Some Pascal-like triangles

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Abstract

We collect some simple facts about analogues of Pascal's triangle where the entries count subsets of the integers with an even or odd sum of its elements. A widely known example is Losanitsch's triangle.

1. Introduction

The entries $\binom{n}{k}$ of Pascal's triangle count the subsets of $\{1,\cdots,n\}$ with k elements which will be called $k-$sets for short.
A $k-$set S will be called *even*, if the sum of its elements is even and *odd* if this sum is odd. By convention the empty set is even.
Let $E_{n,k}$ be the set of all even $k-$subsets of $\{1,\cdots,n\}$ and $e(n,k)=|E_{n,k}|$ the number of its elements and let $O_{n,k}$ be the set of all odd $k-$subsets of $\{1,\cdots,n\}$ and $o(n,k)=|O_{n,k}|$ the number of its elements.
For example $e(5,3)=6$ because $E_{5,3}=\{\{1,2,3\},\{1,2,5\},\{1,3,4\},\{1,4,5\},\{2,3,5\},\{3,4,5\}\}$.
Let us note the trivial fact

$$e(n,k)+o(n,k)=\binom{n}{k}. \tag{1.1}$$

Lemma 1
If k is odd then $o(2n,k)=e(2n,k)$ because $\{s_1,\cdots,s_k\}\leftrightarrow\{2n+1-s_1,\cdots,2n+1-s_k\}$ is a bijection.

Lemma 2

$$\begin{aligned} e(n,k)&=e(n-2,k)+\binom{n-2}{k-1}+o(n-2,k-2),\\ o(n,k)&=o(n-2,k)+\binom{n-2}{k-1}+e(n-2,k-2). \end{aligned} \tag{1.2}$$

Proof

There are 3 possibilities.

a) A $k-$subset of $\{1,\cdots,n\}$ is a $k-$subset of $\{1,\cdots,n-2\}$,

b) it contains precisely one of the numbers $n-1$ and n. The remaining $(k-1)-$set is then an arbitrary subset of $\{1,\cdots,n-2\}$. There are $\binom{n-2}{k-1}$ such subsets.

c) It contains both $n-1$ and $n.$ Since $n+(n-1)$ is odd the remaining $(k-2)-$subset must have the opposite parity of the $k-$subsets.

Let e_k be the column with entries $e(n,k)$ and o_k the column with entries $o(n,k)$ for $n \in \mathbb{N}$. We consider some matrices whose columns are e_k or o_k.

2. Matrices where the columns c_k and c_{k+2} have the same parity

Ia) Let us first consider the matrix $\left(e(n,k)\right) = \left(e_0, e_1, e_2, e_3, \cdots\right)$ whose entries are the number of even sets (Cf. OEIS [3], A282011). The first terms are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 6 & 3 & 0 & 0 \\ 1 & 3 & 6 & 10 & 9 & 3 & 0 \end{pmatrix}$$

Proposition 2.1

The numbers $e(n,k)$ satisfy

$$e(n,k) = e(n-1,k) + e(n-1,k-1) \tag{2.1}$$

if kn is even and

$$e(2n+1,2k+1) = e(2n,2k+1) + e(2n,2k) + (-1)^{k-1}\binom{n}{k}. \tag{2.2}$$

Proof
Consider the difference

$$d(n,k) = e(n,k) - o(n,k) \tag{2.3}$$

and let

$$d_n(x) = \sum_{k=0}^{n} d(n,k)x^k. \tag{2.4}$$

Since the right-hand-side is $\sum_k \sum_{j_1<\cdots<j_k} (-1)^{j_1+\cdots+j_k} x^k = \prod_{j=1}^{n}\left(1+(-1)^j x\right)$

we see that

$$d_n(x) = \prod_{j=1}^{n}\left(1+(-1)^j x\right)$$

which gives by induction

$$d_n(x) = (1+x)^{\left\lfloor \frac{n}{2} \right\rfloor}(1-x)^{\left\lfloor \frac{n+1}{2} \right\rfloor}. \tag{2.5}$$

