 (but the details are not yet fully written).

Thus together with these improvements, we may be able to get an $O(n^{1+\varepsilon})$ time algorithm to get a 2/3-separator as in Theorem 1.2 for any $\varepsilon > 0$.

Finally, let us mention some algorithmic aspect for the case when we just need to find a 2/3-separator of order $O(\sqrt{n})$, i.e, our purpose here is to find a 2/3-separator of order at most $g(t)\sqrt{n}$ for some function g of t. For our convenience, we just say $O(\sqrt{n})$ (without mentioning function $g(t)$) for the order of a 2/3-separator.

Reed and Wood [20] gives an $O(n)$ time algorithm to find a 2/3-separator of order $n^{2/3}$. We may be able to find a 2/3-separator of order $O(\sqrt{n})$ in $O(n)$ time, using the heavy machinery from the whole graph minor theory, and further development in [12]. This is only a rough sketch because the full details are not yet written.

As mentioned in Step 1 in Section IX, we begin with the bramble β_G. If 1 of Theorem 2.1 holds, then we have a desired 2/3-separator. Otherwise, we apply Theorem 2.3 with this bramble β_G to obtain a wall W as in Theorem 2.3.

We then apply the whole graph minor proofs with this wall W. What we need is to give an $O(n)$ time algorithm to construct the structure as in Theorem 3.1.

In addition, we need an $O(n)$ time algorithm to construct a 2/3-separator as in Lemma 8.1, with the expression $7/2\sqrt{n}$ replaced by $\alpha\sqrt{n}$. This can be done by directly applying an $O(n)$ time algorithm for the planar separator theorem [16], because we can delete as many as $\sqrt{n}$ adhesion sets of the unique large vortex (in fact, if we put weights as in the proof of Theorem 8.2, then a 2/3-separator of order $3/2\sqrt{n}$ in the resulting planar graph can be extended to a 2/3-separator of order $\alpha\sqrt{n}$ in the original graph which is planar, up to 3-separations, with one vortex). Hence we can construct a 2/3-separator as in Theorem 8.2 in $O(n)$ time with the expression $7/2\sqrt{n}$ replaced by $\alpha\sqrt{n}$.

Once we have the structure as in Theorem 3.1, we can confirm that we can find a 2/3-separator of order $O(\sqrt{n})$ in $O(n)$ time because;

1) vortices are dealt with by an $O(n)$ time algorithm to construct a 2/3-separator as in Theorem 8.2 (with the expression $7/2\sqrt{n}$ replaced by $\alpha\sqrt{n}$), and
2) we can then directly apply the separator theorem for bounded genus graphs by Djidjev [4], and Gulbert, Hutchinson and Tarjan [8], independently, to the rest of the graph to get a 2/3-separator of order $O(\sqrt{n})$, which can be done in $O(n)$ time.

Note that this algorithm only works if we would get a 2/3-separator of order $g(t)\sqrt{n}$ for some function g of t, and it would not work if we would need to get a 2/3-separator of order $O(t\sqrt{n})$. Thus in order to prove Theorem 1.2, we need a lot more arguments, as we did.

It remains to obtain the structure as in Theorem 3.1 in $O(n)$ time. We can do it, except for one subroutine which is the society theorem, namely (11.11) in [21]. At the moment, we believe we have an $O(n \log n)$ algorithm to find one outcome of the society theorem in [21]. However, if we add one conclusion "there is a $2/3$-separator of order $O(\sqrt{n})$ in G" to (11.11) in [21], we believe we have an $O(n)$ time algorithm for this. Consequently, we believe we have an $O(n)$ algorithm to find a $2/3$-separator of order $O(\sqrt{n})$.

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