A Short Proof of Kuratowski's Graph Planarity Criterion

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ABSTRACT

We present a new short combinatorial proof of the sufficiency part of the well-known Kuratowski's graph planarity criterion. The main steps are to prove that for a minor minimal non-planar graph G and any edge xy:
(1) G-x-y does not contain θ-subgraph;
(2) G-x-y is homeomorphic to the circle;
(3) G is either K_5 or $K_{\{3,3\}}$.

In 1930, K. Kuratowski published his well-known graph planarity criterion [1]: a graph is planar if and only if it does not contain a subgraph, homeomorphic to either K_5 or $K_{\{3,3\}}$. Since then, many new and shorter proofs of this criterion appeared [2]. In this paper we present a short combinatorial proof of the ‘‘if’’part. It is based on contracting edge, similar to that of [2, section 5], but we avoid the reduction to 3-connected graphs. By θ-subgraph we mean a subgraph homeomorphic to $K_{\{3,2\}}$.

Consider a minor minimal non-planar graph G.

Lemma 1. If $xy \in E(G)$, then G-x-y does not contain a θ-subgraph.

Proof. Suppose not. Consider an embedding of G/xy in the plane. Let $G' = G$-x-$y = (G/xy)$-(xy). Let F be the subgraph of G' bounding the face of G' containing the deleted vertex xy of G/xy. Then F cannot contain a θ-subgraph [2, section 1]. But since G' does, there is an edge e in $E(G') - E(F)$. Since for each forest $T \subseteq R^2$, $R^2 - T$ is connected, F contains a cycle

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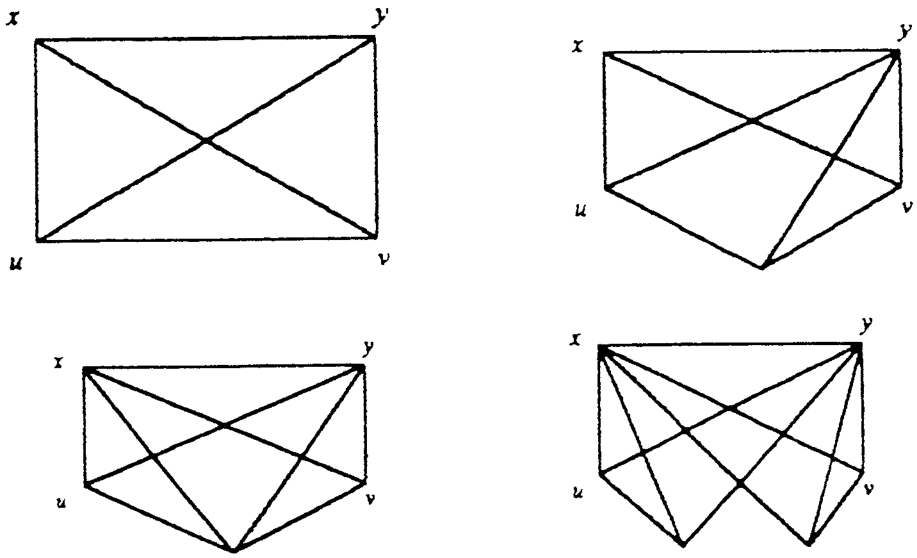

FIGURE 1.

C about which we can assume that its exterior contains e and that its interior contains the deleted vertex xy. It is clear that no pair of vertices on C is connected by a path in G'-$E(C) - E$(ext C). This means that in an embedding of G-ext C, which exists by the minimality of G, C may be assumed to be the outer boundary. This embedding can then be combined with the restriction of that of G/xy to G', which contradicts the non-planarity of G. ∎

Lemma 2. If $xy \in E(G)$, then G-x-y does not have two vertices of degree one.

Proof. If u, v are such vertices, then by minimality of G, they are both of degree more than 2 in G and hence adjacent to x and y. By Lemma 1, there is no edge disjoint from x, y, u, v in G since these vertices contain a θ-subgraph. But each vertex in G-x-y-u-v is of degree more than two and hence joined to at least three among u, v, x, y. Since u and v are of degree three in G, in G there are at most two vertices besides the x, y, u, v and hence G is one of the graphs in Figure 1. The cases are determined by whether, in G-x-y, u and v are adjacent, have a common neighbor or have distinct neighbors. All of them are planar. ∎

Lemma 3. If $xy \in E(G)$, then G-x-y is a cycle.

Proof. Let $G' = G$-x-y. Then every block of G' is either a cycle or just an edge (by Lemma 1). If G' is not a cycle, it has at least two end blocks (as it cannot be an edge). By Lemma 2, one of them is a cycle; denote it by C. There is a unique cut vertex v of G' contained in C. All vertices of C-v are adjacent to x or y (since their degree is more than two).

Since there are not less than two such vertices, we have a θ-subgraph. Hence no edge is disjoint from it by Lemma 1. Also there are no isolated vertices in G' (since they are most of degree two in G, which contradicts the minimality of G). Therefore all other blocks of G are just edges at v. By Lemma 2, there is just one. Since G- (the endpoints of this edge) does not contain a θ-subgraph, G is the 3-prism, which is planar. ∎

Proof of the Criterion. Let x_1, x_2 be two adjacent vertices of a minor minimal non-planar graph G. If a point $u \epsilon G = G - x_1 - x_2$ is connected to x_i but not connected to $x_{(3-i)}$, then the point v, next to u along G', is not connected to x_i (for otherwise, G-(vx_i) is planar by the minimality of G and we can add vx_i to a planar embedding of G-vx_i to get a planar embedding of G. Therefore either every point of G' is connected to both x_1 and x_2 or the points of G', connected to x_1 and x_2 alternate along G'. In the first case G contains a subdivision of K_5, in the second, it contains a subdivision of $K_{\{3,3\}}$. ∎

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