Thus

$$\begin{aligned} d_{2n}(x) &= \left(1-x^2\right)^n, \\ d_{2n+1}(x) &= (1-x)\left(1-x^2\right)^n. \end{aligned} \tag{2.6}$$

By (1.1) and (2.3) we get $e(n,k)=\dfrac{\binom{n}{k}+d(n,k)}{2}$ and therefore

$$e_{2n}(x)=\frac{(1+x)^{2n}+\left(1-x^2\right)^n}{2},$$
$$e_{2n+1}(x)=\frac{(1+x)^{2n+1}+(1-x)\left(1-x^2\right)^n}{2}. \tag{2.7}$$

This implies

$$e_{2n}(x)=(1+x)e_{2n-1}(x),$$
$$e_{2n+1}(x)=(1+x)e_{2n}(x)-x\left(1-x^2\right)^n, \tag{2.8}$$

which is equivalent with Proposition 2.1.

Since $\sum_{k=0}^{n}e(n,n-k)x^k=x^n e_n\left(\frac{1}{x}\right)$ (2.7) implies

Corollary 2.1

$$e(n,n-k)=e(n,k) \text{ for } n\equiv 0,3 \bmod 4,$$
$$e(n,n-k)=o(n,k) \text{ for } n\equiv 1,2 \bmod 4. \tag{2.9}$$

Let us also derive some explicit formulae. From

$$e_n(x)=(1+x)^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{(1+x)^{\left\lfloor \frac{n+1}{2}\right\rfloor}+(1-x)^{\left\lfloor \frac{n+1}{2}\right\rfloor}}{2}=(1+x)^{\left\lfloor \frac{n}{2}\right\rfloor}\sum_j\binom{\left\lfloor \frac{n+1}{2}\right\rfloor}{2j}x^{2j} \tag{2.10}$$

we get

$$e(n,k)=\sum_j\binom{\left\lfloor \frac{n+1}{2}\right\rfloor}{2j}\binom{\left\lfloor \frac{n}{2}\right\rfloor}{k-2j}. \tag{2.11}$$

As special case we get the well-known formula

$$e(2n,n)=\sum_k\binom{n}{2k}^2. \tag{2.12}$$

The numbers $e(n,k)$ are given explicitly by

$$\begin{aligned}
2e(2n,2k) &= \binom{2n}{2k} + (-1)^k \binom{n}{k}, \\
2e(2n+1,2k) &= \binom{2n+1}{2k} + (-1)^k \binom{n}{k} \\
2e(2n,2k+1) &= \binom{2n}{2k+1} \\
2e(2n+1,2k+1) &= \binom{2n+1}{2k+1} + (-1)^{k+1} \binom{n}{k}
\end{aligned} \tag{2.13}$$

Therefore the generating functions are

$$\sum_{n\geq 0} e(n,2k)x^n = \frac{x^{2k}}{2}\left(\frac{1}{(1-x)^{2k+1}} + \frac{(-1)^k(1+x)}{\left(1-x^2\right)^{k+1}}\right) \tag{2.14}$$

$$\sum_{n\geq 0} e(n,2k+1)x^n = \frac{x^{2k+1}}{2}\left(\frac{1}{(1-x)^{2k+2}} + \frac{(-1)^{k+1}}{\left(1-x^2\right)^{k+1}}\right) \tag{2.15}$$

These can also be written in the following way:

$$\begin{aligned}
\sum_{n\geq 0} e(n,4k)x^n &= \frac{x^{4k}e_{4k}(x)}{(1+x)^{4k}(1-x)^{4k+1}}, \\
\sum_{n\geq 0} e(n,4k+1)x^n &= \frac{x^{4k+1}o_{4k+1}(x)}{(1+x)^{4k+1}(1-x)^{4k+2}}, \\
\sum_{n\geq 0} e(n,4k+2)x^n &= \frac{x^{4k+2}o_{4k+2}(x)}{(1+x)^{4k+2}(1-x)^{4k+3}}, \\
\sum_{n\geq 0} e(n,4k+3)x^n &= \frac{x^{4k+3}e_{4k+3}(x)}{(1+x)^{4k+3}(1-x)^{4k+4}}.
\end{aligned} \tag{2.16}$$

Ib) The matrix $\left(o(n,k)\right) = \left(o_0, o_1, o_2, o_3, \cdots\right)$, cf. OEIS [3], A 159916.

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 2 & 4 & 2 & 0 & 0 & 0 \\
0 & 3 & 6 & 4 & 2 & 1 & 0 \\
0 & 3 & 9 & 10 & 6 & 3 & 1
\end{pmatrix}$$

In the same way as above we get

$$o_n(x) = \frac{(1+x)^n - (1+x)^{\left\lfloor \frac{n}{2} \right\rfloor}(1-x)^{\left\lfloor \frac{n+1}{2} \right\rfloor}}{2}. \tag{2.17}$$

$$o(n,k)=\sum_j \binom{\left\lfloor \frac{n+1}{2} \right\rfloor}{2j+1}\binom{\left\lfloor \frac{n}{2} \right\rfloor}{k-2j-1}. \tag{2.18}$$

$$\sum_{n\geq 0} o(n,2k)x^n = \frac{x^{2k}}{2}\left(\frac{1}{(1-x)^{2k+1}}-\frac{(-1)^k(1+x)}{\left(1-x^2\right)^{k+1}}\right) \tag{2.19}$$

$$\sum_{n\geq 0} o(n,2k+1)x^n = \frac{x^{2k+1}}{2}\left(\frac{1}{(1-x)^{2k+2}}+\frac{(-1)^k}{\left(1-x^2\right)^{k+1}}\right). \tag{2.20}$$

The generating functions can also be written as

$$\begin{aligned}
\sum_{n\geq 0} o(n,4k)x^n &= \frac{x^{4k}o_{4k}(x)}{(1+x)^{4k}(1-x)^{4k+1}},\\
\sum_{n\geq 0} o(n,4k+1)x^n &= \frac{x^{4k+1}e_{4k+1}(x)}{(1+x)^{4k+1}(1-x)^{4k+2}},\\
\sum_{n\geq 0} o(n,4k+2)x^n &= \frac{x^{4k+2}e_{4k+2}(x)}{(1+x)^{4k+2}(1-x)^{4k+3}},\\
\sum_{n\geq 0} o(n,4k+3)x^n &= \frac{x^{4k+3}o_{4k+3}(x)}{(1+x)^{4k+3}(1-x)^{4k+4}}.
\end{aligned} \tag{2.21}$$

Ic) The matrix $\left(f(n,k)\right)=\left(e_0,o_1,e_2,o_3,\cdots\right)$.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 1 & 0 & 0 & 0 & 0\\
1 & 2 & 2 & 2 & 1 & 0 & 0\\
1 & 3 & 4 & 4 & 3 & 1 & 0\\
1 & 3 & 6 & 10 & 9 & 3 & 0
\end{pmatrix}$$

Proposition 2.2

$$\begin{aligned}
&f(2n,k)=e(2n,k),\\
&f(2n+1,k)=f(2n,k)+f(2n,k-1)
\end{aligned} \tag{2.22}$$

Proof

$f(2n,k)=e(2n,k)$ for all k. By definition this holds for even k. For odd k it follows from Lemma 1.

To show that $f(2n+1,k)=e(2n,k)+e(2n,k-1)$ consider first an even k.

The $k-$sets which do not contain $2n+1$ are counted by $e(2n,k)$ and the rest by $o(2n,k-1)=e(2n,k-1)$.

If k is odd, then $f(2n+1,k)=o(2n+1,k)$, $o(2n,k)=e(2n,k)$ and the remaining $(k-1)-$set is even.

Id) The opposite matrix $\left(\overline{f}(n,k)\right)=\left(o_0,e_1,o_2,e_3,\cdots\right)$.

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ 0&1&2&1&0&0&0\\ 0&2&4&2&0&0&0\\ 0&2&6&6&2&0&0\\ 0&3&9&10&6&3&1 \end{pmatrix}$$

Consider the polynomials $f_n(x)=\sum_{k=0}^{n} f(n,k)x^k$ and $\overline{f}_n(x)=\sum_{k=0}^{n}\overline{f}(n,k)x^k$.

Since $f(n,k)-\overline{f}(n,k)=(-1)^k\left(e(n,k)-o(n,k)\right)$ we get

$f_n(x)-\overline{f}_n(x)=d(n,-x)$.

This gives

$$f_n(x)=\frac{(1+x)^n+(1-x)^{\left\lfloor\frac{n}{2}\right\rfloor}(1+x)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}{2}=(1+x)^{\left\lfloor\frac{n+1}{2}\right\rfloor}\sum_j\binom{\left\lfloor\frac{n}{2}\right\rfloor}{2j}x^{2j},$$
$$\overline{f}_n(x)=\frac{(1+x)^n-\prod_{j=0}^{n-1}\left(1+(-1)^j x\right)}{2}. \tag{2.23}$$

Thus

$$f(n,k)=\sum_j\binom{\left\lfloor\frac{n}{2}\right\rfloor}{2j}\binom{\left\lfloor\frac{n+1}{2}\right\rfloor}{k-2j} \tag{2.24}$$

and

$$\overline{f}(n,k)=\sum_j\binom{\left\lfloor\frac{n}{2}\right\rfloor}{2j+1}\binom{\left\lfloor\frac{n+1}{2}\right\rfloor}{k-2j-1}. \tag{2.25}$$

(2.23) also implies

$$\begin{aligned} f(n,n-k)&=f(n,k) \quad \text{for } n\equiv 0,1 \bmod 4,\\ f(n,n-k)&=\overline{f}(n,k) \quad \text{for } n\equiv 2,3 \bmod 4. \end{aligned} \tag{2.26}$$

Proposition 2.2 is equivalent with

$$\begin{aligned} f_{2n}(x)&=e_{2n}(x),\\ f_{2n+1}(x)&=(1+x)f_{2n}(x)=(1+x)e_{2n}(x). \end{aligned} \tag{2.27}$$

Let us also sketch another approach. Since the columns c_k and c_{k+2} have the same parity by Lemma 2 the entries of these matrices satisfy
$a(n,k)=a(n-2,k)+\binom{n-2}{k-1}+\binom{n-2}{k-2}-a(n-2,k-2)$ and thus

$$a(n,k)=a(n-2,k)+\binom{n-1}{k-1}-a(n-2,k-2). \tag{2.28}$$

Therefore the polynomials $a_n(x)=\sum_{k=0}^{n}a(n,k)x^k$ satisfy the recursion

$$a_n(x)=\left(1-x^2\right)a_{n-2}(x)+x(1+x)^{n-1}. \tag{2.29}$$

By applying this to n and $n-1$ we get the homogeneous recursion

$$a_n(x)=(1+x)a_{n-1}(x)+\left(1-x^2\right)a_{n-2}(x)-(1+x)\left(1-x^2\right)a_{n-3}(x). \tag{2.30}$$

Observe that

$$z^3-(1+x)z^2-\left(1-x^2\right)z+(1+x)\left(1-x^2\right)=\left(z-1-x\right)\left(z^2-1+x^2\right). \tag{2.31}$$

Therefore

$$\sum_{n\geq 0}e_n(x)z^n=\frac{1-xz-\left(1-x^2\right)z^2}{\left(1-(1+x)z\right)\left(1-\left(1-x^2\right)z^2\right)}=\frac{1}{2}\left(\frac{1}{1-(1+x)z}+\frac{1+(1-x)z}{1-\left(1-x^2\right)z^2}\right) \tag{2.32}$$

which again gives (2.7).
Analogously we get

$$\sum_{n\geq 0}f_n(x)z^n=\frac{1-\left(1+x\right)z^2}{\left(1-(1+x)z\right)\left(1-\left(1-x^2\right)z^2\right)}=\frac{1}{2}\left(\frac{1}{1-(1+x)z}+\frac{1+(1+x)z}{1-\left(1-x^2\right)z^2}\right) \tag{2.33}$$

which gives (2.23).

3. Matrices whose columns c_k and c_{k+2} have opposite parity

Let us now consider another class of triangles where the columns c_k and c_{k+2} have opposite parity.
By Lemma 2 the entries of these matrices satisfy

$$b(n,k)=b(n-2,k)+\binom{n-1}{k-1}+b(n-2,k-2). \tag{3.1}$$

Therefore the polynomials $b_n(x)=\sum_{k=0}^{n}b(n,k)x^k$ satisfy the recursion

$$b_n(x)=\left(1+x^2\right)b_{n-2}(x)+x(1+x)^{n-1}. \tag{3.2}$$

By applying this to n and $n-1$ we get

$$b_n(x) = (1+x)b_{n-1}(x) + \left(1+x^2\right)b_{n-2}(x) - (1+x)\left(1+x^2\right)b_{n-3}(x). \tag{3.3}$$

IIa) The best known special case is *Losanitsch's triangle* $\left(L(n,k)\right) = \left(e_0, o_1, o_2, e_3, e_4, o_5, \cdots\right)$
The first terms are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 & 1 & 0 & 0 \\ 1 & 3 & 6 & 6 & 3 & 1 & 0 \\ 1 & 3 & 9 & 10 & 9 & 3 & 1 \end{pmatrix}$$

By (3.1) we have

$$L(n,k) = L(n-2,k) + \binom{n-1}{k-1} + L(n-2,k-2) \tag{3.4}$$

which is often used to define this triangle.

This matrix has been obtained by the chemist S.M. Losanitsch [2] in his investigation of paraffin. Therefore we call the numbers $L(n,k)$ Losanitsch numbers. The same triangle has also been considered in [1] in the study of some sort of necklaces where these numbers have been called necklace numbers. Further information can be found in OEIS [3], A034851.

Remark

By (2.13) we have $e(n,n) = 1$ if $n \equiv 0,3 \bmod 4$ and $o(n,n) = 1$ else. Therefore Losanitsch's triangle is also characterized by the fact that all columns are e_k or o_k and all elements of the main diagonal are 1.

IIb) The opposite matrix $\left(\overline{L}(n,k)\right) = \left(o_0, e_1, e_2, o_3, o_4, e_5, \cdots\right)$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 & 2 & 0 & 0 \\ 0 & 3 & 6 & 10 & 6 & 3 & 0 \end{pmatrix}$$

This is OEIS [3], A034852 and essentially also A034877.

The polynomials $L_n(x) = \sum_{k=0}^{n} L(n,k)x^k$ and $\overline{L}_n(x) = \sum_{k=0}^{n} \overline{L}(n,k)x^k$ satisfy the recursion (3.2).
Therefore we get

$$L_n(x) - \overline{L}_n(x) = \left(1+x^2\right)\left(L_{n-2}(x) - \overline{L}_{n-2}(x)\right)$$

with initial values $L_0(x) - \overline{L}_0(x) = 1$ and $L_1(x) - \overline{L}_1(x) = 1 + x$.

Let

$$D_n(x) = L_n(x) - \overline{L}_n(x). \tag{3.5}$$

Then

$$\begin{aligned} D_{2n}(x) &= \left(1+x^2\right)^n, \\ D_{2n+1}(x) &= (1+x)\left(1+x^2\right)^n. \end{aligned} \tag{3.6}$$

Therefore we get

$$\begin{aligned} L_n(x) &= \frac{(1+x)^n + D_n(x)}{2}, \\ \overline{L}_n(x) &= \frac{(1+x)^n - D_n(x)}{2}. \end{aligned} \tag{3.7}$$

Thus we get (cf. [1], Theorem 2.8)

$$\begin{aligned} L(2n,2k+1) &= \frac{1}{2}\binom{2n}{2k+1}, \\ L(n,k) &= \frac{1}{2}\left(\binom{n}{k} + \binom{\left\lfloor \frac{n}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor}\right) \text{ else.} \end{aligned} \tag{3.8}$$

Note that

$$L_{2n+1}(x) = (1+x)L_{2n}(x). \tag{3.9}$$

Analogously as above we get

$$\sum_{n\geq 0} L_n(x)z^n = \frac{1-\left(1+x+x^2\right)z^2}{\left(1-(1+x)z\right)\left(1-\left(1+x^2\right)z^2\right)} = \frac{1}{2}\left(\frac{1}{1-(1+x)z} + \frac{1+(1+x)z}{1-\left(1+x^2\right)z^2}\right). \tag{3.10}$$

Further properties of the Losanitsch polynomials can be found in [1] and will not be repeated here. Let us only mention that by (3.6) $L_n(x)$ is palindromic since

$$L(n,k) = L(n,n-k). \tag{3.11}$$

Comparing with (2.16) and (2.21) we get

Proposition 3.1

$$\sum_n L(n,k)x^n = \sum_n L(n,n-k)x^n = \frac{x^k e_k(x)}{(1-x)^{k+1}(1+x)^k}. \tag{3.12}$$

There exists also another interesting relation between the numbers $e(n,k)$ and $L(n,k)$.

Proposition 3.2

$$\sum_n e(n,n-k)x^n = \frac{x^k}{(1-x)^{2\left\lfloor\frac{k+1}{2}\right\rfloor+1}\left(1+x^2\right)^{\left\lfloor\frac{k}{2}\right\rfloor+1}} L_{k+2}(-x). \tag{3.13}$$

Proof

It suffices to show that

$$\sum_n e(n,n-2k)x^n = \frac{x^{2k}}{(1-x)^{2k+1}\left(1+x^2\right)^{k+1}} L_{2k+2}(-x) \tag{3.14}$$

since by Proposition 1.1

$$(1-x)\sum_n e(n,n-2k-1)x^n = \sum_n e(n,n-2k)x^{n+1}$$

and by (3.9)

$$L_{2k+3}(-x) = (1-x)L_{2k+2}(-x).$$

By (2.13) we get

$$e(n,n-2k) = \frac{1}{2}\left(\binom{n}{2k} + (-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor-k}\binom{\left\lfloor\frac{n}{2}\right\rfloor}{k}\right)$$

This implies

$$\sum_n e(n,n-2k)x^n = \frac{x^{2k}}{2}\left(\frac{1}{(1-x)^{2k+1}} + \frac{1-x}{\left(1+x^2\right)^{k+1}}\right) = \frac{x^{2k}}{(1-x)^{2k+1}\left(1+x^2\right)^{k+1}}\left(\left(1+x^2\right)^{k+1} + (1-x)(1-x)^{2k+1}\right).$$

By (3.7) we get $\left(1+x^2\right)^{k+1} + (1-x)(1-x)^{2k+1} = L_{2k+2}(-x)$.

IIc) The matrix $\left(M(n,k)\right) = \left(e_0, e_1, o_2, o_3, e_4, e_5, \cdots\right)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 2 & 1 & 0 & 0 \\ 1 & 2 & 6 & 4 & 3 & 0 & 0 \\ 1 & 3 & 9 & 10 & 9 & 3 & 1 \end{pmatrix}$$

We have

$$M(n,k) = L(n+1,k) - L(n,k-1). \tag{3.15}$$

For

$M(n,4k) = e(n,4k) = e(n+1,4k) - e(n,4k-1),$

$M(n,4k+2) = o(n,4k+2) = o(n+1,4k+2) - o(n,4k),$

$M(n,4k+1) = e(n,4k+1) = o(n+1,4k+1) - e(n,4k),$

$M(n,4k+3) = o(n,4k+3) = e(n+1,4k+3) - o(n,4k+2),$

Observing (3.15) we get

Corollary 3.1

$$\sum_n M(n,n-k)x^n = \frac{x^k}{(1-x^2)^{k+1}} e_{k+1}(x). \tag{3.16}$$

Remark

$$\begin{aligned} M(n,n-k) &= M(n,k) \quad \text{for } n \equiv 0,2 \bmod 4, \\ M(n,n-k) &= \bar{M}(n,k) \quad \text{for } n \equiv 1,3 \bmod 4. \end{aligned} \tag{3.17}$$

IId) The opposite matrix $\left(\bar{M}(n,k)\right) = \left(o_0, o_1, e_2, e_3, o_4, o_5, \cdots\right)$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 3 & 4 & 6 & 2 & 1 & 0 \\ 0 & 3 & 6 & 10 & 6 & 3 & 0 \end{pmatrix}$$

Then $\bar{M}(n,k) = \bar{L}(n+1,k) - \bar{L}(n,k-1)$.

Let $M_n(x) = \sum_{k=0}^{n} M(n,k)x^k$ and $\bar{M}_n(x) = \sum_{k=0}^{n} \bar{M}(n,k)x^k$.
Since $M(n,k) - \bar{M}(n,k) = (-1)^k \left(L(n,k) - \bar{L}(n,k)\right)$ we get $M_n(x) - \bar{M}_n(x) = D_n(-x)$.
Thus

$$\begin{aligned} M_n(x) &= \frac{(1+x)^n + D_n(-x)}{2}, \\ \bar{M}_n(x) &= \frac{(1+x)^n - D_n(-x)}{2}. \end{aligned} \tag{3.18}$$

Finally let us compute the generating function of $\sum_n f(n,n-k)x^n$.

Proposition 3.3

$$\sum_n f(n,n-2k+1)x^n = \frac{x^{2k-1}}{(1-x)^{2k}\left(1+x^2\right)^{k+1}} \sum_{j=0}^{2k} (-1)^j L(2k,j)x^j \tag{3.19}$$

and

$$\sum_n f(n,n-2k)x^n = \frac{x^{2k}}{(1-x)^{2k+1}\left(1+x^2\right)^{k+1}} \frac{\left(1+x^2\right)^{k+1} + (1-x)^{2k+1}(1+x)}{2}. \tag{3.20}$$

Proof
(3.19) follows from $f(n,n-2k+1)=e(n,n-2k+1).$

Since by (2.27) and (2.13)

$$f(n,n-2k)=\frac{1}{2}\left(\binom{n}{2k}+(-1)^{k+\left\lfloor\frac{n}{2}\right\rfloor}\binom{\left\lfloor\frac{n}{2}\right\rfloor}{k}\right)$$

we get

$$\sum_n f(n,n-2k)x^n=\frac{x^{2k}}{2}\left(\frac{1}{(1-x)^{2k+1}}+\frac{(1+x)}{\left(1+x^2\right)^{k+1}}\right)$$

or (3.20).

Final Remarks

There are analogous results for odd primes p.

Let $a(n,k,j)$ be the number of $k-$subsets of $\{1,2,\cdots,n\}$ whose sums are congruent to j modulo p and let ζ be a primitive $p-$th root of unity.

Then

$$\prod_{j=1}^{n}\left(1+\zeta^j x\right)=\sum_k x^k\sum_{j_1<j_2<\cdots<j_k}\zeta^{j_1+\cdots+j_k}=\sum_i x^k\sum_{j=0}^{p-1}a(n,k,j)\zeta^j.$$

Observe that

$$\sum_{\ell=0}^{p-1}\prod_{j=1}^{n}\left(1+\zeta^{\ell j}x\right)=\sum_k x^k\sum_{\ell=0}^{p-1}\sum_{j=0}^{p-1}a(n,k,j)\zeta^{\ell j}=p\sum_k a(n,k,0)x^k.$$

On the other hand we have

$$\sum_{\ell=0}^{p-1}\prod_{j=1}^{n}\left(1+\zeta^{\ell j}x\right)=(1+x)^n+\sum_{\ell=1}^{p-1}\prod_{j=1}^{n}\left(1+\zeta^{\ell j}x\right).$$

Since each product of $1+\zeta^{\ell j}$ over p consecutive values of j equals $1+x^p$ we see that

$$\sum_{\ell=1}^{p-1}\prod_{j=1}^{pn+i}\left(1+\zeta^{\ell j}x\right)=b_i(x)\left(1+x^p\right)^n$$

for some polynomial $b_i(x)$ of degree i.

Therefore the polynomial $a_n(x)=\sum_k a(n,k,0)x^k$ satisfies

$$a_{pn+i}(x)=\frac{(1+x)^{pn+i}+b_i(x)\left(1+x^p\right)^n}{p}. \tag{3.21}$$

Let us only consider the case $p=3$ in more detail.

The first terms of the matrix $\left(a(n,k,0)\right)$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 4 & 4 & 1 & 1 & 0 \\ 1 & 2 & 5 & 8 & 5 & 2 & 1 \end{pmatrix}$$

Here we get

$$a_{3n+i}(x)=\frac{(1+x)^{3n+i}+b_i(x)\left(1+x^3\right)^n}{3} \tag{3.22}$$

with $b_0(x)=2,\ b_1(x)=2-x,\ b_2(x)=2\left(1-x+x^2\right).$

For example

$$a_0(x)=1=\frac{1+2}{3},\quad a_1(x)=1=\frac{(1+x)+(2-x)}{3},\ a_2(x)=1+x^2=\frac{(1+x)^2+2\left(1-x+x^2\right)}{3},$$

$$a_3(x)=1+x+x^2+x^3=\frac{(1+x)^3+2\left(1+x^3\right)}{3},\cdots.$$

For the generating function we get therefore

$$\sum_{n\geq 0}a_n(x)z^n=\frac{1-xz-(1-x)xz^2-\left(1+x^3\right)z^3}{\left(1-(1+x)z\right)\left(1-\left(1+x^3\right)z^3\right)}=\frac{1}{3}\left(\frac{1}{1-(x+1)z}+\frac{2+(2-x)z+2\left(1-x+x^2\right)z^2}{1-\left(1+x^3\right)z^3}\right).$$

In this case we also get

$$a_n(x)=\left(1+x^3\right)a_{n-3}(x)+x(1+x)^{n-2} \tag{3.23}$$

or equivalently

$$a(n,k)=a(n-3,k)+\binom{n-2}{k-1}+a(n-3,k-3). \tag{3.24}$$

To prove this consider the elements $n-2, n-1, n.$
The number of $k-$ sets which contain none of these numbers is $a(n-3,k),$ the number of those which contain precisely one of these numbers is $\binom{n-3}{k-1},$ the number of those which contain precisely two of these numbers is $\binom{n-3}{k-2},$ because $n-2+n-1, n-2+n, n-1+n$ are different modulo 3 and the number of those which contain all of them is $a(n-3,k-3)$ because $n-2+n-1+n=3n-3$ is a multiple of $3.$

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